DPSU-05-2, September 2005
Prog.Theor.Phys.**114**(2005)1245-1260
hep-th/0512155

Calogero-Sutherland-Moser Systems, Ruijsenaars-Schneider-van Diejen Systems and Orthogonal Polynomials

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The equilibrium positions of the multi-particle classical Calogero-Sutherland-Moser (CSM) systems with rational/trigonometric potentials associated with the classical root systems are described by the classical orthogonal polynomials; the Hermite, Laguerre and Jacobi polynomials. The eigenfunctions of the corresponding single-particle quantum CSM systems are also expressed in terms of the same orthogonal polynomials. We show that this interesting property is inherited by the Ruijsenaars-Schneider-van Diejen (RSvD) systems, which are integrable deformation of the CSM systems; the equilibrium positions of the multi-particle classical RSvD systems and the eigenfunctions of the corresponding single-particle quantum RSvD systems are described by the same orthogonal polynomials, the continuous Hahn (special case), Wilson and Askey-Wilson polynomials. They belong to the Askey-scheme of the basic hypergeometric orthogonal polynomials and are deformation of the Hermite, Laguerre and Jacobi polynomials, respectively. The Hamiltonians of these single-particle quantum mechanical systems have two remarkable properties, factorization and shape invariance.

§1. Introduction

Exactly solvable and/or quasi-exactly solvable multi-particle quantum mechanical systems have many remarkable properties. Especially, those of the Calogero-Sutherland-Moser (CSM) systems[1)–3)] and their integrable deformation, called the Ruijsenaars-Schneider-van Diejen (RSvD) systems,[4),5)] have been extensively studied. They have many attractive features at both classical and quantum mechanical levels and are related to topics studied in other fields.

The equilibrium positions of the classical CSM systems with rational/trigonometric potentials associated with the classical root systems are given by the zeros of the classical orthogonal polynomials, namely, the Hermite, Laguerre and Jacobi (Chebyshev, Legendre, Gegenbauer) polynomials.[6)–9)] For example, the Hamiltonian of the A-type multi-particle classical Calogero system is

$$H = \sum_{j=1}^{n} \frac{1}{2m} p_j^2 + \sum_{j=1}^{n} \frac{1}{2} m\omega^2 q_j^2 + \frac{\bar{g}^2}{2m} \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \frac{1}{(q_j - q_k)^2} + \text{const.}\,, \tag{1}$$

and its equilibrium positions $\bar{q}_j$ ($y_j = \sqrt{\frac{m\omega}{\bar{g}}}\bar{q}_j$) are determined by

$$\sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{y_j - y_k} = y_j\,. \qquad (j = 1, \ldots, n) \tag{2}$$

This equation can be converted into a differential equation for $f(y)=\prod_{j=1}^{n}(y-y_j)$. It is easy to show that $\frac{f''}{f}-2y\frac{f'}{f} \xrightarrow{y\to y_j}$ const. Thus the polynomial $f(y)$ satisfies $f''-2yf'=Af$, in which the constant A is determined as $A=-2n$. This is the differential equation for the Hermite polynomial $H_n(y)$. Therefore the equilibrium positions are given by the zeros of the Hermite polynomial. On the other hand, the corresponding single-particle quantum system is described by the Hamiltonian of

$$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2+\text{const.} \tag{3}$$

This is just a harmonic oscillator, and its eigenfunctions are given by (ground state eigenfunction) $\times$ (Hermite polynomials). Interestingly, the equilibrium positions of the multi-particle classical system and the eigenfunctions of the corresponding single-particle quantum system are described by the same orthogonal polynomials, the Hermite polynomials. This is not a fortuitous coincidence. In fact, the B-type (or C-type) Calogero systems and the BC-type Sutherland systems have the same property. In these cases, the relevant orthogonal polynomials are the Laguerre and Jacobi polynomials, respectively.

In this article, we establish that the above interesting property is inherited by the RSvD systems, which are integrable deformation of the CSM systems. Namely, we show that the equilibrium positions of the multi-particle classical RSvD systems with rational/trigonometric potentials associated with the classical root systems and the eigenfunctions of the corresponding single-particle quantum RSvD systems are described by the same orthogonal polynomials; the continuous Hahn (special case), Wilson and Askey-Wilson polynomials, which are deformation of the Hermite, Laguerre and Jacobi polynomials, respectively. These polynomials are members of the Askey-scheme of the basic hypergeometric orthogonal polynomials.[10)–12)] Deformation patterns are shown in Table I. For the trigonometric A_{n-1} systems, the situation is rather trivial; the equilibrium positions are equally-spaced and translationally invariant, and the single-particle system is a free theory. Although we are able to relate them to Chebyshev polynomials of the first kind,[9),13),14)] we do not demonstrate this here.

Table I. Deformation patterns.

potential	root system	CSM system	$\rightsquigarrow$	RSvD system
rational	A_{n-1}	Hermite poly.	$\rightsquigarrow$	continuous Hahn poly.
rational	BC_n	Laguerre poly.	$\rightsquigarrow$	Wilson poly.
trigonometric	BC_n	Jacobi poly.	$\rightsquigarrow$	Askey-Wilson poly.

For these single-particle quantum mechanical systems, the Hamiltonian has two remarkable properties, *factorization* and *shape invariance*.[15)–19)] Shape invariance is an important ingredient of many exactly solvable quantum mechanics. In our case, the shape invariance determines the eigenfunctions and spectrum from the data of the ground state eigenfunction and the energy of the first excited state.[20),21)]

This article is organized as follows. In §2 we recapitulate the essence of the CSM systems with rational/trigonometric potentials associated with the classical

root systems. We briefly outline the argument that the equilibrium positions of the multi-particle classical systems and the eigenfunctions of the corresponding single-particle quantum systems are described by the same orthogonal polynomials; the Hermite, Laguerre and Jacobi polynomials. In §3 we generalize this argument to the RSvD systems. The Hermite, Laguerre and Jacobi polynomials are deformed to the continuous Hahn (special case), Wilson and Askey-Wilson polynomials, respectively. Section 4 is for a summary and comments. In order to study single-particle systems, we repeatedly use the shape invariance of the single-particle Hamiltonian. In appendix some useful techniques for shape invariant single-particle ('discrete') quantum mechanics are summarized along the idea of Crum.[16)]

§2. Calogero-Sutherland-Moser systems

In this section we summarize the basics of the Calogero-Sutherland-Moser systems with rational/trigonometric potentials associated with the classical root systems.

2.1. *Models*

A CSM system is a multi-particle quantum (or classical) mechanical system governed by a Hamiltonian $H(p,q)$ (or classical one, $H^{\text{class}}(p,q)$).[1)–3)] The dynamical variables are the real-valued coordinates $q = {}^t(q_1, \cdots, q_n)$ and their canonically conjugate momenta $p = {}^t(p_1, \cdots, p_n)$. For the quantum case we have $p_j = -i\hbar\frac{\partial}{\partial q_j}$. We keep dimensionful parameters, e.g., mass, angular frequency, the Planck constant, etc. The coordinate q_j has the dimension of length.

