Shape Invariant Potentials in "Discrete Quantum Mechanics" [1]

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Abstract

Shape invariance is an important ingredient of many exactly solvable quantum mechanics. Several examples of shape invariant "discrete quantum mechanical systems" are introduced and discussed in some detail. They arise in the problem of describing the equilibrium positions of Ruijsenaars-Schneider type systems, which are "discrete" counterparts of Calogero and Sutherland systems, the celebrated exactly solvable multi-particle dynamics. Deformed Hermite and Laguerre polynomials are the typical examples of the eigenfunctions of the above shape invariant discrete quantum mechanical systems.

1   Introduction

Many exactly solvable quantum mechanical systems, for example, the harmonic oscillator without/with a centrifugal potential, the coulomb problem, the trigonometric and hyperbolic Pöschl-Teller potential, the symmetric top *etc.* are *shape invariant*[1]. Here we will show that this concept is also useful in a wider context of "discrete quantum mechanics", in which the momentum operator $p = -i\hbar d/dx$ appears as an exponential (hyperbolic) function instead of a polynomial in ordinary quantum mechanics. For demonstration, we present several explicit examples of shape invariant discrete quantum mechanics in some detail. The corresponding eigenfunctions are orthogonal polynomials belonging to the family of Askey-scheme of hypergeometric orthogonal polynomials [2, 3]. The examples, roughly speaking, are related to one and two parameter deformation of the Hermite polynomials and two and three parameter deformation of the Laguerre polynomials. They could be interpreted as describing deformed harmonic oscillator with (Laguerre) and without (Hermite) the centrifugal potential. They also arise in many contexts of theoretical physics [4, 5].

These examples arise within the context of multi-particle exactly solvable quantum mechanical systems, in particular, the Calogero and Sutherland systems [6] and their deformation called the Ruijsenaars-Schneider-van Diejen systems [7]. Their integrability

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is based on the root systems and the associated Weyl (Coxeter) group symmetry. More than twenty years ago, Calogero [8] found out that the equilibrium position of the A-type Calogero system was described by the zeros of the Hermite polynomial. The same fact was discussed by Stieltjes in a slightly different context more than a century ago [9]. The equilibria of the B, BC and D type Calogero systems are described by the zeros of the Laguerre polynomials. The equilibria of the Sutherland (trigonometric potential) systems based on the classical root systems are described by the zeros of the Chebyshev and Jacobi polynomials. The interesting dynamics associated with the equilibria of exactly solvable multi-particle systems and the polynomials describing the equilibria of Calogero-Sutherland (C-S) systems based on the exceptional root systems are presented in [10]. Ruijsenaars-Schneider (R-S) and van Diejen introduced integrable deformation of the C-S systems, with several additional parameters [7]. The equilibrium positions of the R-S and van Diejen systems are discussed by Ragnisco-Sasaki and Odake-Sasaki [11]. As expected certain multi-parameter deformation of the Hermite and Laguerre polynomials describe the equilibrium positions of the rational R-S or the van Diejen systems.

The single particle dynamics related to these polynomials is the main subject of this paper. They are certain *deformation* of the harmonic oscillator without/with the centrifugal potential belonging to the "discrete quantum mechanics". We show that they are exactly solvable thanks to the inherited shape invariance. To the best of our knowledge, they are the first examples of shape invariance discussed in the framework of discrete quantum mechanics. Of course there are plenty of works applying the *factorisation* method [12, 13, 14] to various *difference* equations [15, 16]. However, all of these difference equations contain finite shifts in the *real* direction, which result in polynomials of a *discrete variable*, for example the Charlier, the Meixner and the Hahn polynomials [2]. Apparently they look very different from those occurring in quantum mechanics.

This paper is organised as follows. In section two, the simplest example of a shape invariant discrete quantum mechanics, related to a deformed harmonic oscillator is introduced and discussed in some detail. The eigenfunctions are deformed Hermite polynomials, or a special case of the Meixner-Pollaczek polynomials [3]. The course of the arguments is essentially the same as in ordinary quantum mechanics or the Sturm-Liouville problem, as most clearly formulated in the seminal work of Crum [13]. It is also known as the factorisation method [12], or the supersymmetric quantum mechanics [14]. Here we follow the notion and notation of Crum's paper. In section three we present three explicit examples of shape invariant discrete quantum mechanics, in which a two parameter deformation of the Hermite polynomials, a two and three parameter deformation of Laguerre polynomials are the eigenfunctions. They are a special case of the continuous Hahn polynomial, the continuous dual Hahn polynomial and the Wilson polynomial [3]. Shape invariance is demonstrated elementarily. The meaning of the factorisation at the level of the Hamiltonian and at the level of the polynomial solutions are elaborated in detail. Although these polynomials are perfectly well understood in their own right, we do believe *dynamical* interpretation in terms of the factorised Hamiltonian, the analogues of the creation and the annihilation operators, *etc.* would shed new light on them. Section four is devoted to the clarification of the background of the present research: how these polynomials came to our notice through the polynomials describing the equilibrium points of the van Diejen systems based on the classical root systems. The final section is for comments, a summary and an outlook.

2   Deformed Hermite (Meixner-Pollaczek) Polynomials

The simplest and best known example of *exactly solvable* and *shape invariant* quantum mechanics is the harmonic oscillator

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}(x^2 - 1) = \frac{1}{2}(-i\partial_x + ix)(-i\partial_x - ix), \qquad \partial_x \stackrel{\text{def}}{=} \frac{d}{dx}. \tag{2.1}$$

The Hamiltonian is factorised and the eigenfunctions are the Hermite polynomials, $H_n(x)$:

$$H\phi_n = \mathcal{E}_n\phi_n, \quad \mathcal{E}_n = n, \quad \phi_n(x) \propto H_n(x)\,e^{-x^2/2}, \quad n = 0, 1, \ldots,. \tag{2.2}$$

One naively expects that a certain *deformation* of the Hermite polynomials would constitute the eigenfunctions of a simplest *shape invariant* "discrete quantum mechanics". This is exactly the case and will be explained in some detail in this section.

