International Journal of Modern Physics B

Comments on the Deformed W_N Algebra*

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DPSU-01-2
math.QA/0111230

We obtain an explicit expression for the defining relation of the deformed W_N algebra, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$. Using this expression we can show that, in the $q \to 1$ limit, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = e^{-\frac{2\pi i}{N}} q^{\frac{k+N}{N}}$ reduces to the $\mathfrak{sl}_N$-version of the Lepowsky-Wilson's $\mathcal{Z}$-algebra of level k, $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. In other words $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = e^{-\frac{2\pi i}{N}} q^{\frac{k+N}{N}}$ can be considered as a q-deformation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$.

In the appendix given by H. Awata, S. Odake and J. Shiraishi, we present an interesting relation between $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization.

1. Introduction

One of our motivation for study of elliptic algebras (deformed Virasoro and W algebras, elliptic quantum groups, etc.) is to clarify the symmetry of massive integrable models. Massive integrable models includes quantum field theories with mass scale and solvable statistical lattice models. Typical examples of the latter are models based on $\mathfrak{sl}_2$: Andrews-Baxter-Forester (ABF) model and Baxter's eight vertex model. About these models we know the following:[1]

model	ABF(III)	8 vertex
Boltzmann weight	face type	vertex type
algebra	$\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ $(\mathcal{B} \otimes \{P, e^Q\} = U_{q,p}(\widehat{\mathfrak{sl}}_2))$	$\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$
gradation (energy level of H_C)	homogeneous gradation	principal gradation
space of states	irr. rep. space of DVA	irr. rep. space of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$
free field realization	direct (construction of VO)	indirect (map to ABF)

*Talk at the APCTP-Nankai Joint Symposium on "Lattice Statistics and Mathematical Physics", 8-10 October 2001, Tianjin China.

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In order to obtain more direct free field realization of the eight vertex model and its higher rank generalization, it may be useful to study (deformed) current algebras of $\mathfrak{sl}_N$ in principal gradation. Motivated by this, Hara *et.al* [2] studied free field realization of the Lepowsky-Wilson's $\mathcal{Z}$-algebra[3] and found some relation between the deformed Virasoro algebra (DVA) and $\mathcal{Z}$-algebra. Recently Shiraishi constructed a direct free field realization of the eight vertex model with a specific parameter $p = q^3$, where the type II vertex operator is given by the DVA current.[4]

In this article we extend the relation between DVA and $\mathcal{Z}$-algebra to the higher rank case. In section 2 we present an explicit expression for the defining relation of the deformed W_N algebra. This is a main result of this article. In section 3, by using this explicit expression, we show that the deformed W_N algebra reduces to the $\mathfrak{sl}_N$-version of the Lepowsky-Wilson's $\mathcal{Z}$-algebra in some limit. In the appendix given by Awata, Odake and Shiraishi, we present an interesting relation between the deformed W_N algebra and ζ-function regularization.[5]

2. Deformed W_N Algebra

2.1. *Definition*

Let us recall the definition of the deformed W_N algebra, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$.[6,7] It is defined through a free field realization. This algebra has two parameters(q and t), and we set $t = q^\beta$ and $p = qt^{-1}$. Let us introduce fundamental bosons h^i_n ($n \in \mathbb{Z}$; $i = 1, \cdots, N$; $\sum_{i=1}^N p^{in} h^i_n = 0$) which satisfy

$$[h^i_n, h^j_m] = -\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{(N\delta_{i,j}-1)n}}{1-p^{Nn}} p^{Nn\theta(i<j)}\delta_{n+m,0}\,, \tag{1}$$

where $\theta(P) = 1$ or 0 if the proposition P is true or false, respectively. Exponentiated boson $\Lambda_i(z)$ ($i = 1, \cdots, N$) is defined by

$$\Lambda_i(z) = :\exp\Bigl(\sum_{n\neq 0} h^i_n z^{-n}\Bigr): q^{\sqrt{\beta}h^i_0} p^{\frac{N+1}{2}-i}. \tag{2}$$

Here $:*:$ stands for the usual normal ordering for bosons, *i.e.*, h^i_n with $n \geq 0$ are in the right. By using this $\Lambda_i(z)$, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ current $W^i(z) = \sum_{n\in\mathbb{Z}} W^i_n z^{-n}$ ($i = 1, \cdots, N-1$) is given by

$$W^i(z) = \sum_{1\leq j_1<j_2<\cdots<j_i\leq N} :\Lambda_{j_1}(p^{\frac{i-1}{2}}z)\Lambda_{j_2}(p^{\frac{i-3}{2}}z)\cdots\Lambda_{j_i}(p^{-\frac{i-1}{2}}z):\,, \tag{3}$$

and we set $W^0(z) = W^N(z) = 1$. (Remark that $\Lambda_i(z)$ corresponds to the weight of vector representation of $\mathfrak{sl}_N$ and $W^i(z)$ corresponds to the i-th rank antisymmetric tensor representation.) $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ is defined as an associative algebra over $\mathbb{C}$ generated by W^i_n.[a]

[a] It is also defined as a commutant of the screening currents.[6,8]

The highest weight state $|\lambda\rangle$ is characterized by $W^i_n|\lambda\rangle = 0$ $(n>0)$ and $W^i_0|\lambda\rangle = w^i(\lambda)|\lambda\rangle$ $(w^i(\lambda)\in\mathbb{C})$, and the highest weight representation space is obtained by successive action of W^i_{-n} $(n>0)$.