The Hamiltonian of the CSM systems is

$$H_{\text{CS}}(p,q) = \sum_{j=1}^{n} \frac{1}{2m} p_j^2 + V_{\text{CS}}(q)\,, \tag{4}$$

where the potential $V_{\text{CS}}(q)$ can be written in terms of the prepotential $W(q)$,

$$V_{\text{CS}}(q) = \sum_{j=1}^{n} \frac{1}{2m}\left(\Big(\frac{\partial W(q)}{\partial q_j}\Big)^2 + \hbar\,\frac{\partial^2 W(q)}{\partial q_j^2}\right). \tag{5}$$

The explicit forms of the potential $V_{\text{CS}}(q)$ and the prepotential $W(q)$ are as follows:
(i) rational A_{n-1} :

$$V_{\text{CS}}(q) = \sum_{j=1}^{n} \frac{1}{2} m\omega^2 q_j^2 + \frac{\hbar^2}{2m} \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \frac{g(g-1)}{(q_j-q_k)^2} - \tfrac{1}{2}\hbar\omega n\big(1+g(n-1)\big)\,, \tag{6a}$$

$$W(q) = -\sum_{j=1}^{n} \frac{1}{2} m\omega q_j^2 + \sum_{1\le j<k\le n} g\hbar \log\sqrt{\tfrac{m\omega}{\hbar}}\,\big|q_j - q_k\big|\,, \tag{6b}$$

(ii) rational BC_n :[*)]

$$
\begin{aligned}
V_{\mathrm{CS}}(q) &= \sum_{j=1}^{n}\Big(\frac{1}{2}m\omega^2 q_j^2+\frac{\hbar^2}{2m}\frac{(g_S+g_L)(g_S+g_L-1)}{q_j^2}\Big)\\
&\quad+\frac{\hbar^2}{2m}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\Big(\frac{g_M(g_M-1)}{(q_j-q_k)^2}+\frac{g_M(g_M-1)}{(q_j+q_k)^2}\Big)\\
&\quad-\hbar\omega n\big(g_S+g_L+\tfrac{1}{2}+g_M(n-1)\big)\,, \qquad (7a)\\
W(q) &= -\sum_{j=1}^{n}\frac{1}{2}m\omega q_j^2+\sum_{1\le j<k\le n} g_M\hbar\Big(\log\sqrt{\tfrac{m\omega}{\hbar}}\,\big|q_j-q_k\big|+\log\sqrt{\tfrac{m\omega}{\hbar}}\,\big|q_j+q_k\big|\Big)\\
&\quad+\sum_{j=1}^{n}\Big(g_S\hbar\log\sqrt{\tfrac{m\omega}{\hbar}}\,\big|q_j\big|+g_L\hbar\log\sqrt{\tfrac{m\omega}{\hbar}}\,\big|2q_j\big|\Big)\,, \qquad (7b)
\end{aligned}
$$

(iii) trigonometric BC_n :

$$
\begin{aligned}
V_{\mathrm{CS}}(q) &= \frac{\hbar^2\pi^2}{2mL^2}\sum_{j=1}^{n}\Big(\frac{(g_S+g_L)(g_S+g_L-1)}{\sin^2\frac{\pi}{L}q_j}+\frac{g_L(g_L-1)}{\cos^2\frac{\pi}{L}q_j}\Big)\\
&\quad+\frac{\hbar^2\pi^2}{2mL^2}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\Big(\frac{g_M(g_M-1)}{\sin^2\frac{\pi}{L}(q_j-q_k)}+\frac{g_M(g_M-1)}{\sin^2\frac{\pi}{L}(q_j+q_k)}\Big)\\
&\quad-\frac{\hbar^2\pi^2}{2mL^2}n\Big(\big(g_S+2g_L+g_M(n-1)\big)^2+g_M^2\tfrac{1}{3}(n^2-1)\Big)\,, \qquad (8a)\\
W(q) &= \sum_{1\le j<k\le n} g_M\hbar\Big(\log\big|\sin\tfrac{\pi}{L}(q_j-q_k)\big|+\log\big|\sin\tfrac{\pi}{L}(q_j+q_k)\big|\Big)\\
&\quad+\sum_{j=1}^{n}\Big(g_S\hbar\log\big|\sin\tfrac{\pi}{L}q_j\big|+g_L\hbar\log\big|\sin\tfrac{\pi}{L}2q_j\big|\Big)\,. \qquad (8b)
\end{aligned}
$$

The constant terms in $V_{\mathrm{CS}}(q)$ are the consequences of the expression (5) in terms of the prepotential. A constant shift of $W(q)$ does not affect (5). In the above formulas g, g_S, g_M and g_L are dimensionless coupling constants and we assume they are positive. The other notation is conventional: m is the mass of the particles, ω is the angular frequency, $\hbar$ is the Planck constant (divided by 2π) and L is the circumference. All these parameters are positive.

Note that the Hamiltonian (4) can be expressed as a sum of factorized forms:

$$
H_{\mathrm{CS}}=\sum_{j=1}^{n}\frac{1}{2m}\Big(p_j-i\,\frac{\partial W(q)}{\partial q_j}\Big)\Big(p_j+i\,\frac{\partial W(q)}{\partial q_j}\Big)\,. \qquad (9)
$$

[*)] Since the independent coupling constants are g_M and g_S+g_L, this BC_n model is equivalent to the B_n model (or the C_n model).

2.2. *Equilibrium positions of n-particle classical mechanics*

The classical Hamiltonian $H^{\rm class}(p,q)$ is obtained from the quantum one $H(p,q)$ by the following procedure: (a) regard p_j is a c-number; (b) after replacing the dimensionless coupling constants $g, g_1, g_2, \cdots$ by the dimensionful ones $\bar{g} = g\hbar, \bar{g}_1 = g_1\hbar, \bar{g}_2 = g_2\hbar, \cdots$, assume $\bar{g}, \bar{g}_1, \bar{g}_2, \cdots$ are independent of $\hbar$; (c) take the $\hbar \to 0$ limit. In the same way, $V_{\rm CS}^{\rm class}(q)$ and $W^{\rm class}(q)$ are also obtained.

The canonical equations of motion of the classical systems are

$$\frac{dq_j}{dt} = \frac{\partial H^{\rm class}(p,q)}{\partial p_j}\,, \quad \frac{dp_j}{dt} = -\frac{\partial H^{\rm class}(p,q)}{\partial q_j}\,. \tag{10}$$

The equilibrium positions are given by the stationary solution

$$p = 0\,, \quad q = \bar{q}\,, \tag{11}$$

in which $\bar{q}$ satisfies

$$\frac{\partial H^{\rm class}(0,q)}{\partial q_j}\,\Bigg|_{q=\bar{q}} = 0\,. \qquad (j = 1, \ldots, n) \tag{12}$$

For the CSM system, (12) becomes $\frac{\partial V_{\rm CS}^{\rm class}(q)}{\partial q_j}\,\Big|_{q=\bar{q}} = 0$ and it is equivalent to the condition[22)]

$$\frac{\partial W^{\rm class}(q)}{\partial q_j}\,\Bigg|_{q=\bar{q}} = 0\,. \qquad (j = 1, \ldots, n) \tag{13}$$

For the rational A_{n-1} system, the above condition reduces to (2). Let us consider a polynomial whose zeros give the equilibrium positions:

$$\text{(i) rational } A_{n-1}\ : \qquad f_n(y) = \prod_{j=1}^{n}\Big(y - \sqrt{\frac{m\omega}{\bar{g}}}\,\bar{q}_j\Big)\,, \tag{14a}$$

$$\text{(ii) rational } BC_n\ : \qquad f_n(y) = \prod_{j=1}^{n}\Big(y^2 - \frac{m\omega}{\bar{g}_M}\,\bar{q}_j^{\,2}\Big)\,, \tag{14b}$$

$$\text{(iii) trigonometric } BC_n\ : \quad f_n(\xi) = \prod_{j=1}^{n}\Big(\xi - \cos\big(2\tfrac{\pi}{L}\bar{q}_j\big)\Big)\,. \tag{14c}$$

Then Eq.(13) can be converted into a differential equation for $f_n(x)$, which is satisfied by the Hermite, Laguerre and Jacobi polynomials, respectively. The results are (see, for example, Refs. 6),8),13),22))

$$\text{(i)}\ :\quad f_n(y) = H_n^{\rm monic}(y)\,, \tag{15a}$$

$$\text{(ii)}\ :\quad f_n(y) = L_n^{(\alpha)\,\rm monic}(y^2)\,, \quad \alpha = \frac{\bar{g}_S + \bar{g}_L}{\bar{g}_M} - 1\,, \tag{15b}$$

$$\text{(iii)}\ :\quad f_n(\xi) = P_n^{(\alpha,\beta)\,\rm monic}(\xi)\,, \quad \alpha = \frac{\bar{g}_S + \bar{g}_L}{\bar{g}_M} - 1\,,\ \beta = \frac{\bar{g}_L}{\bar{g}_M} - 1\,. \tag{15c}$$