Let us start with the following Hamiltonian,

$$H \stackrel{\text{def}}{=} \tfrac{1}{2}\sqrt{V(x)}\,e^{-i\partial_x}\sqrt{V^*(x)} + \tfrac{1}{2}\sqrt{V^*(x)}\,e^{i\partial_x}\sqrt{V(x)} - \tfrac{1}{2}(V(x) + V^*(x)), \tag{2.3}$$

$$V(x) = \lambda + ix, \qquad V^*(x) = \lambda - ix\,; \qquad \lambda \in \mathbb{R}_+. \tag{2.4}$$

In the discrete quantum mechanics the momentum operator $p = -i\hbar\partial_x$ (with $\hbar = 1$) appears as exponentiated instead of powers in ordinary quantum mechanics. Thus they cause a finite shift of the wavefunction in the *imaginary* direction. For example,

$$e^{\pm i\partial_x}\phi(x) = \phi(x \pm i).$$

Throughout this paper we adopt the following convention of a complex conjugate function: for an arbitrary function $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$ we define $f^*(x) = \sum_n a_n^* x^n$. Here c^* is the complex conjugation of a number $c \in \mathbb{C}$. Note that $f^*(x)$ is not the complex conjugation of $f(x)$, $(f(x))^* = f^*(x^*)$. This is particularly important when a function is shifted in the imaginary direction.

Factorised Hamiltonian   It is easy to see that the above Hamiltonian (2.3) is *factorised*:

$$H = A^\dagger A, \tag{2.5}$$

$$A = A(x;\lambda) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V^*(x)} - e^{\frac{i}{2}\partial_x}\sqrt{V(x)}\right), \tag{2.6}$$

$$A^\dagger = A(x;\lambda)^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V(x)}\,e^{-\frac{i}{2}\partial_x} - \sqrt{V^*(x)}\,e^{\frac{i}{2}\partial_x}\right). \tag{2.7}$$

Here $\dagger$ denotes the ordinary hermitian conjugation with respect to the ordinary L^2 inner product: $(f, g) = \int_{-\infty}^{\infty}(f(x))^* g(x)dx$. Obviously the Hamiltonian (2.3) is hermitian (self-conjugate) and positive semi-definite.

The ground state ϕ_0 is annihilated by A:

$$A\phi_0(x) = 0, \quad \phi_0(x) \propto \sqrt{\Gamma(\lambda + ix)\Gamma(\lambda - ix)} \implies H\phi_0(x) = 0, \quad \mathcal{E}_0 = 0. \tag{2.8}$$

The above equation reads

$$\sqrt{V^*(x - \tfrac{i}{2})}\, \phi_0(x - \tfrac{i}{2}) = \sqrt{V(x + \tfrac{i}{2})}\, \phi_0(x + \tfrac{i}{2}). \tag{2.9}$$

The other eigenfunctions of the Hamiltonian (2.3) can be obtained in the form

$$H\phi_n = \mathcal{E}_n \phi_n, \quad \phi_n(x) \propto P_n(x)\phi_0(x) \quad \Longrightarrow \tilde{H} P_n = \mathcal{E}_n P_n, \tag{2.10}$$

in which $\tilde{H}$ is a similarity transformed Hamiltonian in terms of the ground state wavefunction ϕ_0:

$$\tilde{H} \stackrel{\text{def}}{=} \phi_0^{-1} \circ H \circ \phi_0$$
$$= \tfrac{1}{2}V(x)e^{-i\partial_x} + \tfrac{1}{2}V^*(x)e^{i\partial_x} - \tfrac{1}{2}(V(x) + V^*(x)). \tag{2.11}$$

The eigenvalue equation (2.10) is now the following *difference* equation

$$(\lambda + ix)P_n(x - i) + (\lambda - ix)P_n(x + i) = 2(\lambda + \mathcal{E}_n)P_n(x), \tag{2.12}$$

which obviously has a polynomial solution, $P_n(x) = a_n x^n+$ lower degree terms, of *definite parity*, $P_n(-x) = (-1)^n P_n(x)$. By comparing the coefficient of the highest degree x^n term, we find easily

$$\mathcal{E}_n = n, \quad n = 1, 2, \dots,. \tag{2.13}$$

The orthogonal polynomials $\{P_n(x), n \geq 0\}$ with respect to the weight function

$$\omega(x; \lambda) = \Gamma(\lambda + ix)\Gamma(\lambda - ix) \propto \phi_0(x)^2 \tag{2.14}$$

satisfy the *three term recurrence*, which reads

$$p_{n+1}(x) = x\, p_n(x) - n(n + 2\lambda - 1)/4\, p_{n-1}(x), \quad n \geq 0, \quad p_0 = 1, \quad p_{-1} = 0, \tag{2.15}$$

for the *monic* polynomial, $p_n(x) = x^n+$ lower degree terms. With proper normalisation

$$P_n(x) = \frac{2^n}{n!} p_n(x) = P_n^{(\lambda)}(x; \tfrac{\pi}{2}), \tag{2.16}$$

it is a special case of the Meixner-Pollaczek polynomial. Here we follow the notation of Koekoek and Swarttouw [3]. Since the second argument $\tfrac{\pi}{2}$ remains fixed throughout our discussion, we will omit it:

$$P_n^{(\lambda)}(x; \tfrac{\pi}{2}) \stackrel{\text{def}}{=} P_n^{(\lambda)}(x).$$

They are polynomials in both x and λ with *real* and *rational* coefficients. The Meixner-Pollaczek polynomial also appears in a "relativistic oscillator" model [4].

Corresponding to the factorisation of H (2.5), $\tilde{H}$ is also factorised:

$$\tilde{H} = BC, \tag{2.17}$$

$$C = C(x) = \tfrac{i}{2}(e^{-\frac{i}{2}\partial_x} - e^{\frac{i}{2}\partial_x}), \tag{2.18}$$

$$B = B(x; \lambda) = -i(V(x)e^{-\frac{i}{2}\partial_x} - V^*(x)e^{\frac{i}{2}\partial_x}) = -i\left((\lambda + ix)e^{-\frac{i}{2}\partial_x} - (\lambda - ix)e^{\frac{i}{2}\partial_x}\right). \tag{2.19}$$

They factorise the eigenvalue equation

$$BC \, P_n^{(\lambda)}(x) = n \, P_n^{(\lambda)}(x) \impliedby C \, P_n^{(\lambda)}(x) = P_{n-1}^{(\lambda+\frac{1}{2})}(x), \quad B \, P_{n-1}^{(\lambda+\frac{1}{2})}(x) = n \, P_n^{(\lambda)}(x), \tag{2.20}$$

giving rise to the *forward* and *backward* shift relations. They read explicitly:

$$C: \quad P_n^{(\lambda)}(x + \tfrac{i}{2}) - P_n^{(\lambda)}(x - \tfrac{i}{2}) = 2i \, P_{n-1}^{(\lambda+\frac{1}{2})}(x), \tag{2.21}$$

$$B: \quad i\left((\lambda - ix)P_{n-1}^{(\lambda+\frac{1}{2})}(x + \tfrac{i}{2}) - (\lambda + ix)P_{n-1}^{(\lambda+\frac{1}{2})}(x - \tfrac{i}{2})\right) = n \, P_n^{(\lambda)}(x). \tag{2.22}$$

