# Recurrence Relations of the Multi-Indexed Orthogonal Polynomials V : Racah and $q$-Racah types 

Satoru Odake<br>Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan


#### Abstract

In previous papers, we discussed the recurrence relations of the multi-indexed orthogonal polynomials of the Laguerre, Jacobi, Wilson and Askey-Wilson types. In this paper we explore those of the Racah and $q$-Racah types. For the $M$-indexed ( $q$-) Racah polynomials, we derive $3+2 M$ term recurrence relations with variable dependent coefficients and $1+2 L$ term $(L \geq M+1)$ recurrence relations with constant coefficients. Based on the latter, the generalized closure relations and the creation and annihilation operators of the quantum mechanical systems described by the multi-indexed $(q-)$ Racah polynomials are obtained.

In appendix we present a proof and some data of the recurrence relations with constant coefficients for the multi-indexed Wilson and Askey-Wilson polynomials.


## 1 Introduction

Ordinary orthogonal polynomials in one variable are characterized by the three term recurrence relations and those satisfying second order differential or difference equations are severely restricted by Bochner's theorem and its generalizations [1, 2]. The exceptional and multi-indexed orthogonal polynomials $\left\{\mathcal{P}_{n}(\eta) \mid n \in \mathbb{Z}_{\geq 0}\right\}$ [3]-[17] are new types of orthogonal polynomials. They satisfy second order differential or difference equations and form a complete set of orthogonal basis in an appropriate Hilbert space in spite of missing degrees. This degree missing is a characteristic feature of them. Instead of the three term recurrence relations, they satisfy some recurrence relations with more terms [18]-[25], and the constraints by Bochner's theorem are avoided. We distinguish the following two cases;
the set of missing degrees $\mathcal{I}=\mathbb{Z}_{\geq 0} \backslash\left\{\operatorname{deg} \mathcal{P}_{n} \mid n \in \mathbb{Z}_{\geq 0}\right\}$ is case-(1): $\mathcal{I}=\{0,1, \ldots, \ell-1\}$, or case-(2): $\mathcal{I} \neq\{0,1, \ldots, \ell-1\}$, where $\ell$ is a positive integer. The situation of case-(1) is called stable in [7]. Our approach to orthogonal polynomials is based on the quantum mechanical formulations: ordinary quantum mechanics ( oQM ) , discrete quantum mechanics with pure imaginary shifts (idQM) [26]-[29] and discrete quantum mechanics with real shifts (rdQM) [30]-[32]. The Askey scheme of the (basic) hypergeometric orthogonal polynomials [33] is well matched to these quantum mechanical formulations: the Jacobi polynomial etc. in oQM, the Askey-Wilson polynomial etc. in idQM and the $q$-Racah polynomial etc. in rdQM. A new type of orthogonal polynomials are obtained by applying the Darboux transformations with appropriate seed solutions to the exactly solvable quantum mechanical systems described by the classical orthogonal polynomials in the Askey scheme. When the virtual state wavefunctions are used as seed solutions, the case-(1) multi-indexed orthogonal polynomials are obtained [9, 11, 13]. When the eigenstate and/or pseudo virtual state wavefunctions are used as seed solutions, the case-(2) multi-indexed orthogonal polynomials are obtained [34]-[36].

In previous papers [19, 22, 24, 25], the recurrence relations for the case-(1) multi-indexed polynomials (Laguerre (L) and Jacobi (J) types in oQM, Wilson (W) and Askey-Wilson (AW) types in idQM) were studied. There are two kinds of recurrence relations: with variable dependent coefficients [19] and with constant coefficients [22, 24]. The recurrence relations with variable dependent coefficients have been proved for L, J, W and AW types, but those with constant coefficients have been conjectured for L, J, W and AW types and proved only for L and J types.