Since $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ has two parameters(q and t), we can take its various limit by relating q and t. In the following limit[b]

$$\text{Limit I}: \begin{cases} q=e^{\hbar}\,, & \hbar\to 0 \\ t=q^{\beta}\,, & \beta:\text{fixed} \quad (\alpha_0=\sqrt{\beta}-\frac{1}{\sqrt{\beta}})\,, \end{cases} \tag{4}$$

$\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ reduces to the W_N algebra with the Virasoro central charge $c=(N-1)(1-N(N+1)\alpha_0^2)$ because the q-Miura transformation of $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)$ becomes the Miura transformation of W_N algebra. Each DWA current $W^i(z)$, however, reduces to some linear combination of W_N currents.

2.2. *Relation*

In order to write down relations between DWA currents, we define the delta function $\delta(z)=\sum_{n\in\mathbb{Z}}z^n$ and the structure function $f^{i,j}(z)=\sum_{\ell\geq 0}f^{i,j}_\ell z^\ell$ $(1\leq i,j\leq N-1)$,

$$f^{i,j}(z)=\exp\Biggl(\sum_{n>0}\frac{1}{n}(1-q^n)(1-t^{-n})\frac{1-p^{\min(i,j)n}}{1-p^n}\frac{1-p^{(N-\max(i,j))n}}{1-p^{Nn}}p^{\frac{|i-j|}{2}n}z^n\Biggr). \tag{5}$$

It has been expected that DWA currents satisfy quadratic relations, $f^{i,j}(\frac{z_2}{z_1})W^i(z_1)$ $W^j(z_2)$ $-W^j(z_2)W^i(z_1)f^{j,i}(\frac{z_1}{z_2})$ = (terms containing delta function), in mode expansion it becomes

$$\begin{aligned}[W^i_n,W^j_m] &= -\sum_{\ell\geq 1}f^{i,j}_\ell\bigl(W^i_{n-\ell}W^j_{m+\ell}-W^j_{m-\ell}W^i_{n+\ell}\bigr) \\ &\qquad +(\text{contribution from the terms containing delta function}).\end{aligned} \tag{6}$$

For $i=1$ and $j\geq 1$ case, the relation is[6,7]

$$\begin{aligned}&f^{1,j}(\tfrac{z_2}{z_1})W^1(z_1)W^j(z_2)-W^j(z_2)W^1(z_1)f^{j,1}(\tfrac{z_1}{z_2}) \qquad (j\geq 1)\\ &=-\frac{(1-q)(1-t^{-1})}{1-p}\Bigl(\delta(p^{\frac{j+1}{2}}\tfrac{z_2}{z_1})W^{j+1}(p^{\frac{1}{2}}z_2)-\delta(p^{-\frac{j+1}{2}}\tfrac{z_2}{z_1})W^{j+1}(p^{-\frac{1}{2}}z_2)\Bigr),\end{aligned} \tag{7}$$

and for $i=2$ and $j\geq 2$ case, the relation is[7]

$$\begin{aligned}&f^{2,j}(\tfrac{z_2}{z_1})W^2(z_1)W^j(z_2)-W^j(z_2)W^2(z_1)f^{j,2}(\tfrac{z_1}{z_2}) \qquad (j\geq 2)\\ &=-\frac{(1-q)(1-t^{-1})}{1-p}\frac{(1-qp)(1-t^{-1}p)}{(1-p)(1-p^2)}\\ &\qquad\times\Bigl(\delta(p^{\frac{j}{2}+1}\tfrac{z_2}{z_1})W^{j+2}(pz_2)-\delta(p^{-\frac{j}{2}-1}\tfrac{z_2}{z_1})W^{j+2}(p^{-1}z_2)\Bigr)\\ &\quad-\frac{(1-q)(1-t^{-1})}{1-p}\Bigl(\delta(p^{\frac{j}{2}}\tfrac{z_2}{z_1}){}^{\circ}_{\circ}W^1(p^{-\frac{1}{2}}z_1)W^{j+1}(p^{\frac{1}{2}}z_2){}^{\circ}_{\circ}\end{aligned}$$

[b] Usually we call this limit as a conformal limit. However there are many other limits in which the resultant algebras are related to conformal field theory.

$$
\begin{aligned}
&\qquad -\delta(p^{-\frac{j}{2}}\tfrac{z_2}{z_1})\,{}^{\circ}_{\circ}W^1(p^{\frac{1}{2}}z_1)W^{j+1}(p^{-\frac{1}{2}}z_2){}^{\circ}_{\circ}\Big)\\
&+\frac{(1-q)^2(1-t^{-1})^2}{(1-p)^2}\Big(\delta(p^{\frac{j}{2}}\tfrac{z_2}{z_1})\Big(\frac{p^2}{1-p^2}W^{j+2}(pz_2)+\frac{1}{1-p^j}W^{j+2}(z_2)\Big)\\
&\qquad -\delta(p^{-\frac{j}{2}}\tfrac{z_2}{z_1})\Big(\frac{p^j}{1-p^j}W^{j+2}(z_2)+\frac{1}{1-p^2}W^{j+2}(p^{-1}z_2)\Big)\Big).
\end{aligned}\tag{8}
$$