Here $H_n(y) = 2^n H_n^{\rm monic}(y)$, $L_n^{(\alpha)}(y^2) = \frac{(-1)^n}{n!} L_n^{(\alpha)\,\rm monic}(y^2)$ and $P_n^{(\alpha,\beta)}(\xi) = \frac{1}{2^n}\binom{\alpha+\beta+2n}{n} P_n^{(\alpha,\beta)\,\rm monic}(\xi)$ are the Hermite, Laguerre and Jacobi polynomials, respectively.[11)]

2.3. *Eigenfunctions of single-particle quantum mechanics*

The Hamiltonian (4) for single-particle case ($n = 1$) reads

$$\text{(i)}: \quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2}\hbar\omega\,, \tag{16a}$$

$$\text{(ii)}: \quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 + \frac{\hbar^2}{2m}\frac{g(g-1)}{x^2} - \hbar\omega\Big(g+\frac{1}{2}\Big)\,, \tag{16b}$$

$$\text{(iii)}: \quad H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2\pi^2}{2mL^2}\Big(\frac{g(g-1)}{\sin^2\frac{\pi}{L}x} + \frac{g'(g'-1)}{\cos^2\frac{\pi}{L}x}\Big) - \frac{\hbar^2\pi^2}{2mL^2}(g+g')^2\,, \tag{16c}$$

where $x = q_1$. These are a harmonic oscillator, a harmonic oscillator with a centrifugal barrier, and the Pöschl-Teller potential,[23] respectively, with some constant energy shifts. By introducing the dimensionless variable y,

$$\text{(i), (ii)}: \quad y = \sqrt{\frac{m\omega}{\hbar}}x\,, \qquad \text{(iii)}: \quad y = \frac{\pi}{L}x\,, \tag{17}$$

these Hamiltonians are reduced to dimensionless ones $\mathcal{H}$:

$$\text{(i)}: \quad H = \hbar\omega\mathcal{H}\,, \qquad \mathcal{H} = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}y^2 - \frac{1}{2}\,, \tag{18a}$$

$$\text{(ii)}: \quad H = \hbar\omega\mathcal{H}\,, \qquad \mathcal{H} = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}y^2 + \frac{g(g-1)}{2y^2} - g - \frac{1}{2}\,, \tag{18b}$$

$$\text{(iii)}: \quad H = \frac{\hbar^2\pi^2}{mL^2}\mathcal{H}\,, \quad \mathcal{H} = \frac{1}{2}\Big(-\frac{d^2}{dy^2} + \frac{g(g-1)}{\sin^2 y} + \frac{g'(g'-1)}{\cos^2 y} - (g+g')^2\Big)\,. \tag{18c}$$

They have two properties, *factorization* (A·3) and *shape invariance* (A·5). In the notation of the appendix, the necessary data of the shape invariance are the invariant (pre)potential $\mathcal{W}(y\,;\boldsymbol{\lambda})$, the set of parameters $\boldsymbol{\lambda}$, the elementary shift of the parameters $\boldsymbol{\delta}$ and the energy of the first excited state $\mathcal{E}_1(\boldsymbol{\lambda})$:

$$\text{(i)}: \quad \mathcal{W}(y) = -\frac{1}{2}y^2, \quad \text{no } \boldsymbol{\lambda}\ (\text{no } \boldsymbol{\delta}), \quad \mathcal{E}_1 = 1, \tag{19a}$$

$$\text{(ii)}: \quad \mathcal{W}(y\,;\boldsymbol{\lambda}) = -\tfrac{1}{2}y^2 + g\log y\,, \quad \boldsymbol{\lambda} = g, \quad \boldsymbol{\delta} = 1, \quad \mathcal{E}_1(\boldsymbol{\lambda}) = 2, \tag{19b}$$

$$\text{(iii)}: \quad \mathcal{W}(y\,;\boldsymbol{\lambda}) = g\log\sin y + g'\log\cos y\,,$$
$$\boldsymbol{\lambda} = (g, g'), \quad \boldsymbol{\delta} = (1,1), \quad \mathcal{E}_1(\boldsymbol{\lambda}) = 2(g+g'+1). \tag{19c}$$

From the formulas (A·16)–(A·17) and (A·11), we obtain the eigenfunctions $P_n\phi_0$ and the corresponding eigenvalues $\mathcal{E}_n$ of $\mathcal{H}$:

$$\text{(i)}: \quad P_n(y) \propto H_n(y), \quad \mathcal{E}_n = n, \tag{20a}$$

$$\text{(ii)}: \quad P_n(y\,;\boldsymbol{\lambda}) \propto L_n^{(g-\frac{1}{2})}(y^2), \quad \mathcal{E}_n(\boldsymbol{\lambda}) = 2n, \tag{20b}$$

$$\text{(iii)}: \quad P_n(y\,;\boldsymbol{\lambda}) \propto P_n^{(g-\frac{1}{2},g'-\frac{1}{2})}(\cos 2y), \quad \mathcal{E}_n(\boldsymbol{\lambda}) = 2n(n+g+g'). \tag{20c}$$

Thus the equilibrium positions of the multi-particle classical systems and the eigenfunctions of the single-particle quantum systems are described by the same orthogonal polynomials; the Hermite, Laguerre and Jacobi polynomials.

§3. Ruijsenaars-Schneider-van Diejen systems

In this section we generalize the results of the previous section to the Ruijsenaars-Schneider-van Diejen (RSvD) systems with the rational/trigonometric potentials associated with the classical root systems.

3.1. *Models*

The RSvD system is an integrable deformation of the CSM system.[4),5)] The Hamiltonian of the RSvD systems is

$$H(p,q)=\frac{1}{2}mc^2\sum_{j=1}^{n}\Big(\sqrt{V_j(q)}\,e^{\frac{1}{mc}p_j}\sqrt{V_j(q)^*}+\sqrt{V_j(q)^*}\,e^{-\frac{1}{mc}p_j}\sqrt{V_j(q)}-V_j(q)-V_j(q)^*\Big)\,, \tag{21}$$

where $V_j(q)$ is given by

$$V_j(q)=w(q_j)\prod_{\substack{k=1\\k\neq j}}^{n}v(q_j-q_k)\times\begin{cases}1 & \text{for } A_{n-1},\\ v(q_j+q_k) & \text{for } BC_n\,.\end{cases} \tag{22}$$

We use the conventional notation that $V_j(q)^*$ is the complex conjugate of $V_j(q)$. Since the operators $e^{\pm\frac{1}{mc}p_j}=e^{\mp i\frac{\hbar}{mc}\frac{\partial}{\partial q_j}}$ cause finite shifts of the wavefunction in the imaginary direction ($e^{\pm\frac{1}{mc}p_j}f(q)=f(q_1,\cdots,q_j\mp i\frac{\hbar}{mc},\cdots,q_n)$), we call these systems 'discrete' dynamical systems. (Sometimes they are misleadingly called 'relativistic' version of the CSM. See Ref. 24) for comments on this point.) The basic potential functions $v(x)$ and $w(x)$ are as follows:
(i) rational A_{n-1} :

$$v(x)=1-i\frac{\hbar}{mc}\frac{g}{x}\,, \tag{23a}$$

$$w(x)=\Big(1+i\,\frac{\omega_1}{c}\,x\Big)\Big(1+i\,\frac{\omega_2}{c}\,x\Big)\,, \tag{23b}$$

(ii) rational BC_n :

$$v(x)=1-i\frac{\hbar}{mc}\frac{g_0}{x}\,, \tag{24a}$$

$$w(x)=\Big(1+i\,\frac{\omega_1}{c}\,x\Big)\Big(1+i\,\frac{\omega_2}{c}\,x\Big)\Big(1-i\frac{\hbar}{mc}\frac{g_1}{x}\Big)\Big(1-i\frac{\hbar}{mc}\frac{g_2}{x-i\frac{\hbar}{2mc}}\Big)\,, \tag{24b}$$

(iii) trigonometric BC_n :

$$v(x)=\frac{\sin\frac{\pi}{L}(x-i\frac{\hbar}{mc}g_0)}{\sin\frac{\pi}{L}x}\,, \tag{25a}$$

$$w(x)=\frac{\sin\frac{\pi}{L}(x-i\frac{\hbar}{mc}g_1)}{\sin\frac{\pi}{L}x}\,\frac{\sin\frac{\pi}{L}(x-i\frac{\hbar}{2mc}-i\frac{\hbar}{mc}g_2)}{\sin\frac{\pi}{L}(x-i\frac{\hbar}{2mc})}\times\frac{\cos\frac{\pi}{L}(x-i\frac{\hbar}{mc}g_1')}{\cos\frac{\pi}{L}x}\,\frac{\cos\frac{\pi}{L}(x-i\frac{\hbar}{2mc}-i\frac{\hbar}{mc}g_2')}{\cos\frac{\pi}{L}(x-i\frac{\hbar}{2mc})}\,. \tag{25b}$$

Here g, g_0, g_1, g_2, g_1' and g_2' are dimensionless coupling constants and c is the (fictitious) speed of light. We assume they are all positive.