Let us define a new set of wavefunctions

$$\phi_{1,n} \stackrel{\text{def}}{=} A\phi_n, \quad n = 1, 2, \ldots, . \tag{2.23}$$

As a consequence of the factorisation, they form eigenfunctions of a new Hamiltonian

$$H_1 = AA^\dagger \tag{2.24}$$

with the same eigenvalues $\{\mathcal{E}_n\}$:

$$H_1\phi_{1,n} = AA^\dagger A\phi_n = A\mathcal{E}_n\phi_n = \mathcal{E}_n\phi_{1,n}, \quad n = 1, 2, \ldots, . \tag{2.25}$$

Shape Invariance   It is straightforward to evaluate the reversed order product $AA^\dagger$:

$$H_1 = \tfrac{1}{2}\sqrt{V_1(x)}\, e^{-i\partial_x}\sqrt{V_1^*(x)} + \tfrac{1}{2}\sqrt{V_1^*(x)}\, e^{i\partial_x}\sqrt{V_1(x)} - \tfrac{1}{2}(V_1(x) + V_1^*(x)) + \mathcal{E}_1, \tag{2.26}$$

$$V_1(x) = \lambda + \tfrac{1}{2} + ix, \qquad V_1^*(x) = \lambda + \tfrac{1}{2} - ix, \qquad \mathcal{E}_1 = 1. \tag{2.27}$$

It has the *same form* as H with a constant term $\mathcal{E}_1 = 1$ added and the parameter λ shifted by $\tfrac{1}{2}$. Thus it is again factorised:

$$H_1 = AA^\dagger = A_1^\dagger A_1 + \mathcal{E}_1, \tag{2.28}$$

$$A_1 = A_1(x; \lambda) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V_1^*(x)} - e^{\frac{i}{2}\partial_x}\sqrt{V_1(x)}\right) = A(x; \lambda + \tfrac{1}{2}), \tag{2.29}$$

$$A_1^\dagger = A_1(x; \lambda)^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V_1(x)}\, e^{-\frac{i}{2}\partial_x} - \sqrt{V_1^*(x)}\, e^{\frac{i}{2}\partial_x}\right) = A^\dagger(x; \lambda + \tfrac{1}{2}). \tag{2.30}$$

The ground state wavefunction is $\phi_{1,1}$ annihilated by A_1:

$$A_1\phi_{1,1}(x) = 0, \quad \phi_{1,1}(x) \propto \sqrt{\Gamma(\lambda + \tfrac{1}{2} + ix)\Gamma(\lambda + \tfrac{1}{2} - ix)} \tag{2.31}$$

$$\implies H_1\phi_{1,1}(x) = \mathcal{E}_1\phi_{1,1}(x), \quad \mathcal{E}_1 = 1. \tag{2.32}$$

Clearly this process can be repeated as many times as the number of discrete levels of the original Hamiltonian H (2.3):

$$H = H_0 \to H_1 \to H_2 \cdots \to H_s \to \cdots . \tag{2.33}$$

All these Hamiltonians share the same spectra $\{\mathcal{E}_n\}$, $n = 0, 1, \ldots$. The eigenfunction of the s-th Hamiltonian H_s are given by

$$H_s\phi_{s,n} = \mathcal{E}_n\phi_{s,n}, \quad \phi_{s,n} \overset{\text{def}}{=} A_{s-1}A_{s-2}\cdots A_1A\,\phi_n, \quad 0 \le s \le n, \tag{2.34}$$

in which

$$H_s = A_{s-1}A_{s-1}^\dagger + \mathcal{E}_{s-1} = A_s^\dagger A_s + \mathcal{E}_s, \tag{2.35}$$

$$A_s = A_s(x; \lambda) \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V_s^*(x)} - e^{\frac{i}{2}\partial_x}\sqrt{V_s(x)}\right) = A(x; \lambda + \tfrac{s}{2}), \tag{2.36}$$

$$A_s^\dagger = A_s(x; \lambda)^\dagger \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V_s(x)}\,e^{-\frac{i}{2}\partial_x} - \sqrt{V_s^*(x)}\,e^{\frac{i}{2}\partial_x}\right) = A(x; \lambda + \tfrac{s}{2})^\dagger, \tag{2.37}$$

$$V_s(x) = \lambda + \tfrac{s}{2} + ix, \qquad V_s^*(x) = \lambda + \tfrac{s}{2} - ix, \qquad \mathcal{E}_s = s, \tag{2.38}$$

$$A_s\phi_{s,s} = 0 \quad \phi_{s,s}(x) \propto \sqrt{\Gamma(\lambda + \tfrac{s}{2} + ix)\Gamma(\lambda + \tfrac{s}{2} - ix)} \tag{2.39}$$

$$\Longrightarrow H_s\phi_{s,s}(x) = \mathcal{E}_s\phi_{s,s}(x), \quad \mathcal{E}_s = s. \tag{2.40}$$

By multiplying $A_s^\dagger$ to $\phi_{s+1,n} \overset{\text{def}}{=} A_s\phi_{s,n}$ we obtain

$$A_s^\dagger\phi_{s+1,n} = A_s^\dagger A_s\phi_{s,n} = (H_s - \mathcal{E}_s)\phi_{s,n} = (\mathcal{E}_n - \mathcal{E}_s)\phi_{s,n}. \tag{2.41}$$

Since the ground state $\phi_{n,n}$ of the n-th Hamiltonian H_n is known explicitly, we can express the n-th eigenfunction $\phi_n \equiv \phi_{0,n}$ of the original Hamiltonian H (2.3) in terms of $A^\dagger$ and $\phi_{n,n}$ by repeated use of the above formula (2.41):

$$\phi_n(x; \lambda) = \frac{A_0^\dagger}{\mathcal{E}_n - \mathcal{E}_0}\frac{A_1^\dagger}{\mathcal{E}_n - \mathcal{E}_1}\cdots\frac{A_{n-1}^\dagger}{\mathcal{E}_n - \mathcal{E}_{n-1}}\phi_{n,n}(x; \lambda) = \tfrac{1}{n!}A_0^\dagger A_1^\dagger\cdots A_{n-1}^\dagger\phi_0(x; \lambda + \tfrac{n}{2}). \tag{2.42}$$

This is another formula giving the eigenfunction $\phi_n(x; \lambda) \propto P_n^{(\lambda)}(x)\phi_0(x; \lambda)$. The situation is depicted in Fig.1. The operator A acts to the right and $A^\dagger$ to the left along the horizontal (*isospectral*) line. They should not be confused with the *annihilation* and *creation* operators, which act along the vertical line of a given Hamiltonian H_s going from one energy level n to another $n \pm 1$. The annihilation and creation operators will be discussed in section 3 in a more general setting.