Here a normal ordering for currents ${}^{\circ}_{\circ}*{}^{\circ}_{\circ}$ is defined by

$$
\begin{aligned}
&{}^{\circ}_{\circ}W^i(rz)W^j(z){}^{\circ}_{\circ}\\
&=\oint\frac{dz'}{2\pi i z'}\Big(\frac{1}{1-\frac{rz}{z'}}f^{i,j}(\tfrac{z}{z'})W^i(z')W^j(z)+\frac{\frac{z'}{rz}}{1-\frac{z'}{rz}}W^j(z)W^i(z')f^{j,i}(\tfrac{z'}{z})\Big)\\
&=\sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}f^{i,j}_{\ell}\Big(r^{m-\ell}W^i_{-m}W^j_{n+m}+r^{\ell-m-1}W^j_{n-m-1}W^i_{m+1}\Big)\cdot z^{-n},
\end{aligned}\tag{9}
$$

where $\frac{1}{1-z}$ stands for $\sum_{n\geq 0}z^n$. Due to this normal ordering, infinite sums in the RHS of (6) become finite sums on the highest weight representation space.

Eqs.(7) and (8) are directly calculated by using the commutation relation of h^i_n. In principle, we can continue this calculation for $i\geq 3$ cases, but in practice it is hopeless. So we use another method: fusion and induction. To write down general formula, we extend the range$(0\leq i\leq N)$ of $W^i(z)$ and that$(1\leq i,j\leq N-1)$ of $f^{i,j}(z)$ to $i\in\mathbb{Z}$ and $i,j\in\mathbb{Z}$ respectively ; $W^i(z)=0$ for $i<0$ or $i>N$, and $f^{i,j}(z)$ is given by (5) for all $i,j\in\mathbb{Z}$.

Explicit expression of the defining relation of $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ is as follows:

$$
\begin{aligned}
&f^{i,j}(\tfrac{z_2}{z_1})W^i(z_1)W^j(z_2)-W^j(z_2)W^i(z_1)f^{j,i}(\tfrac{z_1}{z_2})\qquad(0\leq i\leq j\leq N)\\
&=-\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{i}\prod_{l=1}^{k-1}\gamma(p^{l+\frac{1}{2}})\\
&\quad\times\Big(\delta(p^{\frac{j-i}{2}+k}\tfrac{z_2}{z_1})f^{i-k,j+k}(p^{-\frac{j-i}{2}})W^{i-k}(p^{-\frac{k}{2}}z_1)W^{j+k}(p^{\frac{k}{2}}z_2)\\
&\qquad-\delta(p^{-(\frac{j-i}{2}+k)}\tfrac{z_2}{z_1})f^{i-k,j+k}(p^{\frac{j-i}{2}})W^{i-k}(p^{\frac{k}{2}}z_1)W^{j+k}(p^{-\frac{k}{2}}z_2)\Big),
\end{aligned}\tag{10}
$$

where $\gamma(p^{\frac{1}{2}}z)=\dfrac{(1-qz)(1-t^{-1}z)}{(1-z)(1-pz)}$. We can rewrite the RHS of this relation in terms of the normal ordering ${}^{\circ}_{\circ}*{}^{\circ}_{\circ}$ by repeated use of the following formula, which is obtained from (9) and (10),

$$
\begin{aligned}
&f^{i,j}(r^{-1})W^i(rz)W^j(z)\qquad(0\leq i\leq j\leq N)\\
&={}^{\circ}_{\circ}W^i(rz)W^j(z){}^{\circ}_{\circ}+\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{i}\prod_{l=1}^{k-1}\gamma(p^{l+\frac{1}{2}})\\
&\quad\times\Big(\frac{1}{1-rp^{-(\frac{j-i}{2}+k)}}f^{i-k,j+k}(p^{-\frac{j-i}{2}})W^{i-k}(p^{\frac{j-i+k}{2}}z)W^{j+k}(p^{\frac{k}{2}}z)\\
&\qquad-\frac{1}{1-rp^{\frac{j-i}{2}+k}}f^{i-k,j+k}(p^{\frac{j-i}{2}})W^{i-k}(p^{-\frac{j-i+k}{2}}z)W^{j+k}(p^{-\frac{k}{2}}z)\Big),
\end{aligned}\tag{11}
$$

where $r \in \mathbb{C}$ is a "good" number (such that it does not give poles, see (15)). For example, (8) is easily recovered by (10) with $i=2$ and (11) with $i=1$.