In this paper we explore the recurrence relations for the case-(1) multi-indexed polynomials of Racah (R) and $q$-Racah ( $q \mathrm{R}$ ) types in rdQM. By similar methods used in idQM case, we derive two kinds of recurrence relations: with variable dependent coefficients and with constant coefficients. We present examples of the latter. Through the process of deriving the recurrence relations with constant coefficients and their examples, we have noticed that similar techniques can be applied to $W$ and AW types. In appendix B and C we present a proof and some explicit form of the recurrence relations with constant coefficients for the multi-indexed (Askey-)Wilson polynomials. The recurrence relations with constant coefficients are closely related to the generalized closure relations [25]. The generalized closure relations provide the exact Heisenberg operator solution of a certain operator, from which the creation and annihilation operators of the system are obtained.

This paper is organized as follows. In section 22 the essence of the multi-indexed ( $q$ )Racah polynomials are recapitulated. In section 3 we derive the recurrence relations with variable dependent coefficients. In section 4 we derive the recurrence relations with constant coefficients and present some explicit examples. In section 5 the generalized closure relations and the creation and annihilation operators are presented. Section 6 is for a summary and comments. In Appendix $A$ some basic data of the multi-indexed ( $q$ - $)$ Racah polynomials are summarized. In Appendix B we prove the recurrence relations with constant coefficients for the multi-indexed (Askey-)Wilson polynomials. In Appendix $C$ some data of the recurrence relations with constant coefficients for the multi-indexed (Askey-)Wilson polynomials are presented.

## 2 Multi-indexed ( $q-$ )Racah Orthogonal Polynomials

In this section we recapitulate the multi-indexed Racah (R) and $q$-Racah ( $q \mathrm{R}$ ) orthogonal polynomials [13]. Various quantities depend on a set of parameters $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}, \ldots\right)$ and their dependence is expressed like, $f=f(\boldsymbol{\lambda}), f(x)=f(x ; \boldsymbol{\lambda})$. (We sometimes omit writing $\boldsymbol{\lambda}$-dependence, when it does not cause confusion.) The parameter $q$ is $0<q<1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{\left(\lambda_{1}, \lambda_{2}, \ldots\right)}=\left(q^{\lambda_{1}}, q^{\lambda_{2}}, \ldots\right)$.

## 2.1 ( $q$-)Racah polynomials

The set of parameters $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)$, its shift $\boldsymbol{\delta}$ and $\kappa$ are

$$
\begin{align*}
\mathrm{R}: & \boldsymbol{\lambda}=(a, b, c, d), \quad \boldsymbol{\delta}=(1,1,1,1), \quad \kappa=1 \\
q \mathrm{R}: & q^{\boldsymbol{\lambda}}=(a, b, c, d), \quad \boldsymbol{\delta}=(1,1,1,1), \quad \kappa=q^{-1} . \tag{2.1}
\end{align*}
$$

For $N \in \mathbb{Z}_{>0}$, we take $n_{\max }=x_{\text {max }}=N$ and

$$
\begin{equation*}
\mathrm{R}: a=-N, \quad q \mathrm{R}: a=q^{-N}, \tag{2.2}
\end{equation*}
$$

and assume the following parameter ranges:

$$
\begin{equation*}
\mathrm{R}: 0<d<a+b, \quad 0<c<1+d, \quad q \mathrm{R}: 0<a b<d<1, \quad q d<c<1 \tag{2.3}
\end{equation*}
$$