3 Other examples of shape invariant "discrete quantum mechanics"

The other examples of *shape invariant* "discrete quantum mechanics" are related to a two-parameter deformation of the Hermite polynomial, and two- and three-parameter deformation of the Laguerre polynomial. The Laguerre polynomials are the eigenfunctions of a harmonic oscillator with a centrifugal barrier ($1/x^2$ potential). They all belong to the Askey scheme of hypergeometric orthogonal polynomials [3]. Demonstration of the shape invariance and derivation of shift operators and eigenfunctions, *etc.* go almost parallel with the case of the deformed Hermite polynomials in section 2. We discuss them collectively by adopting an (almost) self-evident notation.
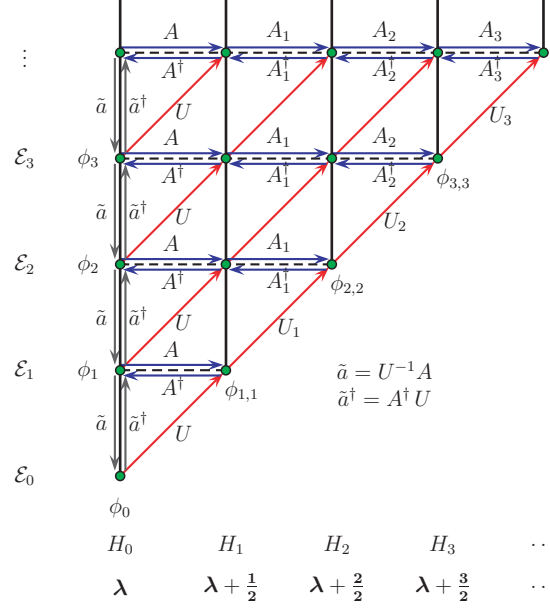
Figure 1. A schematic diagram of the energy levels and the associated Hamiltonian systems together with the definition of the A and $A^\dagger$ operators and the 'creation' and 'annihilation' operators. The parameter set is indicated below each Hamiltonian.

(i) : two-parameter deformation of the Hermite polynomial,

a special case of the **continuous Hahn polynomial**,

$$V(x) = V(x; \boldsymbol{\lambda}) = (a + ix)(b + ix), \tag{3.1}$$

$$\phi_0(x) = \phi_0(x; \boldsymbol{\lambda}) \propto |\Gamma(a + ix)\Gamma(b + ix)|, \tag{3.2}$$

$$\boldsymbol{\lambda} = (a, b), \quad \mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda}) = \tfrac{1}{2}n(n + 2a + 2b - 1), \quad a, b \in \mathbb{R}_+, \tag{3.3}$$

$$P_n(x) = P_n^{(\boldsymbol{\lambda})}(x) \propto p_n(x; a, b, a, b). \tag{3.4}$$

(ii) : two-parameter deformation of the Laguerre polynomial,

the **continuous dual Hahn polynomial**,

$$V(x) = V(x; \boldsymbol{\lambda}) = \frac{(a + ix)(b + ix)(c + ix)}{2ix(2ix + 1)}, \tag{3.5}$$

$$\phi_0(x) = \phi_0(x; \boldsymbol{\lambda}) \propto \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)}{\Gamma(2ix)} \right|, \tag{3.6}$$

$$\boldsymbol{\lambda} = (a, b, c), \quad \mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda}) = \tfrac{1}{2}n, \quad a, b, c \in \mathbb{R}_+, \tag{3.7}$$

$$P_n(x) = P_n^{(\boldsymbol{\lambda})}(x) \propto S_n(x^2; a, b, c). \tag{3.8}$$

(iii) : three-parameter deformation of the Laguerre polynomial,

the **Wilson polynomial**,

$$V(x) = V(x; \boldsymbol{\lambda}) = \frac{(a + ix)(b + ix)(c + ix)(d + ix)}{2ix(2ix + 1)}, \tag{3.9}$$

$$\phi_0(x) = \phi_0(x; \boldsymbol{\lambda}) \propto \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(d + ix)}{\Gamma(2ix)} \right|, \tag{3.10}$$

$$\boldsymbol{\lambda} = (a, b, c, d), \ \mathcal{E}_n = \mathcal{E}_n(\boldsymbol{\lambda}) = \tfrac{1}{2}n(n + a + b + c + d - 1), \ a, b, c, d \in \mathbb{R}_+, \tag{3.11}$$

$$P_n(x) = P_n^{(\boldsymbol{\lambda})}(x) \propto W_n(x^2; a, b, c, d). \tag{3.12}$$

For space reasons we write the ground state wavefunction in terms of the absolute value symbol as in (3.2) which should read

$$|\Gamma(a+ix)\Gamma(b+ix)| = \sqrt{\Gamma(a+ix)\Gamma(b+ix)\Gamma(a-ix)\Gamma(b-ix)}$$

as in (2.8).

