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Simplified Expressions of the Multi-indexed Laguerre and Jacobi Polynomials

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Abstract

The multi-indexed Laguerre and Jacobi polynomials form a complete set of orthogonal polynomials. They satisfy second order differential equations but not three term recurrence relations, because of the 'holes' in their degrees. The multi-indexed Laguerre and Jacobi polynomials have Wronskian expressions originating from multiple Darboux transformations. For the ease of applications, two different forms of simplified expressions of the multi-indexed Laguerre and Jacobi polynomials are derived based on various identities. The parity transformation property of the multi-indexed Jacobi polynomials is derived based on that of the Jacobi polynomial.

Keywords: multi-indexed orthogonal polynomials, Laguerre and Jacobi polynomials, Wronskian formula, determinant formula

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1 Introduction

The exceptional and multi-indexed orthogonal polynomials [10, 9, 22, 2, 15, 11, 17, 18, 19, 21] seem to be a focal point of recent research on exactly solvable quantum mechanics. They belong to a new type of orthogonal polynomials which satisfy second order differential (difference) equations and form complete orthogonal basis in an appropriate Hilbert space. One of their characteristic features is that they do not satisfy three term recurrence relations because of the 'holes' in the degrees. This is how they avoid the constraints due to Bochner [23, 3]. They are constructed from the original quantum mechanical systems, the radial oscillator potential and the Pöschl-Teller potential, by multiple application of Darboux transformations [5, 4] in terms of seed solutions called virtual state wavefunctions which are generated by two types of discrete symmetry transformations [17, 18, 19, 21].

The multi-indexed Laguerre and Jacobi polynomials have Wronskian expressions [17] originating from multiple Darboux transformations [4]. In this note we present two different forms of equivalent determinant expressions without higher derivatives of the Wronskians by using various identities of the Laguerre and Jacobi polynomials [24]. These simplified expressions show explicitly the constituents of the multi-indexed orthogonal polynomials and they are helpful for deeper understanding. In [6] and [7] Durán employed similar simplified expressions as the starting point of his exposition of the exceptional Laguerre and Jacobi polynomials. See also [8] by Durán and Pérez. In their expressions, the matrix elements of the determinants are polynomials. In the original expressions in [17], the matrix elements of the determinants contain the non-polynomial factors, see (10)–(13). In the simplified expressions presented in this paper in § 2.4–2.5, they are also all polynomials.

This short note is organised as follows. In § 2.1 the quantum mechanical settings for the original Laguerre and Jacobi polynomials are recapitulated. That is, the Hamiltonians with the radial oscillator potential and the Pöschl-Teller potential are introduced and their eigenvalues and eigenfunctions are presented. Type I and II discrete symmetry transformations for the Hamiltonians of the Laguerre and Jacobi polynomials are explained in § 2.2. The seed solutions for Darboux transformations, to be called the virtual state wavefunctions of type I and II, for the Laguerre and Jacobi, are listed explicitly. In § 2.3 the Wronskian forms of the multi-indexed Laguerre and Jacobi polynomials derived in [17] and [14] are recapitulated as the starting point. § 2.4 and § 2.5 are the main content of this note. Those who are familiar with the multi-indexed Laguerre and Jacobi polynomials can directly go to this part. The first simplified expressions of the multi-indexed Laguerre and Jacobi polynomials are derived in § 2.4 by using various identities of the original Laguerre and Jacobi polynomials. The second simplified expressions, to be derived in § 2.5, are the consequences of the multi-linearity of determinants and the form of the Schrödinger equation $\psi''(x) = \big(U(x) - \mathcal{E}\big)\psi(x)$. Every even order derivative $\psi^{(2m)}(x)$ in the Wronskian can be replaced by $(-\mathcal{E})^m\psi(x)$. In § 2.6 the parity transformation formula of the multi-indexed Jacobi polynomials is presented. Section 3 is for a summary and comments.

2 Multi-indexed Laguerre and Jacobi Polynomials

The foundation of the theory of multi-indexed orthogonal polynomials is the exactly solvable one dimensional quantum mechanical system $\mathcal{H}$:

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n = 0, 1, \ldots), \quad \mathcal{H} = -\frac{d^2}{dx^2} + U(x), \tag{1}$$

and its iso-spectrally deformed system

$$\mathcal{H}_{\mathcal{D}}\phi_{\mathcal{D}\,n}(x) = \mathcal{E}_n\phi_{\mathcal{D}\,n}(x) \quad (n = 0, 1, \ldots), \quad \mathcal{H}_{\mathcal{D}} = -\frac{d^2}{dx^2} + U_{\mathcal{D}}(x), \tag{2}$$

in terms of multiple application of Darboux transformations [17, 4]. In this note we discuss two systems which have the Laguerre and Jacobi polynomials as the main parts of the eigenfunctions. We follow the notation of [17] with slight modifications for simplification sake.

2.1 Original Laguerre and Jacobi polynomials

2.1.1 radial oscillator potential

The Hamiltonian with the radial oscillator potential

$$U(x) \stackrel{\text{def}}{=} x^2 + \frac{g(g-1)}{x^2} - 2g - 1, \quad 0 < x < \infty, \quad g > \frac{1}{2}, \tag{3}$$

has the Laguerre polynomials $L_n^{(\alpha)}(\eta)$ as the main part of the eigenfunctions:

$$\phi_n(x;g) \stackrel{\text{def}}{=} \phi_0(x;g)L_n^{(g-\frac{1}{2})}\bigl(\eta(x)\bigr), \quad \mathcal{E}_n \stackrel{\text{def}}{=} 4n, \quad \eta(x) \stackrel{\text{def}}{=} x^2,$$
$$\phi_0(x;g) \stackrel{\text{def}}{=} e^{-\frac{1}{2}x^2}x^g, \quad L_n^{(\alpha)}(\eta) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{k!}(\alpha+k+1)_{n-k}\eta^k.$$