In order to prove (10) we present some formulas. Direct calculation shows

$$f^{1,j}(p^{\pm\frac{i+1}{2}}z)f^{i,j}(z) = f^{i+1,j}(p^{\pm\frac{1}{2}}z) \times \begin{cases} 1 & i<j \\ \gamma(p^{\pm\frac{i-j+1}{2}}z) & i \geq j \end{cases} \quad (j \geq 1), \tag{12}$$

$$f^{1,i}(p^{\pm(\frac{j-i}{2}+k)}z)f^{1,j}(z) = f^{1,i-k}(p^{\pm\frac{j-i+k}{2}}z)f^{1,j+k}(p^{\pm\frac{k}{2}}z) \qquad (i,j,i-k,j+k \geq 1), \tag{13}$$

$$f^{1,i}(p^{\pm\frac{j+i}{2}}z)f^{1,j}(z) = f^{1,j+i}(p^{\pm\frac{i}{2}}z)\gamma(p^{\pm\frac{j}{2}}z) \quad (i,j \geq 1), \tag{14}$$

and $f^{a,b}$ in the RHS of (10) is regular. By computing $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ in the free field realization, we can show that (10) implies

$$\begin{aligned} &\text{Poles of } f^{i,j}(\tfrac{z_2}{z_1})W^i(z_1)W^j(z_2) \ (0 \leq i \leq j \leq N) \text{ are} \\ &\tfrac{z_2}{z_1} = p^{\pm(\frac{j-i}{2}+k)} \ (1 \leq k \leq \min(i, N-j)), \end{aligned} \tag{15}$$

because $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ is a Taylor series in $\frac{z_2}{z_1}$, and for any states of the highest weight representation space, $|\psi\rangle$ and $|\phi\rangle$, $\langle\psi|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\phi\rangle$ differs from $\langle\lambda|f^{i,j}(\frac{z_2}{z_1})W^i(z_1)W^j(z_2)|\lambda\rangle$ only for finite number of terms(Laurant polynomials in z_1 and z_2), which do not create other poles. (See also Appendix C of ref.[9] where different notation is used.) Therefore $f^{a,b}W^aW^b$ in the RHS of (10) is regular and we can reverse its order, $f^{a,b}(p^c)W^aW^b = f^{b,a}(p^{-c})W^bW^a$. From (7) (or by using the free field realization), we have the following fusion relation

$$\begin{aligned} &\lim_{z_1 \to p^{\pm\frac{j+1}{2}}z_2} (1-p^{\pm\frac{j+1}{2}}\tfrac{z_2}{z_1})f^{1,j}(\tfrac{z_2}{z_1})W^1(z_1)W^j(z_2) \\ &= \mp\frac{(1-q)(1-t^{-1})}{1-p}W^{j+1}(p^{\pm\frac{1}{2}}z_2) \quad (1 \leq j \leq N), \end{aligned} \tag{16}$$

and if (10) is correct, (10) implies

$$\begin{aligned} &\lim_{z_2 \to p^{\mp\frac{j+i}{2}}z_1} (1-p^{\mp\frac{j+i}{2}}\tfrac{z_1}{z_2})f^{j,i}(\tfrac{z_1}{z_2})W^j(z_2)W^i(z_1) \\ &= \pm\frac{(1-q)(1-t^{-1})}{1-p}\prod_{l=1}^{i-1}\gamma(p^{l+\frac{1}{2}})\cdot W^{j+i}(p^{\mp\frac{i}{2}}z_1) \quad (0 \leq i \leq j \leq N). \end{aligned} \tag{17}$$

Proof of (10) : (i) The case $i=0$ and $i \leq \forall j \leq N$, and the case $j=N$ and $0 \leq \forall i \leq j$ are trivial. (ii) The case $i=1$ and $i \leq \forall j \leq N$, i.e. (7), is already proved. (iii) Let us assume (10) holds for $i(<N)$ and $i \leq \forall j \leq N$. We will show (10) holds for $i+1$ and $i+1 \leq \forall j \leq N$. (For $i=N-1$, we have $i+1=N \leq j=N$. Therefore it is sufficient to consider $i<N-1$ and $j<N$.) Multiply $f^{1,i}(\frac{z_1}{z_3})f^{1,j}(\frac{z_2}{z_3})W^1(z_3)$ from left to (10) with $i \geq 1$ (whose second $f^{a,b}(p^c)W^aW^b$ term in the RHS is replaced by reversed order one $f^{b,a}(p^{-c})W^bW^a$), rewrite $f^{1,j}(\frac{z_2}{z_3})W^1(z_3)W^j(z_2) = f^{j,1}(\frac{z_3}{z_2})W^j(z_2)W^1(z_3)+\cdots$ by using (7), multiply $\frac{1-p}{(1-q)(1-t^{-1})}(1-p^{-\frac{i+1}{2}}\frac{z_2}{z_1})$, and take a limit $z_3 \to p^{-\frac{i+1}{2}}z_1$. By using (12)–(16) and

(17) (with $j \to j+1$), studying poles carefully and replacing $z_1 \to p^{\frac{1}{2}} z_1$, we obtain (10) with $i \to i+1$ $(2 \le i+1 \le j < N)$. (iv) Therefore we have proved (10) by induction on i. $\square$