Note that the Hamiltonian (21) can be expressed as a sum of factorized forms,

$$H=\frac{1}{2}mc^2\sum_{j=1}^{n}\Big(\sqrt{V_j(q)}\,e^{\frac{1}{2mc}p_j}-\sqrt{V_j(q)^*}\,e^{-\frac{1}{2mc}p_j}\Big)\Big(e^{\frac{1}{2mc}p_j}\sqrt{V_j(q)^*}-e^{-\frac{1}{2mc}p_j}\sqrt{V_j(q)}\Big)\,. \tag{26}$$

We also remark that in the $c\to\infty$ limit, the RSvD systems reduce to the CSM systems,

$$\lim_{c\to\infty}H(p,q)=H_{\mathrm{CS}}(p,q), \tag{27}$$

where the correspondence of the parameters is

$$\text{(i)}:\quad \omega_1+\omega_2=\omega,\quad g=g\,, \tag{28a}$$

$$\text{(ii)}:\quad \omega_1+\omega_2=\omega,\quad g_0=g_M,\quad g_1+g_2=g_S+g_L\,, \tag{28b}$$

$$\text{(iii)}:\quad g_0=g_M,\quad g_1+g_2=g_S+g_L,\quad g_1'+g_2'=g_L\,. \tag{28c}$$

3.2. *Equilibrium positions of n-particle classical mechanics*

The classical Hamiltonian $H^{\mathrm{class}}(p,q)$ (and also $V_j^{\mathrm{class}}(q)$, $v^{\mathrm{class}}(x)$, $w^{\mathrm{class}}(x)$) is obtained by the prescription given in §2.2. For the RSvD system, the equation for the equilibrium positions (12) is equivalent to the condition[25)]

$$V_j^{\mathrm{class}}(\bar q)=V_j^{\mathrm{class}}(\bar q)^*>0\,.\qquad (j=1,\dots,n) \tag{29}$$

This equation *without the inequality sign* can be rewritten in a Bethe-ansatz-like form,

$$\prod_{\substack{k=1\\k\neq j}}^{n}\frac{v^{\mathrm{class}}(\bar q_j-\bar q_k)\,v^{\mathrm{class}}(\bar q_j+\bar q_k)}{v^{\mathrm{class}}(\bar q_j-\bar q_k)^*\,v^{\mathrm{class}}(\bar q_j+\bar q_k)^*}=\frac{w^{\mathrm{class}}(\bar q_j)^*}{w^{\mathrm{class}}(\bar q_j)}\,.\qquad (j=1,\dots,n) \tag{30}$$

(For the A_{n-1} type systems, $v^{\mathrm{class}}(\bar q_j+\bar q_k)$ and $v^{\mathrm{class}}(\bar q_j+\bar q_k)^*$ should be omitted.)

Let us consider a polynomial whose zeros give the equilibrium positions:

$$\text{(i) rational } A_{n-1}:\qquad f_n(y)=\prod_{j=1}^{n}\Big(y-\sqrt{\frac{m\omega_1}{\bar g}}\,\bar q_j\Big)\,, \tag{31a}$$

$$\text{(ii) rational } BC_n:\qquad f_n(y)=\prod_{j=1}^{n}\Big(y^2-\frac{m\omega_1}{\bar g_0}\,\bar q_j^2\Big)\,, \tag{31b}$$

$$\text{(iii) trigonometric } BC_n:\quad f_n(\xi)=\prod_{j=1}^{n}\Big(\xi-\cos\big(2\tfrac{\pi}{L}\bar q_j\big)\Big)\,. \tag{31c}$$

Then Eq.(29) can be converted into a functional equation for $f_n(x)$.[13),21)] For the simplest case (i) with $\omega_2=0$, this functional equation can be solved explicitly by using its generating function. However, such a straightforward solution of the functional equation cannot be easily obtained in the other cases. For this reason, we

solved (29) numerically and guessed a recursion relation for $\{f_n(x)\}$. We hypothesize that the polynomials $\{f_n(x)\}$ satisfy the three-term recurrence $f_{n+1}(x) - (x - a_n)f_n(x) + b_n f_{n-1}(x) = 0$ ($n \geq 0$, $f_{-1}(x) = 0$, $f_0(x) = 1$) for some constants a_n and b_n. The three-term recurrence implies that $\{f_n(x)\}$ are orthogonal polynomials. Then we can show that the polynomials $\{f_n(x)\}$ with this three-term recurrence solve the functional equation. The results are[13),21)] (see also Refs. 25)–27))

$$\text{(i)}: \quad f_n(y) = p_n^{\text{monic}}\Big(\sqrt{\frac{mc^2}{\omega_1\bar{g}}}\,y\,;\,\frac{mc^2}{\omega_1\bar{g}},\frac{mc^2}{\omega_2\bar{g}},\frac{mc^2}{\omega_1\bar{g}},\frac{mc^2}{\omega_2\bar{g}}\Big)\,, \tag{32a}$$

$$\text{(ii)}: \quad f_n(y) = W_n^{\text{monic}}\Big(\frac{mc^2}{\omega_1\bar{g}_0}\,y^2;\frac{mc^2}{\omega_1\bar{g}_0},\frac{mc^2}{\omega_2\bar{g}_0},\frac{\bar{g}_1}{\bar{g}_0},\frac{\bar{g}_2}{\bar{g}_0}\Big)\,, \tag{32b}$$

$$\text{(iii)}: \quad f_n(\xi) = p_n^{\text{monic}}\big(\xi\,;e^{-\frac{2\pi\bar{g}_1}{mcL}},e^{-\frac{2\pi\bar{g}_2}{mcL}},-e^{-\frac{2\pi\bar{g}_1'}{mcL}},-e^{-\frac{2\pi\bar{g}_2'}{mcL}}\;\big|\;e^{-\frac{2\pi\bar{g}_0}{mcL}}\big)\,. \tag{32c}$$

With proper normalization, these orthogonal polynomials are known as:[11)]

(i): the continuous Hahn polynomials, (33a)

$$p_n(y;a_1,a_2,b_1,b_2) = \frac{1}{n!}(n+a_1+a_2+b_1+b_2-1)_n\,p_n^{\text{monic}}(y;a_1,a_2,b_1,b_2),$$

(ii): the Wilson polynomials, (33b)

$$W_n(y^2;a_1,a_2,a_3,a_4) = (-1)^n(n+\textstyle\sum_{\alpha=1}^4 a_\alpha - 1)_n\,W_n^{\text{monic}}(y^2;a_1,a_2,a_3,a_4),$$

(iii): the Askey-Wilson polynomials, (33c)

$$p_n(\xi\,;a_1,a_2,a_3,a_4|q) = 2^n(a_1a_2a_3a_4q^{n-1};q)_n\,p_n^{\text{monic}}(\xi\,;a_1,a_2,a_3,a_4|q).$$

These orthogonal polynomials are deformation of the Hermite, Laguerre and Jacobi polynomials, respectively.