Here are some identities of the Laguerre polynomials [24] to be used in § 2.4–2.5,

$$\partial_\eta L_n^{(\alpha)}(\eta) = -L_{n-1}^{(\alpha+1)}(\eta), \tag{4}$$

$$L_n^{(\alpha)}(\eta) - L_n^{(\alpha-1)}(\eta) = L_{n-1}^{(\alpha)}(\eta), \tag{5}$$

$$\eta L_{n-1}^{(\alpha+1)}(\eta) - \alpha L_{n-1}^{(\alpha)}(\eta) = -nL_n^{(\alpha-1)}(\eta). \tag{6}$$

As is clear from (3), the lower boundary, $x = 0$, is the regular singular point with the characteristic exponents g and $1-g$. The upper boundary point, $x = \infty$, is an irregular singular point. Here $(a)_n \stackrel{\text{def}}{=} \prod_{j=1}^{n}(a+j-1)$ is the shifted factorial or the so-called Pochhammer's symbol.

2.1.2 Pöschl-Teller potential

The Hamiltonian with the Pöschl-Teller potential

$$U(x) \stackrel{\text{def}}{=} \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, \quad 0 < x < \frac{\pi}{2}, \quad g > \frac{1}{2},\ h > \frac{1}{2},$$

has the Jacobi polynomials $P_n^{(\alpha,\beta)}(\eta)$ as the main part of the eigenfunctions:

$$\phi_n\big(x;(g,h)\big) \stackrel{\text{def}}{=} \phi_0\big(x;(g,h)\big) P_n^{(g-\frac{1}{2},h-\frac{1}{2})}\big(\eta(x)\big), \quad \eta(x) \stackrel{\text{def}}{=} \cos 2x,$$

$$\phi_0\big(x;(g,h)\big) \stackrel{\text{def}}{=} (\sin x)^g (\cos x)^h, \quad \mathcal{E}_n(g,h) \stackrel{\text{def}}{=} 4n(n+g+h),$$

$$P_n^{(\alpha,\beta)}(\eta) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{1}{k!} \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k} \Big(\frac{1-\eta}{2}\Big)^k.$$

Here are some identities of the Jacobi polynomials [24] to be used in in § 2.4–2.5,

$$\partial_\eta P_n^{(\alpha,\beta)}(\eta) = \tfrac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(\eta), \tag{7}$$

$$(n+\beta) P_n^{(\alpha,\beta-1)}(\eta) - \beta P_n^{(\alpha-1,\beta)}(\eta) = (n+\alpha+\beta) \tfrac{1+\eta}{2} P_{n-1}^{(\alpha,\beta+1)}(\eta), \tag{8}$$

$$(n+\alpha) P_n^{(\alpha-1,\beta)}(\eta) - \alpha P_n^{(\alpha,\beta-1)}(\eta) = -(n+\alpha+\beta) \tfrac{1-\eta}{2} P_{n-1}^{(\alpha+1,\beta)}(\eta). \tag{9}$$

The two boundary points, $x = 0, \frac{\pi}{2}$ are the regular singular points with the characteristic exponents g, $1-g$ and h, $1-h$, respectively.

2.2 Discrete symmetry transformations and virtual state wavefunctions

The seed solutions (virtual state wavefunctions) for Darboux transformations can be constructed by applying discrete symmetry transformations of the Hamiltonian to the eigenfunctions. They have negative energies and have no node and they are not square integrable, see [17] for more detail.

2.2.1 Laguerre (L) system

Type I transformation It is obvious that the transformation $x \to ix$ is the symmetry of the radial oscillator system. The seed solutions are

$$\text{L1}: \ \mathcal{H}\tilde{\phi}^{\text{I}}_{\text{v}}(x) = \tilde{\mathcal{E}}^{\text{I}}_{\text{v}} \tilde{\phi}^{\text{I}}_{\text{v}}(x), \quad \tilde{\phi}^{\text{I}}_{\text{v}}(x) \stackrel{\text{def}}{=} \tilde{\phi}^{\text{I}}_0(x) \xi^{\text{I}}_{\text{v}}\big(\eta(x)\big), \quad \tilde{\phi}^{\text{I}}_0(x) \stackrel{\text{def}}{=} e^{\frac{1}{2}x^2} x^g,$$

$$\xi^{\text{I}}_{\text{v}}(\eta) \stackrel{\text{def}}{=} L_{\text{v}}^{(g-\frac{1}{2})}(-\eta), \quad \tilde{\mathcal{E}}^{\text{I}}_{\text{v}} \stackrel{\text{def}}{=} -4(g+\text{v}+\tfrac{1}{2}), \quad \text{v} \in \mathbb{Z}_{\geq 0}.$$

Hereafter we use v for the degree of the seed polynomial ξ_{v} in order to distinguish it with the degree n of eigenpolynomial P_n.

Type II transformation The exchange of the characteristic exponents of the regular singular point $x=0$, $g\to 1-g$, is another discrete symmetry transformation. It generates the seed solutions,

$$\begin{aligned}\text{L2}:\ &\mathcal{H}\tilde{\phi}^{\text{II}}_{\text{v}}(x)=\tilde{\mathcal{E}}^{\text{II}}_{\text{v}}\tilde{\phi}^{\text{II}}_{\text{v}}(x),\quad \tilde{\phi}^{\text{II}}_{\text{v}}(x)\stackrel{\text{def}}{=}\tilde{\phi}^{\text{II}}_0(x)\xi^{\text{II}}_{\text{v}}\bigl(\eta(x)\bigr),\quad \tilde{\phi}^{\text{II}}_0(x)\stackrel{\text{def}}{=}e^{-\frac12x^2}x^{1-g},\\ &\xi^{\text{II}}_{\text{v}}(\eta)\stackrel{\text{def}}{=}L^{(\frac12-g)}_{\text{v}}(\eta),\quad \tilde{\mathcal{E}}^{\text{II}}_{\text{v}}\stackrel{\text{def}}{=}-4(g-\text{v}-\tfrac12),\quad \text{v}=0,1,\ldots,[g-\tfrac12]',\end{aligned}$$

in which $[a]'$ denotes the greatest integer less than a.

