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Dual Christoffel transformations

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Abstract

Crum's theorem and its modification à la Krein-Adler are formulated for the discrete quantum mechanics with real shifts, whose eigenfunctions consist of orthogonal polynomials of a discrete variable. The modification produces the associated polynomials with a finite number of *degrees* deleted. This in turn provides the well known Christoffel transformation for the dual orthogonal polynomials with the corresponding *positions* deleted.

1 Introduction

In a previous paper [1] we have developed a new paradigm for the orthogonal polynomials of a discrete variable [2, 3, 4, 5], which is called *the discrete quantum mechanics with real shifts*. Various well-known orthogonal polynomials, for example, the (q)-Racah, (q)-(dual)-Hahn, (q)-Meixner, etc. [5] are derived in [1] as the main parts of the eigenvectors of specific *tri-diagonal* real symmetric (Jacobi) matrices of finite or infinite dimensions. The orthogonality of the eigenpolynomials is evident by construction and the orthogonality weight function is obtained as the square of the groundstate eigenvector. The dual pair of orthogonal polynomials (*the Leonard pair*) [6, 7] is naturally defined and the *three term recurrence relation* for the dual polynomials is built in automatically. It should be emphasised that the present paradigm offers a unified understanding [8, 9] of the (q)-Askey scheme of hypergeometric orthogonal polynomials [3, 5]. Various properties of the discrete quantum mechanical systems with real shifts, for example, the duality, the orthogonality, the exact solvability in the

Schrödinger as well as the Heisenberg picture [10], the dynamical symmetry algebras (the Askey-Wilson algebra [11] and its degenerations [12]), the quasi-exact solvability, etc. can be understood in a unified fashion, not one by one specifically.

In the present paper we start with the formulation of the discrete quantum mechanics version of the well-known theorem by Crum [13] for one-dimensional quantum mechanics or Sturm-Liouville theory. Crum's seminal paper asserts, in the language of quantum mechanics, the existence of an infinite family of associated Hamiltonian systems (*i.e.* the Hamiltonian together with its eigenvalues and eigenfunctions) which are essentially *iso-spectral* with each other. We will show that the situation is the same in the discrete quantum mechanics with real shifts. The associated Hamiltonians are tri-diagonal real symmetric (Jacobi) matrices, too. See Fig. 1 for the general structure of the associated Hamiltonian systems. Crum's theorem for *the discrete quantum mechanics with pure imaginary shifts* [14, 15, 8] has been formulated in [16]. This covers the orthogonal polynomials with absolutely continuous weight functions, for example, the Wilson and the Askey-Wilson polynomials and their various degenerate forms. See also [17] in this context.

Next we will present the discrete quantum mechanics (with real shifts) version of Krein-Adler [18] modification of Crum's theorem [19]. The Krein-Adler modification goes as follows: starting from a given Hamiltonian with eigenvalues $\{\mathcal{E}(n)\}$ and the corresponding eigenfunctions $\{\phi_n\}$, $n = 0, 1, \ldots$, one can again construct an associated Hamiltonian system by *deleting* a finite number of energy levels $\{\mathcal{E}(d_j)\}$, $d_j \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, \ell$, which satisfy certain conditions (4.1). We will demonstrate that the situation is the same in the discrete quantum mechanics. The associated Hamiltonian is again a tri-diagonal real symmetric (Jacobi) matrix and the eigenfunctions form a complete set of orthogonal functions. See Fig. 2a for its schematic structure to be contrasted with the original Crum's case in Fig. 2b, which corresponds to the very special choice of deletion $\{d_1, d_2, \ldots, d_\ell\} = \{0, 1, 2, \ldots, \ell-1\}$. If the starting system is exactly solvable, as those examples given in [1], the modified system is also exactly solvable.

Crum's theorem and its modification will be presented generically, at the beginning. That is, no extra condition is imposed on the functions $B(x)$ and $D(x)$ in the Hamiltonian, other than the positivity, the boundary (asymptotic) conditions (2.3). Many formulas will be drastically simplified when the eigenfunctions consist of polynomials and/or the system is shape invariant [20], which is the case for most practical applications. We will present these simplified formulas, too. Due to the lack of the generic 'oscillation theorem' in discrete quantum mechanics, we do not have categorical proof of the hermiticity of the Hamiltonian $\bar{\mathcal{H}}$ (4.39) obtained by the application of the modified version of Crum's theorem. For each particular case of specific polynomials the hermiticity is verified.

It is important to stress that the modified Crum's theorem provides a unified theory of *dual Christoffel transformations* [21] for orthogonal polynomials of a discrete variable. The dual Christoffel transformation has the merits that the formulas are universal, concise and algorithmic compared with the original Christoffel transformation, which should be performed specifically for each case, the polynomials and the set of deletion $\mathcal{D}$. This situation is explained in some detail in section 6.

As an illustration of the modified Crum's theorem, we will derive in Appendix unified expressions of the eigenpolynomials for the special case of deletion $\{d_1, d_2, \ldots, d_\ell\} = \{1, 2, \ldots, \ell\}$. The same formulation and examples for the discrete quantum mechanics with pure imaginary shifts are reported in [19]. As is well known these orthogonal polynomials of a discrete variable have many important applications in many arenas of physics and mathematics [2, 3, 4]. To name a few recent applications, the linear quantum registers [22], quantum communications [23] and the birth and death processes. As shown in [24], the explicit examples of 18 orthogonal polynomials in [1], the (q)-Racah, (q)-(dual)-Hahn etc, provide *exactly solvable birth and death processes* [4, 25]. That is, for the given birth and death rates $\{B(x), D(x)\}$ which define the Hamiltonian (2.1), the corresponding transition probabilities are given explicitly, not in a general spectral representation form of Karlin-McGregor [26]. By applying the present modification of Crum's theorem to these polynomials, one can generate an (in)finite variety of exactly solvable birth and death processes.

This article is organised as follows. In section 2 we recapitulate the essence of the discrete quantum mechanics with real shifts in order to introduce necessary notions and notation. In section 3, the discrete quantum mechanics version of Crum's theorem is formulated in its full generality. The modification of Crum's theorem à la Krein-Adler is developed in section 4. Section 5 provides various simplifications of the formulas in the presence of shape invariance and/or polynomial eigenfunctions. In section 6, the Christoffel transformations for orthogonal polynomials of a discrete variable in discrete quantum mechanics are shown to be *dual* to the modification of Crum's theorem developed in the preceding sections. The final section is for a summary and comments. Appendix gives the simplest examples of the modified Hamiltonian systems obtained by deleting the lowest lying ℓ excited states for various exactly solvable Hamiltonians discussed in [1].

2 Discrete Quantum Mechanics with Real Shifts

Let us review the discrete quantum mechanics with real shifts formulated in [1]. The Hamiltonian $\mathcal{H}=(\mathcal{H}_{x,y})$ is a very special type of *tri-diagonal* real symmetric (Jacobi) matrices and its rows and columns are indexed by non-negative integers x and y $(x,y=0,1,\ldots,x_{\max})$, in which $x_{\max}$ is either finite $(x_{\max}=N)$ or infinite $(x_{\max}=\infty)$. The Hamiltonian $\mathcal{H}$ has a form

$$\mathcal{H}\stackrel{\text{def}}{=}-\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)}-\sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)}+B(x)+D(x), \tag{2.1}$$

$$\mathcal{H}_{x,y}=-\sqrt{B(x)D(x+1)}\,\delta_{x+1,y}-\sqrt{B(x-1)D(x)}\,\delta_{x-1,y}+\big(B(x)+D(x)\big)\delta_{x,y}, \tag{2.2}$$

in which $\partial=\frac{d}{dx}$ is the differentiation operator $(\partial f)(x)=\frac{df(x)}{dx}$, $(e^{\pm\partial}f)(x)=f(x\pm1)$. The two functions $B(x)$ and $D(x)$ are real and *positive* but vanish at the boundary:

$$B(x)>0,\quad D(x)>0,\quad D(0)=0\ ;\quad B(x_{\max})=0\ \text{ for the finite case.} \tag{2.3}$$

The problem at hand is to find the *complete set of eigenvalues* and the *corresponding eigenvectors* of the hermitian matrix $\mathcal{H}$ $(n_{\max}=N$ or $\infty)$:

$$\mathcal{H}\phi_n(x)=\mathcal{E}(n)\phi_n(x)\quad(n=0,1,\ldots,n_{\max}), \tag{2.4}$$

which is the Schrödinger equation of the discrete quantum mechanics with real shifts. It has non-degenerate spectrum thanks to the generic property of Jacobi matrices,

$$\mathcal{E}(0)<\mathcal{E}(1)<\mathcal{E}(2)<\cdots. \tag{2.5}$$

The Hamiltonian (2.1) can be expressed as a product of a bi-diagonal *lower triangular* matrix $\mathcal{A}^\dagger$ and a bi-diagonal *upper triangular* matrix $\mathcal{A}$:

$$\mathcal{H}=\mathcal{A}^\dagger\mathcal{A},\qquad \mathcal{A}=(\mathcal{A}_{x,y}),\quad \mathcal{A}^\dagger=((\mathcal{A}^\dagger)_{x,y})=(\mathcal{A}_{y,x})\quad(x,y=0,1,\ldots,x_{\max}), \tag{2.6}$$

$$\mathcal{A}\stackrel{\text{def}}{=}\sqrt{B(x)}-e^{\partial}\sqrt{D(x)},\quad \mathcal{A}^\dagger=\sqrt{B(x)}-\sqrt{D(x)}\,e^{-\partial}, \tag{2.7}$$

$$\mathcal{A}_{x,y}=\sqrt{B(x)}\,\delta_{x,y}-\sqrt{D(x+1)}\,\delta_{x+1,y},\quad(\mathcal{A}^\dagger)_{x,y}=\sqrt{B(x)}\,\delta_{x,y}-\sqrt{D(x)}\,\delta_{x-1,y}. \tag{2.8}$$

The zero mode $\mathcal{A}\phi_0(x) = 0$ is easily obtained:

$$\phi_0(x) = \sqrt{\prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)}}, \tag{2.9}$$

with the normalisation $\phi_0(0) = 1$ (convention: $\prod_{k=n}^{n-1} * = 1$). Note that the groundstate wavefunction $\phi_0(x)$ of (2.1) is positive ($\phi_0(x) > 0$) throughout the range of x ($x = 0, 1, \ldots, x_{\max}$). Then the Hamiltonian is *positive semi-definite*:

$$\mathcal{E}(0) = 0. \tag{2.10}$$

The finite ℓ^2-norm condition of $\phi_0(x)$

$$\sum_{x=0}^{x_{\max}} \phi_0(x)^2 < \infty, \tag{2.11}$$

imposes conditions on the asymptotic forms of $B(x)$ and $D(x)$ for the infinite case.

The orthogonality relation is

$$(\phi_n, \phi_m) \stackrel{\text{def}}{=} \sum_{x=0}^{x_{\max}} \phi_n(x)\phi_m(x) = \frac{1}{d_n^2}\,\delta_{nm} \quad (n, m = 0, 1, \ldots, n_{\max}). \tag{2.12}$$

Here $1/d_n^2$ is the normalisation constant. For the infinite case ($x_{\max} = \infty$), the eigenfunctions should satisfy the asymptotic condition $\phi_n(x_{\max}) = 0$ and the finite ℓ^2-norm condition.

If the eigenfunction has the factorised form,

$$\phi_n(x) = \phi_0(x) P_n(\eta(x)), \tag{2.13}$$

where $P_n(\eta)$ is a polynomial of degree n in the sinusoidal coordinate $\eta = \eta(x)$ with the normalisation,

$$P_n(0) = 1 \quad (n = 0, 1, \ldots, n_{\max}), \quad P_{-1}(\eta) \stackrel{\text{def}}{=} 0, \quad \eta(0) = 0. \tag{2.14}$$

The sinusoidal coordinate $\eta(x)$ is the central notion of exactly solvable quantum mechanics [1, 8, 10]. It undergoes sinusoidal motion through the time evolution governed by the Hamiltonian $\mathcal{H}$, see (2.28) of [10]. It is also established that the Heisenberg operator equation for the sinusoidal coordinate $\eta(x)$ is exactly solvable [1, 8, 10]. In the discrete quantum mechanics with real shifts, there are five sinusoidal coordinates, see (4.72)–(4.76) of [1]. Then the above orthogonality relation becomes that of the orthogonal polynomials

$$\sum_{x=0}^{x_{\max}} \phi_0(x)^2 P_n(\eta(x)) P_m(\eta(x)) = \frac{1}{d_n^2}\,\delta_{nm} \quad (n, m = 0, 1, \ldots, n_{\max}). \tag{2.15}$$

Here the discrete weightfunction $\phi_0(x)^2$ is given explicitly (2.9). The orthogonal polynomials $P_n(\eta(x))$ are the 'eigenfunctions' of the similarity transformed Hamiltonian $\widetilde{\mathcal{H}} = (\widetilde{\mathcal{H}}_{x,y})$ $(x, y = 0, 1, \ldots, x_{\text{max}})$

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1 - e^{\partial}) + D(x)(1 - e^{-\partial}), \tag{2.16}$$

$$\widetilde{\mathcal{H}}_{x,y} = B(x)(\delta_{x,y} - \delta_{x+1,y}) + D(x)(\delta_{x,y} - \delta_{x-1,y}), \tag{2.17}$$

$$\begin{aligned} \widetilde{\mathcal{H}} P_n(\eta(x)) &= \mathcal{E}(n) P_n(\eta(x)) \quad (n = 0, 1, \ldots, n_{\text{max}}) \\ &= B(x)\big(P_n(\eta(x)) - P_n(\eta(x+1))\big) + D(x)\big(P_n(\eta(x)) - P_n(\eta(x-1))\big). \end{aligned} \tag{2.18}$$

The *dual polynomial* $Q_x(\mathcal{E})$, which is a degree x polynomial in $\mathcal{E}$ with the normalisation condition

$$Q_x(0) = 1 \quad (x = 0, 1, \ldots, x_{\text{max}}), \quad Q_{-1}(\mathcal{E}) \stackrel{\text{def}}{=} 0, \quad \mathcal{E}(0) = 0, \tag{2.19}$$

is defined by the *three term recurrence relation*

$$B(x)\big(Q_x(\mathcal{E}) - Q_{x+1}(\mathcal{E})\big) + D(x)\big(Q_x(\mathcal{E}) - Q_{x-1}(\mathcal{E})\big) = \mathcal{E} Q_x(\mathcal{E}) \quad (x = 0, 1, \ldots, x_{\text{max}}). \tag{2.20}$$

For $\mathcal{E} = \mathcal{E}(n)$, the above difference equation for $P_n(\eta(x))$ (2.18) is identical with the three term recurrence relation under the identification

$$P_n(\eta(x)) = Q_x(\mathcal{E}(n)) \quad (x = 0, 1, \ldots, x_{\text{max}}), \quad (n = 0, 1, \ldots, n_{\text{max}}). \tag{2.21}$$

This establishes the *duality* [6, 7]. The orthogonality relation for the dual orthogonal polynomials $Q_x(\mathcal{E}(n))$ takes a dual form to (2.15)

$$\sum_{n=0}^{n_{\text{max}}} d_n^2 Q_x(\mathcal{E}(n)) Q_y(\mathcal{E}(n)) = \frac{1}{\phi_0(x)^2}\, \delta_{xy} \quad (x, y = 0, 1, \ldots, x_{\text{max}}). \tag{2.22}$$

For more details of the duality in the context of discrete quantum mechanics with real shifts, see § 3 of [1].

For all the examples presented in [1], the eigenfunctions have the form (2.13) and the square of the groundstate wavefunction $\phi_0(x)^2$ can be analytically continued to the whole complex x plane and it has (at least) a simple zero at integral points outside $[0, x_{\text{max}}]$,

$$\phi_0(x)^2 = 0 \quad \text{for } x \in \mathbb{Z} \backslash \{0, 1, \ldots, x_{\text{max}}\}, \tag{2.23}$$

due to the factors of $\phi_0(x)^2$ such as $(q\,;q)_x^{-1} = (q^{1+x}; q)_\infty (q\,;q)_\infty^{-1}$, $(q^{-N}; q)_x$, etc. Thanks to this situation, various expressions in the subsequent sections whose arguments appear to go beyond the defined range of $[0, x_{\text{max}}]$ cause no harm.

3 Crum's Theorem

Crum's theorem [13] describes the relationship between the original and the associated Hamiltonian systems, which are iso-spectral except for the lowest energy state. The relationship among various associated Hamiltonian systems is quite general, as depicted in Fig. 1, and it is shared by the most general ordinary quantum mechanical systems [13] as well as discrete quantum mechanical systems with real and pure imaginary shifts [16, 17]. As shown below, the factorised form of the Hamiltonian (2.6) is essential.