3. Relation to $\mathscr{Z}$-Algebra

Affine Lie algebra $\widehat{\mathfrak{sl}}_N$ is an associative algebra over $\mathbb{C}$ with the Chevally generators, $e_i^{\pm}$ and h_i $(i = 0, 1, \cdots, N-1)$, which satisfy

$$[h_i, h_j] = 0, \quad [h_i, e_j^{\pm}] = \pm a_{ij} e_j^{\pm}, \quad [e_i^+, e_j^-] = \delta_{ij} h_i, \tag{18}$$

and the Serre relation $(\mathrm{ad}\, e_i^{\pm})^{1-a_{ij}} e_j^{\pm} = 0$ $(i \ne j)$, where $(a_{ij})_{0 \le i,j \le N-1}$ is the Cartan matrix of $A_{N-1}^{(1)}$ Dynkin diagram. [10] This algebra admits various gradations and we denote its grading operator as d and ρ for the homogeneous and principal gradation respectively, which satisfy

$$\begin{array}{ll} \text{homogeneous gradation} & : [d, e_i^{\pm}] = \pm e_i^{\pm} \delta_{i0}, \\ \text{principal gradation} & : [\rho, e_i^{\pm}] = \pm e_i^{\pm}. \end{array} \tag{19}$$

In current basis $\widehat{\mathfrak{sl}}_N$ is given as follows:

homogeneous gradation

generators : H_n^i, $E_n^{\pm,i}$ $(n \in \mathbb{Z}, 1 \le i \le N-1)$, k : center, d : grading operator.

relations :

$$\begin{aligned} &[H_n^i, H_m^j] = k \bar{a}_{ij} n \delta_{n+m,0}, \quad [H_n^i, E_m^{\pm,j}] = \pm \bar{a}_{ij} E_{n+m}^{\pm,j}, \\ &[E_n^{+,i}, E_m^{-,j}] = \delta^{ij} (H_{n+m}^i + kn\delta_{n+m,0}), \quad [d, X_n] = nX_n \ \ (X = H^i, E^{\pm,i}), \end{aligned} \tag{20}$$

and $[E_n^{\pm,i}, E_m^{\pm,j}] = [E_{n-1}^{\pm,i}, E_{m+1}^{\pm,j}]$ and the Serre relations which we omit to write explicitly, where $(\bar{a}_{ij})_{1 \le i,j \le N-1}$ is the Cartan matrix of A_{N-1} Dynkin diagram.

principal gradation Let us set $\omega = e^{\frac{2\pi i}{N}}$. Symbol $\equiv$ stands for $\equiv$ (mod N).

generators : β_n $(n \in \mathbb{Z}, n \not\equiv 0)$, $x_n^{(\mu)}$ $(n \in \mathbb{Z}, 1 \le \mu \le N-1$, μ is understood as mod $N)$, k : center, ρ : grading operator.

relations :

$$\begin{aligned} &[\beta_n, \beta_m] = kn\delta_{n+m,0} \quad (n, m \not\equiv 0), \quad [\beta_n, x_m^{(\nu)}] = (1 - \omega^{-\nu n}) x_{n+m}^{(\nu)}, \\ &[x_n^{(\mu)}, x_m^{(\nu)}] = \begin{cases} (\omega^{-\mu m} - \omega^{-\nu n}) x_{n+m}^{(\mu+\nu)} & (\mu + \nu \not\equiv 0) \\ (\omega^{-\mu m} - \omega^{-\nu n}) \beta_{n+m} + kn\omega^{\mu n} \delta_{n+m,0} & (\mu + \nu \equiv 0), \end{cases} \\ &[\rho, X_n] = X_n \ \ (X = \beta, x^{(\mu)}), \end{aligned} \tag{21}$$

and the Serre relations.

Since these two current basis are basis of the same Lie algebra $\widehat{\mathfrak{sl}}_N$, they are related by linear transformation,

$$\beta_{Nm+\nu} = \sum_{i=1}^{N-\nu} E_m^{i,i+\nu} + \sum_{i=N-\nu+1}^{N} E_{m+1}^{i,i+\nu-N},$$

$$x^{(\mu)}_{Nm+\nu} = \sum_{i=1}^{N-\nu} \omega^{\mu(i+\nu-1)} E^{i,i+\nu}_m + \sum_{i=N-\nu+1}^{N} \omega^{\mu(i+\nu-1)} E^{i,i+\nu-N}_{m+1}, \qquad (22)$$
$$x^{(\mu)}_{Nm} = \sum_{i=1}^{N-1} \frac{1-\omega^{\mu i}}{1-\omega^{\mu}} H^i_m - \frac{k}{1-\omega^{\mu}}\delta_{m,0},$$

where $m \in \mathbb{Z}$ and $1 \le \mu, \nu \le N-1$. Here, for simplicity of the presentation, we have introduced $\widehat{\mathfrak{gl}}_N$ generators $E^{i,j}_n$ ($n \in \mathbb{Z}$, $1 \le i,j \le N$), which satisfy $[E^{i,j}_n, E^{i',j'}_m] = \delta^{ji'} E^{i,j'}_{n+m} - \delta^{ij'} E^{i',j}_{n+m} + \delta^{ij'}\delta^{ji'} kn\delta_{n+m,0}$, and the generators in the homogeneous picture are expressed as $E^{+,i}_n = E^{i,i+1}_n$, $E^{-,i}_n = E^{i+1,i}_n$ and $H^i_n = E^{i,i}_n - E^{i+1,i+1}_n$. We remark $E^{i,j}_n = [E^{i,l}_m, E^{l,j}_{n-m}]$ ($i \ne j$) and this RHS is independent on l and m.