3.3. *Eigenfunctions of single-particle quantum mechanics*

The Hamiltonian (21) for the single-particle case ($n = 1$) reads

$$H = \frac{mc^2}{2}\Big(\sqrt{w(x)}\,e^{-i\frac{\hbar}{mc}\frac{d}{dx}}\sqrt{w(x)^*} + \sqrt{w(x)^*}\,e^{i\frac{\hbar}{mc}\frac{d}{dx}}\sqrt{w(x)} - w(x) - w(x)^*\Big)\,, \tag{34}$$

where $x = q_1$. By introducing a dimensionless variable y (or z), a rescaled potential $V(y)$ (or $V(z)$), and a set of parameters $\boldsymbol{\lambda}$ (and a parameter q) given by

$$\text{(i)}: \quad y = \frac{mc}{\hbar}x\,, \quad \boldsymbol{\lambda} = (a_1,a_2), \quad V(y) = V\big(y\,;\boldsymbol{\lambda}\big) = (a_1+iy)(a_2+iy)\,, \tag{35a}$$

$$\text{(ii)}: \quad y = \frac{mc}{\hbar}x\,, \quad \boldsymbol{\lambda} = (a_1,a_2,a_3,a_4),$$

$$V(y) = V\big(y\,;\boldsymbol{\lambda}\big) = \frac{(a_1+iy)(a_2+iy)(a_3+iy)(a_4+iy)}{2iy(2iy+1)}\,, \tag{35b}$$

$$\text{(iii)}: \quad y = \frac{\pi}{L}x, \quad z = e^{2iy} = e^{2\pi i\frac{x}{L}}, \quad \boldsymbol{\lambda} = (a_1,a_2,a_3,a_4), \quad q = e^{-\frac{2\pi\hbar}{mcL}},$$

$$V(z) = V\big(z\,;\boldsymbol{\lambda},q\big) = \frac{(1-a_1z)(1-a_2z)(1-a_3z)(1-a_4z)}{(1-z^2)(1-qz^2)}\,, \tag{35c}$$

the function $w(x)$ takes the form

$$\text{(i)} : w(x) = \frac{1}{a_1 a_2} V(y\,;\boldsymbol{\lambda})\,, \quad \boldsymbol{\lambda} = \Big(\frac{mc^2}{\hbar\omega_1}, \frac{mc^2}{\hbar\omega_2}\Big)\,, \tag{36a}$$

$$\text{(ii)} : w(x) = \frac{4}{a_1 a_2} V(y\,;\boldsymbol{\lambda})\,, \quad \boldsymbol{\lambda} = \Big(\frac{mc^2}{\hbar\omega_1}, \frac{mc^2}{\hbar\omega_2}, g_1, g_2 + \tfrac{1}{2}\Big)\,, \tag{36b}$$

$$\text{(iii)} : w(x)^* = (a_1 a_2 a_3 a_4 q^{-1})^{-\frac{1}{2}} V(z\,;\boldsymbol{\lambda}, q), \quad \boldsymbol{\lambda} = (q^{g_1}, q^{g_2+\frac{1}{2}}, -q^{g_1'}, -q^{g_2'+\frac{1}{2}})\,. \tag{36c}$$

Then the Hamiltonian (34) reduces to the dimensionless one $\mathcal{H}$, with

$$\text{(i)} : \quad H = \frac{mc^2}{a_1 a_2}\mathcal{H}\,, \quad \text{where } \mathcal{H} \text{ is given by (A·18)}\,, \tag{37a}$$

$$\text{(ii)} : \quad H = \frac{4mc^2}{a_1 a_2}\mathcal{H}\,, \quad \text{where } \mathcal{H} \text{ is given by (A·18)}\,, \tag{37b}$$

$$\text{(iii)} : \quad H = mc^2 (a_1 a_2 a_3 a_4 q^{-1})^{-\frac{1}{2}}\mathcal{H}\,, \quad \text{where } \mathcal{H} \text{ is given by (A·22)}\,. \tag{37c}$$

These Hamiltonians have two properties, *factorization* and *shape invariance*.[20),21)] In the notation used in the appendix, the necessary data for the shape invariance are the ground state wavefunction ϕ_0, the elementary shift $\boldsymbol{\delta}$ of the parameters and the first energy level $\mathcal{E}_1(\boldsymbol{\lambda})$:

$$\text{(i)} : \quad \phi_0(y\,;\boldsymbol{\lambda}) \propto \big|\Gamma(a_1 + iy)\Gamma(a_2 + iy)\big|\,, \quad \boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2})\,, \quad \mathcal{E}_1(\boldsymbol{\lambda}) = a_1 + a_2\,, \tag{38a}$$

$$\text{(ii)} : \quad \phi_0(y\,;\boldsymbol{\lambda}) \propto \left|\frac{\Gamma(a_1 + iy)\Gamma(a_2 + iy)\Gamma(a_3 + iy)\Gamma(a_4 + iy)}{\Gamma(2iy)}\right|\,,$$
$$\boldsymbol{\delta} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), \quad \mathcal{E}_1(\boldsymbol{\lambda}) = \tfrac{1}{2}(a_1 + a_2 + a_3 + a_4)\,, \tag{38b}$$

$$\text{(iii)} : \quad \phi_0(z\,;\boldsymbol{\lambda}, q) \propto \left|\frac{(z^2;q)_\infty}{(a_1 z, a_2 z, a_3 z, a_4 z; q)_\infty}\right|\,,$$
$$\delta = \frac{1}{2}, \quad \delta' = -\frac{1}{2}, \quad \mathcal{E}_1(\boldsymbol{\lambda}, q) = \tfrac{1}{2}(q^{-1} - 1)(1 - a_1 a_2 a_3 a_4)\,. \tag{38c}$$

From the formulas (A·16)–(A·17)/(A·37)–(A·38) and (A·11)/(A·32), we obtain the eigenfunctions $P_n\phi_0$ and the corresponding eigenvalues $\mathcal{E}_n$ of $\mathcal{H}$:

$$\text{(i)} : \quad P_n(y;\boldsymbol{\lambda}) \propto p_n(y\,;a_1, a_2, a_1, a_2), \quad \mathcal{E}_n(\boldsymbol{\lambda}) = \tfrac{1}{2}n(n + 2a_1 + 2a_2 - 1)\,, \tag{39a}$$

$$\text{(ii)} : \quad P_n(y\,;\boldsymbol{\lambda}) \propto W_n(y^2; a_1, a_2, a_3, a_4),$$
$$\mathcal{E}_n(\boldsymbol{\lambda}) = \tfrac{1}{2}n(n + a_1 + a_2 + a_3 + a_4 - 1)\,, \tag{39b}$$

$$\text{(iii)} : \quad P_n(z;\boldsymbol{\lambda}, q) \propto p_n(\mathrm{Re}z\,;a_1, a_2, a_3, a_4|q),$$
$$\mathcal{E}_n(\boldsymbol{\lambda}, q) = \tfrac{1}{2}(q^{-n} - 1)(1 - a_1 a_2 a_3 a_4 q^{n-1})\,, \tag{39c}$$

where the first p_n, W_n and the second p_n are the continuous Hahn, Wilson and Askey-Wilson polynomials in the notation used in Ref. 11).

We thus find that the equilibrium positions of the multi-particle classical systems and the eigenfunctions of the single-particle quantum systems are described by the same orthogonal polynomials, namely, the continuous Hahn (special case), Wilson and Askey-Wilson polynomials.

§4. Summary and comments

We have established an interesting property of the Ruijsenaars-Schneider-van Diejen systems with the rational/trigonometric potentials associated with the classical root systems. The equilibrium positions of the multi-particle classical systems and the eigenfunctions of the single-particle quantum systems are described by the same orthogonal polynomials; the continuous Hahn (special case), Wilson and Askey-Wilson polynomials. This property is inherited from the Calogero-Sutherland-Moser systems, in which the relevant orthogonal polynomials are the Hermite, Laguerre and Jacobi polynomials. These polynomials are members of the Askey-scheme of the basic hypergeometric orthogonal polynomials. The CSM and RSvD systems admit elliptic potentials and finding eigenfunctions of such elliptic systems is a good challenge. If this property is inherited by the elliptic RSvD systems, study of classical equilibrium positions may shed light on the quantum problem of finding eigenfunctions, which is quite non-trivial.