3.1 Deletion of the ground state

For later convenience, let us attach the superscript $^{[0]}$ to all the quantities of the original Hamiltonian system, $\mathcal{H}^{[0]} \stackrel{\text{def}}{=} \mathcal{H}$, $\phi_n^{[0]}(x) \stackrel{\text{def}}{=} \phi_n(x)$, $\mathcal{A}^{[0]} \stackrel{\text{def}}{=} \mathcal{A}$, $\mathcal{A}^{[0]\dagger} \stackrel{\text{def}}{=} \mathcal{A}^\dagger$, $B^{[0]}(x) \stackrel{\text{def}}{=} B(x)$, $D^{[0]}(x) \stackrel{\text{def}}{=} D(x)$. Let us define an associated Hamiltonian $\mathcal{H}^{[1]}$ by simply changing the order of $\mathcal{A}$ and $\mathcal{A}^\dagger$:

$$\mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A}^{[0]}\mathcal{A}^{[0]\dagger}. \tag{3.1}$$

By construction $\mathcal{A}$ and $\mathcal{A}^\dagger$ intertwine $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$:

$$\mathcal{A}^{[0]}\mathcal{H}^{[0]} = \mathcal{H}^{[1]}\mathcal{A}^{[0]}, \qquad \mathcal{A}^{[0]\dagger}\mathcal{H}^{[1]} = \mathcal{H}^{[0]}\mathcal{A}^{[0]\dagger}. \tag{3.2}$$

The matrix elements of the associated Hamiltonian $\mathcal{H}^{[1]}$ are

$$\mathcal{H}^{[1]}_{x,y} = -\sqrt{B(x+1)D(x+1)}\,\delta_{x+1,y} - \sqrt{B(x)D(x)}\,\delta_{x-1,y} + \big(B(x)+D(x+1)(1-\delta_{x,x_{\text{max}}})\big)\delta_{x,y}. \tag{3.3}$$

For the finite case ($x^{[0]}_{\text{max}} = N$, $B(N) = 0$), the matrix elements in the $(N+1)$-st entry vanish, namely $\mathcal{H}^{[1]}$ has the following form:

$$\mathcal{H}^{[1]} = \begin{pmatrix} \mathcal{H}^{[1]\prime} & \mathbf{0} \\ {}^t\mathbf{0} & 0 \end{pmatrix}, \tag{3.4}$$

where $\mathcal{H}^{[1]\prime}$ is an $N\times N$ matrix and $\mathbf{0}$ is an N-dimensional zero column vector. Therefore the eigenvalue problem of $\mathcal{H}^{[1]}$ reduces to that of $\mathcal{H}^{[1]\prime}$ and the trivial one dimensional part with zero eigenvalue corresponding to the deleted groundstate of the original Hamiltonian. To get rid of this trivial part we define $x^{[1]}_{\text{max}}$ as $x^{[1]}_{\text{max}} = N-1$. For the infinite case ($x_{\text{max}} = \infty$), $x^{[1]}_{\text{max}}$ is defined as $x^{[1]}_{\text{max}} = \infty$. To treat the finite and the infinite cases in parallel and to avoid notational cumbersomeness, we write $\mathcal{H}^{[1]\prime}$ as $\mathcal{H}^{[1]}$ by adopting the following convention.

Namely $\mathcal{H}^{[1]}$ represents an $(x^{[1]}_{\max}+1)\times(x^{[1]}_{\max}+1)$ matrix $\mathcal{H}^{[1]}=(\mathcal{H}^{[1]}_{x,y})$ $(x,y=0,1,\ldots,x^{[1]}_{\max})$ with $x^{[1]}_{\max}=N-1$ or ∞.

We will show that the associated Hamiltonian system $\mathcal{H}^{[1]}$ is *iso-spectral* to the original Hamiltonian system $\mathcal{H}^{[0]}$ and the eigenfunctions are in one to one correspondence, *except for* the groundstate. Thanks to the first of the above relation (3.2), it is trivial to verify that the eigenfunctions of the associated Hamiltonian system $\mathcal{H}^{[1]}$ are generated algebraically by multiplying $\mathcal{A}^{[0]}$ to the eigenfunction of the original system:

$$\phi^{[1]}_n(x)\stackrel{\text{def}}{=}\mathcal{A}^{[0]}\phi^{[0]}_n(x)\qquad(n=1,2,\ldots,n_{\max}),\tag{3.5}$$

$$\mathcal{H}^{[1]}\phi^{[1]}_n(x)=\mathcal{E}(n)\phi^{[1]}_n(x)\quad(n=1,2,\ldots,n_{\max}).\tag{3.6}$$

The orthogonality relation is

$$\begin{aligned}(\phi^{[1]}_n,\phi^{[1]}_m)&\stackrel{\text{def}}{=}\sum_{x=0}^{x^{[1]}_{\max}}\phi^{[1]}_n(x)\phi^{[1]}_m(x)\qquad(n,m=1,2,\ldots,n_{\max})\\&=(\mathcal{A}^{[0]}\phi^{[0]}_n,\mathcal{A}^{[0]}\phi^{[0]}_m)=(\phi^{[0]}_n,\mathcal{A}^{[0]\dagger}\mathcal{A}^{[0]}\phi^{[0]}_m)=(\phi^{[0]}_n,\mathcal{H}^{[0]}\phi^{[0]}_m)=\mathcal{E}(n)\frac{1}{d_n^2}\,\delta_{nm}.\end{aligned}\tag{3.7}$$

For the finite case this gives all the N eigenvectors of $\mathcal{H}^{[1]}$. For the infinite case, suppose the associated Hamiltonian $\mathcal{H}^{[1]}$ has an eigenfunction $\phi'(x)$ with an eigenvalue $\mathcal{E}'$ other than those listed above:

$$\mathcal{H}^{[1]}\phi'(x)=\mathcal{E}'\phi'(x).\tag{3.8}$$

Again, thanks to the second of the relation (3.2), it is trivial to verify

$$\mathcal{H}^{[0]}\mathcal{A}^{[0]\dagger}\phi'(x)=\mathcal{E}'\mathcal{A}^{[0]\dagger}\phi'(x).\tag{3.9}$$

Due to the *completeness* of the spectrum of the original Hamiltonian $\mathcal{H}^{[0]}$, the provisional eigenvalue $\mathcal{E}'$ must belong to the spectrum (2.5) $\mathcal{E}(n)$ for $n=1,2.\ldots,n_{\max}$. In other words, $\mathcal{E}'$ cannot be vanishing, $\mathcal{E}'\neq 0$. Suppose that is the case ($\mathcal{E}'=0$), then ϕ' is annihilated by $\mathcal{A}^{[0]\dagger}$. But it is easy to see that there exists no non-zero solution (the finite case) or no finite ℓ^2-norm solution (the infinite case) of the equation $\mathcal{A}^{[0]\dagger}\phi'(x)=0$. Thus we have established that the associated Hamiltonian system $\mathcal{H}^{[1]}$ is essentially *iso-spectral* to the original Hamiltonian system $\mathcal{H}^{[0]}$ and the eigenfunctions are in one to one correspondence, *except for* the groundstate with the wavefunction $\phi^{[0]}_0(x)$.

If the groundstate energy $\mathcal{E}(1)$ is subtracted from the associated Hamiltonian $\mathcal{H}^{[1]}$, it is again positive semi-definite and can be factorised as above:

$$\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} + \mathcal{E}(1), \tag{3.10}$$

$$\mathcal{A}^{[1]} \stackrel{\text{def}}{=} \sqrt{B^{[1]}(x)} - e^{\partial}\sqrt{D^{[1]}(x)}, \quad \mathcal{A}^{[1]\dagger} = \sqrt{B^{[1]}(x)} - \sqrt{D^{[1]}(x)}\, e^{-\partial}, \tag{3.11}$$

$$B^{[1]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[0]}(x+1)D^{[0]}(x+1)}\, \frac{\phi_1^{[1]}(x+1)}{\phi_1^{[1]}(x)}, \tag{3.12}$$

$$D^{[1]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[0]}(x)D^{[0]}(x)}\, \frac{\phi_1^{[1]}(x-1)}{\phi_1^{[1]}(x)}. \tag{3.13}$$

Since $\phi_1^{[1]}$ is the groundstate of $\mathcal{H}^{[1]}$, it does not vanish inside the interval $[0, x_{\max}^{[1]}]$. Thus $B^{[1]}(x)$ and $D^{[1]}(x)$ are non-singular and positive. Note that $D^{[1]}(0)$ is not yet defined because $\phi_1^{[1]}(-1)$ is not defined. Due to the factor $D^{[0]}(x)$ in (3.13) and property $D^{[0]}(0) = 0$, we define $D^{[1]}(0) \stackrel{\text{def}}{=} 0$. In concrete examples the expression of $\phi_1^{[1]}(x)^2$ can be analytically continued and it vanishes at $x = -1$. This fact also supports the definition $D^{[1]}(0) = 0$. The groundstate wavefunction $\phi_1^{[1]}(x)$ has the following form

$$\phi_1^{[1]}(x) = \mathcal{A}^{[0]}\phi_1^{[0]}(x) = \sqrt{B^{[0]}(x)}\, \phi_1^{[0]}(x) - \sqrt{D^{[0]}(x+1)}\, \phi_1^{[0]}(x+1)(1 - \delta_{x, x_{\max}^{[0]}}). \tag{3.14}$$

Due to the vanishing factor $B^{[0]}(x_{\max}^{[0]}) = 0$ for the finite case and $\phi_n^{[0]}(x_{\max}^{[0]}) = 0$ for the infinite case, this expression of the groundstate wavefunction $\phi_1^{[1]}(x)$ vanishes at $x = x_{\max}^{[0]}$,

$$\phi_1^{[1]}(x_{\max}^{[0]}) = 0. \tag{3.15}$$

Thus we have $B^{[1]}(x_{\max}^{[1]}) = 0$ for the finite case.

It is easy to verify that $\phi_1^{[1]}(x)$ is annihilated by $\mathcal{A}^{[1]}$,

$$\mathcal{A}^{[1]}\phi_1^{[1]}(x) = 0. \tag{3.16}$$

The equation (3.10) is shown by elementary calculation.

3.2 Repetition

Starting from (3.10), the second associated Hamiltonian system $\mathcal{H}^{[2]}$ can be defined by reversing the order of $\mathcal{A}^{[1]\dagger}$ and $\mathcal{A}^{[1]}$. This process can go on repeatedly.

Here we list the definition of the s-th quantities step by step for $s \geq 1$,

$$\mathcal{H}^{[s]} \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]}\mathcal{A}^{[s-1]\,\dagger} + \mathcal{E}(s-1), \quad x_{\max}^{[s]} \stackrel{\text{def}}{=} N - s \text{ or } \infty, \tag{3.17}$$

$$\phi_n^{[s]}(x) \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]}\phi_n^{[s-1]}(x) \quad (n=s,s+1,\ldots,n_{\max}), \tag{3.18}$$
$$\mathcal{A}^{[s]} \stackrel{\text{def}}{=} \sqrt{B^{[s]}(x)} - e^{\partial}\sqrt{D^{[s]}(x)}, \quad \mathcal{A}^{[s]\dagger} = \sqrt{B^{[s]}(x)} - \sqrt{D^{[s]}(x)}\,e^{-\partial}, \tag{3.19}$$
$$B^{[s]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[s-1]}(x+1)D^{[s-1]}(x+1)}\,\frac{\phi_s^{[s]}(x+1)}{\phi_s^{[s]}(x)}, \tag{3.20}$$
$$D^{[s]}(x) \stackrel{\text{def}}{=} \sqrt{B^{[s-1]}(x)D^{[s-1]}(x)}\,\frac{\phi_s^{[s]}(x-1)}{\phi_s^{[s]}(x)}. \tag{3.21}$$

Like in the $s=1$ case we set $D^{[s]}(0)=0$. Recall that $\mathcal{H}^{[s]}$ is an $(x_{\max}^{[s]}+1)\times(x_{\max}^{[s]}+1)$ matrix $\mathcal{H}^{[s]}=(\mathcal{H}_{x,y}^{[s]})$ $(x,y=0,1,\ldots,x_{\max}^{[s]})$. Then we can show the following for $s\geq 0$,

$$\mathcal{H}^{[s]}\phi_n^{[s]}(x) = \mathcal{E}(n)\phi_n^{[s]}(x) \quad (n=s,s+1,\ldots,n_{\max}), \tag{3.22}$$
$$\mathcal{A}^{[s]}\phi_s^{[s]}(x) = 0, \tag{3.23}$$
$$\mathcal{H}^{[s]} = \mathcal{A}^{[s]\dagger}\mathcal{A}^{[s]} + \mathcal{E}(s), \tag{3.24}$$
$$(\phi_n^{[s]},\phi_m^{[s]}) \stackrel{\text{def}}{=} \sum_{x=0}^{x_{\max}^{[s]}} \phi_n^{[s]}(x)\phi_m^{[s]}(x) = \prod_{j=0}^{s-1}(\mathcal{E}(n)-\mathcal{E}(j))\cdot\frac{1}{d_n^2}\,\delta_{nm} \quad (n,m=s,s+1,\ldots,n_{\max}), \tag{3.25}$$

and $B^{[s]}(x_{\max}^{[s]})=0$ for the finite case. We have also for $s\geq 1$

$$\phi_n^{[s-1]}(x) = \frac{\mathcal{A}^{[s-1]\dagger}}{\mathcal{E}(n)-\mathcal{E}(s-1)}\,\phi_n^{[s]}(x) \quad (n=s,s+1,\ldots,n_{\max}). \tag{3.26}$$

The situation of Crum's theorem is illustrated in Fig. 1.

As in the original Crum's case [13], the eigenfunction $\phi_n^{[s]}(x)$ can be expressed in terms of determinants. Let us define the Casorati determinant for n functions $f_j(x)$ as

$$\mathrm{W}[f_1,\ldots,f_n](x) \stackrel{\text{def}}{=} \det\Bigl(f_k(x+j-1)\Bigr)_{1\leq j,k\leq n}, \tag{3.27}$$

(for $n=0$, we set $\mathrm{W}[\cdot](x)=1$), which satisfies

$$\mathrm{W}[gf_1,gf_2,\ldots,gf_n](x) = \prod_{k=0}^{n-1}g(x+k)\cdot\mathrm{W}[f_1,f_2,\ldots,f_n](x), \tag{3.28}$$
$$\begin{aligned}&\mathrm{W}\bigl[\mathrm{W}[f_1,f_2,\ldots,f_n,g],\mathrm{W}[f_1,f_2,\ldots,f_n,h]\,\bigr](x)\\ &= \mathrm{W}[f_1,f_2,\ldots,f_n](x+1)\,\mathrm{W}[f_1,f_2,\ldots,f_n,g,h](x) \qquad (n\geq 0).\end{aligned} \tag{3.29}$$

By using the Casorati determinant, we obtain

$$\phi_n^{[s]}(x) = (-1)^s\prod_{k=0}^{s-1}\sqrt{B^{[k]}(x)}\cdot\frac{\mathrm{W}[\phi_0,\phi_1,\ldots,\phi_{s-1},\phi_n](x)}{\mathrm{W}[\phi_0,\phi_1,\ldots,\phi_{s-1}](x+1)} \tag{3.30}$$

$$= (-1)^s \prod_{k=0}^{s-1} \sqrt{D^{[k]}(x+s-k)} \cdot \frac{\mathrm{W}[\phi_0, \phi_1, \ldots, \phi_{s-1}, \phi_n](x)}{\mathrm{W}[\phi_0, \phi_1, \ldots, \phi_{s-1}](x)}. \tag{3.31}$$

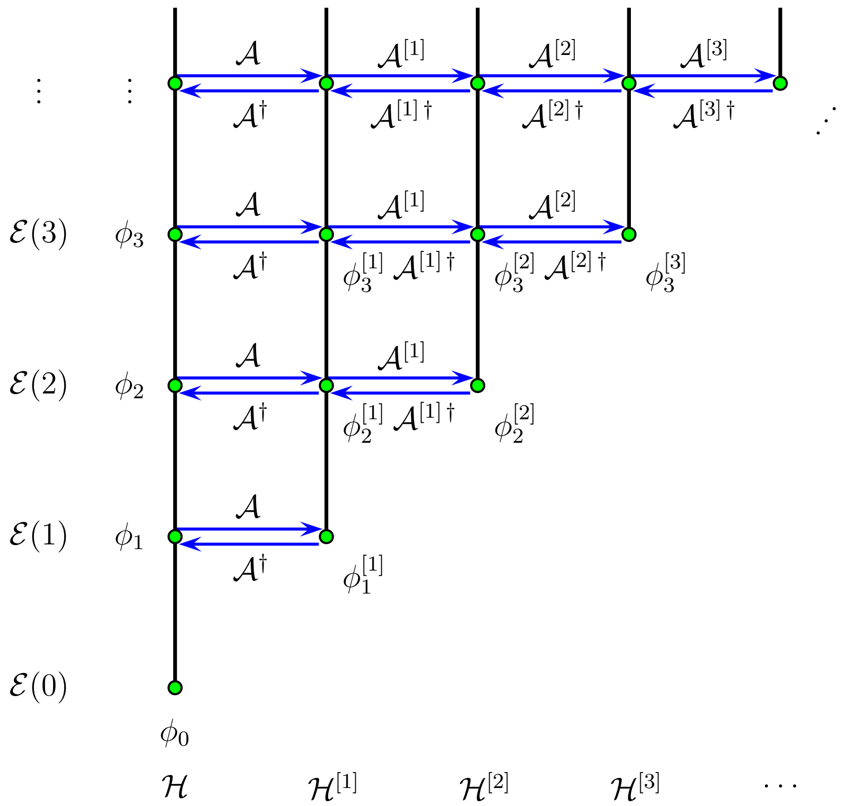


Figure 1: Schematic picture of Crum's theorem

4 Adler's Modification of Crum's Theorem

Crum's theorem describes the construction of an associated Hamiltonian system which is iso-spectral to the original one with the lowest energy state deleted. Adler's modification [18] of Crum's theorem is the construction of an associated Hamiltonian system which is iso-spectral to the original one with finitely many states deleted.