Factorised Hamiltonian   Factorisation and consequently the positive semi-definiteness of the generic Hamiltonian hold exactly the same as before:

$$H \stackrel{\text{def}}{=} \tfrac{1}{2}\sqrt{V(x)}\,e^{-i\partial_x}\sqrt{V^*(x)} + \tfrac{1}{2}\sqrt{V^*(x)}\,e^{i\partial_x}\sqrt{V(x)} - \tfrac{1}{2}(V(x)+V^*(x)), \qquad (3.13)$$

$$H = A^\dagger A, \tag{3.14}$$

$$A = A(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V^*(x)} - e^{\frac{i}{2}\partial_x}\sqrt{V(x)}\right), \tag{3.15}$$

$$A^\dagger = A(x;\boldsymbol{\lambda})^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V(x)}\,e^{-\frac{i}{2}\partial_x} - \sqrt{V^*(x)}\,e^{\frac{i}{2}\partial_x}\right). \tag{3.16}$$

Here $\boldsymbol{\lambda}$ denotes the (set of) parameters. The ground state ϕ_0 is annihilated by A:

$$A\phi_0(x) = 0 \quad \Longrightarrow H\phi_0(x) = 0, \quad \mathcal{E}_0 = 0. \tag{3.17}$$

Verification of the ground state wavefunction (3.2) for (i), (3.6) for (ii) and (3.10) for (iii) is straightforward. The similarity transformed Hamiltonian $\tilde{H} = \phi_0^{-1} \circ H \circ \phi_0$, (2.11) determines the other eigenfunctions of the Hamiltonian in the form $\phi_n(x) \propto P_n(x)\phi_0(x)$, $\tilde{H}P_n = \mathcal{E}_n P_n$ (2.10). They are the *difference equation* of the form

$$V(x)P_n(x-i) + V^*(x)P_n(x+i) - (V(x)+V^*(x))P_n(x) = 2\mathcal{E}_n P_n(x). \tag{3.18}$$

Explicitly they read for the three cases listed above:

$$
\begin{aligned}
\text{(i)} \quad &: \quad (a+ix)(b+ix)P_n(x-i) + (a-ix)(b-ix)P_n(x+i) \\
&\quad = 2(ab - x^2 + \mathcal{E}_n)P_n(x).
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\text{(ii)} \quad &: \quad (a+ix)(b+ix)(c+ix)(2ix-1)P_n(x-i) + \\
&\quad (a-ix)(b-ix)(c-ix)(2ix+1)P_n(x+i) \\
&\quad = -2ix\left[(ab+ac+bc-2abc+2(a+b+c)x^2 - x^2 + 2(4x^2+1)\mathcal{E}_n\right]P_n(x).
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
\text{(iii)} \quad &: \quad (a+ix)(b+ix)(c+ix)(d+ix)(2ix-1)P_n(x-i) + \\
&\quad (a-ix)(b-ix)(c-ix)(d-ix)(2ix+1)P_n(x+i) \\
&\quad = -2ix\left[(abc+abd+acd+bcd-2abcd+2(ab+ac+ad+bc+bd+cd)x^2\right. \\
&\quad \left. -(a+b+c+d)x^2 - 2x^4 + 2(4x^2+1)\mathcal{E}_n\right]P_n(x).
\end{aligned}
\tag{3.21}
$$

They admit a degree n polynomial solution in x for (i), and in x^2 for (ii) and (iii). By comparing the coefficients of the leading degree, one obtains the energy eigenvalues $\mathcal{E}_n$ (3.3), (3.7), (3.11). The solutions form orthogonal polynomials satisfying three term recurrence, which will not be shown here, see [3].

  The corresponding factorisation of $\tilde{H}$,

$$\tilde{H} = BC, \quad C = \tfrac{i}{2}(e^{-\frac{i}{2}\partial_x} - e^{\frac{i}{2}\partial_x}), \quad B = -i(V(x)e^{-\frac{i}{2}\partial_x} - V^*(x)e^{\frac{i}{2}\partial_x}), \tag{3.22}$$

provides the *forward* and *backward* shift operators of the polynomials. The C operator, corresponding to the A operator, shifts to the right along the isospectral line: $C : P_n^{(\boldsymbol{\lambda})}(x) \to P_{n-1}^{(\boldsymbol{\lambda}+\frac{1}{2})}(x)$ with $\frac{1}{2} = (\frac{1}{2}, \frac{1}{2}, \cdots)$. The B operator, corresponding to the $A^\dagger$ operator, shifts to the left along the isospectral line: $B : P_{n-1}^{(\boldsymbol{\lambda}+\frac{1}{2})}(x) \to P_n^{(\boldsymbol{\lambda})}(x)$. For each case they are:

(i) : $BCp_n(x; \boldsymbol{\lambda}) = \mathcal{E}_n p_n(x; \boldsymbol{\lambda})$

$\Longleftarrow Cp_n(x; \boldsymbol{\lambda}) = (\mathcal{E}_n/n)p_{n-1}(x; \boldsymbol{\lambda}+\tfrac{1}{2}), \quad Bp_{n-1}(x; \boldsymbol{\lambda}+\tfrac{1}{2}) = np_n(x; \boldsymbol{\lambda}),$ (3.23)

$p_n(x + \tfrac{i}{2}; \boldsymbol{\lambda}) - p_n(x - \tfrac{i}{2}; \boldsymbol{\lambda}) = i(n + 2a + 2b - 1)p_{n-1}(x; \boldsymbol{\lambda}+\tfrac{1}{2}),$ (3.24)

$(a - ix)(b - ix)p_{n-1}(x + \tfrac{i}{2}; \boldsymbol{\lambda}+\tfrac{1}{2}) - (a + ix)(b + ix)p_{n-1}(x - \tfrac{i}{2}; \boldsymbol{\lambda}+\tfrac{1}{2})$
$= -inp_n(x; \boldsymbol{\lambda}).$ (3.25)

(ii) : $BCS_n(x^2; \boldsymbol{\lambda}) = \mathcal{E}_n S_n(x^2; \boldsymbol{\lambda})$

$\Longleftarrow CS_n(x^2; \boldsymbol{\lambda}) = -2\mathcal{E}_n xS_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}), \quad -2B\, xS_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}) = S_n(x^2; \boldsymbol{\lambda}),$ (3.26)

$S_n((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}) - S_n((x - \tfrac{i}{2})^2; \boldsymbol{\lambda}) = -2inxS_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}),$ (3.27)

$(a - ix)(b - ix)(c - ix)S_{n-1}((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}+\tfrac{1}{2})$
$-(a + ix)(b + ix)(c + ix)S_{n-1}((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}+\tfrac{1}{2})$
$= -2ixS_n(x^2; \boldsymbol{\lambda}).$ (3.28)

(iii) : $BCW_n(x^2; \boldsymbol{\lambda}) = \mathcal{E}_n W_n(x^2; \boldsymbol{\lambda})$

$\Longleftarrow CW_n(x^2; \boldsymbol{\lambda}) = -2\mathcal{E}_n xW_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}), \quad -2B\, xW_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}) = W_n(x^2; \boldsymbol{\lambda}),$ (3.29)