The $(q-)$ Racah polynomials $P_{n}(\eta)\left(n=0,1, \ldots, n_{\max }\right)$ are

$$
\check{P}_{n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} P_{n}(\eta(x ; \boldsymbol{\lambda}) ; \boldsymbol{\lambda})=\left\{\begin{array}{cl}
{ }_{4} F_{3}\left(\left.\begin{array}{c}
-n, n+\tilde{d},-x, x+d \\
a, b, c
\end{array} \right\rvert\, 1\right) & : \mathrm{R}  \tag{2.4}\\
{ }_{4} \phi_{3}\left(\begin{array}{c}
q^{-n}, \tilde{d} q^{n}, q^{-x}, d q^{x} \\
a, b, c
\end{array} q ; q\right) & : q \mathrm{R}
\end{array}\right.
$$

$$
\begin{gather*}
= \begin{cases}R_{n}(\eta(x ; \boldsymbol{\lambda}) ; a-1, \tilde{d}-a, c-1, d-c) \\
R_{n}\left(\eta(x ; \boldsymbol{\lambda})+1+d ; a q^{-1}, \tilde{d} a^{-1}, c q^{-1}, d c^{-1} \mid q\right) & : q \mathrm{R}\end{cases} \\
\eta(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left\{\begin{array}{ll}
x(x+d) & : \mathrm{R} \\
\left(q^{-x}-1\right)\left(1-d q^{x}\right) & : q \mathrm{R}
\end{array}, \quad \tilde{d} \stackrel{\text { def }}{=} \begin{cases}a+b+c-d-1 & : \mathrm{R} \\
a b c d^{-1} q^{-1} & : q \mathrm{R}\end{cases} \right. \tag{2.5}
\end{gather*}
$$

where $R_{n}(x(x+\gamma+\delta+1) ; \alpha, \beta, \gamma, \delta)$ and $R_{n}\left(q^{-x}+\gamma \delta q^{x+1} ; \alpha, \beta, \gamma, \delta \mid q\right)$ are the Racah and $q$ Racah polynomials in the conventional parametrization [33], respectively. Our parametrization respects the correspondence between the ( $q-$ )Racah and (Askey-)Wilson polynomials, and symmetries in $(a, b, c, d)$ are transparent. Note that the sinusoidal coordinates $\eta(x ; \boldsymbol{\lambda})$ depend on parameters $\boldsymbol{\lambda}$ only through $d$. The normalization of $\eta(x)$ and $P_{n}(\eta)$ is

$$
\begin{equation*}
\eta(0 ; \boldsymbol{\lambda})=0, \quad \check{P}_{n}(0 ; \boldsymbol{\lambda})=P_{n}(0 ; \boldsymbol{\lambda})=1 . \tag{2.6}
\end{equation*}
$$

The three term recurrence relations are

$$
\begin{equation*}
\eta P_{n}(\eta ; \boldsymbol{\lambda})=A_{n}(\boldsymbol{\lambda}) P_{n+1}(\eta ; \boldsymbol{\lambda})+B_{n}(\boldsymbol{\lambda}) P_{n}(\eta ; \boldsymbol{\lambda})+C_{n}(\boldsymbol{\lambda}) P_{n-1}(\eta ; \boldsymbol{\lambda}), \tag{2.7}
\end{equation*}
$$

where $A_{n}, B_{n}$ and $C_{n}$ are given in (A.2). As a consequence of (2.6), $B_{n}$ is equal to $-A_{n}-C_{n}$.
In the quantum mechanical formulation [30], the polynomials $P_{n}(\eta)$ appear in the eigenvectors $\phi_{n}(x)$,

$$
\begin{equation*}
\phi_{n}(x ; \boldsymbol{\lambda})=\phi_{0}(x ; \boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }}, \tag{2.8}
\end{equation*}
$$

and the orthogonality relations are

$$
\begin{equation*}
\sum_{x=0}^{x_{\max }} \phi_{n}(x ; \boldsymbol{\lambda}) \phi_{m}(x ; \boldsymbol{\lambda})=\frac{\delta_{n m}}{d_{n}(\boldsymbol{\lambda})^{2}} \quad\left(n, m=0,1, \ldots, n_{\max }\right) \tag{2.9}
\end{equation*}
$$