Next let us consider the splitting of the Cartan part:

$$(\ \widehat{\mathfrak{sl}}_N \text{ generator}) = (\text{exponential of Cartan generators}) \times (\text{new generator}), \qquad (23)$$

where new generator commutes with Cartan generators. For homogeneous gradation, the algebra generated by these new generators is known as the ($\mathfrak{sl}_N$ version of) parafermion algebra of level k. For principal gradation, we name it as ($\mathfrak{sl}_N$-version of) $\mathcal{Z}$ algebra of level k, $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. ($N = 2$ case was studied by Lepowsky and Wilson.[3]) Explicitly the generators of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$, z^μ_n ($n \in \mathbb{Z}$, $1 \le \mu \le N-1$, μ is understood as mod N), are obtained by

$$x^{(\mu)}(\zeta) = : \exp\Big(-\frac{1}{k}\sum_{n\ne 0}\frac{1}{n}(1-\omega^{\mu n})\beta_n\zeta^{-n}\Big) : z^\mu(\zeta), \qquad (24)$$

where $:*:$ stands for the normal ordering for boson β_n and we have introduced currents $x^{(\mu)}(\zeta) = \sum_{n\in\mathbb{Z}} x^{(\mu)}_n \zeta^{-n}$ and $z^\mu(\zeta) = \sum_{n\in\mathbb{Z}} z^\mu_n\zeta^{-n}$. Then (21) implies the relation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$,

$$\begin{aligned} & g^{\mu,\nu}(\tfrac{\zeta_2}{\zeta_1}) z^\mu(\zeta_1) z^\nu(\zeta_2) - z^\nu(\zeta_2) z^\mu(\zeta_1) g^{\nu,\mu}(\tfrac{\zeta_1}{\zeta_2}) \\ &= \begin{cases} \delta(\omega^\mu \frac{\zeta_2}{\zeta_1}) z^{\mu+\nu}(\omega^\mu\zeta_2) - \delta(\omega^{-\nu}\frac{\zeta_2}{\zeta_1}) z^{\mu+\nu}(\zeta_2) & (\mu+\nu \not\equiv 0) \\ k\, D\delta(\omega^\mu\frac{\zeta_2}{\zeta_1}) & (\mu+\nu\equiv 0)\,, \end{cases} \end{aligned} \qquad (25)$$

where $D = \zeta\frac{d}{d\zeta}$, $D\delta(\zeta) = \sum_{n\in\mathbb{Z}} n\zeta^n$ and the structure function $g^{\mu,\nu}(\zeta)$ is given by

$$g^{\mu,\nu}(\zeta) = \exp\Big(-\frac{1}{k}\sum_{\substack{n>0\\ n\not\equiv 0}}\frac{1}{n}(1-\omega^{\mu n})(1-\omega^{-\nu n})\zeta^n\Big). \qquad (26)$$

Next we present an interesting relation between $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ and $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$. Let us consider the following limit[c]:

$$\text{Limit II} : \begin{cases} q = e^\hbar\,, & \hbar \to 0 \\ t = \omega^{-1} q^{\frac{k+N}{N}}\,, & k : \text{fixed}\,. \end{cases} \qquad (27)$$

[c] For this choice of $t = \omega^{-1}q^{\frac{k+N}{N}}$, we cannot take Limit I because $\beta = \frac{k+N}{N} - \frac{2\pi i}{N\hbar}$ depends on $\hbar$.

We assume that DWA currents $W^i(z)$ have the $\hbar$-expansion

$$W^i(p^{\frac{1-i}{2}}\zeta) = \hbar\omega^{\frac{i}{2}} z^i(\zeta) + O(\hbar^2). \tag{28}$$

Then, under the Limit II, we can show that the relation of $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ (10) reduces to that of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$ (25). (Eq.(10) begins from $\hbar^2$ term and its coefficient is (25). We remark that in this derivation we do not use free field realization at all.) In other words, $\mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t}$ with $t = \omega^{-1}q^{\frac{k+N}{N}}$ can be considered as a q-deformation of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_N)_k$, which we denote as $\mathrm{DZA}(\widehat{\mathfrak{sl}}_N)_k$,

$$\mathrm{DZA}(\widehat{\mathfrak{sl}}_N)_k = \mathrm{DWA}(\widehat{\mathfrak{sl}}_N)_{q,t=\omega^{-1}q^{\frac{k+N}{N}}}. \tag{29}$$

Concerning the free field realization, however, our assumption (28) does not hold on the Fock space except for $N=2$ case. But calculation of some correlation functions supports the assumption (28); We have checked $\langle\lambda|W^1(\zeta_1)\cdots W^1(\zeta_n)|\lambda\rangle = O(\hbar^n)$ for $n \le 6$. We guess that the assumption (28) holds on the level of correlation functions, or, on the irreducible representation space obtained by taking some BRST cohomology. For $N=2$ case, (28) holds on the Fock space, and screening currents and vertex operators of DVA (after some modification of zero mode) reduce to those of ZA.