2.2.2 Jacobi (J) system

Type I transformation The exchange of the characteristic exponents of the regular singular point $x=\frac{\pi}{2}$, $h\to 1-h$, is a discrete symmetry transformation:

$$\begin{aligned}\text{J1}:\ &\mathcal{H}\tilde{\phi}^{\text{I}}_{\text{v}}(x)=\tilde{\mathcal{E}}^{\text{I}}_{\text{v}}\tilde{\phi}^{\text{I}}_{\text{v}}(x),\quad \tilde{\phi}^{\text{I}}_{\text{v}}(x)\stackrel{\text{def}}{=}\tilde{\phi}^{\text{I}}_0(x)\xi^{\text{I}}_{\text{v}}\bigl(\eta(x)\bigr),\quad \tilde{\phi}^{\text{I}}_0(x)\stackrel{\text{def}}{=}(\sin x)^g(\cos x)^{1-h},\\ &\xi^{\text{I}}_{\text{v}}(\eta)\stackrel{\text{def}}{=}P^{(g-\frac12,\frac12-h)}_{\text{v}}(\eta),\ \tilde{\mathcal{E}}^{\text{I}}_{\text{v}}\stackrel{\text{def}}{=}-4(g+\text{v}+\tfrac12)(h-\text{v}-\tfrac12),\ \ \text{v}=0,1,\ldots,[h-\tfrac12]'.\end{aligned}$$

Type II transformation Likewise the exchange of the characteristic exponents of the regular singular point $x=0$, $g\to 1-g$, is a discrete symmetry transformation:

$$\begin{aligned}\text{J2}:\ &\mathcal{H}\tilde{\phi}^{\text{II}}_{\text{v}}(x)=\tilde{\mathcal{E}}^{\text{II}}_{\text{v}}\tilde{\phi}^{\text{II}}_{\text{v}}(x),\quad \tilde{\phi}^{\text{II}}_{\text{v}}(x)\stackrel{\text{def}}{=}\tilde{\phi}^{\text{II}}_0(x)\xi^{\text{II}}_{\text{v}}\bigl(\eta(x)\bigr),\quad \tilde{\phi}^{\text{II}}_0(x)\stackrel{\text{def}}{=}(\sin x)^{1-g}(\cos x)^{h},\\ &\xi^{\text{II}}_{\text{v}}(\eta)\stackrel{\text{def}}{=}P^{(\frac12-g,h-\frac12)}_{\text{v}}(\eta),\ \tilde{\mathcal{E}}^{\text{II}}_{\text{v}}\stackrel{\text{def}}{=}-4(g-\text{v}-\tfrac12)(h+\text{v}+\tfrac12),\ \ \text{v}=0,1,\ldots,[g-\tfrac12]'.\end{aligned}$$

2.3 Wronskian forms of the multi-indexed Laguerre and Jacobi polynomials

We deform the original Hamiltonian system (1) by applying multiple Darboux transformations [17, 4] in terms of M_{I} type I seed solutions and M_{II} type II seed solutions specified by the degree set $\mathcal{D}\stackrel{\text{def}}{=}\{d_1,\ldots,d_M\}$ (ordered set), $M\stackrel{\text{def}}{=}M_{\text{I}}+M_{\text{II}}$, $M_{\text{I}}\stackrel{\text{def}}{=}\#\{d_j|d_j\in\mathcal{D},\text{ type I}\}$, $M_{\text{II}}\stackrel{\text{def}}{=}\#\{d_j|d_j\in\mathcal{D},\text{ type II}\}$. The multi-indexed orthogonal polynomials $\{P_{\mathcal{D},n}(\eta)\}$ are the main parts of the eigenfunctions $\{\phi_{\mathcal{D}\,n}(x)\}$ of the deformed Hamiltonian system $\mathcal{H}_{\mathcal{D}}$ (2):

$$\begin{aligned}U_{\mathcal{D}}(x)&\stackrel{\text{def}}{=}U(x)-2\partial_x^2\log\bigl|\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M}](x)\bigr|,\\ \phi_{\mathcal{D}\,n}(x)&\stackrel{\text{def}}{=}\frac{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M},\phi_n](x)}{\mathrm{W}[\tilde{\phi}_{d_1},\ldots,\tilde{\phi}_{d_M}](x)}\quad(n=0,1,\ldots)\\ &\stackrel{\text{def}}{=}c_{\mathcal{F}}^M\psi_{\mathcal{D}}(x)P_{\mathcal{D},n}\bigl(\eta(x)\bigr),\quad \psi_{\mathcal{D}}(x)\stackrel{\text{def}}{=}\frac{\hat{\phi}_0(x)}{\Xi_{\mathcal{D}}\bigl(\eta(x)\bigr)},\end{aligned}$$

in which $\hat{\phi}_0(x)$ and $c_{\mathcal{F}}$ are defined by

$$\hat{\phi}_0(x) \stackrel{\text{def}}{=} \begin{cases} \phi_0(x; g+M_{\text{I}}-M_{\text{II}}) & : \text{L} \\ \phi_0\big(x; (g+M_{\text{I}}-M_{\text{II}}, h-M_{\text{I}}+M_{\text{II}})\big) & : \text{J} \end{cases}, \quad c_{\mathcal{F}} \stackrel{\text{def}}{=} \begin{cases} 2 & : \text{L} \\ -4 & : \text{J} \end{cases}.$$

The superscript I and II of the seed solutions are suppressed for simplicity of notation. The Wronskian of n-functions $\{f_1, \ldots, f_n\}$ is defined by formula

$$\text{W}[f_1, \ldots, f_n](x) \stackrel{\text{def}}{=} \det\Big(\frac{d^{j-1} f_k(x)}{dx^{j-1}}\Big)_{1\le j,k\le n}.$$