where the ground state eigenvector $\phi_{0}(x)$ and the normalization constant $d_{n}(\boldsymbol{\lambda})$ are given in (A.3) and (A.4), respectively. The Hamiltonian of this rdQM system is a tridiagonal matrix $\mathcal{H}=\left(\mathcal{H}_{x, y}\right)_{0 \leq x, y \leq x_{\max }}$,

$$
\begin{equation*}
\mathcal{H}=-\sqrt{B(x)} e^{\partial} \sqrt{D(x)}-\sqrt{D(x)} e^{-\partial} \sqrt{B(x)}+B(x)+D(x) \tag{2.10}
\end{equation*}
$$

where potential functions $B(x ; \boldsymbol{\lambda})$ and $D(x ; \boldsymbol{\lambda})$ are given in (A.1) and matrices $e^{ \pm \partial}$ are $\left(e^{ \pm \partial}\right)_{x, y}=\delta_{x \pm 1, y}$ and the unit matrix $\mathbf{1}=\left(\delta_{x, y}\right)$ is suppressed. The notation $f(x) \operatorname{Ag}(x)$, where $f(x)$ and $g(x)$ are functions of $x$ and $A$ is a matrix $A=\left(A_{x, y}\right)$, stands for a matrix whose $(x, y)$-element is $f(x) A_{x, y} g(y)$. The Schrödinger equation is

$$
\begin{equation*}
\mathcal{H}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right) \tag{2.11}
\end{equation*}
$$

where the energy eigenvalue $\mathcal{E}_{n}$ is given in (A.5) $\left(0=\mathcal{E}_{0}<\mathcal{E}_{1}<\cdots<\mathcal{E}_{n_{\max }}\right.$ ). By similarity transformation, (2.11) is rewritten as

$$
\begin{align*}
& \widetilde{\mathcal{H}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}(\boldsymbol{\lambda}) \circ \phi_{0}(x ; \boldsymbol{\lambda})=B(x ; \boldsymbol{\lambda})\left(1-e^{\partial}\right)+D(x ; \boldsymbol{\lambda})\left(1-e^{-\partial}\right),  \tag{2.12}\\
& \widetilde{\mathcal{H}}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right), \tag{2.13}
\end{align*}
$$

namely $(q-)$ Racah polynomials $\check{P}_{n}(x)$ satisfy second order difference equations. The three term recurrence relations of $P_{n}(\eta)(2.7)$ imply those of the eigenvectors $\phi_{n}(x)$,

$$
\begin{equation*}
\eta(x ; \boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})=A_{n}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})+B_{n}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})+C_{n}(\boldsymbol{\lambda}) \phi_{n-1}(x ; \boldsymbol{\lambda}) . \tag{2.14}
\end{equation*}
$$

Let $\mathcal{R}$ be the ring of polynomials in $x$ (the Racah case) or the ring of Laurent polynomials in $q^{x}$ (the $q$-Racah case). Let us introduce automorphisms $\mathcal{I}_{\boldsymbol{\lambda}}$ in $\mathcal{R}$ by

$$
\begin{equation*}
\mathcal{I}_{\boldsymbol{\lambda}}(x)=-x-d \quad: \mathrm{R}, \quad \mathcal{I}_{\boldsymbol{\lambda}}\left(q^{x}\right)=q^{-x} d^{-1} \quad: q \mathrm{R} \tag{2.15}
\end{equation*}
$$

which are involutions $\mathcal{I}_{\lambda}^{2}=\mathrm{id}$. We have the following lemma [13].
Lemma 1 If a (Laurent) polynomial $\check{f}$ in $x\left(q^{x}\right)$ is invariant under $\mathcal{I}_{\boldsymbol{\lambda}}$, it is a polynomial in the sinusoidal coordinate $\eta(x ; \boldsymbol{\lambda})$ :

$$
\begin{equation*}
\mathcal{I}_{\lambda}(\check{f}(x))=\check{f}(x) \Leftrightarrow \check{f}(x)=f(\eta(x ; \boldsymbol{\lambda})), \quad f(\eta) \text { : a polynomial in } \eta \text {. } \tag{2.16}
\end{equation*}
$$

Note that the involutions $\mathcal{I}_{\boldsymbol{\lambda}}$ depend on parameters $\boldsymbol{\lambda}$ only through $d$.