Finally we mention the character of $\mathrm{DZA}(\widehat{\mathfrak{sl}}_2)_k$ for $k \in \mathbb{Z}_{\ge 2}$, i.e. that of $\mathrm{DVA}_{q,t} = \mathrm{DWA}(\widehat{\mathfrak{sl}}_2)_{q,t}$ with $t = e^{-\pi i}q^{\frac{k+2}{2}}$. We write $W^1(\zeta)$ and $w^1(\lambda)$ as $T(\zeta)$ and λ respectively, e.g., the highest weight state is defined by $T_n|\lambda\rangle = \lambda|\lambda\rangle\delta_{n0}$ $(n \ge 0)$. Since degenerate representations of DVA occur at $\lambda = \lambda_{r,s} = t^{\frac{r}{2}}q^{-\frac{s}{2}} + t^{-\frac{r}{2}}q^{\frac{s}{2}}$,[1] let us consider $\lambda = \lambda_{1,j+\frac{k+2}{2}}$ $(j = -\frac{k}{2}, -\frac{k}{2}+1, \cdots, \frac{k}{2})$ representations. Grading operator ρ satisfies $[\rho, T_n] = nT_n$ and $-\rho|\lambda\rangle = \big(\frac{2j^2+k}{4(k+2)} - \frac{1}{8}\big)|\lambda\rangle$. The character of DZA is defined by $\chi^{\mathrm{DZA}}_j(\tau) = \mathrm{tr}\, y^{-\rho}$ where $y = e^{2\pi i\tau}$ and the trace is taken over irreducible DZA spin j representation space. Shiraishi and present author conjectured[d]

$$\chi^{\mathrm{DZA}}_j(\tau) = y^{\frac{2j^2+k}{4(k+2)}-\frac{1}{8}} \frac{1}{(y;y)_\infty} \sum_{m\in\mathbb{Z}} (-1)^m y^{m(j+\frac{k+2}{2}m)} = y^{\frac{2j^2+k}{4(k+2)}-\frac{1}{8}} \chi^{(2,k+2)}_{1,j+\frac{k+2}{2}}(\tau). \tag{30}$$

Here $\chi^{(p',p'')}_{r,s}(\tau)$ is the Rocha-Caridi character formula[1]

$$\chi^{(p',p'')}_{r,s}(\tau) = \frac{1}{(y;y)_\infty} \sum_{m\in\mathbb{Z}} \Big(y^{(p''r-p's+mp'p'')m} - y^{(r+mp')(s+mp'')} \Big), \tag{31}$$

which gives the character of the Virasoro minimal representation when p' and p'' are coprime, $(p',p'')=1$. In the above case, $p'=2$ and $p''=k+2$ imply that $(p',p'')=1$ for odd k but $(p',p'')=2$ for even k. When q is not a root of unity, by studying the Kac determinant of DVA[1], we can check that (30) is true. We remark that the character of $\mathrm{DZA}(\widehat{\mathfrak{sl}}_2)_k$, χ^{DZA}_j, coincides with that of $\mathrm{ZA}(\widehat{\mathfrak{sl}}_2)_k$ which is obtained by using the result of ref.[11], BRST structure of principal $\widehat{\mathfrak{sl}}_2$.

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[d] We remark that this character appears in the calculation of the one-point local height probability of the Kashiwara-Miwa model (M. Jimbo, T. Miwa and M. Okado, *Nucl. Phys.* **B275[FS17]** (1986) 517-545).

Acknowledgements

The author would like to thank H. Awata and J. Shiraishi for valuable discussions, and M. Jimbo for helpful comments and explaining poles of $f^{i,j}W^iW^j$ and efficient graphical computational method for DWA correlation functions. He is also grateful to M.L. Ge for kind invitation to this APCTP-Nankai joint symposium and thanks members of Nankai Institute for their kind hospitality. This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, No.12640261.

Appendix A. DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization (by H. Awata, S. Odake and J. Shiraishi)

In this appendix we present an interesting relation between DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ and ζ-function regularization.[5]

In string theory[12], the physical state condition is given by $(L_0-1)|\mathrm{phys}\rangle=0$ (and its antichiral counterpart), where L_0 is the zero mode of the Virasoro generator. This condition and the space-time dimension are derived by careful study of string theory (Lotentz invariance in the light-cone gauge, nilpotency of BRST charge, etc.), but there is a shortcut method, ζ function regularization method.