This interesting property is obtained as a result of explicit computation and at present we do not know any deeper reason or meaning behind it. In §6 of Ref. 9), we presented the 'phenomenological' observation that the Hermite, Laguerre and Jacobi polynomials of degree n can be obtained from the prepotential of CSM systems with $n+1$ particles. Is this also true for the continuous Hahn, Wilson and Askey-Wilson polynomials? Although there is no prepotential for the RSvD systems, the corresponding quantity may be $\log\phi_0$. The ground state eigenfunction $\phi_0(\{q_j\})$ of (21) (see (26)) is

$$\text{(i)} \ : \ \phi_0(\{q_j\};\boldsymbol{\lambda},g) \propto \left| \prod_{j=1}^{n} \Gamma(a_1+iy_j)\Gamma(a_2+iy_j) \cdot \prod_{1\le j<k\le n} \frac{\Gamma(g+i(y_j-y_k))}{\Gamma(i(y_j-y_k))} \right|, \tag{40a}$$

$$\text{(ii)} \ : \ \phi_0(\{q_j\};\boldsymbol{\lambda},g_0) \propto \left| \prod_{j=1}^{n} \frac{\prod_{\alpha=1}^{4}\Gamma(a_\alpha+iy_j)}{\Gamma(2iy_j)} \cdot \prod_{1\le j<k\le n}\prod_{\epsilon=\pm1} \frac{\Gamma(g_0+i(y_j+\epsilon\, y_k))}{\Gamma(i(y_j+\epsilon\, y_k))} \right|, \tag{40b}$$

$$\text{(iii)} \ : \ \phi_0(\{q_j\};\boldsymbol{\lambda},g_0,q) \propto \left| \prod_{j=1}^{n} \frac{(z_j^2;q)_\infty}{\prod_{\alpha=1}^{4}(a_\alpha z_j;q)_\infty} \cdot \prod_{1\le j<k\le n}\prod_{\epsilon=\pm1} \frac{(z_j z_k^\epsilon;q)_\infty}{(a_0 z_j z_k^\epsilon;q)_\infty} \right|, \tag{40c}$$

where $\boldsymbol{\lambda}$ is given in (35a)–(36c) and $y_j = \frac{mc}{\hbar}q_j$, $z_j = e^{2\pi i\frac{q_j}{L}}$ and $a_0 = q^{g_0}$. For the CSM systems, the second factor of ϕ_0 is simply $q_j \pm q_j$ or $\sin(q_j \pm q_j)$. The observation of Ref. 9) is that the first factor of ϕ_0 gives the measure and the second factor gives the desired polynomial. For the RSvD systems, the first factor of ϕ_0 gives the measure, but the second factor does not give the desired polynomial. Therefore, the observation of Ref. 9) does not apply to the RSvD systems in its naivest form.

In the treatment of the Hamiltonians of these single-particle quantum mechanics, we have emphasized factorization, shape invariance and construction of the isospectral Hamiltonians. Although the examples given in this article possess rational and trigonometric potentials, this method and the basic idea can be applied to a wider

class of potentials, e.g. elliptic potentials. In ordinary quantum mechanics there exists Crum's theorem,[16] which describes the construction of the associated isospectral Hamiltonians $\mathcal{H}_s$ and their eigenfunctions $\phi_{s,n}$ for general systems without shape invariance. The construction of $\mathcal{H}_s$ and $\phi_{s,n}$ given in this article for the 'discrete' cases is based on shape invariance. A 'discrete' analogue of Crum's theorem, that is, the construction of the associated isospectral Hamiltonians and their eigenfunctions without shape invariance, would be very helpful, if it exists.

Acknowledgements

S. O. and R. S. are supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, No. 13135205 and 14540259, respectively.

Appendix

—— Shape Invariance in Single-Particle ('Discrete') Quantum Mechanics ——

In this appendix we collect useful techniques for the shape invariant single-particle ('discrete') quantum mechanics along the idea of Crum.[16] All the quantities in this appendix are dimensionless.

A.1. *Ordinary quantum mechanics*

Let us consider an ordinary quantum mechanical system with the single coordinate y governed by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\frac{d^2}{dy^2} + \mathcal{V}(y), \tag{A·1}$$

where the potential $\mathcal{V}$ may contain a set of parameters $\boldsymbol{\lambda}$. We assume that $\mathcal{H}$ has a square integrable ground state and a discrete spectrum,

$$\mathcal{H}\phi_n = \mathcal{E}_n\phi_n \quad (n = 0, 1, 2, \ldots), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots, \tag{A·2}$$

where the constant term of $\mathcal{V}$ is chosen such that $\mathcal{E}_0 = 0$. Since the ground state eigenfunction ϕ_0 has no node, it can be expressed as $\phi_0(y) = e^{\mathcal{W}(y)}$, and the excited state eigenfunctions are expressed as $\phi_n(y) = P_n(y)\phi_0(y)$. Then the Hamiltonian $\mathcal{H}$ can be factorized as

$$\mathcal{H} = \mathcal{A}(y)^\dagger\mathcal{A}(y) = \frac{1}{2}\Big(-\frac{d^2}{dy^2} + \Big(\frac{d\mathcal{W}(y)}{dy}\Big)^2 + \frac{d^2\mathcal{W}(y)}{dy^2}\Big), \tag{A·3}$$

$$\mathcal{A}(y) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(-i\frac{d}{dy} + i\,\frac{d\,\mathcal{W}(y)}{dy}\Big), \quad \mathcal{A}(y)^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(-i\frac{d}{dy} - i\,\frac{d\,\mathcal{W}(y)}{dy}\Big). \tag{A·4}$$

The ground state ϕ_0 is annihilated by $\mathcal{A}$, $\mathcal{A}\phi_0 = 0$.

Let us write the parameter dependence explicitly, as $\mathcal{A}(y) = \mathcal{A}(y\,;\boldsymbol{\lambda})$. We assume the following *shape invariance*

$$\mathcal{A}(y\,;\boldsymbol{\lambda})\mathcal{A}(y\,;\boldsymbol{\lambda})^\dagger = \mathcal{A}(y\,;\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(y\,;\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{A·5}$$

where $\boldsymbol{\delta}$ stands for a set of constants indicating the elementary shift in the parameters. Isospectral Hamiltonians can be constructed in the following way.*) Starting from $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{H}_0 = \mathcal{H}$ and $\phi_{0,n} = \phi_n$, let us define $\mathcal{A}_s$, $\mathcal{H}_s$ and $\phi_{s,n}$ $(n \geq s \geq 0)$ recursively:

$$\mathcal{A}_{s+1}(y\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_s(y\,;\boldsymbol{\lambda}+\boldsymbol{\delta})\,, \tag{A·6}$$

$$\mathcal{H}_{s+1}(y\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_s(y\,;\boldsymbol{\lambda})\mathcal{A}_s(y\,;\boldsymbol{\lambda})^\dagger + \mathcal{E}_s(\boldsymbol{\lambda})\,, \tag{A·7}$$

$$\phi_{s+1,n}(y\,;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}_s(y\,;\boldsymbol{\lambda})\phi_{s,n}(y\,;\boldsymbol{\lambda})\,. \tag{A·8}$$