There are two different but equivalent Wronskian definitions of the denominator polynomial $\Xi_{\mathcal{D}}(\eta)$ and the multi-indexed orthogonal polynomial $P_{\mathcal{D},n}(\eta)$. The first is based on the Wronskians of the 'polynomials' [17]:

$$\Xi_{\mathcal{D}}(\eta) \stackrel{\text{def}}{=} \text{W}[\mu_{d_1}, \ldots, \mu_{d_M}](\eta) \times \begin{cases} \eta^{(M_{\text{I}}+g-\frac12)M_{\text{II}}} e^{-M_{\text{I}}\eta} & : \text{L} \\ \big(\frac{1-\eta}{2}\big)^{(M_{\text{I}}+g-\frac12)M_{\text{II}}} \big(\frac{1+\eta}{2}\big)^{(M_{\text{II}}+h-\frac12)M_{\text{I}}} & : \text{J} \end{cases}, \tag{10}$$

$$P_{\mathcal{D},n}(\eta) \stackrel{\text{def}}{=} \text{W}[\mu_{d_1}, \ldots, \mu_{d_M}, P_n](\eta) \times \begin{cases} \eta^{(M_{\text{I}}+g+\frac12)M_{\text{II}}} e^{-M_{\text{I}}\eta} & : \text{L} \\ \big(\frac{1-\eta}{2}\big)^{(M_{\text{I}}+g+\frac12)M_{\text{II}}} \big(\frac{1+\eta}{2}\big)^{(M_{\text{II}}+h+\frac12)M_{\text{I}}} & : \text{J} \end{cases}, \tag{11}$$

$$\mu_{\text{v}}(\eta) \stackrel{\text{def}}{=} \begin{cases} e^{\eta} \times L_{\text{v}}^{(g-\frac12)}(-\eta) & : \text{L, v type I} \\ \eta^{\frac12-g} \times L_{\text{v}}^{(\frac12-g)}(\eta) & : \text{L, v type II} \\ \big(\frac{1+\eta}{2}\big)^{\frac12-h} \times P_{\text{v}}^{(g-\frac12,\frac12-h)}(\eta) & : \text{J, v type I} \\ \big(\frac{1-\eta}{2}\big)^{\frac12-g} \times P_{\text{v}}^{(\frac12-g,h-\frac12)}(\eta) & : \text{J, v type II} \end{cases}, \tag{12}$$

$$P_n(\eta) \stackrel{\text{def}}{=} \begin{cases} L_n^{(g-\frac12)}(\eta) & : \text{L} \\ P_n^{(g-\frac12,h-\frac12)}(\eta) & : \text{J} \end{cases}. \tag{13}$$

The second is based on the Wronskians of the virtual state wavefunctions $\tilde{\phi}_{\text{v}}(x)$ and the eigenfunction $\phi_n(x)$ [14]:

$$\Xi_{\mathcal{D}}(\eta) = c_{\mathcal{F}}^{-\frac12 M(M-1)} \text{W}[\tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}](x) \times \begin{cases} \eta^{-M'(M'+g-\frac12)} e^{-M'\eta} & : \text{L} \\ \big(\frac{1-\eta}{2}\big)^{-M'(M'+g-\frac12)} \big(\frac{1+\eta}{2}\big)^{-M'(M'-h+\frac12)} & : \text{J} \end{cases}, \tag{14}$$

$$P_{\mathcal{D},n}(\eta) = c_{\mathcal{F}}^{-\frac12 M(M+1)} \text{W}[\tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}, \phi_n](x) \times \begin{cases} \eta^{-(M'+\frac12)(M'+g)} e^{-(M'-\frac12)\eta} & : \text{L} \\ \big(\frac{1-\eta}{2}\big)^{-(M'+\frac12)(M'+g)} \big(\frac{1+\eta}{2}\big)^{-(M'-\frac12)(M'-h)} & : \text{J} \end{cases}, \tag{15}$$

in which $M' \stackrel{\text{def}}{=} \frac12(M_{\text{I}} - M_{\text{II}})$ and $\eta = \eta(x)$.

2.4 Simplified forms of the multi-indexed Laguerre and Jacobi polynomials, A:

In this and the subsequent subsections we will derive simplified expressions of the multi-indexed Laguerre and Jacobi polynomials $P_{\mathcal{D},n}(\eta)$ and the corresponding denominator polynomials $\Xi_{\mathcal{D}}(\eta)$ starting from the Wronskian expressions (10)–(15) in the previous subsection.

In this subsection we simplify the Wronskians of the 'polynomials' (10)–(13). Let us first transform the higher derivatives of the 'seed polynomials' $\mu_{\rm v}(\eta)$ in (12). For the L system, we obtain

$$
\begin{aligned}
&\text{L1}: \quad \partial_\eta\bigl(e^\eta L_{\rm v}^{(g-\frac12)}(-\eta)\bigr)=e^\eta L_{\rm v}^{(g+1-\frac12)}(-\eta),\\
&\text{L2}: \quad \partial_\eta\bigl(\eta^{\frac12-g}L_{\rm v}^{(\frac12-g)}(\eta)\bigr)=({\rm v}-g+\tfrac12)\eta^{\frac12-(g+1)}L_{\rm v}^{(\frac12-(g+1))}(\eta),
\end{aligned}
$$

by using (4)–(5) and (4)–(6), respectively. By repeating these we arrive at for $j\in\mathbb{Z}_{\geq1}$,