Let us choose a set of ℓ distinct non-negative integers[1] $\mathcal{D} \overset{\text{def}}{=} \{d_1, d_2, \ldots, d_\ell\} \subset \mathbb{Z}_{\geq 0}^{\ell}$, satisfying the condition [18]

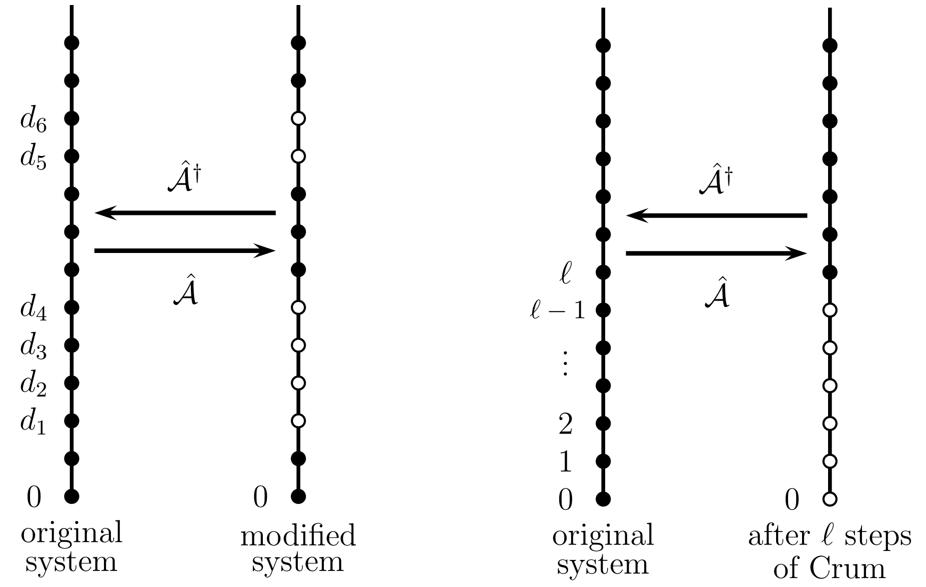
$$\prod_{j=1}^{\ell} (m - d_j) \geq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}. \tag{4.1}$$

[1]Although this notation d_j conflicts with the notation of the normalisation constant d_n in (2.12), we think this does not cause any confusion because the latter appears as $\frac{1}{d_n^2}\,\delta_{nm}$.

This condition means that the set $\mathcal{D}$ consists of several clusters, each containing an *even number* of *contiguous* integers

$$d_{k_1}, d_{k_1}+1, \cdots, d_{k_2}\ ;\ d_{k_3}, d_{k_3}+1, \cdots, d_{k_4}\ ;\ d_{k_5}, d_{k_5}+1, \cdots, d_{k_6}\ ;\ \cdots, \tag{4.2}$$

where $d_{k_2}+1 < d_{k_3},\ d_{k_4}+1 < d_{k_5},\ \cdots$. If $d_{k_1} = 0$ for the lowest lying cluster, it could contain an even or odd number of contiguous integers. The set $\mathcal{D}$ specifies the energy levels to be *deleted*. Deleting an arbitrary number of contiguous energy levels starting from the groundstate ($\mathcal{D} = \{0, 1, 2, \ldots, \ell - 1\}$) is achieved by Crum's theorem discussed in § 3.



$\hat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\ldots d_\ell} \cdots \mathcal{A}_{d_1 d_2}\mathcal{A}_{d_1}$

Figure 2a: Generic case

$\hat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}^{[\ell-1]} \ldots \mathcal{A}^{[1]}\mathcal{A}^{[0]}$

Figure 2b: Crum's case

We will construct associated Hamiltonian systems corresponding to the successive deletions $\mathcal{H}_{d_1\ldots}$ (and $\mathcal{A}_{d_1\ldots}$, $\mathcal{A}^\dagger_{d_1\ldots}$, etc.) step by step, algebraically. It should be noted that some Hamiltonians in the intermediate steps could be non-hermitian.

4.1 First step

For given d_1, the original Hamiltonian $\mathcal{H}$ can be expressed in two different ways:

$$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A} = \mathcal{A}^\dagger_{d_1}\mathcal{A}_{d_1} + \mathcal{E}(d_1), \tag{4.3}$$

$$\mathcal{A}_{d_1} \stackrel{\text{def}}{=} \sqrt{B_{d_1}(x)} - e^{\partial}\sqrt{D_{d_1}(x)}, \quad \mathcal{A}^\dagger_{d_1} \stackrel{\text{def}}{=} \sqrt{B_{d_1}(x)} - \sqrt{D_{d_1}(x)}\, e^{-\partial}, \tag{4.4}$$

$$B_{d_1}(x) \stackrel{\text{def}}{=} \sqrt{B(x)D(x+1)}\,\frac{\phi_{d_1}(x+1)}{\phi_{d_1}(x)}, \quad D_{d_1}(x) \stackrel{\text{def}}{=} \sqrt{B(x-1)D(x)}\,\frac{\phi_{d_1}(x-1)}{\phi_{d_1}(x)}, \tag{4.5}$$

and we have

$$\mathcal{A}_{d_1}\phi_{d_1}(x) = 0. \tag{4.6}$$

As in § 3 we set $D_{d_1}(0) = 0$. Note that $B_{d_1}(x_{\max}) = 0$ for the finite case. Unless $d_1 = 0$, $B_{d_1}(x)$ and $D_{d_1}(x)$ are not always positive due to the zeros of $\phi_{d_1}(x)$. It is important to note that $\mathcal{A}_{d_1}^{\dagger}$ in (4.4) is a 'formal adjoint' of $\mathcal{A}_{d_1}$ due to the above mentioned sign changes of $B_{d_1}(x)$ and $D_{d_1}(x)$. We stick to this notation, since the algebraic structure of various expressions appearing in the deletion processes are best described by using the 'formal adjoint'. By changing the order of $\mathcal{A}_{d_1}$ and $\mathcal{A}_{d_1}^{\dagger}$, let us define a new Hamiltonian system

$$\mathcal{H}_{d_1} \stackrel{\text{def}}{=} \mathcal{A}_{d_1}\mathcal{A}_{d_1}^{\dagger} + \mathcal{E}(d_1), \quad x_{\max}^{d_1} \stackrel{\text{def}}{=} N-1 \text{ or } \infty. \tag{4.7}$$

As in § 3, the Hamiltonian $\mathcal{H}_{d_1}$ represents an $(x_{\max}^{d_1}+1)\times(x_{\max}^{d_1}+1)$ matrix $\mathcal{H}_{d_1} = (\mathcal{H}_{d_1;x,y})$ $(x, y = 0, 1, \ldots, x_{\max}^{d_1})$. It is easy to show that the 'eigenfunctions' of this Hamiltonian are given by

$$\phi_{d_1\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1}\phi_n(x) \qquad (n \in \{0, 1, \ldots, n_{\max}\}\backslash\{d_1\}), \tag{4.8}$$

$$\mathcal{H}_{d_1}\phi_{d_1\,n}(x) = \mathcal{E}(n)\phi_{d_1\,n}(x) \quad (n \in \{0, 1, \ldots, n_{\max}\}\backslash\{d_1\}). \tag{4.9}$$

Thus the energy level d_1 is now deleted, $\phi_{d_1\,d_1}(x) \equiv 0$, from the set of 'eigenfunctions' $\{\phi_{d_1\,n}(x)\}$ of the new Hamiltonian $\mathcal{H}_{d_1}$.

4.2 Repetition

Suppose we have determined the Hamiltonian $\mathcal{H}_{d_1\ldots d_s}$ together with the eigenfunctions $\phi_{d_1\ldots d_s\,n}(x)$ with s deletions. They have the following properties:

$$\mathcal{H}_{d_1\ldots d_s} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\ldots d_s}\mathcal{A}_{d_1\ldots d_s}^{\dagger} + \mathcal{E}(d_s), \quad x_{\max}^{d_1\ldots d_s} \stackrel{\text{def}}{=} N-s \text{ or } \infty, \tag{4.10}$$

$$\mathcal{A}_{d_1\ldots d_s} \stackrel{\text{def}}{=} \sqrt{B_{d_1\ldots d_s}(x)} - e^{\partial}\sqrt{D_{d_1\ldots d_s}(x)},$$

$$\mathcal{A}_{d_1\ldots d_s}^{\dagger} \stackrel{\text{def}}{=} \sqrt{B_{d_1\ldots d_s}(x)} - \sqrt{D_{d_1\ldots d_s}(x)}\,e^{-\partial}, \tag{4.11}$$

$$B_{d_1\ldots d_s}(x) \stackrel{\text{def}}{=} \begin{cases} \sqrt{B_{d_1\ldots d_{s-1}}(x+1)D_{d_1\ldots d_{s-1}}(x+1)}\,\dfrac{\phi_{d_1\ldots d_s}(x+1)}{\phi_{d_1\ldots d_s}(x)} & (s \geq 2) \\ \sqrt{B(x)D(x+1)}\,\dfrac{\phi_{d_1}(x+1)}{\phi_{d_1}(x)} & (s = 1) \end{cases}, \tag{4.12}$$

$$D_{d_1\dots d_s}(x) \stackrel{\text{def}}{=} \begin{cases} \sqrt{B_{d_1\dots d_{s-1}}(x)D_{d_1\dots d_{s-1}}(x)}\,\dfrac{\phi_{d_1\dots d_s}(x-1)}{\phi_{d_1\dots d_s}(x)} & (s\geq 2) \\ \sqrt{B(x-1)D(x)}\,\dfrac{\phi_{d_1}(x-1)}{\phi_{d_1}(x)} & (s=1) \end{cases}, \tag{4.13}$$

$$\phi_{d_1\dots d_s\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1\dots d_s}\phi_{d_1\dots d_{s-1}\,n}(x), \tag{4.14}$$

$$\mathcal{H}_{d_1\dots d_s}\phi_{d_1\dots d_s\,n}(x) = \mathcal{E}(n)\phi_{d_1\dots d_s\,n}(x), \tag{4.15}$$

where $n \in \{0,1,\dots,n_{\text{max}}\}\backslash\{d_1,\dots,d_s\}$. As before we set $D_{d_1\dots d_s}(0) = 0$. We also have $B_{d_1\dots d_s}(x^{[s]}_{\text{max}}) = 0$ for the finite case. We note that the following relations hold:

$$B_{d_1\dots d_s}(x) = D_{d_1\dots d_s}(x+1)\Big(\frac{\phi_{d_1\dots d_s}(x+1)}{\phi_{d_1\dots d_s}(x)}\Big)^2 \quad (s\geq 1), \tag{4.16}$$

$$B_{d_1\dots d_s}(x)D_{d_1\dots d_s}(x+1) = \begin{cases} B_{d_1\dots d_{s-1}}(x+1)D_{d_1\dots d_{s-1}}(x+1) & (s\geq 2) \\ B(x)D(x+1) & (s=1) \end{cases}, \tag{4.17}$$

$$B_{d_1\dots d_s}(x) + D_{d_1\dots d_s}(x) + \mathcal{E}(d_s) = \begin{cases} B_{d_1\dots d_{s-1}}(x) + D_{d_1\dots d_{s-1}}(x+1) + \mathcal{E}(d_{s-1}) & (s\geq 2) \\ B(x)+D(x) & (s=1) \end{cases}. \tag{4.18}$$

We have also

$$\phi_{d_1\dots d_{s-1}\,n}(x) = \frac{\mathcal{A}^\dagger_{d_1\dots d_s}}{\mathcal{E}(n)-\mathcal{E}(d_s)}\phi_{d_1\dots d_s\,n}(x) \quad (n\in\{0,1,\dots,n_{\text{max}}\}\backslash\{d_1,\dots,d_s\}). \tag{4.19}$$

Next we will define a new Hamiltonian system with one more deletion of the level d_{s+1}. We can show the following:

$$\mathcal{H}_{d_1\dots d_s} = \mathcal{A}^\dagger_{d_1\dots d_s\,d_{s+1}}\mathcal{A}_{d_1\dots d_s\,d_{s+1}} + \mathcal{E}(d_{s+1}), \quad \mathcal{A}_{d_1\dots d_s\,d_{s+1}}\phi_{d_1\dots d_s\,d_{s+1}}(x) = 0, \tag{4.20}$$

$$\begin{aligned} \mathcal{A}_{d_1\dots d_s\,d_{s+1}} &\stackrel{\text{def}}{=} \sqrt{B_{d_1\dots d_s\,d_{s+1}}(x)} - e^{\partial}\sqrt{D_{d_1\dots d_s\,d_{s+1}}(x)}, \\ \mathcal{A}^\dagger_{d_1\dots d_s\,d_{s+1}} &\stackrel{\text{def}}{=} \sqrt{B_{d_1\dots d_s\,d_{s+1}}(x)} - \sqrt{D_{d_1\dots d_s\,d_{s+1}}(x)}\,e^{-\partial}, \end{aligned} \tag{4.21}$$

$$B_{d_1\dots d_s\,d_{s+1}}(x) \stackrel{\text{def}}{=} \sqrt{B_{d_1\dots d_s}(x+1)D_{d_1\dots d_s}(x+1)}\,\frac{\phi_{d_1\dots d_s\,d_{s+1}}(x+1)}{\phi_{d_1\dots d_s\,d_{s+1}}(x)}, \tag{4.22}$$

$$D_{d_1\dots d_s\,d_{s+1}}(x) \stackrel{\text{def}}{=} \sqrt{B_{d_1\dots d_s}(x)D_{d_1\dots d_s}(x)}\,\frac{\phi_{d_1\dots d_s\,d_{s+1}}(x-1)}{\phi_{d_1\dots d_s\,d_{s+1}}(x)}. \tag{4.23}$$

These determine a new Hamiltonian system with $s+1$ deletions:

$$\mathcal{H}_{d_1\dots d_{s+1}} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\dots d_{s+1}}\mathcal{A}^\dagger_{d_1\dots d_{s+1}} + \mathcal{E}(d_{s+1}), \quad x^{d_1\dots d_{s+1}}_{\text{max}} \stackrel{\text{def}}{=} N-s-1 \text{ or } \infty, \tag{4.24}$$

$$\phi_{d_1\dots d_{s+1}\,n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{d_1\dots d_{s+1}}\phi_{d_1\dots d_s\,n}(x), \tag{4.25}$$

$$\mathcal{H}_{d_1\dots d_{s+1}}\phi_{d_1\dots d_{s+1}\,n}(x) = \mathcal{E}(n)\phi_{d_1\dots d_{s+1}\,n}(x), \tag{4.26}$$

where $n \in \{0, 1, \ldots, n_{\max}\} \backslash \{d_1, \ldots, d_{s+1}\}$.