First we illustrate this method by taking a bosonic string theory as an example. In the light-cone gauge the Virasoro generator L_n is given by $L_n=\sum_{i=1}^{24}\sum_{m\in\mathbb{Z}}\frac{1}{2}:\alpha^i_{n-m}\alpha^i_m:$ where α^i_n $(n\in\mathbb{Z},\, i=1,\cdots,24)$ satisfies $[\alpha^i_n,\alpha^j_m]=n\delta^{ij}\delta_{n+m,0}$ and $:*:$ stands for the normal ordering. The Virasoro zero mode without the normal ordering is $L_0^{\mathrm{noNO}}=\sum_{i=1}^{24}\sum_{n\in\mathbb{Z}}\frac{1}{2}\alpha^i_{-n}\alpha^i_n=\sum_{i=1}^{24}\sum_{n\in\mathbb{Z}}\frac{1}{2}:\alpha^i_{-n}\alpha^i_n:+12$"$\sum_{n>0}n$". Of course the sum "$\sum_{n>0}n$" is divergent and this expression is meaningless. But we replace the sum "$\sum_{n>0}n$" by $\zeta(-1)$, where $\zeta(z)$ is the Riemann ζ function. Then the above physical state condition is equivalent to the condition that the Virasoro zero mode without the normal ordering annihilates the physical state:

$$L_0^{\mathrm{noNO}}|\mathrm{phys}\rangle=0,\quad L_0^{\mathrm{noNO}}=L_0+12\zeta(-1), \tag{A.1}$$

because of the value $\zeta(-1)=-\frac{1}{12}$. We might say that the Virasoro generator "knows" the value $\zeta(-1)$.

Next let us mimic the above procedure for DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ case. DWA current without the normal ordering becomes $W^{i\ \mathrm{noNO}}(z)=$ "$f^{i,i}(1)^{-\frac{1}{2}}$" $W^i(z)$, where "$f^{i,i}(1)$" is divergent for generic β (recall $t=q^\beta$ and $q=e^\hbar$). Let a^i_{2m} be coefficients of the following $\hbar$-expansion $(1-q^n)(1-t^{-n})\frac{1-p^{in}}{1-p^n}\frac{1-p^{(N-i)n}}{1-p^{Nn}}=\sum_{m>0}a^i_{2m}(n\hbar)^{2m}$. Then $f^{i,i}(z)$ is $f^{i,i}(z)=\exp\bigl(\sum_{n>0}\frac{1}{n}\sum_{m>0}a^i_{2m}(n\hbar)^{2m}z^n\bigr)$. We define ζ-regularized $f^{i,i}_{\zeta\text{-reg}}(1)$ by exchanging these summations over n and m and replacing $\sum_{n>0}n^{2m-1}$ with $\zeta(1-2m)$ as follows:

$$f^{i,i}_{\zeta\text{-reg}}(1)=\exp\Bigl(\sum_{m>0}a^i_{2m}\zeta(1-2m)\hbar^{2m}\Bigr). \tag{A.2}$$

In the Limit I (4), DWA current behaves as $W^i(z)=\binom{N}{i}+O(\hbar^2)$, which can be shown by using free field realization.[7] So we require that $\beta=\frac{N+1}{N}$ or $\frac{N}{N+1}$, which corresponds to the vanishing Virasoro central charge, and the zero mode of the i-th DWA current without normal ordering takes the above value $\binom{N}{i}$ on the vacuum state $|\text{vac}\rangle$, which is characterized by $h^i_n|\text{vac}\rangle=0$ $(n\geq 0,\,\forall i)$,

$$W_0^{i\ \text{noNO}}|\text{vac}\rangle=\binom{N}{i}|\text{vac}\rangle,\quad W^{i\ \text{noNO}}(z)=f^{i,i}_{\zeta\text{-reg}}(1)^{-\frac{1}{2}}\,W^i(z). \tag{A.3}$$

Since we can show $W^i_0|\text{vac}\rangle=\begin{bmatrix}N\\ i\end{bmatrix}|\text{vac}\rangle$, this requirement implies

$$f^{i,i}_{\zeta\text{-reg}}(1)^{\frac{1}{2}}=\binom{N}{i}^{-1}\begin{bmatrix}N\\ i\end{bmatrix}, \tag{A.4}$$

where $\begin{bmatrix}N\\ i\end{bmatrix}=\frac{[N]!}{[i]![N-i]!}$, $[n]!=[n]\cdots[1]$ and $[n]=\frac{p^{\frac{n}{2}}-p^{-\frac{n}{2}}}{p^{\frac{1}{2}}-p^{-\frac{1}{2}}}$. We can check that this equation really holds by using formulas $\log(\sinh x)=\log x+\sum_{n>0}(-1)^{n-1}\frac{2^{2n-1}B_n}{(2n)!n}$ x^{2n} $(0<|x|<\pi)$ and $\zeta(1-2m)=(-1)^m\frac{B_m}{2m}$ $(m=1,2,\cdots)$. Here B_n is the Bernoulli number defined by $\frac{x}{e^x-1}+\frac{x}{2}=1+\sum_{n>0}(-1)^{n-1}\frac{B_n}{(2n)!}x^{2n}$ $(|x|<2\pi)$. Therefore we might say that DWA$(\widehat{\mathfrak{sl}}_N)_{q,t}$ (with $t=q^{\frac{N+1}{N}},q^{\frac{N}{N+1}}$) for each N "knows" all the values $\zeta(1-2m)$ $(m=1,2,\cdots)$.

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