$$
\begin{aligned}
\partial_\eta^{j-1}\bigl(e^\eta L_{\rm v}^{(g-\frac12)}(-\eta)\bigr)&=e^\eta\times L_{\rm v}^{(g+j-\frac32)}(-\eta),\\
\partial_\eta^{j-1}\bigl(\eta^{\frac12-g}L_{\rm v}^{(\frac12-g)}(\eta)\bigr)&=(-1)^{j-1}(g-\tfrac12-{\rm v})_{j-1}\eta^{\frac32-g-j}L_{\rm v}^{(\frac32-g-j)}(\eta)\\
&=\eta^{\frac12-g-K}\times(-1)^{j-1}(g-\tfrac12-{\rm v})_{j-1}\eta^{K+1-j}L_{\rm v}^{(\frac32-g-j)}(\eta),
\end{aligned}
$$

in which K is a positive integer. For the J system, we obtain

$$
\begin{aligned}
&\text{J1}: \ \partial_\eta\bigl((\tfrac{1+\eta}{2})^{\frac12-h}P_{\rm v}^{(g-\frac12,\frac12-h)}(\eta)\bigr)=\tfrac12({\rm v}-h+\tfrac12)(\tfrac{1+\eta}{2})^{\frac12-(h+1)}P_{\rm v}^{(g+1-\frac12,\frac12-(h+1))}(\eta),\\
&\text{J2}: \ \partial_\eta\bigl((\tfrac{1-\eta}{2})^{\frac12-g}P_{\rm v}^{(\frac12-g,h-\frac12)}(\eta)\bigr)=-\tfrac12({\rm v}-g+\tfrac12)(\tfrac{1-\eta}{2})^{\frac12-(g+1)}P_{\rm v}^{(\frac12-(g+1),h+1-\frac12)}(\eta),
\end{aligned}
$$

by using (7)&(8) and (7)&(9), respectively. Repeated applications of these formulas lead to

$$
\begin{aligned}
\partial_\eta^{j-1}\bigl((\tfrac{1+\eta}{2})^{\frac12-h}P_{\rm v}^{(g-\frac12,\frac12-h)}(\eta)\bigr)&=\frac{(-1)^{j-1}}{2^{j-1}}(h-\tfrac12-{\rm v})_{j-1}(\tfrac{1+\eta}{2})^{\frac32-h-j}P_{\rm v}^{(g+j-\frac32,\frac32-h-j)}(\eta)\\
&=(\tfrac{1+\eta}{2})^{\frac12-h-K}\times\frac{(-1)^{j-1}}{2^{j-1}}(h-\tfrac12-{\rm v})_{j-1}(\tfrac{1+\eta}{2})^{K+1-j}P_{\rm v}^{(g+j-\frac32,\frac32-h-j)}(\eta),\\
\partial_\eta^{j-1}\bigl((\tfrac{1-\eta}{2})^{\frac12-g}P_{\rm v}^{(\frac12-g,h-\frac12)}(\eta)\bigr)&=\frac{1}{2^{j-1}}(g-\tfrac12-{\rm v})_{j-1}(\tfrac{1-\eta}{2})^{\frac32-g-j}P_{\rm v}^{(\frac32-g-j,h+j-\frac32)}(\eta)\\
&=(\tfrac{1-\eta}{2})^{\frac12-g-K}\times\frac{1}{2^{j-1}}(g-\tfrac12-{\rm v})_{j-1}(\tfrac{1-\eta}{2})^{K+1-j}P_{\rm v}^{(\frac32-g-j,h+j-\frac32)}(\eta).
\end{aligned}
$$

The higher derivatives of the eigenpolynomials $P_n(\eta)$ in (13) are replaced simply through (4) and (7), respectively,

$$
\partial_\eta^{j-1}L_n^{(g-\frac12)}(\eta)=(-1)^{j-1}L_{n+1-j}^{(g+j-\frac32)}(\eta),
$$

$$\partial_\eta^{j-1}P_n^{(g-\frac{1}{2},h-\frac{1}{2})}(\eta)=\frac{1}{2^{j-1}}(n+g+h)_{j-1}P_{n+1-j}^{(g+j-\frac{3}{2},h+j-\frac{3}{2})}(\eta),$$

in which we adopt the convention $L_n^{(\alpha)}(\eta)=P_n^{(\alpha,\beta)}(\eta)=0$ $(n\in\mathbb{Z}_{<0})$.

Let us define M dimensional column vectors $\vec{X}_{\rm v}^{(M)}=\big(X_{{\rm v},j}^{(M)}\big)_{j=1}^{M}$ and $\vec{Z}_n^{(M)}=\big(Z_{n,j}^{(M)}\big)_{j=1}^{M}$ by

$$X_{{\rm v},j}^{(M)}(\eta)\stackrel{\text{def}}{=}\begin{cases}L_{\rm v}^{(g+j-\frac{3}{2})}(-\eta) & :\text{L, v type I}\\ (-1)^{j-1}(g-\frac{1}{2}-{\rm v})_{j-1}\eta^{M-j}L_{\rm v}^{(\frac{3}{2}-g-j)}(\eta) & :\text{L, v type II}\\ \frac{(-1)^{j-1}}{2^{j-1}}(h-\frac{1}{2}-{\rm v})_{j-1}(\frac{1+\eta}{2})^{M-j}P_{\rm v}^{(g+j-\frac{3}{2},\frac{3}{2}-h-j)}(\eta) & :\text{J, v type I}\\ \frac{1}{2^{j-1}}(g-\frac{1}{2}-{\rm v})_{j-1}(\frac{1-\eta}{2})^{M-j}P_{\rm v}^{(\frac{3}{2}-g-j,h+j-\frac{3}{2})}(\eta) & :\text{J, v type II}\end{cases},$$

$$Z_{n,j}^{(M)}(\eta)\stackrel{\text{def}}{=}\begin{cases}(-1)^{j-1}L_{n+1-j}^{(g+j-\frac{3}{2})}(\eta) & :\text{L}\\ \frac{1}{2^{j-1}}(n+g+h)_{j-1}P_{n+1-j}^{(g+j-\frac{3}{2},h+j-\frac{3}{2})}(\eta) & :\text{J}\end{cases}.$$