By induction we can show that the eigenfunction is expressed in terms of the Casorati determinant:

$$\phi_{d_1 \ldots d_s n}(x) = (-1)^s \sqrt{\prod_{k=1}^{s} B_{d_1 \ldots d_k}(x)} \cdot \frac{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}, \phi_n](x)}{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x+1)} \tag{4.27}$$

$$= (-1)^s \sqrt{\prod_{k=1}^{s} D_{d_1 \ldots d_k}(x+s+1-k)} \cdot \frac{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}, \phi_n](x)}{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x)}. \tag{4.28}$$

We can also show the following:

$$\prod_{k=1}^{s} B_{d_1 \ldots d_k}(x) = \sqrt{\prod_{k=1}^{s} B(x+k-1) D(x+k)} \cdot \frac{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x+1)}{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x)}, \tag{4.29}$$

$$\prod_{k=1}^{s} D_{d_1 \ldots d_k}(x+s+1-k) = \sqrt{\prod_{k=1}^{s} B(x+k-1) D(x+k)} \cdot \frac{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x)}{\mathrm{W}[\phi_{d_1}, \ldots, \phi_{d_s}](x+1)}. \tag{4.30}$$

4.3 Last step

After deleting all the $\mathcal{D} = \{d_1, \cdots, d_\ell\}$ energy levels, the resulting Hamiltonian system $\mathcal{H}_{\mathcal{D}} \equiv \mathcal{H}_{d_1 \ldots d_\ell}$, $\mathcal{A}_{\mathcal{D}} \equiv \mathcal{A}_{d_1 \ldots d_\ell}$, etc has the following form:

$$\mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{D}} \mathcal{A}_{\mathcal{D}}^{\dagger} + \mathcal{E}(d_\ell), \quad x_{\max}^{\mathcal{D}} \stackrel{\text{def}}{=} N - \ell \text{ or } \infty, \tag{4.31}$$

$$\mathcal{A}_{\mathcal{D}} \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x)} - e^{\partial} \sqrt{D_{\mathcal{D}}(x)}, \quad \mathcal{A}_{\mathcal{D}}^{\dagger} \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x)} - \sqrt{D_{\mathcal{D}}(x)}\, e^{-\partial}, \tag{4.32}$$

$$B_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \sqrt{B_{d_1 \ldots d_{\ell-1}}(x+1) D_{d_1 \ldots d_{\ell-1}}(x+1)}\, \frac{\phi_{\mathcal{D}}(x+1)}{\phi_{\mathcal{D}}(x)}, \tag{4.33}$$

$$D_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \sqrt{B_{d_1 \ldots d_{\ell-1}}(x) D_{d_1 \ldots d_{\ell-1}}(x)}\, \frac{\phi_{\mathcal{D}}(x-1)}{\phi_{\mathcal{D}}(x)}, \tag{4.34}$$

$$\phi_{\mathcal{D}\, n}(x) \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{D}} \phi_{d_1 \ldots d_{\ell-1} n}(x) \quad (n \in \{0, 1, \ldots, n_{\max}\} \backslash \mathcal{D}), \tag{4.35}$$

$$\mathcal{H}_{\mathcal{D}} \phi_{\mathcal{D}\, n}(x) = \mathcal{E}(n) \phi_{\mathcal{D}\, n}(x) \quad (n \in \{0, 1, \ldots, n_{\max}\} \backslash \mathcal{D}). \tag{4.36}$$

Now $\mathcal{H}_{\mathcal{D}}$ has the lowest energy level μ:

$$\mu \stackrel{\text{def}}{=} \min\{n \,|\, n \in \{0, 1, \ldots, n_{\max}\} \backslash \mathcal{D}\}, \tag{4.37}$$

with the groundstate wavefunction $\bar{\phi}_\mu(x)$

$$\bar{\phi}_\mu(x) \stackrel{\text{def}}{=} \phi_{\mathcal{D}\, \mu}(x) \equiv \phi_{d_1 \cdots d_\ell\, \mu}(x). \tag{4.38}$$

Then the Hamiltonian system can be expressed simply in terms of the groundstate wavefunction $\bar{\phi}_\mu(x)$, which we will denote by new symbols $\bar{\mathcal{H}}$, $\bar{\mathcal{A}}$, etc:

$$\bar{\mathcal{H}} \equiv \mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \bar{\mathcal{A}}^\dagger \bar{\mathcal{A}} + \mathcal{E}(\mu), \quad \bar{x}_{\text{max}} \equiv x^{\mathcal{D}}_{\text{max}} = N - \ell \text{ or } \infty, \quad \bar{\mathcal{A}}\bar{\phi}_\mu(x) = 0, \tag{4.39}$$

$$\bar{\mathcal{A}} \equiv \mathcal{A}_{\mathcal{D}\,\mu} \stackrel{\text{def}}{=} \sqrt{\bar{B}(x)} - e^{\partial}\sqrt{\bar{D}(x)}, \quad \bar{\mathcal{A}}^\dagger \equiv \mathcal{A}^\dagger_{\mathcal{D}\,\mu} \stackrel{\text{def}}{=} \sqrt{\bar{B}(x)} - \sqrt{\bar{D}(x)}\, e^{-\partial}, \tag{4.40}$$

$$\bar{B}(x) \equiv B_{\mathcal{D}\,\mu}(x) \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x+1)D_{\mathcal{D}}(x+1)}\,\frac{\bar{\phi}_\mu(x+1)}{\bar{\phi}_\mu(x)}, \tag{4.41}$$

$$\bar{D}(x) \equiv D_{\mathcal{D}\,\mu}(x) \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x)D_{\mathcal{D}}(x)}\,\frac{\bar{\phi}_\mu(x-1)}{\bar{\phi}_\mu(x)}, \tag{4.42}$$

$$\bar{\mathcal{H}}\bar{\phi}_n(x) = \mathcal{E}(n)\bar{\phi}_n(x), \quad \bar{\phi}_n(x) \equiv \phi_{\mathcal{D}\,n}(x) \quad (n \in \{0,1,\ldots,n_{\text{max}}\}\backslash\mathcal{D}), \tag{4.43}$$

$$\bar{B}(x) + \bar{D}(x) + \mathcal{E}(\mu) = B_{\mathcal{D}}(x) + D_{\mathcal{D}}(x+1) + \mathcal{E}(d_\ell). \tag{4.44}$$

Again we set $\bar{D}(0) = 0$. We have $\bar{B}(\bar{x}_{\text{max}}) = 0$ for the finite case. Note that the final Hamiltonian $\bar{\mathcal{H}}$ is an $(\bar{x}_{\text{max}}+1)\times(\bar{x}_{\text{max}}+1)$ matrix $\bar{\mathcal{H}} = (\bar{\mathcal{H}}_{x,y})$ $(x, y = 0,1,\ldots,\bar{x}_{\text{max}})$.

In terms of the Casorati determinants we obtain the expressions of the final eigenfunction $\bar{\phi}_n(x)$ and the final functions $\bar{B}(x)$ and $\bar{D}(x)$ $(\ell \geq 0)$

$$\bar{\phi}_n(x) = (-1)^\ell \sqrt{\prod_{k=1}^{\ell} B_{d_1\ldots d_k}(x)} \cdot \frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_n](x)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x+1)} \tag{4.45}$$

$$= (-1)^\ell \sqrt{\prod_{k=1}^{\ell} D_{d_1\ldots d_k}(x+\ell+1-k)} \cdot \frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_n](x)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x)}, \tag{4.46}$$

$$\bar{B}(x) = \sqrt{B(x+\ell)D(x+\ell+1)}\,\frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x+1)}\,\frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_\mu](x+1)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_\mu](x)}, \tag{4.47}$$

$$\bar{D}(x) = \sqrt{B(x-1)D(x)}\,\frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x+1)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x)}\,\frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_\mu](x-1)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell},\phi_\mu](x)}. \tag{4.48}$$

From (4.29)–(4.30) we also have $(\ell \geq 0)$

$$\prod_{k=1}^{\ell} B_{d_1\ldots d_k}(x) = \sqrt{\prod_{k=1}^{\ell} B(x+k-1)D(x+k)} \cdot \frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x+1)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x)}, \tag{4.49}$$

$$\prod_{k=1}^{\ell} D_{d_1\ldots d_k}(x+\ell+1-k) = \sqrt{\prod_{k=1}^{\ell} B(x+k-1)D(x+k)} \cdot \frac{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x)}{\mathrm{W}[\phi_{d_1},\ldots,\phi_{d_\ell}](x+1)}. \tag{4.50}$$

Therefore the functions $\bar{B}(x)$, $\bar{D}(x)$ and $\bar{\phi}_n(x)$ are symmetric with respect to $d_1,\ldots,d_\ell$, and the final Hamiltonian $\bar{\mathcal{H}}$ is independent of the order of $\{d_j\}$.

Let us state the discrete quantum mechanics version of Adler's theorem; *If the set of deleted energy levels $\mathcal{D}=\{d_1,\ldots,d_\ell\}$ satisfy the condition* (4.1)*, the modified Hamiltonian is given by $\bar{\mathcal{H}}=\mathcal{H}_{d_1\ldots d_\ell}=\bar{\mathcal{A}}^\dagger\bar{\mathcal{A}}+\mathcal{E}(\mu)$ with the potential functions given by* (4.47)–(4.48) *and its eigenfunctions are given by* (4.45)–(4.46). The discrete QM version of Crum's theorem in § 3 corresponds to the choice $\{d_1,\ldots,d_\ell\}=\{0,1,\ldots,\ell-1\}$ and the new groundstate is at the level $\mu=\ell$ and there is no vacant energy level above that. We have not yet proven the hermiticity of the resulting Hamiltonian $\bar{\mathcal{H}}$ and the reality of the eigenfunctions $\bar{\phi}_n(x)$ categorically for the discrete quantum mechanics, even when the condition (4.1) is satisfied by the deleted levels. This is due to the lack of the difference equation analogue of the oscillation theorem. It should be stressed that in most practical cases, in particular, in the cases of polynomial eigenfunctions, the hermiticity of the Hamiltonian $\bar{\mathcal{H}}$ is satisfied.

The orthogonality relation of the complete set of eigenfunctions is

$$(\bar{\phi}_n,\bar{\phi}_m)\stackrel{\text{def}}{=}\sum_{x=0}^{\bar{x}_{\max}}\bar{\phi}_n(x)\bar{\phi}_m(x)=\prod_{j=1}^{\ell}\bigl(\mathcal{E}(n)-\mathcal{E}(d_j)\bigr)\cdot\frac{1}{d_n^2}\,\delta_{nm}\quad(n,m\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}).\tag{4.51}$$

Note that the coefficient of δ_{nm} is positive for $\mathcal{D}$ satisfying (4.1).

5 Simplifications

In the preceding sections, Crum's theorem and its modification are presented in their most generic forms. These formulas are simplified substantially when the system is *shape invariant* [20] and/or the eigenfunctions consist of polynomials.

5.1 Formulas for Crum's theorem

5.1.1 Shape invariance

Shape invariance is a well known sufficient condition for *exact solvability* in ordinary quantum mechanics. The situation is exactly the same in discrete quantum mechanics. Shape invariance simply means that the s-th associated Hamiltonian system has the same *form* as the original one with certain shifts of the contained parameters.

The Hamiltonian may contain several parameters $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots)$ and we indicate them symbolically as $\mathcal{H}=\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A}=\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}(n)=\mathcal{E}(n;\boldsymbol{\lambda})$, $\phi_n(x)=\phi_n(x;\boldsymbol{\lambda})$, etc. Let us consider the case that the potential functions of the first associated Hamiltonian $B^{[1]}(x)=$

$B^{[1]}(x\,;\boldsymbol{\lambda})$ and $D^{[1]}(x)=D^{[1]}(x\,;\boldsymbol{\lambda})$ have the same forms as the original functions B and D with a different set of parameters and up to a multiplicative positive constant $\kappa\in\mathbb{R}_{>0}$:

$$B^{[1]}(x\,;\boldsymbol{\lambda})=\kappa B(x\,;\boldsymbol{\lambda}'),\quad D^{[1]}(x\,;\boldsymbol{\lambda})=\kappa D(x\,;\boldsymbol{\lambda}'). \tag{5.1}$$

Here the new set of parameters $\boldsymbol{\lambda}'$ is uniquely determined by $\boldsymbol{\lambda}$. In concrete examples we can choose an appropriate set of parameters such that $\boldsymbol{\lambda}'=\boldsymbol{\lambda}+\boldsymbol{\delta}$, with a shift of parameters denoted by $\boldsymbol{\delta}$. In the following we assume this. In fact, (5.1) is equivalent to the definition of the shape invariance by Odake and Sasaki [14, 1, 8],

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger=\kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}(1;\boldsymbol{\lambda}). \tag{5.2}$$

It should be stressed that the above definition of shape invariance is more stringent and thus more constraining than the original definition of Gendenshtein [20]. This condition is rewritten as

$$\sqrt{B(x+1;\boldsymbol{\lambda})D(x+1;\boldsymbol{\lambda})}=\kappa\sqrt{B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})D(x+1;\boldsymbol{\lambda}+\boldsymbol{\delta})}, \tag{5.3}$$

$$B(x;\boldsymbol{\lambda})+D(x+1;\boldsymbol{\lambda})=\kappa\big(B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})+D(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)+\mathcal{E}(1;\boldsymbol{\lambda}). \tag{5.4}$$

The shape invariance condition (5.2) combined with Crum's theorem implies

$$\mathcal{A}^{[s]}(\boldsymbol{\lambda})=\kappa^{\frac{s}{2}}\mathcal{A}(\boldsymbol{\lambda}+s\boldsymbol{\delta}),\quad \mathcal{A}^{[s]\dagger}(\boldsymbol{\lambda})=\kappa^{\frac{s}{2}}\mathcal{A}(\boldsymbol{\lambda}+s\boldsymbol{\delta})^\dagger, \tag{5.5}$$

$$\mathcal{H}^{[s]}(\boldsymbol{\lambda})=\kappa^s\mathcal{H}(\boldsymbol{\lambda}+s\boldsymbol{\delta})+\mathcal{E}(s;\boldsymbol{\lambda}), \tag{5.6}$$

and the entire energy spectrum and the excited states wavefunctions are expressed in terms of $\mathcal{E}(1;\boldsymbol{\lambda})$ and $\phi_0(x\,;\boldsymbol{\lambda})$ as follows:

$$\mathcal{E}(n;\boldsymbol{\lambda})=\sum_{s=0}^{n-1}\kappa^s\mathcal{E}(1;\boldsymbol{\lambda}+s\boldsymbol{\delta}), \tag{5.7}$$

$$\phi_n(x\,;\boldsymbol{\lambda})\propto\mathcal{A}(\boldsymbol{\lambda})^\dagger\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda}+2\boldsymbol{\delta})^\dagger\cdots\mathcal{A}(\boldsymbol{\lambda}+(n-1)\boldsymbol{\delta})^\dagger\phi_0(x\,;\boldsymbol{\lambda}+n\boldsymbol{\delta}). \tag{5.8}$$

Therefore the shape invariance is a sufficient condition for exact solvability. We have also

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda})=\frac{1}{\sqrt{B(0;\boldsymbol{\lambda})}}\,f_n(\boldsymbol{\lambda})\phi_{n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big), \tag{5.9}$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger\phi_{n-1}\big(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\big)=\sqrt{B(0;\boldsymbol{\lambda})}\,b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}), \tag{5.10}$$

where $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of the energy eigenvalue, $\mathcal{E}(n;\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$. It is interesting to note that the polynomial eigenfunctions are not the direct consequence of the shape invariance. When the eigenfunctions consist of polynomials (2.13), the above formula (5.8) could be called the Rodrigues formula for the polynomials. Relations (5.9)–(5.10) would translate into the *forward* and *backward shift relations* [1, 8] for the polynomial eigenfunctions.