$W_n((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}) - W_n((x - \tfrac{i}{2})^2; \boldsymbol{\lambda})$
$= -2in(n + a + b + c + d - 1)\, xW_{n-1}(x^2; \boldsymbol{\lambda}+\tfrac{1}{2}),$ (3.30)

$(a - ix)(b - ix)(c - ix)(d - ix)W_{n-1}((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}+\tfrac{1}{2})$
$-(a + ix)(b + ix)(c + ix)(d + ix)W_{n-1}((x + \tfrac{i}{2})^2; \boldsymbol{\lambda}+\tfrac{1}{2})$
$= -2ixW_n(x^2; \boldsymbol{\lambda}).$ (3.31)

Shape Invariance   For shape invariance, we first need to find the operators A_1, $A_1^\dagger$ and a real constant $\mathcal{E}_1$ satisfying

$$H_1 = AA^\dagger = A_1^\dagger A_1 + \mathcal{E}_1,$$ (3.32)

$$A_1 = A_1(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V_1^*(x)} - e^{\frac{i}{2}\partial_x}\sqrt{V_1(x)}\right),$$ (3.33)

$$A_1^\dagger = A_1(x; \boldsymbol{\lambda})^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V_1(x)}\,e^{-\frac{i}{2}\partial_x} - \sqrt{V_1^*(x)}\,e^{\frac{i}{2}\partial_x}\right).$$ (3.34)

In other words, given $V(x) = V(x; \boldsymbol{\lambda})$, find a new potential $V_1(x) = V_1(x; \boldsymbol{\lambda})$ satisfying

$$V_1^*(x - i)V_1(x) = V^*(x - \tfrac{i}{2})V(x - \tfrac{i}{2}),$$ (3.35)

$$V_1(x) + V_1^*(x) = V(x + \tfrac{i}{2}) + V^*(x - \tfrac{i}{2}) + 2\mathcal{E}_1.$$ (3.36)

If V_1 has the same form as V with a shifted set of parameters $\boldsymbol{\lambda}'$,

$$V_1(x; \boldsymbol{\lambda}) = V(x; \boldsymbol{\lambda}'), \tag{3.37}$$

it is *shape invariant*. Suppose V_1 has the form $V_1(x) = V(x - \frac{i}{2})g(x)$, the above conditions (3.35), (3.36) get slightly simplified:

$$g^*(x - i)g(x) = 1, \tag{3.38}$$

$$V(x - \tfrac{i}{2})g(x) + V^*(x + \tfrac{i}{2})g^*(x) = V(x + \tfrac{i}{2}) + V^*(x - \tfrac{i}{2}) + 2\mathcal{E}_1. \tag{3.39}$$

The following choice satisfies the above conditions (3.38), (3.38) for the three cases (3.2)–(3.12):

$$\text{(i)} \quad : \quad g(x) = 1, \quad \mathcal{E}_1(\boldsymbol{\lambda}) = a + b, \tag{3.40}$$

$$V_1(x; \boldsymbol{\lambda}) = V(x - \tfrac{i}{2}; a, b) = V(x; a + \tfrac{1}{2}, b + \tfrac{1}{2}). \tag{3.41}$$

$$\text{(ii)} \quad : \quad g(x) = \frac{2ix + 2}{2ix}, \quad \mathcal{E}_1(\boldsymbol{\lambda}) = \tfrac{1}{2}, \tag{3.42}$$

$$V_1(x; \boldsymbol{\lambda}) = V(x - \tfrac{i}{2}; a, b, c)g(x) = V(x; a + \tfrac{1}{2}, b + \tfrac{1}{2}, c + \tfrac{1}{2}). \tag{3.43}$$

$$\text{(iii)} \quad : \quad g(x) = \frac{2ix + 2}{2ix}, \quad \mathcal{E}_1(\boldsymbol{\lambda}) = \tfrac{1}{2}(a + b + c + d), \tag{3.44}$$

$$V_1(x; \boldsymbol{\lambda}) = V(x - \tfrac{i}{2}; a, b, c, d)g(x) = V(x; a + \tfrac{1}{2}, b + \tfrac{1}{2}, c + \tfrac{1}{2}, d + \tfrac{1}{2}). \tag{3.45}$$

With $\boldsymbol{\frac{1}{2}} = (\frac{1}{2}, \frac{1}{2}, \cdots)$, we have collectively

$$V_1(x; \boldsymbol{\lambda}) = V(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{1}{2}}), \ A_1(x; \boldsymbol{\lambda}) = A(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{1}{2}}), \ H_1(x; \boldsymbol{\lambda}) = H(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{1}{2}}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{3.46}$$

Starting from $V_0 = V$, $H_0 = H$, $\phi_{0,n} = \phi_n$, let us define V_s, H_s, $\phi_{s,n}$ ($n \geq s \geq 0$) step by step:

$$V_{s+1}(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} V_s(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{1}{2}}), \tag{3.47}$$

$$H_{s+1}(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} A_s(x; \boldsymbol{\lambda})A_s(x; \boldsymbol{\lambda})^\dagger + \mathcal{E}_s(\boldsymbol{\lambda}), \tag{3.48}$$

$$\phi_{s+1,n}(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} A_s(x; \boldsymbol{\lambda})\phi_{s,n}(x; \boldsymbol{\lambda}), \tag{3.49}$$

$$\mathcal{E}_s(\boldsymbol{\lambda}) = \mathcal{E}_{s-1}(\boldsymbol{\lambda}) + \mathcal{E}_1(\boldsymbol{\lambda} + (\boldsymbol{s} - \boldsymbol{1})/\boldsymbol{2}). \tag{3.50}$$

Here A_s, $A_s^\dagger$ are defined by

$$A_s(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\partial_x}\sqrt{V_s^*(x; \boldsymbol{\lambda})} - e^{\frac{i}{2}\partial_x}\sqrt{V_s(x; \boldsymbol{\lambda})}\right), \tag{3.51}$$

$$A_s(x; \boldsymbol{\lambda})^\dagger \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\sqrt{V_s(x; \boldsymbol{\lambda})}\, e^{-\frac{i}{2}\partial_x} - \sqrt{V_s^*(x; \boldsymbol{\lambda})}\, e^{\frac{i}{2}\partial_x}\right). \tag{3.52}$$