The Wronskians in (10)–(11) are replaced by ordinary determinants consisting of these column vectors:

$$\begin{aligned}\mathrm{W}[\mu_{d_1},\ldots,\mu_{d_M}](\eta)&=\Big|\vec{X}_{d_1}^{(M)}(\eta)\ \cdots\ \vec{X}_{d_M}^{(M)}(\eta)\Big|\\&\quad\times\begin{cases}\big(e^{\eta}\big)^{M_{\rm I}}\big(\eta^{\frac{3}{2}-g-M}\big)^{M_{\rm II}} & :\text{L}\\ \big((\frac{1+\eta}{2})^{\frac{3}{2}-h-M}\big)^{M_{\rm I}}\big((\frac{1-\eta}{2})^{\frac{3}{2}-g-M}\big)^{M_{\rm II}} & :\text{J}\end{cases},\\ \mathrm{W}[\mu_{d_1},\ldots,\mu_{d_M},P_n](\eta)&=\Big|\vec{X}_{d_1}^{(M+1)}(\eta)\ \cdots\ \vec{X}_{d_M}^{(M+1)}(\eta)\ \vec{Z}_n^{(M+1)}(\eta)\Big|\\&\quad\times\begin{cases}\big(e^{\eta}\big)^{M_{\rm I}}\big(\eta^{\frac{1}{2}-g-M}\big)^{M_{\rm II}} & :\text{L}\\ \big((\frac{1+\eta}{2})^{\frac{1}{2}-h-M}\big)^{M_{\rm I}}\big((\frac{1-\eta}{2})^{\frac{1}{2}-g-M}\big)^{M_{\rm II}} & :\text{J}\end{cases}.\end{aligned}$$

We arrive at the main result, the simple expressions of $\Xi_{\mathcal{D}}(\eta)$ and $P_{\mathcal{D},n}(\eta)$

$$\Xi_{\mathcal{D}}(\eta)=A\times\Big|\vec{X}_{d_1}^{(M)}(\eta)\ \cdots\ \vec{X}_{d_M}^{(M)}(\eta)\Big|, \tag{16}$$

$$P_{\mathcal{D},n}(\eta)=A\times\Big|\vec{X}_{d_1}^{(M+1)}(\eta)\ \cdots\ \vec{X}_{d_M}^{(M+1)}(\eta)\ \vec{Z}_n^{(M+1)}(\eta)\Big|, \tag{17}$$

$$A=\begin{cases}\eta^{-M_{\rm II}(M_{\rm II}-1)} & :\text{L}\\ (\frac{1+\eta}{2})^{-M_{\rm I}(M_{\rm I}-1)}(\frac{1-\eta}{2})^{-M_{\rm II}(M_{\rm II}-1)} & :\text{J}\end{cases}.$$

It should be stressed that the components of the matrices in (16) and (17) are all polynomials in η. This is a good contrast with the starting Wronskians $\mathrm{W}[\mu_{d_1},\ldots,\mu_{d_M}](\eta)$ and $\mathrm{W}[\mu_{d_1},\ldots,\mu_{d_M},P_n](\eta)$ in (10), (11), in which μ_{d_j}'s have non-polynomial factors (12).

2.5 Simplified forms of the multi-indexed Laguerre and Jacobi polynomials, B:

Here we will simplify the Wronskians of the virtual state wavefunctions and the eigenpolynomials (14), (15). By replacing the even order derivatives of the virtual state wavefunctions and eigenfunctions in the Wronskians (14), (15), by the rule $\psi^{(2m)}(x) \to (-\mathcal{E})^m\psi(x)$, we obtain

$$
\mathrm{W}[\tilde\phi_{d_1},\ldots,\tilde\phi_{d_M},\phi_n](x) = \det(a_{j,k})_{1\le j,k\le M+1}, \tag{18}
$$

$$
\begin{cases} a_{2l-1,k} = (-\tilde{\mathcal{E}}_{d_k})^{l-1}\tilde\phi_{d_k}(x)\ (1\le k\le M), & a_{2l-1,M+1} = (-\mathcal{E}_n)^{l-1}\phi_n(x) \quad (1\le l\le [\frac{M+2}{2}]) \\ a_{2l,k} = (-\tilde{\mathcal{E}}_{d_k})^{l-1}\tilde\phi'_{d_k}(x)\ (1\le k\le M), & a_{2l,M+1} = (-\mathcal{E}_n)^{l-1}\phi'_n(x) \quad (1\le l\le [\frac{M+1}{2}]) \end{cases},
$$

in which $[a]$ denotes the greatest integer not exceeding a. The first derivatives in the $2l$-th row can be simplified by adding $-\frac{\phi_0'(x)}{\phi_0(x)}\times(2l-1)$-th row,

$$
\begin{aligned}
\phi'_n(x) &\to \Bigl(\frac{d}{dx}-\frac{\phi'_0(x)}{\phi_0(x)}\Bigr)\phi_n(x) = \frac{c_{\mathcal{F}}}{\eta'(x)}\,\phi_0(x)\zeta_n\bigl(\eta(x)\bigr) = \phi_0(x)\zeta_n\bigl(\eta(x)\bigr)\times A,\\
\tilde\phi'_{\rm v}(x) &\to \Bigl(\frac{d}{dx}-\frac{\phi'_0(x)}{\phi_0(x)}\Bigr)\tilde\phi_{\rm v}(x) = \frac{c_{\mathcal{F}}}{\eta'(x)}\,\tilde\phi_0(x)\tilde\zeta_{\rm v}\bigl(\eta(x)\bigr) = \tilde\phi_0(x)\tilde\zeta_{\rm v}\bigl(\eta(x)\bigr)\times A,\\
A &= \frac{c_{\mathcal{F}}}{\eta'(x)} = \begin{cases} x^{-1} & : \mathrm{L} \\ (\sin x\cos x)^{-1} & : \mathrm{J} \end{cases},
\end{aligned} \tag{19}
$$