5.1.2 Polynomial eigenfunctions

Here we consider a generic Hamiltonian (2.1). That is the shape invariance is not assumed. Let us define a function $\eta(x)$ as a ratio of $\phi_1(x)$ and $\phi_0(x)$,

$$\frac{\phi_1(x)}{\phi_0(x)} = a + b\,\eta(x), \tag{5.11}$$

where a and b ($b \neq 0$) are real constants. Although $\eta(x)$ is not well defined without specifying a and b, this ambiguity (affine transformation of $\eta(x)$) does not affect the following discussion. Then (3.12) and (3.13) imply

$$B^{[1]}(x) = B(x+1)\,\frac{\eta(x+1)-\eta(x+2)}{\eta(x)-\eta(x+1)}, \quad D^{[1]}(x) = D(x)\,\frac{\eta(x-1)-\eta(x)}{\eta(x)-\eta(x+1)}. \tag{5.12}$$

Let us assume further that the n-th eigenfunction ϕ_n/ϕ_0 is a degree n polynomial in this $\eta(x)$ for all $n \geq 2$:

$$\phi_n(x) = \phi_0(x) P_n\big(\eta(x)\big), \quad P_n(y) = \sum_{k=0}^{n} a_{n,k}\, y^k, \quad a_{n,n} \neq 0. \tag{5.13}$$

Obviously $a_{0,0} = 1$, $a = a_{1,0}$ and $b = a_{1,1}$. The orthogonality of the eigenfunctions $\{\phi_n\}$ implies that $\{P_n\big(\eta(x)\big)\}$ are orthogonal polynomials in $\eta(x)$ with respect to the weight function $\phi_0(x)^2$. Then we can show the following:

$$\frac{\phi^{[s]}_{s+1}(x)}{\phi^{[s]}_s(x)} = \frac{a_{s+1,s}}{a_{s,s}} + \frac{a_{s+1,s+1}}{a_{s,s}}\,\eta^{[s]}(x), \quad \eta^{[s]}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{s} \eta(x+k), \tag{5.14}$$

$$B^{[s]}(x) = B^{[s-1]}(x+1)\,\frac{\eta^{[s-1]}(x+1)-\eta^{[s-1]}(x+2)}{\eta^{[s-1]}(x)-\eta^{[s-1]}(x+1)}, \tag{5.15}$$

$$D^{[s]}(x) = D^{[s-1]}(x)\,\frac{\eta^{[s-1]}(x-1)-\eta^{[s-1]}(x)}{\eta^{[s-1]}(x)-\eta^{[s-1]}(x+1)}, \tag{5.16}$$

$$\phi^{[s]}_n(x) = \phi^{[s]}_s(x)\check{\mathcal{P}}^{[s]}_n(x) \quad (n \geq s), \tag{5.17}$$

where $\check{\mathcal{P}}^{[s]}_n(x)$ is a symmetric polynomial of degree $n-s$ in $\eta(x), \eta(x+1), \ldots, \eta(x+s)$ and satisfies the recurrence relation

$$\check{\mathcal{P}}^{[s]}_n(x) = \frac{a_{s-1,s-1}}{a_{s,s}}\,\frac{\check{\mathcal{P}}^{[s-1]}_n(x)-\check{\mathcal{P}}^{[s-1]}_n(x+1)}{\eta^{[s-1]}(x)-\eta^{[s-1]}(x+1)}. \tag{5.18}$$

Obviously $\check{\mathcal{P}}_n^{[0]}(x) = P_n(\eta(x))$ and $\check{\mathcal{P}}_s^{[s]}(x) = 1$. From (5.18), $\check{\mathcal{P}}_n^{[s]}(x)$ is explicitly expressed as

$$\check{\mathcal{P}}_n^{[s]}(x) = \frac{1}{a_{s,s}} \sum_{k=s}^{n} a_{n,k} \sum_{\substack{k_0,\ldots,k_s \geq 0 \\ \sum_{j=0}^{s} k_j = k-s}} \prod_{j=0}^{s} \eta(x+j)^{k_j} \quad (n \geq s \geq 0). \tag{5.19}$$

For the shape invariant case, we have $\check{\mathcal{P}}_n^{[s]}(x) \propto P_{n-s}(\eta(x;\boldsymbol{\lambda}+s\boldsymbol{\delta});\boldsymbol{\lambda}+s\boldsymbol{\delta})$.

5.1.3 Casorati Determinants

Here we prepare several formulas including various Casorati determinants and the sinusoidal coordinates.

For any polynomial in η, $\{P_n(\eta)\}$ ($P_n(\eta) = c_n\eta^n +$(lower degree terms)), a set of variables $\{\eta_j\}$, and a set of non-negative distinct integers $\{n_k\}$, it is easy to show

$$\det\bigl(P_{n_k}(\eta_j)\bigr)_{1\leq j,k\leq m} = \prod_{1\leq j<k\leq m} (\eta_k - \eta_j) \cdot R(\eta_1,\ldots,\eta_m), \tag{5.20}$$

where $R(\eta_1,\ldots,\eta_m)$ is a symmetric polynomial of degree $\sum_{k=1}^{m} n_k - \frac{1}{2}m(m-1)$ in $\eta_1,\ldots,\eta_m$. Especially, for $(n_1,\ldots,n_m) = (0,1,\ldots,\ell)$, the polynomial R becomes a constant and we have

$$\det\bigl(P_{k-1}(\eta_j)\bigr)_{1\leq j,k\leq \ell+1} = \prod_{k=0}^{\ell} c_k \cdot \prod_{1\leq j<k\leq \ell+1} (\eta_k - \eta_j). \tag{5.21}$$

For all the sinusoidal coordinates $\eta(x)$ studied in [1, 9], we have

$$\eta(x+\alpha) + \eta(x-\alpha) = (\text{a polynomial of degree 1 in } \eta(x)), \tag{5.22}$$

$$\eta(x+\alpha)\eta(x-\alpha) = (\text{a polynomial of degree 2 in } \eta(x)), \tag{5.23}$$

with $^\forall\alpha \in \mathbb{R}$. Hence a symmetric polynomial in $\eta(x),\eta(x+1),\ldots,\eta(x+\ell)$ becomes a polynomial in $\eta(x+\frac{\ell}{2})$. Moreover the sinusoidal coordinate $\eta(x;\boldsymbol{\lambda})$ satisfies

$$\eta(x+\alpha;\boldsymbol{\lambda}) = (\text{a polynomial of degree 1 in } \eta(x;\boldsymbol{\lambda}+2\alpha\boldsymbol{\delta})), \tag{5.24}$$

with $^\forall\alpha \in \mathbb{R}$. Therefore a symmetric polynomial in $\eta(x;\boldsymbol{\lambda}),\eta(x+1;\boldsymbol{\lambda}),\ldots,\eta(x+\ell;\boldsymbol{\lambda})$ becomes a polynomial in $\eta(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})$. The sinusoidal coordinate $\eta(x;\boldsymbol{\lambda})$ satisfies

$$\frac{\eta(x+\alpha;\boldsymbol{\lambda}) - \eta(x;\boldsymbol{\lambda})}{\eta(\alpha;\boldsymbol{\lambda})} = \varphi(x;\boldsymbol{\lambda}+(\alpha-1)\boldsymbol{\delta}). \tag{5.25}$$

For the definition of the auxiliary function $\varphi(x;\boldsymbol{\lambda})$, see eqs. (4.12) and (4.23) in [1]. Let us define the ℓ-th auxiliary function $\varphi_\ell(x;\boldsymbol{\lambda})$ as

$$\varphi_\ell(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \prod_{0\le j<k\le \ell-1} \frac{\eta(x+k;\boldsymbol{\lambda})-\eta(x+j;\boldsymbol{\lambda})}{\eta(k-j;\boldsymbol{\lambda})} = \prod_{0\le j<k\le \ell-1} \varphi(x+j;\boldsymbol{\lambda}+(k-j-1)\boldsymbol{\delta}). \quad (5.26)$$

Then, for any polynomials of degree n in $\eta(x;\boldsymbol{\lambda})$, $\check{P}_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})$, we have

$$\frac{\mathrm{W}[\check{P}_{n_1},\check{P}_{n_2},\ldots,\check{P}_{n_m}](x;\boldsymbol{\lambda})}{\varphi_m(x;\boldsymbol{\lambda})} = \begin{pmatrix} \text{a polynomial of degree } \sum_{k=1}^m n_k - \frac{1}{2}m(m-1) \\ \text{in } \eta(x;\boldsymbol{\lambda}+(m-1)\boldsymbol{\delta}) \end{pmatrix}, \quad (5.27)$$

and (5.21) implies

$$\mathrm{W}[\check{P}_0,\check{P}_1,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda}) = \prod_{k=0}^{\ell} c_k(\boldsymbol{\lambda}) \cdot \prod_{k=1}^{\ell}\prod_{j=1}^{k} \eta(j;\boldsymbol{\lambda}) \cdot \varphi_{\ell+1}(x;\boldsymbol{\lambda}). \quad (5.28)$$

Note that, for the orthogonal polynomials studied in [1], we have (A.16).

5.2 Formulas for modified Crum's theorem

In this subsection we consider the simplification of the formulas in § 4 when the eigenfunctions consist of the orthogonal polynomials $\{P_n\}$:

$$\phi_n(x) = \phi_0(x)P_n(\eta(x)), \quad \check{P}_n(x) \stackrel{\text{def}}{=} P_n(\eta(x)), \quad (5.29)$$

where $P_n(\eta)$ is a polynomial of degree n in η, satisfying the boundary condition $\eta(0)=0$ (2.14).

First we recall that the ground state $\phi_0(x)$ is annihilated by $\mathcal{A}$,

$$\sqrt{B(x)}\,\phi_0(x) = \sqrt{D(x+1)}\,\phi_0(x+1). \quad (5.30)$$

By using this and (3.28), eqs. (4.49)–(4.50) become

$$\prod_{k=1}^{\ell} B_{d_1\ldots d_k}(x) = \prod_{k=1}^{\ell} B(x+k-1)\cdot \frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)}, \quad (5.31)$$

$$\prod_{k=1}^{\ell} D_{d_1\ldots d_k}(x+\ell+1-k) = \prod_{k=1}^{\ell} D(x+k)\cdot \frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}. \quad (5.32)$$

From these and (4.45)–(4.48) we obtain

$$\bar{\phi}_n(x) = (-1)^\ell \phi_0(x)\sqrt{\prod_{k=1}^{\ell} B(x+k-1)}\cdot \frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_n](x)}{\sqrt{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}} \quad (5.33)$$

$$= (-1)^{\ell}\phi_0(x+\ell)\sqrt{\prod_{k=1}^{\ell} D(x+k)}\cdot\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_n](x)}{\sqrt{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}}, \tag{5.34}$$

$$\bar{B}(x) = B(x+\ell)\,\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}\,\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x+1)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x)}, \tag{5.35}$$

$$\bar{D}(x) = D(x)\,\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x+1)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x)}\,\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x-1)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x)}. \tag{5.36}$$

The eigenfunction $\bar{\phi}_n(x)$ of this $\mathcal{D}$-deleted system has the following structure,

$$\bar{\phi}_n(x) = \bar{\phi}_\mu(x)\times\frac{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_n](x)}{\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x)}. \tag{5.37}$$

In the following we assume that $\eta(x)$ satisfies (5.22)–(5.25). Then the above formulas (5.33)–(5.36) are simplified thanks to (5.27):

$$\bar{\phi}_n(x;\boldsymbol{\lambda}) = \bar{\phi}_\mu(x;\boldsymbol{\lambda})\times\frac{\mathcal{P}_n(\eta_\ell(x))}{\mathcal{P}_\mu(\eta_\ell(x))}, \qquad \eta_\ell(x) \stackrel{\text{def}}{=} \eta(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}), \tag{5.38}$$

$$\bar{B}(x) = B(x+\ell)\,\frac{\mathcal{P}(\eta_{\ell-1}(x))}{\mathcal{P}(\eta_{\ell-1}(x+1))}\frac{\mathcal{P}_\mu(\eta_\ell(x+1))}{\mathcal{P}_\mu(\eta_\ell(x))}\frac{\varphi_\ell(x;\boldsymbol{\lambda})}{\varphi_\ell(x+1;\boldsymbol{\lambda})}\frac{\varphi_{\ell+1}(x+1;\boldsymbol{\lambda})}{\varphi_{\ell+1}(x;\boldsymbol{\lambda})}, \tag{5.39}$$

$$\bar{D}(x) = D(x)\,\frac{\mathcal{P}(\eta_{\ell-1}(x+1))}{\mathcal{P}(\eta_{\ell-1}(x))}\frac{\mathcal{P}_\mu(\eta_\ell(x-1))}{\mathcal{P}_\mu(\eta_\ell(x))}\frac{\varphi_\ell(x+1;\boldsymbol{\lambda})}{\varphi_\ell(x;\boldsymbol{\lambda})}\frac{\varphi_{\ell+1}(x-1;\boldsymbol{\lambda})}{\varphi_{\ell+1}(x;\boldsymbol{\lambda})}, \tag{5.40}$$

where the deformed polynomials $\mathcal{P}(\eta_{\ell-1})$, $\mathcal{P}_n(\eta_\ell)$ and $\mathcal{P}_\mu(\eta_\ell)$ are defined as (we suppress writing $\boldsymbol{\lambda}$-dependence explicitly)

$$\mathcal{P}(\eta_{\ell-1}(x)) \stackrel{\text{def}}{=} \varphi_\ell(x;\boldsymbol{\lambda})^{-1}\,\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell}](x), \tag{5.41}$$

$$\mathcal{P}_\mu(\eta_\ell(x)) \stackrel{\text{def}}{=} \varphi_{\ell+1}(x;\boldsymbol{\lambda})^{-1}\,\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_\mu](x), \tag{5.42}$$

$$\mathcal{P}_n(\eta_\ell(x)) \stackrel{\text{def}}{=} \varphi_{\ell+1}(x;\boldsymbol{\lambda})^{-1}\mathrm{W}[\check{P}_{d_1},\ldots,\check{P}_{d_\ell},\check{P}_n](x), \tag{5.43}$$

and they are of degree $|\mathcal{D}|-\frac{1}{2}\ell(\ell-1)$, $|\mathcal{D}|+\mu-\frac{1}{2}\ell(\ell+1)$ and $|\mathcal{D}|+n-\frac{1}{2}\ell(\ell+1)$ in $\eta_{\ell-1}$, η_ℓ and η_ℓ, respectively. Here $|\mathcal{D}|$ is defined by $|\mathcal{D}| \stackrel{\text{def}}{=} \sum_{j=1}^{\ell} d_j$. By construction the deformed polynomials $\mathcal{P}(\eta_{\ell-1}(x))$ and $\mathcal{P}_\mu(\eta_\ell(x))$, which appear in the denominator of (5.38) and in (5.39)-(5.40), are positive definite for $x=0,1,\ldots,\bar{x}_{\max}$.

The ratio of the polynomials $\mathcal{P}_n(\eta_\ell(x))/\mathcal{P}_\mu(\eta_\ell(x))$ are eigenfunctions of the similarity transformed Hamiltonian (a second order difference operator) $\widetilde{\bar{\mathcal{H}}}$:

$$\widetilde{\bar{\mathcal{H}}} \stackrel{\text{def}}{=} \bar{\phi}_\mu(x;\boldsymbol{\lambda})^{-1}\circ\bar{\mathcal{H}}\circ\bar{\phi}_\mu(x;\boldsymbol{\lambda}) = \bar{B}(x)(1-e^{\partial})+\bar{D}(x)(1-e^{-\partial}), \tag{5.44}$$

$$\widetilde{\bar{\mathcal{H}}}\,\frac{\mathcal{P}_n(\eta_\ell(x))}{\mathcal{P}_\mu(\eta_\ell(x))} = \mathcal{E}(n)\frac{\mathcal{P}_n(\eta_\ell(x))}{\mathcal{P}_\mu(\eta_\ell(x))} \quad (n\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}). \tag{5.45}$$

The orthogonality relations of $\{\bar{\phi}_n\}$ (4.51) are now rewritten as the orthogonality relations of the polynomials $\{\mathcal{P}_n(\eta_\ell(x))\}$,

$$\sum_{x=0}^{\bar{x}_{\max}} \bar{\psi}(x;\boldsymbol{\lambda})^2 \mathcal{P}_n(\eta_\ell(x))\mathcal{P}_m(\eta_\ell(x)) = \prod_{j=1}^{\ell}\bigl(\mathcal{E}(n)-\mathcal{E}(d_j)\bigr)\cdot\frac{1}{d_n^2}\,\delta_{nm} \quad (n,m\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}), \tag{5.46}$$

in which the new weightfunction is defined by

$$\bar{\psi}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\bar{\phi}_\mu(x;\boldsymbol{\lambda})}{\mathcal{P}_\mu(\eta_\ell(x))}. \tag{5.47}$$

We have demonstrated the following: When the modified Crum's theorem is applied to the shape invariant (exactly solvable) Hamiltonian systems corresponding to the (q)-Askey scheme of hypergeometric orthogonal polynomials $P_n(\eta(x))$ [1], the resulting system is no longer shape invariant but it is exactly solvable since all the eigenvalues and the corresponding eigenfunctions are explicitly given. The eigenfunctions consist of deformed polynomials $\mathcal{P}_n(\eta_\ell(x))$ (5.43). They inherit the sinusoidal coordinate of the original system and they are orthogonal with respect to the weight function given as the square of the deformed groundstate eigenfunction $\bar{\psi}(x;\boldsymbol{\lambda})^2$ (5.47). These deformed orthogonal polynomials $\mathcal{P}_n(\eta_\ell(x))$, although forming a complete set of eigenpolynomials, have vacancies in the degrees corresponding to the deleted levels $\mathcal{D}=\{d_1,d_2,\ldots,d_\ell\}$. Therefore they *do not satisfy three term recurrence relations* and they are not called orthogonal polynomials in the strict sense. When another (modified) Crum's theorem is applied to such a system, most of the simplification formulas (due to polynomial eigenfunctions) in this section are still valid with some adjustments.