### 2.2 Multi-indexed ( $q-$ )Racah polynomials

Let us introduce the twist operation $\mathfrak{t}$ and the twisted shift $\tilde{\boldsymbol{\delta}}$,

$$
\begin{equation*}
\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(\lambda_{4}-\lambda_{1}+1, \lambda_{4}-\lambda_{2}+1, \lambda_{3}, \lambda_{4}\right), \quad \tilde{\boldsymbol{\delta}} \stackrel{\text { def }}{=}(0,0,1,1) . \tag{2.17}
\end{equation*}
$$

Note that $\eta(x ; \mathfrak{t}(\boldsymbol{\lambda}))=\eta(x ; \boldsymbol{\lambda})$ and $\eta(x ; \boldsymbol{\lambda}+\beta \tilde{\boldsymbol{\delta}})=\eta(x ; \boldsymbol{\lambda}+\beta \boldsymbol{\delta})(\beta \in \mathbb{R})$. The virtual state polynomial $\xi_{\mathrm{v}}(\eta)$ is defined as

$$
\begin{equation*}
\check{\xi}_{\mathrm{v}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \xi_{\mathrm{v}}(\eta(x ; \boldsymbol{\lambda}) ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{P}_{\mathrm{v}}(x ; \mathfrak{t}(\boldsymbol{\lambda}))=P_{\mathrm{v}}(\eta(x ; \boldsymbol{\lambda}) ; \mathfrak{t}(\boldsymbol{\lambda})) . \tag{2.18}
\end{equation*}
$$

Let $\mathcal{D}=\left\{d_{1}, d_{2}, \ldots, d_{M}\right\}\left(d_{1}<d_{2}<\cdots<d_{M}, d_{j} \in \mathbb{Z}_{\geq 1}\right)$ be the multi-index set, which specifies the virtual state vectors used in the $M$-step Darboux transformations. (Although this notation $d_{j}$ conflicts with the notation of the normalization constant $d_{n}(\boldsymbol{\lambda})$ in (2.9), we
think this does not cause any confusion because the latter appears as $\frac{1}{d_{n}(\boldsymbol{\lambda})^{2}} \delta_{n m}$.) We restrict the parameter range for $M$ virtual states deletion,

$$
\begin{equation*}
\mathrm{R}: d+\max (\mathcal{D})+1<a+b, \quad q \mathrm{R}: a b<d q^{\max (\mathcal{D})+1} \tag{2.19}
\end{equation*}
$$

Although these parameter ranges are important for the well-definedness of the quantum systems, they are irrelevant to the recurrence relations considered in this paper, which are polynomial equations and valid independent of the parameter ranges (except for the zeros of the denominators). So we do not bother about the range of parameters (except for orthogonality relations, positivity of some quantities and some part of $\S(5)$.

The denominator polynomials $\Xi_{\mathcal{D}}(\eta)$ and the multi-indexed $(q-)$ Racah polynomials $P_{\mathcal{D}, n}(\eta)$ $\left(n=0,1, \ldots, n_{\max }\right)$ are defined as