As a consequence of the shape invariance (A·5), we obtain for $n \geq s \geq 0$

$$\mathcal{A}_s(y\,;\boldsymbol{\lambda}) = \mathcal{A}(y\,;\boldsymbol{\lambda}+s\boldsymbol{\delta})\,, \tag{A·9}$$

$$\mathcal{H}_s(y\,;\boldsymbol{\lambda}) = \mathcal{A}_s(y\,;\boldsymbol{\lambda})^\dagger\mathcal{A}_s(y\,;\boldsymbol{\lambda}) + \mathcal{E}_s(\boldsymbol{\lambda}) = \mathcal{H}(y\,;\boldsymbol{\lambda}+s\boldsymbol{\delta}) + \mathcal{E}_s(\boldsymbol{\lambda})\,, \tag{A·10}$$

$$\mathcal{E}_{s+1}(\boldsymbol{\lambda}) = \mathcal{E}_s(\boldsymbol{\lambda}) + \mathcal{E}_1(\boldsymbol{\lambda}+s\boldsymbol{\delta}) \quad \Big(\Rightarrow \mathcal{E}_n(\boldsymbol{\lambda}) = \sum_{s=0}^{n-1} \mathcal{E}_1(\boldsymbol{\lambda}+s\boldsymbol{\delta})\Big)\,, \tag{A·11}$$

$$\mathcal{H}_s(y\,;\boldsymbol{\lambda})\phi_{s,n}(y\,;\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\phi_{s,n}(y\,;\boldsymbol{\lambda})\,, \tag{A·12}$$

$$\mathcal{A}_s(y\,;\boldsymbol{\lambda})\phi_{s,s}(y\,;\boldsymbol{\lambda}) = 0\,, \tag{A·13}$$

$$\mathcal{A}_s(y\,;\boldsymbol{\lambda})^\dagger\phi_{s+1,n}(y\,;\boldsymbol{\lambda}) = \big(\mathcal{E}_n(\boldsymbol{\lambda}) - \mathcal{E}_s(\boldsymbol{\lambda})\big)\phi_{s,n}(y\,;\boldsymbol{\lambda})\,. \tag{A·14}$$

As is clear from (A·10), all the Hamiltonians $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_s, \ldots$ have the same shape and only the parameters are shifted. From (A·8) and (A·14) we obtain formulas relating the eigenfunctions along the horizontal line (the *isospectral line*) of Fig. 1,

$$\phi_{s,n}(y\,;\boldsymbol{\lambda}) = \mathcal{A}_{s-1}(y\,;\boldsymbol{\lambda})\cdots\mathcal{A}_1(y\,;\boldsymbol{\lambda})\mathcal{A}_0(y\,;\boldsymbol{\lambda})\phi_n(y\,;\boldsymbol{\lambda})\,, \tag{A·15}$$

$$\phi_n(y\,;\boldsymbol{\lambda}) = \frac{\mathcal{A}_0(y\,;\boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda})-\mathcal{E}_0(\boldsymbol{\lambda})}\,\frac{\mathcal{A}_1(y\,;\boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda})-\mathcal{E}_1(\boldsymbol{\lambda})}\cdots\frac{\mathcal{A}_{n-1}(y\,;\boldsymbol{\lambda})^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda})-\mathcal{E}_{n-1}(\boldsymbol{\lambda})}\,\phi_{n,n}(y\,;\boldsymbol{\lambda})\,, \tag{A·16}$$

and from (A·10) we have

$$\phi_{n,n}(y\,;\boldsymbol{\lambda}) \propto \phi_0(y\,;\boldsymbol{\lambda}+n\boldsymbol{\delta}). \tag{A·17}$$

It should be emphasized that all the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ in the above formulas are explicitly known thanks to the shape invariance. The formula (A·16) with (A·17) can be understood as the Rodrigues-type formula. The relation (A·11) implies that $\{\mathcal{E}_n(\boldsymbol{\lambda})\}_{n\geq 0}$ can be calculated from $\mathcal{E}_1(\boldsymbol{\lambda})$, and hence that the spectrum is determined completely by the shape invariance.

As seen above, the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ act isospectrally, that is, horizontally in Fig. 1. On the other hand, the annihilation and creation operators map from one eigenstate to another (*i.e.* vertically) of a given Hamiltonian. In order to define the annihilation and creation operators, we need unitary operators $\mathcal{U}_s$. (For details see for example Refs. 18), 20), 28).)

The scheme described above is illustrated in Fig. 1.

*) Isospectral Hamiltonians can be constructed *without* shape invariance and some formulas e.g., (A·15) and (A·16), are also valid in that case. However, there is no explicit systematic method to obtain $\mathcal{A}_s$ (and $\mathcal{E}_n$) in general.

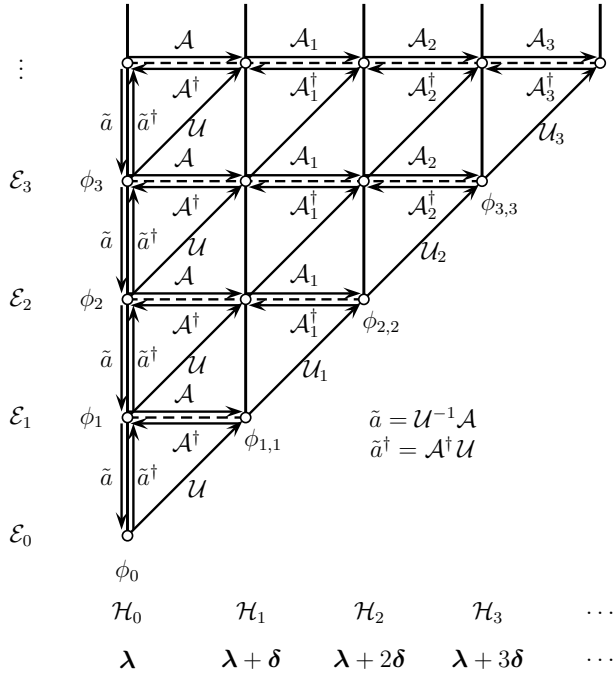

Fig. 1. A schematic diagram of the energy levels and the associated Hamiltonian systems together with the definition of the $\mathcal{A}$ and $\mathcal{A}^\dagger$ operators and the 'creation' ($\tilde{a}^\dagger$) and 'annihilation' ($\tilde{a}$) operators. The parameter set is indicated below each Hamiltonian.

A.2. *'Discrete' quantum mechanics*

Here we discuss two different types of shape invariance mechanisms. The first, with an additive shift of the parameters, appears in the ordinary quantum mechanics discussed in the preceding subsection. The second one, involving a multiplicative shift of the parameters, is new.

A.2.1. Additive shift

First, let us consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\Big(\sqrt{V(y)}\,e^{-i\frac{d}{dy}}\sqrt{V(y)^*} + \sqrt{V(y)^*}\,e^{i\frac{d}{dy}}\sqrt{V(y)} - V(y) - V(y)^*\Big) \tag{A·18}$$

and its eigenfunctions given in (A·2). The function $V(y)$ depends on a set of parameters $\boldsymbol{\lambda}$. This Hamiltonian $\mathcal{H}$ is factorizable:

$$\mathcal{H} = \mathcal{H}(y\,;\boldsymbol{\lambda}) = \mathcal{A}(y\,;\boldsymbol{\lambda})^\dagger\mathcal{A}(y\,;\boldsymbol{\lambda})\,, \tag{A·19}$$

$$\mathcal{A} = \mathcal{A}(y\,;\boldsymbol{\lambda}) \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(e^{-\frac{i}{2}\frac{d}{dy}}\sqrt{V(y\,;\boldsymbol{\lambda})^*} - e^{\frac{i}{2}\frac{d}{dy}}\sqrt{V(y\,;\boldsymbol{\lambda})}\Big)\,, \tag{A·20}$$

$$\mathcal{A}^\dagger = \mathcal{A}(y\,;\boldsymbol{\lambda})^\dagger \overset{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(\sqrt{V(y\,;\boldsymbol{\lambda})}\,e^{-\frac{i}{2}\frac{d}{dy}} - \sqrt{V(y\,;\boldsymbol{\lambda})^*}\,e^{\frac{i}{2}\frac{d}{dy}}\Big)\,. \tag{A·21}$$

The ground state ϕ_0 is annihilated by $\mathcal{A}$, $\mathcal{A}\phi_0 = 0$.