6 Dual Christoffel Transformations

In this section we will show that the simple case (*i.e.* $\mu=0$) of the modified Crum's theorem applied to an orthogonal polynomial $P_n(\eta(x))$ provides the well known Christoffel transformation [21, 3] for the corresponding dual orthogonal polynomial $Q_x(\mathcal{E}(n))$ (2.21). Therefore we will call it the *dual Christoffel transformation*. Corresponding to the deletion of the *levels* or *degrees* $n\in\mathcal{D}=\{d_1,\ldots,d_\ell\}$ of the polynomials $P_n(\eta(x))$, the *positions* $n\in\mathcal{D}=\{d_1,\ldots,d_\ell\}$ are deleted from the dual polynomial $Q_x(\mathcal{E}(n))$. Here we assume that the set of deleted levels does not contain 0, the original ground state, *i.e.* $0\notin\mathcal{D}$. This also means that ℓ is *even* and the modified groundstate has also the label 0, $\mu=0$. The dual

Christoffel transformation needs the framework of the discrete quantum mechanics and it cannot be applied to the general orthogonal polynomials.

The deformed dual polynomials $\mathcal{Q}_x(\mathcal{E})$ are defined by the three term recurrence relations in terms of $\bar{B}(x)$ (5.35) and $\bar{D}(x)$ (5.36):

$$\bar{B}(x)\big(\mathcal{Q}_x(\mathcal{E})-\mathcal{Q}_{x+1}(\mathcal{E})\big)+\bar{D}(x)\big(\mathcal{Q}_x(\mathcal{E})-\mathcal{Q}_{x-1}(\mathcal{E})\big)=\mathcal{E}\,\mathcal{Q}_x(\mathcal{E})\quad(x=0,1,\ldots,\bar{x}_{\max}),$$
$$\mathcal{Q}_0(\mathcal{E})=1,\quad \mathcal{Q}_{-1}(\mathcal{E})=0. \tag{6.1}$$

One simple consequence of the above recurrence relation is the universal normalisation

$$\mathcal{Q}_x(0)=1\quad(x=0,1,\ldots,\bar{x}_{\max}). \tag{6.2}$$

Assume that $\eta(x)$ satisfies (5.22)–(5.25). At the eigenvalues $\mathcal{E}=\mathcal{E}(n)$, the Schrödinger equation (5.45) for the ratio of the deformed polynomials $\mathcal{P}_n(\eta_\ell(x))/\mathcal{P}_0(\eta_\ell(x))$ is identical with the above three term recurrence relation and the *duality relation* holds:

$$\frac{\mathcal{P}_n(\eta_\ell(x))}{\mathcal{P}_0(\eta_\ell(x))}=p_n\mathcal{Q}_x(\mathcal{E}(n)),\quad x\in[0,\bar{x}_{\max}],\quad n\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}, \tag{6.3}$$

in which

$$p_n\stackrel{\text{def}}{=}\frac{\mathcal{P}_n(0)}{\mathcal{P}_0(0)}=(-1)^\ell\prod_{j=1}^{\ell}\frac{\mathcal{E}(n)-\mathcal{E}(d_j)}{\mathcal{E}(d_j)},\quad n\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}, \tag{6.4}$$

by the normalisation condition. We have verified the second equality for all the examples in [1], see (A.16)–(A.17). Thus $\mathcal{Q}_x(\mathcal{E})$ are genuine orthogonal polynomials, satisfying the orthogonality relation dual to (5.46):

$$\sum_{n\in\{0,1,\ldots,n_{\max}\}\backslash\mathcal{D}} d_n^2\prod_{j=1}^{\ell}\big(\mathcal{E}(n)-\mathcal{E}(d_j)\big)\cdot\mathcal{Q}_x(\mathcal{E}(n))\mathcal{Q}_y(\mathcal{E}(n))$$
$$=\frac{\prod_{j=1}^{\ell}\mathcal{E}(d_j)^2}{\bar{\phi}_0(x)^2}\,\delta_{xy}\quad(x,y=0,1,\ldots,\bar{x}_{\max}). \tag{6.5}$$

Note that $\prod_{j=1}^{\ell}\big(\mathcal{E}(n)-\mathcal{E}(d_j)\big)>0$ due to (4.1). It is obvious that the values of $\mathcal{Q}_x(\mathcal{E}(n))$ at '*positions*' $n\in\mathcal{D}=\{d_1,d_2,\ldots,d_\ell\}$ do not enter the orthogonality relation or the Schrödinger equation (5.45). In other words, the values of $\mathcal{Q}_x(\mathcal{E}(n))$ at $n\in\mathcal{D}$ are not defined but the degree x runs from 0 to $\bar{x}_{\max}$ without any hole. This transformation of the dual orthogonal polynomials, $Q_x(\mathcal{E}(n))\to\mathcal{Q}_x(\mathcal{E}(n))$ is the well known *Christoffel transformation* [21, 3, 27, 28, 29]. The transformation of the orthogonality weight function is read from (6.5):

$$d_n^2\to d_n^2\prod_{j=1}^{\ell}\big(\mathcal{E}(n)-\mathcal{E}(d_j)\big). \tag{6.6}$$

The inverse of the Christoffel transformation is called the Geronimus transformation [27], which adds new 'positions'. Its effects can be practically incorporated by the dual Christoffel transformations with redefinion of the parameters.

The dual Christoffel transformation has the merits that the formulas determining $\bar{B}(x)$ and $\bar{D}(x)$ (5.35) and (5.36) are universal, concise and algorithmic compared with the original Christoffel transformation, which should be performed specifically for each case, the polynomials and the set of deletion $\mathcal{D}$. While the dual Christoffel transformation requires the framework of the discrete quantum mechanics, the original Christoffel transformation is defined for general orthogonal polynomials, *i. e.,* those having the three term recurrence relations only without the difference equation, nor the sinusoidal coordinate. The elementary Christoffel transformation is defined by

$$\widetilde{P}_n(x) \stackrel{\text{def}}{=} \frac{P_{n+1}(x) - A_n P_n(x)}{x-a}, \quad A_n \stackrel{\text{def}}{=} \frac{P_{n+1}(a)}{P_n(a)}, \tag{6.7}$$

for a parameter $a \in \mathbb{R}$, which is not constrained by the sinusoidal coordinate. Corresponding to the ℓ-deletion above, one considers multiple (ℓ times) applications of the elementary Christoffel transformations. The normalised weight function for the polynomials $\{P_n(x)\}$ is, in general, not known explicitly and it usually contains continuous measure. Thus determination of the positivity of the resulting weight function generically involves a moment problem. For the polynomials belonging to the discrete quantum mechanics [1, 5], explicit forms of multiple Christoffel transformations are known for some specific cases. For example, see [28, 29] for the Racah polynomials and others.

7 Summary and Comments

Crum's theorem and its modification à la Adler are formulated for the discrete quantum mechanics with real shifts. They are slightly more complicated than those in the ordinary quantum mechanics or in the discrete quantum mechanics with pure imaginary shifts, partly because the size of the Hamiltonian matrix, or the range of the x variable (x_{max}), is reduced by one in each deletion. Another source of complications is the fact that in the generic cases the sinusoidal coordinate $\eta(x)$ depends on the parameters $\boldsymbol{\lambda}$, which changes at each deletion. In the discrete quantum mechanics with real shifts, the modified Crum's theorem generates the Christoffel transformation of the dual orthogonal polynomials. Very special and simple examples, in which all the excited states from the first to the ℓ-th are deleted (see Fig. 3), are presented explicitly in Appendix for more than two dozens of orthogonal polynomials discussed in [1].

One of the motivations of the present research is the connection with the *infinitely many exceptional orthogonal polynomials* [30, 31, 32, 33, 34]. As explained in some detail in § 4 of [19], the insight obtained from the explicit examples of the application of modified Crum's theorem was instrumental for the discovery of the infinitely many exceptional Laguerre and Jacobi polynomials in the ordinary quantum mechanics [32] and the exceptional Wilson and Askey-Wilson polynomials in the discrete quantum mechanics with the pure imaginary shifts [34]. The explicit examples in the present paper are also very helpful for the discovery of the (infinitely) many *exceptional orthogonal polynomials of a discrete variable*, for example, the exceptional q-Racah polynomials, etc. They will be derived and discussed in detail elsewhere [35].

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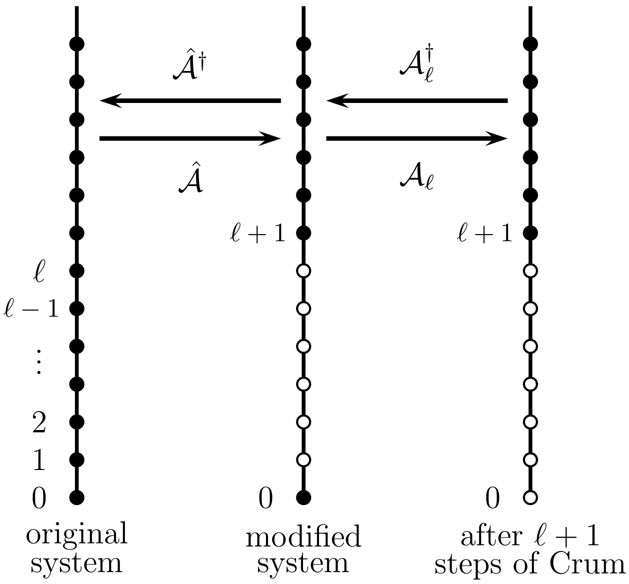
A Special Examples

In Appendix we present very special and simple examples of an application of Adler's theorem, in which the eigenstates $\phi_1, \phi_2, \ldots, \phi_\ell$ are deleted. Similar examples of the application of Adler's theorem in discrete quantum mechanics with pure imaginary shifts were given in Appendix B of [19]. In the ordinary quantum mechanics, some simple examples of the same sort were demonstrated in Appendix A of [19] and [36].

In this case, $\mathcal{D} = \{d_1, d_2, \ldots, d_\ell\} = \{1, 2, \ldots, \ell\}$, that is, the modified groundstate level is the same as that of the original theory $\mu = 0$. This means $|\mathcal{D}| = \frac{1}{2}\ell(\ell+1)$ and the polynomial $\mathcal{P}_\mu$ (5.42) is a constant. The degrees of the deformed polynomial $\mathcal{P}$ (5.41) and $\mathcal{P}_n$ (5.43) are ℓ and $\ell + n$, and they are proportional to the deforming polynomial ξ_ℓ and the deformed polynomial $P_{\ell,n}$ (A.24), respectively. The duality (6.3) is now a relation between two polynomials $\mathcal{P}_n(\eta_\ell)$ and $\mathcal{Q}_x(\mathcal{E})$.

The main results are the unified expressions of the functions $B_\ell(x)$ (A.29) and $D_\ell(x)$ (A.30), which specify the Hamiltonian $\mathcal{H}_\ell$ together with the eigenfunctions $\phi_{\ell,0}$ (A.31) and

$\phi_{\ell,n}(x)$ (A.32). The unified expressions of the deformed eigenpolynomials $\check{P}_{\ell,n}(x)$ are given in (A.43). An important ingredient specific to each type of polynomials is the *deforming polynomial* $\check{\xi}_\ell(x)$, which is listed in A.3. The situation is illustrated in Fig. 3, which should be compared with Fig. 2a depicting the generic case discussed in section 4.



$\hat{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}_{d_1\ldots d_\ell}\cdots\mathcal{A}_{d_1d_2}\mathcal{A}_{d_1}$

Figure 3: Special case

The black circles denote the energy levels, whereas the white circles denote *deleted* energy levels. We write $\bar{\mathcal{H}} = \mathcal{H}_{12\ldots\ell}$, $\bar{\phi}_n = \phi_{12\ldots\ell n}$, $\bar{\mathcal{A}} = \mathcal{A}_{12\ldots\ell}$, $\bar{B} = B_{12\ldots\ell}$, $\bar{D} = D_{12\ldots\ell}$ etc. as $\mathcal{H}_\ell$, $\phi_{\ell,n}$, $\mathcal{A}_\ell$, B_ℓ, D_ℓ etc. This Hamiltonian $\mathcal{H}_\ell = \mathcal{A}_\ell^\dagger\mathcal{A}_\ell$ is hermitian for even ℓ but may be non-hermitian for odd ℓ. Since algebraic formulas such as the Casoratians are valid for even and odd ℓ, we present various formulas without restricting to the even ℓ. The original systems are shape invariant but the $(\phi_1,\ldots,\phi_\ell)$-deleted systems $\mathcal{H}_\ell$ are not. The rightmost vertical line in Fig. 3 corresponds to the Hamiltonian system $\mathcal{H}'_\ell = \mathcal{A}_\ell\mathcal{A}_\ell^\dagger = \mathcal{H}^{[\ell+1]}$, which is shape invariant and it is obtained from $\mathcal{H}_\ell$ by one more step of Crum's method.

We apply Adler's theorem to the shape invariant, therefore solvable, systems whose eigenfunctions are described by the orthogonal polynomials studied in [1]; *i.e.* (q)-Racah, (q)-(dual)-Hahn, etc. Hereafter we display the parameter dependence explicitly by $\boldsymbol{\lambda}$, which represents the set of the parameters, and we follow the notation of [1]. This is to emphasise that the basic structure of Crum's theorem and its modification discussed in the main sections are well-founded for the most generic systems.

A.1 The original systems

Here we summarise various properties of the original Hamiltonian systems studied in [1] to be compared with the specially modified systems to be presented in A.2. All the examples are shape invariant and exactly solvable. Let us start with the Hamiltonians, Schrödinger equations and eigenfunctions ($x = 0, 1, \ldots, x_{\max}$, $x_{\max} = N$ or ∞, $n_{\max} = N$ or ∞):

$$\mathcal{A}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{B(x;\boldsymbol{\lambda})} - e^{\partial}\sqrt{D(x;\boldsymbol{\lambda})}, \quad \mathcal{A}(\boldsymbol{\lambda})^{\dagger} = \sqrt{B(x;\boldsymbol{\lambda})} - \sqrt{D(x;\boldsymbol{\lambda})}\, e^{-\partial}, \tag{A.1}$$

$$\mathcal{H}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\mathcal{A}(\boldsymbol{\lambda}), \tag{A.2}$$

$$\mathcal{H}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \mathcal{E}(n;\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) \quad (n = 0, 1, \ldots, n_{\max}), \tag{A.3}$$

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) \quad \bigl(P_n(0;\boldsymbol{\lambda}) = 1, \ \ \check{P}_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})\bigr). \tag{A.4}$$

The explicit forms of the set of parameters $\boldsymbol{\lambda}$, the potential functions $B(x;\boldsymbol{\lambda})$ and $D(x;\boldsymbol{\lambda})$, the energy eigenvalues $\mathcal{E}(n;\boldsymbol{\lambda})$, the sinusoidal coordinate $\eta(x;\boldsymbol{\lambda})$, the ground state wavefunctions $\phi_0(x;\boldsymbol{\lambda})$ and the eigenpolynomials $P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})$ are given in [1]. The groundstate wavefunction $\phi_0(x;\boldsymbol{\lambda})$ is annihilated by $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})\phi_0(x;\boldsymbol{\lambda}) = 0$, and given by

$$\phi_0(x;\boldsymbol{\lambda}) = \sqrt{\prod_{y=0}^{x-1} \frac{B(y;\boldsymbol{\lambda})}{D(y+1;\boldsymbol{\lambda})}}. \tag{A.5}$$

The systems are shape invariant,

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^{\dagger} = \kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^{\dagger}\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}(1;\boldsymbol{\lambda}), \tag{A.6}$$

where the explicit forms of $\boldsymbol{\delta}$ and κ are given in [1]. The action of $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^{\dagger}$ on the eigenfunctions is

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}) = \frac{1}{\sqrt{B(0;\boldsymbol{\lambda})}}\, f_n(\boldsymbol{\lambda})\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr), \tag{A.7}$$

$$\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\phi_{n-1}\bigl(x;\boldsymbol{\lambda}+\boldsymbol{\delta}\bigr) = \sqrt{B(0;\boldsymbol{\lambda})}\, b_{n-1}(\boldsymbol{\lambda})\phi_n(x;\boldsymbol{\lambda}), \tag{A.8}$$

where $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of the energy eigenvalue, $\mathcal{E}(n;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})$, and their explicit forms are given in [1]. The forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ are defined by

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{B(0;\boldsymbol{\lambda})}\, \phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x;\boldsymbol{\lambda})$$

$$
\begin{aligned}
&= B(0;\boldsymbol{\lambda})\varphi(x;\boldsymbol{\lambda})^{-1}(1-e^{\partial}), &&\text{(A.9)}\\
\mathcal{B}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{B(0;\boldsymbol{\lambda})}}\,\phi_0(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}(\boldsymbol{\lambda})^{\dagger}\circ\phi_0(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\\
&= \frac{1}{B(0;\boldsymbol{\lambda})}\bigl(B(x;\boldsymbol{\lambda})-D(x;\boldsymbol{\lambda})e^{-\partial}\bigr)\varphi(x;\boldsymbol{\lambda}), &&\text{(A.10)}
\end{aligned}
$$

and their action on the polynomials is

$$
\mathcal{F}(\boldsymbol{\lambda})P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{n-1}(\eta(x;\boldsymbol{\lambda}+\boldsymbol{\delta});\boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{A.11}
$$

$$
\mathcal{B}(\boldsymbol{\lambda})P_{n-1}(\eta(x;\boldsymbol{\lambda}+\boldsymbol{\delta});\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}). \tag{A.12}
$$

For the definition of the auxiliary function $\varphi(x;\boldsymbol{\lambda})$ see eqs. (4.12) and (4.23) in [1]. Their explicit forms are also given in § 5 of [1].