As a consequence of the shape invariance, we obtain for $n \geq s \geq 0$:

$$V_s(x; \boldsymbol{\lambda}) = V(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{s}{2}}), \tag{3.53}$$

$$A_s(x; \boldsymbol{\lambda}) = A(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{s}{2}}), \quad A_s(x; \boldsymbol{\lambda})^\dagger = A(x; \boldsymbol{\lambda} + \boldsymbol{\tfrac{s}{2}})^\dagger, \tag{3.54}$$

$$H_s(x; \boldsymbol{\lambda}) = A_s(x; \boldsymbol{\lambda})^\dagger A_s(x; \boldsymbol{\lambda}) + \mathcal{E}_s(\boldsymbol{\lambda}), \tag{3.55}$$

$$H_s(x; \boldsymbol{\lambda})\phi_{s,n}(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_{s,n}(x; \boldsymbol{\lambda}), \tag{3.56}$$

$$A_s(x; \boldsymbol{\lambda})\phi_{s,s}(x; \boldsymbol{\lambda}) = 0, \tag{3.57}$$

$$A_s(x; \boldsymbol{\lambda})^\dagger \phi_{s+1,n}(x; \boldsymbol{\lambda}) = (\mathcal{E}_n(\boldsymbol{\lambda}) - \mathcal{E}_s(\boldsymbol{\lambda}))\phi_{s,n}(x; \boldsymbol{\lambda}). \tag{3.58}$$

The eigenfunction $\phi_{s,n}$ has $n - s$ nodes, or the corresponding polynomial is of degree $n - s$ in x for (i), and in x^2 for (ii) and (iii). In the latter case, the zeros on a half line, say $x \geq 0$, count as the nodes.

From these we obtain formulas

$$\phi_{s,n}(x; \boldsymbol{\lambda}) = A_{s-1}(x; \boldsymbol{\lambda}) \cdots A_1(x; \boldsymbol{\lambda}) A_0(x; \boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda}), \tag{3.59}$$

$$\phi_n(x; \boldsymbol{\lambda}) = \frac{A_0(x; \boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda}) - \mathcal{E}_0(\boldsymbol{\lambda})} \frac{A_1(x; \boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda}) - \mathcal{E}_1(\boldsymbol{\lambda})} \cdots \frac{A_{n-1}(x; \boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda}) - \mathcal{E}_{n-1}(\boldsymbol{\lambda})} \phi_{n,n}(x; \boldsymbol{\lambda}), \tag{3.60}$$

corresponding to (2.34) and (2.42) discussed in section 2. The former (3.59) gives the eigenfunction $\phi_{s,n}$ of the s-th Hamiltonian H_s along the isospectral line with energy $\mathcal{E}_n$, starting from ϕ_n of the original Hamiltonian H by repeated application of the A operators. The latter (3.60), on the other hand, expresses the n-th eigenfunction ϕ_n of the original Hamiltonian, starting from the explicitly known ground state $\phi_{n,n}$ of the n-th Hamiltonian H_n by repeated application of the $A^\dagger$ operators. The latter formula (3.60), a simple generalisation of the well-known formula for the harmonic oscillator $|n\rangle \propto (a^\dagger)^n |0\rangle$, could also be understood as the generic form of the Rodrigue's formula for the orthogonal polynomials.

In order to define the annihilation and creation operators, let us introduce normalised basis $\{\hat{\phi}_{s,n}\}_{n \geq s}$ for each Hamiltonian H_s. Ordinarily, the phase of each element of an orthonormal basis could be completely arbitrary. In the present case, however, the eigenfunctions are orthogonal polynomials. That is, they are real and the relations among different degree members are governed by the three term recurrence relations. So the phases of $\{\hat{\phi}_{s,n}\}_{n \geq s}$ are fixed. Let us introduce a unitary (in fact an orthogonal) operator U_s mapping the s-th orthonormal basis $\{\hat{\phi}_{s,n}\}_{n \geq s}$ to the $(s+1)$-th $\{\hat{\phi}_{s+1,n}\}_{n \geq s+1}$ (see Fig. 1 and for example [16, 17]):

$$U_s \hat{\phi}_{s,n} = \hat{\phi}_{s+1,n+1}, \quad U_s^\dagger \hat{\phi}_{s+1,n+1} = \hat{\phi}_{s,n}. \tag{3.61}$$

We denote that $U_0 = U$. Roughly speaking U increases the parameters from $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda} + \frac{1}{2}$:

$$U : P_n^{(\boldsymbol{\lambda})}(x) \to P_n^{(\boldsymbol{\lambda} + \frac{1}{2})}(x), \quad U^\dagger : P_n^{(\boldsymbol{\lambda} + \frac{1}{2})}(x) \to P_n^{(\boldsymbol{\lambda})}(x).$$

Let us introduce an annihilation $\tilde{a}$ and a creation operator $\tilde{a}^\dagger$ for the Hamiltonian H as follows:

$$\tilde{a} = \tilde{a}(x; \boldsymbol{\lambda}) \overset{\text{def}}{=} U^\dagger A(x; \boldsymbol{\lambda}), \quad \tilde{a}^\dagger = \tilde{a}(x; \boldsymbol{\lambda})^\dagger \overset{\text{def}}{=} A(x; \boldsymbol{\lambda})^\dagger U. \tag{3.62}$$

It is straightforward to derive

$$H = \tilde{a}^\dagger \tilde{a}, \tag{3.63}$$

$$[\tilde{a}, \tilde{a}^\dagger] \hat{\phi}_n(x; \lambda) = (\mathcal{E}_{n+1}(\boldsymbol{\lambda}) - \mathcal{E}_n(\boldsymbol{\lambda})) \hat{\phi}_n(x; \lambda). \tag{3.64}$$

For the harmonic oscillator (2.1) $U =$ id. and we recover the known result. For the linear spectrum $\mathcal{E}_n = n, n/2$ of the deformed Hermite polynomial (2.3) and the two parameter deformation of the Laguerre polynomial (3.5)–(3.8), $\tilde{a}$ and $\tilde{a}^\dagger$ have the same (up to rescaling) commutation relations as those of the harmonic oscillator. Essentially the same arguments and results hold for the annihilation $\tilde{a}_s = U_s^\dagger A_s(x; \boldsymbol{\lambda})$ and the creation operator $\tilde{a}_s^\dagger = A_s(x; \boldsymbol{\lambda}) U_s$ for the Hamiltonian H_s.