As in §A.1 we assume the shape invariance (A·5); that is, $\mathcal{A}\mathcal{A}^\dagger$ has the same shape as $\mathcal{A}^\dagger\mathcal{A}$ with the additive shift of the parameters $\boldsymbol{\lambda}$ combined with the ex-

citation energy of the first level $\mathcal{E}_1(\boldsymbol{\lambda})$. Then the isospectral Hamiltonians can be constructed in the same way as in the previous subsection and equations (A·6)–(A·17) also hold here.[20)]

A.2.2. Multiplicative shift

Next let us consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\Big(\sqrt{V(z)}\,q^{D_z}\sqrt{V(z)^*} + \sqrt{V(z)^*}\,q^{-D_z}\sqrt{V(z)} - V(z) - V(z)^*\Big) \tag{A·22}$$

and its eigenfunctions (A·2). The variable z is related to y as $z = e^{2iy}$ and $D_z = z\frac{d}{dz}$. (In our notation, $V(z)$ here corresponds to $V(y)^*$ in (A·18).) This form is more convenient for the trigonometric potentials. The function $V(z)$ depends on a set of parameters $\boldsymbol{\lambda}$ and a parameter q, which controls the multiplicative shift. This Hamiltonian $\mathcal{H}$ is factorizable:

$$\mathcal{H} = \mathcal{H}(z\,;\boldsymbol{\lambda},q) = \mathcal{A}(z\,;\boldsymbol{\lambda},q)^\dagger\mathcal{A}(z\,;\boldsymbol{\lambda},q)\,, \tag{A·23}$$

$$\mathcal{A} = \mathcal{A}(z\,;\boldsymbol{\lambda},q) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(q^{\frac{1}{2}D_z}\sqrt{V(z\,;\boldsymbol{\lambda},q)^*} - q^{-\frac{1}{2}D_z}\sqrt{V(z\,;\boldsymbol{\lambda},q)}\Big)\,, \tag{A·24}$$

$$\mathcal{A}^\dagger = \mathcal{A}(z\,;\boldsymbol{\lambda},q)^\dagger \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\Big(\sqrt{V(z\,;\boldsymbol{\lambda},q)}\,q^{\frac{1}{2}D_z} - \sqrt{V(z\,;\boldsymbol{\lambda},q)^*}\,q^{-\frac{1}{2}D_z}\Big)\,. \tag{A·25}$$

The ground state ϕ_0 is annihilated by $\mathcal{A}$, $\mathcal{A}\phi_0 = 0$.

We assume the following type of shape invariance

$$\mathcal{A}(z\,;\boldsymbol{\lambda},q)\mathcal{A}(z\,;\boldsymbol{\lambda},q)^\dagger = q^{2\delta'}\mathcal{A}(z\,;q^\delta\boldsymbol{\lambda},q)^\dagger\mathcal{A}(z\,;q^\delta\boldsymbol{\lambda},q) + \mathcal{E}_1(\boldsymbol{\lambda},q)\,, \tag{A·26}$$

where δ and δ' are constants. As in §A.1 isospectral Hamiltonians can be constructed.[21)] Starting from $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{H}_0 = \mathcal{H}$ and $\phi_{0,n} = \phi_n$, let us define $\mathcal{A}_s$, $\mathcal{H}_s$ and $\phi_{s,n}$ $(n \geq s \geq 0)$ recursively as

$$\mathcal{A}_{s+1}(z\,;\boldsymbol{\lambda},q) \stackrel{\text{def}}{=} q^{\delta'}\mathcal{A}_s(z\,;q^\delta\boldsymbol{\lambda},q)\,, \tag{A·27}$$

$$\mathcal{H}_{s+1}(z\,;\boldsymbol{\lambda},q) \stackrel{\text{def}}{=} \mathcal{A}_s(z\,;\boldsymbol{\lambda},q)\mathcal{A}_s(z\,;\boldsymbol{\lambda},q)^\dagger + \mathcal{E}_s(\boldsymbol{\lambda},q)\,, \tag{A·28}$$

$$\phi_{s+1,n}(z\,;\boldsymbol{\lambda},q) \stackrel{\text{def}}{=} \mathcal{A}_s(z\,;\boldsymbol{\lambda},q)\phi_{s,n}(z\,;\boldsymbol{\lambda},q)\,. \tag{A·29}$$

As a consequence of the multiplicative shape invariance (A·26), we obtain for $n \geq s \geq 0$

$$\mathcal{A}_s(z\,;\boldsymbol{\lambda},q) = q^{s\delta'}\mathcal{A}(z\,;q^{s\delta}\boldsymbol{\lambda},q)\,, \tag{A·30}$$

$$\begin{aligned}\mathcal{H}_s(z\,;\boldsymbol{\lambda},q) &= \mathcal{A}_s(z\,;\boldsymbol{\lambda},q)^\dagger\mathcal{A}_s(z\,;\boldsymbol{\lambda},q) + \mathcal{E}_s(\boldsymbol{\lambda},q)\\ &= q^{2s\delta'}\mathcal{H}(z\,;q^{s\delta}\boldsymbol{\lambda},q) + \mathcal{E}_s(\boldsymbol{\lambda},q)\,,\end{aligned} \tag{A·31}$$

$$\mathcal{E}_{s+1}(\boldsymbol{\lambda},q) = \mathcal{E}_s(\boldsymbol{\lambda},q) + q^{2s\delta'}\mathcal{E}_1(q^{s\delta}\boldsymbol{\lambda},q)\ \Big(\Rightarrow \mathcal{E}_n(\boldsymbol{\lambda},q) = \sum_{s=0}^{n-1} q^{2s\delta'}\mathcal{E}_1(q^{s\delta}\boldsymbol{\lambda},q)\Big), \tag{A·32}$$

$$\mathcal{H}_s(z\,;\boldsymbol{\lambda},q)\phi_{s,n}(z\,;\boldsymbol{\lambda},q) = \mathcal{E}_n(\boldsymbol{\lambda},q)\phi_{s,n}(z\,;\boldsymbol{\lambda},q)\,, \tag{A·33}$$

$$\mathcal{A}_s(z\,;\boldsymbol{\lambda},q)\phi_{s,s}(z\,;\boldsymbol{\lambda},q) = 0\,, \tag{A·34}$$

$$\mathcal{A}_s(z\,;\boldsymbol{\lambda},q)^\dagger\phi_{s+1,n}(z\,;\boldsymbol{\lambda},q) = \big(\mathcal{E}_n(\boldsymbol{\lambda},q) - \mathcal{E}_s(\boldsymbol{\lambda},q)\big)\phi_{s,n}(z\,;\boldsymbol{\lambda},q)\,. \tag{A·35}$$

It should be noted that the Hamiltonian $\mathcal{H}$ is rescaled by a factor $q^{2\delta'}$ and the parameters $\boldsymbol{\lambda}$ are multiplied by a factor q^{δ} at each step. From (A·29) and (A·35) we obtain the formulas

$$\phi_{s,n}(z\,;\boldsymbol{\lambda},q)=\mathcal{A}_{s-1}(z\,;\boldsymbol{\lambda},q)\cdots\mathcal{A}_1(z\,;\boldsymbol{\lambda},q)\mathcal{A}_0(z\,;\boldsymbol{\lambda},q)\phi_n(z\,;\boldsymbol{\lambda},q)\,, \tag{A·36}$$

$$\begin{aligned}\phi_n(z\,;\boldsymbol{\lambda},q)=&\frac{\mathcal{A}_0(z\,;\boldsymbol{\lambda},q)^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda},q)-\mathcal{E}_0(\boldsymbol{\lambda},q)}\,\frac{\mathcal{A}_1(z\,;\boldsymbol{\lambda},q)^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda},q)-\mathcal{E}_1(\boldsymbol{\lambda},q)}\cdots\\ &\times\cdots\frac{\mathcal{A}_{n-1}(z\,;\boldsymbol{\lambda},q)^\dagger}{\mathcal{E}_n(\boldsymbol{\lambda},q)-\mathcal{E}_{n-1}(\boldsymbol{\lambda},q)}\,\phi_{n,n}(z\,;\boldsymbol{\lambda},q)\,,\end{aligned} \tag{A·37}$$

and from (A·31) we have

$$\phi_{n,n}(z\,;\boldsymbol{\lambda},q)\propto\phi_0(z\,;q^{n\delta}\boldsymbol{\lambda},q). \tag{A·38}$$

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