The orthogonality relation is

$$
\sum_{x=0}^{x_{\max}}\phi_0(x;\boldsymbol{\lambda})^2P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda})P_m(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = \frac{1}{d_n(\boldsymbol{\lambda})^2}\,\delta_{nm}\quad(n,m=0,1,\ldots,n_{\max}), \tag{A.13}
$$

where the explicit forms of the normalisation constants $d_n(\boldsymbol{\lambda})$ are given in [1].

The coefficient of the leading term $c_n(\boldsymbol{\lambda})$, which appears in $P_n(y;\boldsymbol{\lambda}) = c_n(\boldsymbol{\lambda})P_n^{\text{monic}}(y;\boldsymbol{\lambda})$, is given by

$$
c_n(\boldsymbol{\lambda}) = \frac{(-1)^n\kappa^{-\frac12 n(n-1)}}{\prod_{j=1}^n\eta(j;\boldsymbol{\lambda})}\prod_{j=0}^{n-1}\frac{\mathcal{E}(n;\boldsymbol{\lambda})-\mathcal{E}(j;\boldsymbol{\lambda})}{B(0;\boldsymbol{\lambda}+j\boldsymbol{\delta})}, \tag{A.14}
$$

for all the examples in [1]. Especially $n=1$ gives the relation

$$
\frac{B(0;\boldsymbol{\lambda})}{\mathcal{E}(1;\boldsymbol{\lambda})} = \frac{-c_1(\boldsymbol{\lambda})^{-1}}{\eta(1;\boldsymbol{\lambda})}, \tag{A.15}
$$

which expresses the important relation among the basic quantities $B(0;\boldsymbol{\lambda})$, $\mathcal{E}(1;\boldsymbol{\lambda})$, $\eta(1;\boldsymbol{\lambda})$ and $c_1(\boldsymbol{\lambda})$, see eq. (4.55) of [1]. The Casorati determinant (5.27) becomes

$$
\begin{aligned}
&\varphi_m(x;\boldsymbol{\lambda})^{-1}\,\mathrm{W}[\check{P}_{n_1},\ldots,\check{P}_{n_m}](x;\boldsymbol{\lambda})\\
&=\prod_{k=0}^{m-1}c_k(\boldsymbol{\lambda})\cdot\frac{\prod_{1\le j<k\le m}(\mathcal{E}(n_k;\boldsymbol{\lambda})-\mathcal{E}(n_j;\boldsymbol{\lambda}))}{\prod_{0\le j<k\le m-1}(\mathcal{E}(k;\boldsymbol{\lambda})-\mathcal{E}(j;\boldsymbol{\lambda}))}\cdot\prod_{k=1}^{m-1}\prod_{j=1}^{k}\eta(j;\boldsymbol{\lambda})\cdot\check{P}_{(n_1,\ldots,n_m)}(x;\boldsymbol{\lambda})\\
&=(-1)^{\binom{m}{2}}\kappa^{-\binom{m}{3}}\prod_{1\le j<k\le m}\frac{\mathcal{E}(n_k;\boldsymbol{\lambda})-\mathcal{E}(n_j;\boldsymbol{\lambda})}{B(0;\boldsymbol{\lambda}+(j-1)\boldsymbol{\delta})}\cdot\check{P}_{(n_1,\ldots,n_m)}(x;\boldsymbol{\lambda}), \qquad\text{(A.16)}
\end{aligned}
$$

for all the examples in [1]. Here $\check{P}_{(n_1,\ldots,n_m)}(x;\boldsymbol{\lambda})$ is a polynomial of degree $\sum_{k=1}^m n_k-\frac12 m(m-1)$ in $\eta(x;\boldsymbol{\lambda}+(m-1)\boldsymbol{\delta})$ and satisfies the normalisation

$$
\check{P}_{(n_1,\ldots,n_m)}(0;\boldsymbol{\lambda}) = 1. \tag{A.17}
$$

A.2 The $(\phi_1,\ldots,\phi_\ell)$-deleted systems

For this very special case, the potential functions (5.35)–(5.36), the Hamiltonian (4.39)–(4.40) and the Schrödinger equation (4.43) of the modified system are:

$$B_\ell(x;\boldsymbol{\lambda})=\bar{B}(x;\boldsymbol{\lambda})=B(x+\ell;\boldsymbol{\lambda})\frac{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda})}{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x+1;\boldsymbol{\lambda})}\frac{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_0](x+1;\boldsymbol{\lambda})}{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_0](x;\boldsymbol{\lambda})},\tag{A.18}$$

$$D_\ell(x;\boldsymbol{\lambda})=\bar{D}(x;\boldsymbol{\lambda})=D(x;\boldsymbol{\lambda})\frac{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x+1;\boldsymbol{\lambda})}{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda})}\frac{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_0](x-1;\boldsymbol{\lambda})}{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_0](x;\boldsymbol{\lambda})},\tag{A.19}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})=\bar{\mathcal{A}}(\boldsymbol{\lambda})=\sqrt{B_\ell(x;\boldsymbol{\lambda})}-e^{\partial}\sqrt{D_\ell(x;\boldsymbol{\lambda})},$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger=\bar{\mathcal{A}}(\boldsymbol{\lambda})^\dagger=\sqrt{B_\ell(x;\boldsymbol{\lambda})}-\sqrt{D_\ell(x;\boldsymbol{\lambda})}\,e^{-\partial},\tag{A.20}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})=\bar{\mathcal{H}}(\boldsymbol{\lambda})=\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda}),\quad x^\ell_{\text{max}}\stackrel{\text{def}}{=}N-\ell\ \text{or}\ \infty,\tag{A.21}$$

$$\mathcal{H}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda})\quad(n=0,\ell+1,\ell+2,\ldots,n_{\text{max}}).\tag{A.22}$$

Note that the Hamiltonian $\mathcal{H}_\ell$ is an $(x^\ell_{\text{max}}+1)\times(x^\ell_{\text{max}}+1)$ matrix $\mathcal{H}_\ell=(\mathcal{H}_{\ell;x,y})$ $(x,y=0,1,\ldots,x^\ell_{\text{max}})$. The explicit expression of $D_\ell(x;\boldsymbol{\lambda})$ can be analytically continued and gives $D_\ell(0;\boldsymbol{\lambda})=0$. We have $B_\ell(x^\ell_{\text{max}})=0$ for the finite case. From (5.33) we have

$$\phi_{\ell,n}(x;\boldsymbol{\lambda})=\bar{\phi}_n(x;\boldsymbol{\lambda})=(-1)^\ell\phi_0(x;\boldsymbol{\lambda})\prod_{k=0}^{\ell-1}\sqrt{B(x+k;\boldsymbol{\lambda})}\\ \times\frac{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell,\check{P}_n](x;\boldsymbol{\lambda})}{\sqrt{\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x;\boldsymbol{\lambda})\mathrm{W}[\check{P}_1,\ldots,\check{P}_\ell](x+1;\boldsymbol{\lambda})}}.\tag{A.23}$$

Let us introduce the deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\xi_\ell(\eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta});\boldsymbol{\lambda})$, which is a polynomial of degree ℓ in $\eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})$, and the deformed polynomial $\check{P}_{\ell,n}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}P_{\ell,n}(\eta(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta});\boldsymbol{\lambda})$ $(n=0,\ell+1,\ell+2,\ldots)$, which is a polynomial of degree n in $\eta(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})$, in terms of (A.16) as follows:

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\check{P}_{(1,2,\ldots,\ell)}(x;\boldsymbol{\lambda}),\quad\check{P}_{\ell,n}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\check{P}_{(1,2,\ldots,\ell,n)}(x;\boldsymbol{\lambda}).\tag{A.24}$$

Then we have

$$\check{\xi}_\ell(0;\boldsymbol{\lambda})=1,\quad\check{P}_{\ell,n}(0;\boldsymbol{\lambda})=1.\tag{A.25}$$

We set $\check{P}_{\ell,n}(x;\boldsymbol{\lambda})=0$ for $n=1,\ldots,\ell$. For the q-Racah case, which is the most generic case, the deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda})$ has the following form,

$$q\text{-Racah}:\ \check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})),\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\boldsymbol{\lambda},\tag{A.26}$$

and this is indeed a polynomial in $\eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})$ because $\eta(-x;-\boldsymbol{\lambda}-(\ell-1)\boldsymbol{\delta}) = \eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})(dq^{\ell-1})^{-1}$. For the other cases the deforming polynomials are obtained from this in certain limits. However these limits are not so trivial. So we present the explicit forms of $\check{\xi}_\ell(x;\boldsymbol{\lambda})$ in A.3. Note that the deforming polynomial ξ_ℓ is related to $P_{\ell,n}$ as

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_{\ell-1,\ell}(x;\boldsymbol{\lambda}), \tag{A.27}$$

and satisfies the recurrence relation:

$$\begin{aligned} B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_{\ell+1}(x;\boldsymbol{\lambda}) &= B(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\varphi(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x;\boldsymbol{\lambda}) \\ &\quad - D(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\varphi(x-1;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x+1;\boldsymbol{\lambda}). \end{aligned} \tag{A.28}$$

By using these quantities, eqs. (A.18), (A.19) and (A.23) become

$$B_\ell(x;\boldsymbol{\lambda}) = \kappa^\ell B(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x;\boldsymbol{\lambda})}{\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}, \tag{A.29}$$

$$D_\ell(x;\boldsymbol{\lambda}) = \kappa^\ell D(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}{\check{\xi}_\ell(x;\boldsymbol{\lambda})}, \tag{A.30}$$

$$\phi_{\ell,0}(x;\boldsymbol{\lambda}) = \frac{C(\ell,\boldsymbol{\lambda})}{\sqrt{B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}}\frac{\phi_0(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\sqrt{\check{\xi}_\ell(x;\boldsymbol{\lambda})\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}}, \tag{A.31}$$

$$\phi_{\ell,n}(x;\boldsymbol{\lambda}) = \phi_{\ell,0}(x;\boldsymbol{\lambda})(-1)^\ell\prod_{j=1}^{\ell}\frac{\mathcal{E}(n;\boldsymbol{\lambda})-\mathcal{E}(j;\boldsymbol{\lambda})}{\mathcal{E}(j;\boldsymbol{\lambda})}\cdot\check{P}_{\ell,n}(x;\boldsymbol{\lambda}), \tag{A.32}$$

where $C(\ell,\boldsymbol{\lambda})$ is defined by

$$C(\ell,\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}\,(-1)^\ell\kappa^{-\frac14\ell(\ell-1)}\prod_{j=1}^{\ell}\frac{\mathcal{E}(j;\boldsymbol{\lambda})}{\sqrt{B(0;\boldsymbol{\lambda}+(j-1)\boldsymbol{\delta})}}. \tag{A.33}$$

For *even* ℓ, the deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \xi_\ell(\eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta});\boldsymbol{\lambda})$ is positive at integer points $x=0,1,\ldots,x_{\max}^\ell+1$. It has either no zero in the interval $0\le x\le x_{\max}^\ell+1$ or when it has zeros, an even number of zeros appear between two contiguous integers. The deformed polynomial $P_{\ell,n}(y;\boldsymbol{\lambda})$ $(n\ge\ell+1)$ has $n-\ell$ zeros in the interval $0\le y\le\eta(x_{\max}^\ell;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})$ for even ℓ. These we have verified by numerical calculation for lower ℓ for all the examples in [1]. To the best of our knowledge, no general proof of this property has been reported. Note that the normalisation of ξ_ℓ does not affect $\mathcal{H}_\ell$. This system is not shape invariant. We have not chosen the normalisation like as $\phi_0(0;\boldsymbol{\lambda})=1$, namely $\phi_{\ell,0}(0;\boldsymbol{\lambda}) = \frac{C(\ell,\boldsymbol{\lambda})}{\sqrt{B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}\sqrt{\check{\xi}_\ell(1;\boldsymbol{\lambda})}} \neq 1$.

The operators $\mathcal{A}_\ell(\boldsymbol{\lambda})$ and $\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger$ connect the modified system $\mathcal{H}_\ell(\boldsymbol{\lambda})=\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\mathcal{A}_\ell(\boldsymbol{\lambda})$ to the shape invariant system $\mathcal{H}'_\ell(\boldsymbol{\lambda})=\mathcal{A}_\ell(\boldsymbol{\lambda})\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger=\kappa^{\ell+1}\mathcal{H}(\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})+\mathcal{E}(\ell+1;\boldsymbol{\lambda})=\mathcal{H}^{[\ell+1]}(\boldsymbol{\lambda})$, which is denoted by the rightmost vertical line in Fig. 3. The n-th level ($n\geq\ell+1$) of the modified system $\mathcal{H}_\ell$ is *iso-spectral* with the $n-\ell-1$-th level of the new shape invariant system $\mathcal{H}'_\ell$:

$$\mathcal{A}_\ell(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})\phi_{n-\ell-1}\big(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\big), \tag{A.34}$$

$$\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\phi_{n-\ell-1}\big(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\big)=b_{\ell,n-1}(\boldsymbol{\lambda})\phi_{\ell,n}(x;\boldsymbol{\lambda}). \tag{A.35}$$

Here, $f_{\ell,n}(\boldsymbol{\lambda})$ and $b_{\ell,n-1}(\boldsymbol{\lambda})$ are the factors of the energy eigenvalue, $\mathcal{E}(n;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})b_{\ell,n-1}(\boldsymbol{\lambda})$, and are defined by

$$f_{\ell,n}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}f_n(\boldsymbol{\lambda})\frac{\kappa^{-\frac{\ell}{2}}C(\ell,\boldsymbol{\lambda})}{B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})},\quad b_{\ell,n-1}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}b_{n-1}(\boldsymbol{\lambda})\frac{B(0;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})}{\kappa^{-\frac{\ell}{2}}C(\ell,\boldsymbol{\lambda})}. \tag{A.36}$$

Note that $\mathcal{E}(n;\boldsymbol{\lambda})=\kappa^{\ell+1}\mathcal{E}(n-\ell-1;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})+\mathcal{E}(\ell+1;\boldsymbol{\lambda})$ for $n\geq\ell+1$. The forward and backward shift operators $\mathcal{F}_\ell(\boldsymbol{\lambda})$ and $\mathcal{B}_\ell(\boldsymbol{\lambda})$ and the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})$, which act on the polynomial eigenfunctions, are defined by:

$$\begin{aligned}\mathcal{F}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\phi_0\big(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\big)^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})\circ\phi_{\ell,0}(x;\boldsymbol{\lambda})\\&=\frac{\kappa^{\frac{\ell}{2}}C(\ell,\boldsymbol{\lambda})}{\varphi(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}\big(1-e^{\partial}\big),\end{aligned} \tag{A.37}$$

$$\begin{aligned}\mathcal{B}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\phi_{\ell,0}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{A}_\ell(\boldsymbol{\lambda})^\dagger\circ\phi_0\big(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}\big)\\&=\frac{\kappa^{\frac{\ell}{2}}}{C(\ell,\boldsymbol{\lambda})}\Big(B(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x;\boldsymbol{\lambda})-D(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x+1;\boldsymbol{\lambda})e^{-\partial}\Big)\varphi(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta}),\end{aligned} \tag{A.38}$$

$$\begin{aligned}\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\phi_{\ell,0}(x;\boldsymbol{\lambda})^{-1}\circ\mathcal{H}_\ell(\boldsymbol{\lambda})\circ\phi_{\ell,0}(x;\boldsymbol{\lambda})=\mathcal{B}_\ell(\boldsymbol{\lambda})\mathcal{F}_\ell(\boldsymbol{\lambda})\\&=\kappa^\ell\Big(B(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x;\boldsymbol{\lambda})}{\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}(1-e^{\partial})+D(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\frac{\check{\xi}_\ell(x+1;\boldsymbol{\lambda})}{\check{\xi}_\ell(x;\boldsymbol{\lambda})}(1-e^{-\partial})\Big).\end{aligned} \tag{A.39}$$