$$
\begin{align*}
\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) & \stackrel{\text { def }}{=} \Xi_{\mathcal{D}}(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \\
& \stackrel{\text { def }}{=} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})^{-1} \varphi_{M}(x ; \boldsymbol{\lambda})^{-1} \operatorname{det}\left(\check{\xi}_{d_{k}}\left(x_{j} ; \boldsymbol{\lambda}\right)\right)_{1 \leq j, k \leq M},  \tag{2.20}\\
\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) & \stackrel{\text { def }}{=} P_{\mathcal{D}, n}(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \\
& \stackrel{\text { def }}{=} \mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{-1} \varphi_{M+1}(x ; \boldsymbol{\lambda})^{-1} \\
& \times\left|\begin{array}{cccc}
\check{\xi}_{d_{1}}\left(x_{1} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{1} ; \boldsymbol{\lambda}\right) & r_{1}\left(x_{1}\right) \check{P}_{n}\left(x_{1} ; \boldsymbol{\lambda}\right) \\
\check{\xi}_{d_{1}}\left(x_{2} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{2} ; \boldsymbol{\lambda}\right) & r_{2}\left(x_{2}\right) \check{P}_{n}\left(x_{2} ; \boldsymbol{\lambda}\right) \\
\vdots & \cdots & \vdots & \vdots \\
\check{\xi}_{d_{1}}\left(x_{M+1} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{M+1} ; \boldsymbol{\lambda}\right) & r_{M+1}\left(x_{M+1}\right) \check{P}_{n}\left(x_{M+1} ; \boldsymbol{\lambda}\right)
\end{array}\right| \tag{2.21}
\end{align*}
$$

where $x_{j} \stackrel{\text { def }}{=} x+j-1$ and $r_{j}\left(x_{j}\right)=r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)(1 \leq j \leq M+1)$ are given in (A.10). The constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})\left(\underline{\text { A.11) }}\right.$ ) and $\mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda})$ (A.12) correspond to the normalization

$$
\begin{equation*}
\check{\Xi}_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=\Xi_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=1, \quad \check{P}_{\mathcal{D}, n}(0 ; \boldsymbol{\lambda})=P_{\mathcal{D}, n}(0 ; \boldsymbol{\lambda})=1 . \tag{2.22}
\end{equation*}
$$

The denominator polynomial $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and the multi-indexed orthogonal polynomial $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ are polynomials in $\eta$ and their degrees are $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}}+n$, respectively (we assume $c_{\mathcal{D}}^{\Xi}$ (A.16) and $c_{\mathcal{D}, n}^{P}\left(\widehat{\text { A.17) }}\right.$ do not vanish). Here $\ell_{\mathcal{D}}$ is

$$
\begin{equation*}
\ell_{\mathcal{D}} \stackrel{\text { def }}{=} \sum_{j=1}^{M} d_{j}-\frac{1}{2} M(M-1) \tag{2.23}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})=\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.24}
\end{equation*}
$$

as a consequence of the shape invariance of the system [9, 11, 13]. Other determinant expressions of $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ can be found in [17].

The isospectral deformation is realized by multi-step Darboux transformations with virtual state vectors as seed solutions. The multi-indexed polynomials $P_{\mathcal{D}, n}(\eta)$ appear in the eigenvectors $\phi_{\mathcal{D} n}(x)$ of the deformed Hamiltonian, which is also a tridiagonal matrix $\mathcal{H}_{\mathcal{D}}=\left(\mathcal{H}_{\mathcal{D} ; x, y}\right)_{0 \leq x, y \leq x_{\text {max }}}$,

$$
\begin{align*}
& \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }},  \tag{2.25}\\
& \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sqrt{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \frac{\phi_{0}(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}},  \tag{2.26}\\
& \mathcal{H}_{\mathcal{D}}=-\sqrt{B_{\mathcal{D}}(x)} e^{\partial} \sqrt{D_{\mathcal{D}}(x)}-\sqrt{D_{\mathcal{D}}(x)} e^{-\partial} \sqrt{B_{\mathcal{D}}(x)}+B_{\mathcal{D}}(x)+D_{\mathcal{D}}(x),  \tag{2.27}\\
& \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right), \tag{2.28}
\end{align*}
$$

where potential functions $B_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $D_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ are given in (A.18). The orthogonality relations are

$$
\begin{equation*}
\sum_{x=0}^{x_{\max }} \frac{\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{2}}{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, m}(x ; \boldsymbol{\lambda})=\frac{\delta_{n m}}{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}} \quad\left(n, m=0,1, \ldots, n_{\max }\right) \tag{2.29}
\end{equation*}
$$