in which $\zeta_n(\eta)$ and $\tilde\zeta_{\rm v}(\eta)$ are polynomials in η defined by

$$
\zeta_n(\eta) \stackrel{\text{def}}{=} \begin{cases} -2\eta L_{n-1}^{(g+\frac12)}(\eta) & : \mathrm{L} \\ -\frac12(n+g+h)(1-\eta^2)P_{n-1}^{(g+\frac12,h+\frac12)}(\eta) & : \mathrm{J} \end{cases},
$$

$$
\tilde\zeta_{\rm v}(\eta) \stackrel{\text{def}}{=} \begin{cases} 2\eta L_{\rm v}^{(g+\frac12)}(-\eta) & : \mathrm{L,I} \\ -2(g-\frac12-{\rm v})L_{\rm v}^{(-g-\frac12)}(\eta) & : \mathrm{L,II} \\ (h-\frac12-{\rm v})(1-\eta)P_{\rm v}^{(g+\frac12,-h-\frac12)}(\eta) & : \mathrm{J,I} \\ -(g-\frac12-{\rm v})(1+\eta)P_{\rm v}^{(-g-\frac12,h+\frac12)}(\eta) & : \mathrm{J,II} \end{cases}.
$$

Use is made of (4)–(6) for L and (7)–(9) for J.

By extracting the functions $\phi_0(x)$, $\tilde\phi_0(x)$ from each column of the matrix $a_{j,k}$ (18) and the factor A of (19) from the even rows, we arrive at another set of simplified determinant expressions for the multi-indexed polynomials:

$$
P_{\mathcal{D},n}(\eta) = c_{\mathcal{F}}^{-\frac12 M(M+1)}\det(a_{j,k})_{1\le j,k\le M+1}\times A, \tag{20}
$$

$$\begin{cases} a_{2l-1,k}=(-\tilde{\mathcal{E}}_{d_k})^{l-1}\xi_{d_k}(\eta)\ (1\leq k\leq M), & a_{2l-1,M+1}=(-\mathcal{E}_n)^{l-1}P_n(\eta)\quad (1\leq l\leq [\frac{M+2}{2}]) \\ a_{2l,k}=(-\tilde{\mathcal{E}}_{d_k})^{l-1}\tilde{\zeta}_{d_k}(\eta)\ (1\leq k\leq M), & a_{2l,M+1}=(-\mathcal{E}_n)^{l-1}\zeta_n(\eta)\quad (1\leq l\leq [\frac{M+1}{2}]) \end{cases},$$

$$A=\begin{cases} \eta^{-([M']+1)([M']+M-2[\frac{M}{2}])} & :\mathrm{L} \\ \bigl(\frac{1-\eta}{2}\bigr)^{-([M']+1)([M']+M-2[\frac{M}{2}])}\bigl(\frac{1+\eta}{2}\bigr)^{-([-M']+1)([-M']+M-2[\frac{M}{2}])} & :\mathrm{J} \end{cases}.$$

In a similar way, we obtain

$$\Xi_{\mathcal{D}}(\eta)=c_{\mathcal{F}}^{-\frac{1}{2}M(M-1)}\det(a_{j,k})_{1\leq j,k\leq M}\times A, \tag{21}$$

$$\begin{cases} a_{2l-1,k}=(-\tilde{\mathcal{E}}_{d_k})^{l-1}\xi_{d_k}(\eta)\quad (1\leq l\leq [\frac{M+1}{2}]) \\ a_{2l,k}=(-\tilde{\mathcal{E}}_{d_k})^{l-1}\tilde{\zeta}_{d_k}(\eta)\quad (1\leq l\leq [\frac{M}{2}]) \end{cases},\quad A=\begin{cases} \eta^{-[M']([M']+M-2[\frac{M}{2}])} & :\mathrm{L} \\ \bigl(\frac{1-\eta}{2}\frac{1+\eta}{2}\bigr)^{-[M']([M']+M-2[\frac{M}{2}])} & :\mathrm{J} \end{cases}.$$

Again all the components of the matrices $a_{j,k}$ in (20) and (21) are polynomials in η.

2.6 Parity transformation of the multi-indexed Jacobi polynomials

The Jacobi polynomial has the parity transformation property [24]

$$P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x). \tag{22}$$

We will show that this property is inherited by the multi-indexed Jacobi polynomials. It is based on the property of the Wronskian

$$\mathrm{W}[f_1,\ldots,f_n](-\eta)=(-1)^{\frac{1}{2}n(n-1)}\mathrm{W}[g_1,\ldots,g_n](\eta),\quad g_k(\eta)\stackrel{\text{def}}{=}f_k(-\eta).$$

In this subsection, we indicate the types of the virtual states explicitly by (v,t), in which t stands for type I or II. Based on (22), we obtain

$$\mu_{(\mathrm{v},\mathrm{t})}\bigl(-\eta;(g,h)\bigr)=(-1)^{\mathrm{v}}\mu_{(\mathrm{v},\bar{\mathrm{t}})}\bigl(\eta;(h,g)\bigr),\quad \bar{\mathrm{t}}\stackrel{\text{def}}{=}\begin{cases} \mathrm{II} & :\mathrm{t}=\mathrm{I} \\ \mathrm{I} & :\mathrm{t}=\mathrm{II} \end{cases},$$
$$P_n\bigl(-\eta;(g,h)\bigr)=(-1)^nP_n\bigl(\eta;(h,g)\bigr).$$