4 Dynamical Background

Here we will show briefly a logical (dynamical) path that led to the shape invariant difference equations introduced in section 2. The equilibrium position of an n-particle A-type Calogero system is determined by

$$\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{x_j-x_k}=x_j\,,\quad j=1,\ldots,n, \tag{4.1}$$

after adjustment of the coupling constants and rescaling of the variables [8, 9]. They describe the zeros of the Hermite polynomial, since $H_n(x)\stackrel{\text{def}}{=}2^n\prod_{j=1}^n(x-x_j)$ satisfies the differential equation

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0 \tag{4.2}$$

and the three term recurrence [9] $n\geq 0$,

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0,\quad H_0=1,\quad H_{-1}=0. \tag{4.3}$$

The equilibrium position of an n-particle A-type van Diejen system (or the rational R-S system with a linear confining potential) is determined by

$$\prod_{\substack{k=1\\k\neq j}}^{n}\frac{x_j-x_k-i\sqrt{\delta}}{x_j-x_k+i\sqrt{\delta}}=\frac{1-i\sqrt{\delta}\,x_j}{1+i\sqrt{\delta}\,x_j}\,,\quad j=1,\ldots,n, \tag{4.4}$$

after adjustment of the coupling constants and rescaling of the variables [11]. The corresponding *deformed* polynomial $H_n(x,\delta)\stackrel{\text{def}}{=}2^n\prod_{j=1}^n(x-x_j)$ satisfies the difference equation

$$\left(x+\frac{i}{\sqrt{\delta}}\right)H_n(x+i\sqrt{\delta},\delta)-\left(x-\frac{i}{\sqrt{\delta}}\right)H_n(x-i\sqrt{\delta},\delta)=\frac{2i}{\sqrt{\delta}}(1+n\delta)H_n(x,\delta), \tag{4.5}$$

and the three term recurrence [11] $n\geq 0$,

$$H_{n+1}(x,\delta)-2xH_n(x,\delta)+\Big(2n+n(n-1)\delta\Big)H_{n-1}(x,\delta)=0,\quad H_0=1,\quad H_{-1}=0. \tag{4.6}$$

It is elementary to show that (4.4), (4.5), (4.6) reduce to (4.1), (4.2), (4.3) in the zero deformation limit $\delta\to 0$; $\lim_{\delta\to 0}H_n(x,\delta)=H_n(x)$. The relationship between the deformed Hermite polynomial and the special case of the Meixner-Pollaczek polynomial discussed in section 2 is

$$H_n(x,\delta)=n!\sqrt{\delta}^{\,n}P_n^{(\frac{1}{\delta})}\left(\frac{x}{\sqrt{\delta}}\,;\frac{\pi}{2}\right). \tag{4.7}$$

It is instructive to write down the classical Hamiltonian of the n-particle A-type van Diejen system

$$H(p,q)=\sum_{j=1}^{n}\left(\cosh p_j\,\sqrt{V_j(q)\,V_j^*(q)}-\frac{1}{2}\Big(V_j(q)+V_j^*(q)\Big)\right), \tag{4.8}$$

$$V_j(q)=\left(1+\frac{iq_j}{a}\right)\prod_{\substack{k=1\\k\neq j}}^{n}\left(1-\frac{ig}{q_j-q_k}\right), \tag{4.9}$$

in which a and g are coupling constants. In fact, the Hamiltonian (2.3) is the single particle ($n = 1$) case with $p_j = -i\partial/\partial q_j$ and proper ordering. We refer to [11] for similar orientation of the dynamical systems introduced in section 3.

5   Comments and Discussion

Several examples of shape invariant difference equations are discussed in some detail. They are related to deformation of the harmonic oscillator without/with a centrifugal potential and their eigenfunctions are deformed Hermite and Laguerre polynomials belonging to the Askey-scheme of hypergeometric orthogonal polynomials. They arise in the problems of determining the equilibrium positions of the Ruijsenaars-Schneider-van Diejen systems, which are integrable deformation of the celebrated Calogero-Sutherland systems of exactly solvable multi-particle quantum mechanics. Here we have treated rational potentials $V(x)$ only. Obviously the method works for a wider range of potentials, the trigonometric, hyperbolic, elliptic, *etc*. We have not discussed the shape invariance of the discrete systems corresponding to the equilibria of the trigonometric Ruijsenaars-Schneider systems. These have various kinds of deformed Jacobi polynomials as eigenfunctions [11]. We will report on this subject elsewhere.

In the literature, shape invariance is discussed almost always within the context of 'supersymmetric quantum mechanics'[1]. We presume that this might be the psychological barrier for considering the shape invariance in discrete quantum mechanics, or shape invariant difference equations. As is well-known, supersymmetry is the 'square root' of the space-time symmetry, say the Poincaré symmetry, which is usually lost if the space-time is discretised..... In fact, it is important to realise that main results of 'supersymmetric quantum mechanics' are already contained in Crum's theorem without supersymmetry or shape invariance. Among them are: the existence of *associated isospectral* Hamiltonians $H_1, \ldots, H_s, \ldots$, as many as the number of discrete levels of H_0 and the formulas of their eigenfunctions. For example, $\phi_{s,n}$ of H_s with the eigenvalue $\mathcal{E}_n$ is expressed in terms of the eigenfunctions $\phi_0, \phi_1, \ldots, \phi_{s-1}, \phi_n$ of H_0 (Fig. 1):

$$\phi_{s,n} = \frac{\mathcal{W}_{s,n}}{\mathcal{W}_s}, \quad n \geq s \geq 1, \tag{5.1}$$

$$\mathcal{W}_s = \mathcal{W}[\phi_0, \ldots, \phi_{s-1}], \quad \mathcal{W}_{s,n} = \mathcal{W}[\phi_0, \ldots, \phi_{s-1}, \phi_n]. \tag{5.2}$$

Here the Wronskian $\mathcal{W}$ is defined as usual $\mathcal{W}[f_1, \ldots, f_n] = \det\left(f_j^{(i-1)}\right)_{1 \leq i,j \leq n}$. This formula, translated in the discrete dynamics context, corresponds to (3.59).

We do believe dynamical (Hamiltonian) interpretation of discrete quantum systems (difference equations) would be useful and fruitful. One of our starting points, equations determining a certain equilibrium, *eg.* (4.4), are called Bethe ansatz like equations. The present problems could be considered as those of finding associated polynomial solutions of Bethe ansatz like equations. They occur in many branches of theoretical physics, for example, the quasi-exactly solvable single and multi-particle quantum systems [18] on top of the well-known integrable spin chains [19].

Acknowledgements

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