Their action on the polynomials is

$$\mathcal{F}_\ell(\boldsymbol{\lambda})\check{P}_{\ell,n}(x;\boldsymbol{\lambda})=f_{\ell,n}(\boldsymbol{\lambda})\check{P}_{n-\ell-1}(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}), \tag{A.40}$$

$$\mathcal{B}_\ell(\boldsymbol{\lambda})\check{P}_{n-\ell-1}(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})=b_{\ell,n-1}(\boldsymbol{\lambda})\check{P}_{\ell,n}(x;\boldsymbol{\lambda}), \tag{A.41}$$

$$\widetilde{\mathcal{H}}_\ell(\boldsymbol{\lambda})\check{P}_{\ell,n}(x;\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_{\ell,n}(x;\boldsymbol{\lambda})\quad(n=0,\ell+1,\ell+2,\ldots,n_{\max}). \tag{A.42}$$

For $n\geq\ell+1$, the above formula (A.38) provides a simple expression of the modified eigenpolynomial $P_{\ell,n}$ in terms of ξ_ℓ and the original eigenpolynomial P_n:

$$\kappa^{-\frac{\ell}{2}}C(\ell,\boldsymbol{\lambda})b_{\ell,n-1}(\boldsymbol{\lambda})\check{P}_{\ell,n}(x;\boldsymbol{\lambda})$$

$$= B(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x;\boldsymbol{\lambda})\varphi(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{P}_{n-\ell-1}(x;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta})$$
$$- D(x;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{\xi}_\ell(x+1;\boldsymbol{\lambda})\varphi(x-1;\boldsymbol{\lambda}+\ell\boldsymbol{\delta})\check{P}_{n-\ell-1}(x-1;\boldsymbol{\lambda}+(\ell+1)\boldsymbol{\delta}). \qquad \text{(A.43)}$$

The orthogonality relation is ($x^\ell_{\max} = N-\ell$ or ∞ and $n_{\max} = N$ or ∞):

$$\sum_{x=0}^{x^\ell_{\max}} \phi_{\ell,0}(x;\boldsymbol{\lambda})^2 \check{P}_{\ell,n}(x;\boldsymbol{\lambda})\check{P}_{\ell,m}(x;\boldsymbol{\lambda}) = \frac{1}{d_{\ell,n}(\boldsymbol{\lambda})^2}\,\delta_{nm} \quad (n,m=0,\ell+1,\ell+2,\ldots,n_{\max}). \quad \text{(A.44)}$$

Here the normalisation constant $d_{\ell,n}(\boldsymbol{\lambda})$ is

$$d_{\ell,n}(\boldsymbol{\lambda})^2 = d_n(\boldsymbol{\lambda})^2 \prod_{j=1}^{\ell} \frac{\mathcal{E}(n;\boldsymbol{\lambda})-\mathcal{E}(j;\boldsymbol{\lambda})}{\mathcal{E}(j;\boldsymbol{\lambda})^2}, \qquad \text{(A.45)}$$

which is a consequence of (5.46) and positive for even ℓ. The weight function $\phi_{\ell,0}(x;\boldsymbol{\lambda})^2$ is positive definite for even ℓ.

A.3 Explicit forms of the deforming polynomial ξ_ℓ

As shown in the preceding subsection, the various quantities, the functions $B_\ell(x)$ (A.29), $D_\ell(x)$ (A.30), which specify the Hamiltonian $\mathcal{H}_\ell$ and the eigenfunctions $\phi_{\ell,0}$ (A.31), $\phi_{\ell,n}(x)$ (A.32) are determined by the *deforming polynomial* $\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \xi_\ell(\eta(x;\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta});\boldsymbol{\lambda})$. This deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda})$ is positive at integer points $x = 0,1,\ldots,x^\ell_{\max}$ for *even* ℓ. Here we give the list of the explicit forms of the deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda})$. The list contains the name of the polynomial (*e.g.* Racah) and the corresponding section number of [1] together with that of Koekoek-Swarttouw's review [5], the definition of the *twist operator* $\mathfrak{t}$ and the form of the deforming polynomial $\check{\xi}_\ell(x;\boldsymbol{\lambda})$ in terms of the original polynomial $\check{P}_\ell(x;\boldsymbol{\lambda})$ (*e.g.* the Racah polynomial):

name (§ of [1], § of [5]):

Racah (5.1.1, 1.2): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})), \qquad \text{(A.46)}$$

Hahn (5.1.2, 1.5): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})), \qquad \text{(A.47)}$$

dual Hahn (5.1.3, 1.6): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})+(0,2,0)), \qquad \text{(A.48)}$$

Krawtchouk (5.1.4, 1.10): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (p,-N)$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})),\tag{A.49}$$

q-Racah (5.1.5, 3.2): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})),\tag{A.50}$$

q-Hahn (5.1.6, 3.6): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-(b,a,N)$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(x-N+\ell-1;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta}))\frac{(-1)^\ell a^\ell q^{\frac{1}{2}\ell(\ell-1)}(b;q)_\ell}{(a;q)_\ell},\tag{A.51}$$

dual q-Hahn (5.1.7, 3.7):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x,a^{-1}b^{-1}q^{-x+2-\ell}}{a^{-1}q^{-\ell+1},q^{N-\ell+1}}\Bigm|q\,;bq^N\Bigr),\tag{A.52}$$

quantum q-Krawtchouk (5.1.8, 3.14):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},0,q^{-x+N-\ell+1}}{p^{-1}q^{-\ell},q^{N-\ell+1}}\Bigm|q\,;q\Bigr)(pq\,;q)_\ell,\tag{A.53}$$

dual quantum q-Krawtchouk (5.1.8, —):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x}{q^{N-\ell+1}}\Bigm|q\,;pq^{N+1}\Bigr),\tag{A.54}$$

q-Krawtchouk (5.1.9, 3.15): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(x-N+\ell-1;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})+(-2,0))(-1)^\ell q^{\ell^2}p^\ell,\tag{A.55}$$

dual q-Krawtchouk (in the standard parametrization) (5.1.9, 3.17):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_3\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x,c^{-1}q^{-x+N-\ell+1}}{q^{N-\ell+1}}\Bigm|q\,;cq^\ell\Bigr),\tag{A.56}$$

affine q-Krawtchouk (5.1.10, 3.16):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^{-x+N-\ell+1}}{q^{N-\ell+1}}\Bigm|q\,;p^{-1}\Bigr)\frac{(-1)^\ell p^\ell q^{\frac{1}{2}\ell(\ell+1)}}{(pq\,;q)_\ell},\tag{A.57}$$

alternative q-Hahn (5.3.1, —): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-(b,a,N)$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(x-N+\ell-1;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta}))\frac{(-1)^\ell q^{-\frac{1}{2}\ell(\ell-1)}(a;q)_\ell}{a^\ell(b;q)_\ell},\tag{A.58}$$

alternative q-Krawtchouk (5.3.2, —): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(x-N+\ell-1;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})+(-2,0))(-1)^\ell p^{-\ell}q^{-\ell^2},\tag{A.59}$$

alternative affine q-Krawtchouk (5.3.3, —):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})={}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^{-x+N-\ell+1}}{q^{N-\ell+1}}\Bigm|q\,;pq^{x+\ell+1}\Bigr)\frac{1}{(pq\,;q)_\ell},\tag{A.60}$$

Meixner (5.2.1, 1.9): $\quad\mathfrak{t}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}(-\beta,c)$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda})=\check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})),\tag{A.61}$$

Charlier (5.2.2, 1.12): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_\ell(-x;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})), \tag{A.62}$$

little q-Jacobi (5.2.3, 3.12): $\quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\boldsymbol{\lambda}$,

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = \check{P}_\ell(x+b'+\ell;\mathfrak{t}(\boldsymbol{\lambda}+(\ell-1)\boldsymbol{\delta})-(2,2))a^{-\ell}b^{-\ell}q^{-\ell(\ell+1)}, \quad b=q^{b'}, \tag{A.63}$$

dual little q-Jacobi (5.2.3, —):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x,a^{-1}b^{-1}q^{-x-\ell}}{b^{-1}q^{-\ell},0}\Bigm|q\,;q\Bigr), \tag{A.64}$$

q-Meixner (5.2.4, 3.13):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x}{b^{-1}q^{-\ell}}\Bigm|q\,;-b^{-1}c^{-1}q^{1-x}\Bigr), \tag{A.65}$$

little q-Laguerre/Wall (5.2.5, 3.20):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_1\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell}}{a^{-1}q^{-\ell}}\Bigm|q\,;a^{-1}q^x\Bigr)(-1)^\ell a^{-\ell}q^{-\frac12\ell(\ell+1)}(aq\,;q)_\ell, \tag{A.66}$$

Al-Salam-Carlitz II (5.2.6, 3.25):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x}{0}\Bigm|q\,;a^{-1}q^{1-x}\Bigr), \tag{A.67}$$

alternative q-Charlier (5.2.7, 3.22):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_2\phi_0\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},-a^{-1}q^{-\ell}}{-}\Bigm|q\,;-aq^{x+2\ell}\Bigr)(-a)^{-\ell}q^{-\ell^2}, \tag{A.68}$$

dual alternative q-Charlier (5.2.7, —):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_3\phi_2\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x,-a^{-1}q^{1-x-\ell}}{0,0}\Bigm|q\,;q\Bigr), \tag{A.69}$$

q-Charlier (5.2.8, 3.23):

$$\check{\xi}_\ell(x;\boldsymbol{\lambda}) = {}_2\phi_0\Bigl(\genfrac{}{}{0pt}{}{q^{-\ell},q^x}{-}\Bigm|q\,;-a^{-1}q^{\ell+1-x}\Bigr). \tag{A.70}$$

The twist operator $\mathfrak{t}$ is not listed when it is not used for the definition of ξ_ℓ.

A.4 Supplementary data on dual orthogonal polynomials

Here we present the extra data for various dual orthogonal polynomials which were not listed in [1]. They are the dual quantum q-Krawtchouk in § 5.1.8, the dual little q-Jacobi in § 5.2.3, the dual alternative q-Charlier in § 5.2.7 in [1].

• dual quantum q-Krawtchouk:

$$q^{\boldsymbol{\lambda}} = (p,q^N), \quad \boldsymbol{\delta}=(0,-1), \quad \kappa=q^{-1}, \quad p>q^{-N}, \tag{A.71}$$

$$\mathcal{E}(n;\boldsymbol{\lambda}) = q^{-n}-1, \quad \eta(x;\boldsymbol{\lambda}) = 1-q^x, \tag{A.72}$$

$$P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = {}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, q^{-x}}{q^{-N}}\Bigm|\, q\,;pq^{x+1}\Bigr), \tag{A.73}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{(q\,;q)_N}{(q\,;q)_x(q\,;q)_{N-x}}\,\frac{p^{-x}q^{-Nx}}{(p^{-1}q^{-x}\,;q)_x}, \tag{A.74}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{(q\,;q)_N}{(q\,;q)_n(q\,;q)_{N-n}}\,\frac{p^{-n}q^{n(n-1-N)}}{(p^{-1}q^{-N}\,;q)_n} \times (p^{-1}q^{-N}\,;q)_N, \tag{A.75}$$

$$R_1(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2 z', \quad z' \stackrel{\text{def}}{=} z+1, \tag{A.76}$$

$$R_0(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2 z'^2, \tag{A.77}$$

$$R_{-1}(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2\bigl(-z'^2 + p^{-1}(1+p+q^{-N-1})z' - p^{-1}q^{-N}(1+q^{-1})\bigr), \tag{A.78}$$

$$\varphi(x;\boldsymbol{\lambda}) = q^x, \quad f_n(\boldsymbol{\lambda}) = q^{-n}-1, \quad b_n(\boldsymbol{\lambda}) = 1. \tag{A.79}$$

- dual little q-Jacobi:

$$q^{\boldsymbol{\lambda}} = (a,b), \quad \boldsymbol{\delta} = (0,1), \quad \kappa = q, \quad 0 < a < q^{-1}, \quad b < q^{-1}, \tag{A.80}$$

$$\mathcal{E}(n;\boldsymbol{\lambda}) = 1-q^n, \quad \eta(x;\boldsymbol{\lambda}) = (q^{-x}-1)(1-abq^{x+1}), \tag{A.81}$$

$$P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = {}_3\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, q^{-x},\, abq^{x+1}}{bq}\Bigm|\, q\,;a^{-1}q^n\Bigr), \tag{A.82}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{(bq,abq\,;q)_x\, a^x q^{x^2}}{(q,aq\,;q)_x}\,\frac{1-abq^{2x+1}}{1-abq}, \tag{A.83}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{(bq\,;q)_n}{(q\,;q)_n}(aq)^n \times \frac{(aq\,;q)_\infty}{(abq^2;q)_\infty}, \tag{A.84}$$

$$R_1(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2 z', \quad z' \stackrel{\text{def}}{=} z-1, \tag{A.85}$$

$$R_0(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2 z'^2, \tag{A.86}$$

$$R_{-1}(z;\boldsymbol{\lambda}) = (q^{-\frac12}-q^{\frac12})^2\bigl((1+abq)z'^2 + (1+a)z'\bigr), \tag{A.87}$$

$$\varphi(x;\boldsymbol{\lambda}) = \frac{q^{-x}-abq^{x+2}}{1-abq^2}, \quad f_n(\boldsymbol{\lambda}) = 1-q^n, \quad b_n(\boldsymbol{\lambda}) = 1. \tag{A.88}$$

- dual alternative q-Charlier:

$$q^{\boldsymbol{\lambda}} = a, \quad \boldsymbol{\delta} = 1, \quad \kappa = q, \quad a > 0, \tag{A.89}$$

$$\mathcal{E}(n;\boldsymbol{\lambda}) = 1-q^n, \quad \eta(x;\boldsymbol{\lambda}) = (q^{-x}-1)(1+aq^x), \tag{A.90}$$

$$P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = q^{nx}{}_2\phi_1\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, q^{-x}}{0}\Bigm|\, q\,;-a^{-1}q^{1-x}\Bigr) = {}_3\phi_0\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\, q^{-x},\, -aq^x}{-}\Bigm|\, q\,;-a^{-1}q^n\Bigr), \tag{A.91}$$

$$\phi_0(x;\boldsymbol{\lambda})^2 = \frac{a^x q^{\frac12 x(3x-1)}(-a\,;q)_x}{(q\,;q)_x}\,\frac{1+aq^{2x}}{1+a}, \tag{A.92}$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{a^n q^{\frac12 n(n+1)}}{(q\,;q)_n} \times \frac{1}{(-aq\,;q)_\infty}, \tag{A.93}$$

$$R_1(z;\boldsymbol{\lambda}) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 z', \quad z' \stackrel{\text{def}}{=} z - 1, \tag{A.94}$$

$$R_0(z;\boldsymbol{\lambda}) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 z'^2, \tag{A.95}$$

$$R_{-1}(z;\boldsymbol{\lambda}) = (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 \big((1-a)z'^2 + z'\big), \tag{A.96}$$

$$\varphi(x;\boldsymbol{\lambda}) = \frac{q^{-x} + aq^{x+1}}{1+aq}, \quad f_n(\boldsymbol{\lambda}) = 1 - q^n, \quad b_n(\boldsymbol{\lambda}) = 1. \tag{A.97}$$

We note that the little q-Jacobi polynomial eq. (5.193) and the alternative q-Charlier polynomial eq. (5.248) in [1] can be rewritten as

$$\text{little } q\text{-Jacobi}: \qquad P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = {}_3\phi_1\Big(\begin{matrix} q^{-n}, q^{-x}, abq^{n+1} \\ bq \end{matrix} \;\Big|\; q\,; a^{-1}q^x\Big), \tag{A.98}$$

$$\text{alternative } q\text{-Charlier}: \quad P_n(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) = {}_3\phi_0\Big(\begin{matrix} q^{-n}, q^{-x}, -aq^n \\ - \end{matrix} \;\Big|\; q\,; -a^{-1}q^x\Big). \tag{A.99}$$

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