For the multi-index set $\mathcal{D}=\{(d_1,\mathrm{t}_1),\ldots,(d_M,\mathrm{t}_M)\}$ of the virtual state wavefunctions, let us define the 'mirror reflected' multi-index set $\mathcal{D}'\stackrel{\text{def}}{=}\{(d_1,\bar{\mathrm{t}}_1),\ldots,(d_M,\bar{\mathrm{t}}_M)\}$. Corresponding to $M_{\mathrm{I}}\stackrel{\text{def}}{=}\#\{d_j|(d_j,\mathrm{I})\in\mathcal{D}\}$, $M_{\mathrm{II}}\stackrel{\text{def}}{=}\#\{d_j|(d_j,\mathrm{II})\in\mathcal{D}\}$, we have $M'_{\mathrm{I}}\stackrel{\text{def}}{=}\#\{d_j|(d_j,\mathrm{I})\in\mathcal{D}'\}=M_{\mathrm{II}}$, $M'_{\mathrm{II}}\stackrel{\text{def}}{=}\#\{d_j|(d_j,\mathrm{II})\in\mathcal{D}'\}=M_{\mathrm{I}}$. By parity transformation $\eta\to-\eta$, the multi-indexed Jacobi polynomial $P_{\mathcal{D},n}\bigl(\eta;(g,h)\bigr)$ is mapped to $P_{\mathcal{D}',n}\bigl(\eta;(h,g)\bigr)$ with the 'mirror reflected' multi-index set $\mathcal{D}'$:

$$P_{\mathcal{D},n}\bigl(-\eta;(g,h)\bigr)$$

$$
\begin{aligned}
&= \big(\tfrac{1-\eta}{2}\big)^{(M_{\text{II}}+h+\frac12)M_{\text{I}}}\big(\tfrac{1+\eta}{2}\big)^{(M_{\text{I}}+g+\frac12)M_{\text{II}}}\mathrm{W}[\mu_{(d_1,\mathrm{t}_1)},\ldots,\mu_{(d_M,\mathrm{t}_M)},P_n]\big(-\eta;(g,h)\big)\\
&= \big(\tfrac{1-\eta}{2}\big)^{(M_{\text{II}}+h+\frac12)M_{\text{I}}}\big(\tfrac{1+\eta}{2}\big)^{(M_{\text{I}}+g+\frac12)M_{\text{II}}}(-1)^{\frac12 M(M+1)}\\
&\quad\times \mathrm{W}[(-1)^{d_1}\mu_{(d_1,\bar{\mathrm{t}}_1)},\ldots,(-1)^{d_M}\mu_{(d_M,\bar{\mathrm{t}}_M)},(-1)^nP_n]\big(\eta;(h,g)\big)\\
&= \big(\tfrac{1-\eta}{2}\big)^{(M_{\text{II}}+h+\frac12)M_{\text{I}}}\big(\tfrac{1+\eta}{2}\big)^{(M_{\text{I}}+g+\frac12)M_{\text{II}}}(-1)^{\frac12 M(M+1)}(-1)^{d_1+\cdots+d_M+n}\\
&\quad\times \mathrm{W}[\mu_{(d_1,\bar{\mathrm{t}}_1)},\ldots,\mu_{(d_M,\bar{\mathrm{t}}_M)},P_n]\big(\eta;(h,g)\big)\\
&= (-1)^{n+\frac12 M(M+1)+\sum_{k=1}^M d_k}P_{\mathcal{D}',n}\big(\eta;(h,g)\big).
\end{aligned}
$$

Similarly, the denominator polynomial $\Xi_{\mathcal{D}}\big(\eta;(g,h)\big)$ is mapped to $\Xi_{\mathcal{D}'}\big(\eta;(h,g)\big)$ with a sign factor:

$$
\Xi_{\mathcal{D}}\big(-\eta;(g,h)\big)=(-1)^{\frac12 M(M-1)+\sum_{k=1}^M d_k}\,\Xi_{\mathcal{D}'}\big(\eta;(h,g)\big).
$$

For the special case of 'mirror symmetric' multi-index set $\mathcal{D}'=\mathcal{D}$ (as a set), *i.e.* $\{d_j|(d_j,\mathrm{I})\in\mathcal{D}\}=\{d_j|(d_j,\mathrm{II})\in\mathcal{D}\}$ (as a set), we have $P_{\mathcal{D}',n}(\eta)=\pm P_{\mathcal{D},n}(\eta)$. In fact, this formula turns out to be

$$
P_{\mathcal{D}',n}(\eta)=(-1)^{(\frac{M}{2})^2}P_{\mathcal{D},n}(\eta).
$$

For this special case the parity transformation gives

$$
P_{\mathcal{D},n}\big(-\eta;(g,h)\big)=(-1)^nP_{\mathcal{D},n}\big(\eta;(h,g)\big),\quad \Xi_{\mathcal{D}}\big(-\eta;(g,h)\big)=\Xi_{\mathcal{D}}\big(\eta;(h,g)\big).
$$

3 Summary and Comments

The multi-indexed Laguerre and Jacobi polynomials are defined by the Wronskian expressions originating from multiple Darboux transformations. Two simplified determinant expressions of them, (16)–(17) and (20)–(21), which do not contain derivatives, are derived based on the properties of the Wronskian and identities of the Laguerre and Jacobi polynomials. For (20)–(21), the Schrödinger equation is used. For (16)–(17), various identities of the Laguerre and Jacobi polynomials are used, which are essentially forward shift relations. Although the calculation in § 2.4 is performed for polynomials, it can be done for wavefunctions just like [20], in which simplified determinant expressions are presented for the multi-indexed polynomials obtained by multiple Darboux transformations with pseudo virtual states wavefunctions as seed solutions. The parity transformation property of the multi-indexed Jacobi polynomials is also derived.

Multi-indexed orthogonal polynomials have been constructed for the classical orthogonal polynomials in the Askey scheme [1, 12], *i.e.*, the Wilson, Askey-Wilson, Meixner, little q-Jacobi and (q-) Racah polynomials, etc [18, 19, 21]. These polynomials belong to 'discrete' quantum mechanics [16], in which the Schrödinger equations are second order difference equations. The Casoratian expressions of these multi-indexed polynomials can also be simplified by using various identities as demonstrated here. These simplifications will be published elsewhere [13].

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