where the normalization constant $d_{\mathcal{D}, n}(\boldsymbol{\lambda})$ is given in (A.14). The normalization of $\psi_{\mathcal{D}}(x)$ and $\phi_{\mathcal{D} n}(x)$ is $\psi_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=\phi_{\mathcal{D} n}(0 ; \boldsymbol{\lambda})=1$. By similarity transformation, (2.28) is rewritten as

$$
\begin{align*}
& \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \\
&= B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{\partial}\right) \\
&+D\left(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\Xi}}^{\boldsymbol{\delta}}\right) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{-\partial}\right),  \tag{2.30}\\
& \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right), \tag{2.31}
\end{align*}
$$

namely multi-indexed $(q-)$ Racah polynomials $\check{P}_{\mathcal{D}, n}(x)$ satisfy second order difference equations.

In the following we set

$$
\begin{equation*}
P_{n}(\eta ; \boldsymbol{\lambda})=P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})=0 \quad(n<0) \tag{2.32}
\end{equation*}
$$

and $A_{-1}(\boldsymbol{\lambda})=0$. We remark that the coefficients of $P_{n}(\eta)$ and $P_{\mathcal{D}, n}(\eta)$ are rational functions of the parameters $(a, b, c, d)$, in which the number $N$ appears only through the parameter $a$
(2.2). If we treat the parameter $a$ as an indeterminate, $\check{P}_{n}(x)$ and $\check{P}_{\mathcal{D}, n}(x)$ are defined for $n \in$ $\mathbb{Z}_{\geq 0}$ and $x \in \mathbb{C}$. For the choice (2.2), however, $\check{P}_{n}(x)$ and $\check{P}_{\mathcal{D}, n}(x)$ are ill-defined for $n>n_{\max }$ and $x \in \mathbb{C} \backslash\left\{0,1, \ldots, x_{\max }\right\}$, because $\check{P}_{n}(x)$ (2.4) have the form ${ }_{4} F_{3}(\cdots)=\sum_{k=0}^{n} \frac{(-x)_{k}}{(a)_{k}}(\cdots)$ and ${ }_{4} \phi_{3}(\cdots)=\sum_{k=0}^{n} \frac{\left(q^{-x} ; q\right)_{k}}{(a ; q)_{k}}(\cdots)$. For the choice (2.2)) (we take the limit from an indeterminate $a$ to $a$ in (2.2) ), $\check{P}_{\mathcal{D}, n}(x)$ are well-defined for $n \in\left\{0,1, \ldots, n_{\max }\right\}$ and $x \in \mathbb{C}$, or $n \in \mathbb{Z}_{>n_{\max }}$ and $x \in\left\{0,1, \ldots, x_{\max }\right\}$, for which the factors $(x+a)_{j-1}$ or $\left(a q^{x} ; q\right)_{j-1}$ in $r_{j}\left(x_{j}\right)$ A.10) contribute. In order for $\check{P}_{n}(x)$ and $\check{P}_{\mathcal{D}, n}(x)$ to be orthogonal polynomials, parameters should satisfy (2.2)-(2.3) and (2.19), and $n$ should be $n \in\left\{0,1, \ldots, n_{\max }\right\}$.

## 3 Recurrence Relations with Variable Dependent Coefficients

In this section we present $3+2 M$ term recurrence relations with variable dependent coefficients. The quantum mechanical formulation is used to derive them. For simplicity of the arguments, we assume that $N$ is sufficiently large $(N \gg n)$, or the parameter $a$ is treated as an indeterminate.

For the discrete quantum mechanics with real shifts, the multi-step Darboux transformations in terms of the virtual state vectors were given in [13]. For the ( $q$-) Racah systems, the general expression for the eigenvector of the deformed system is (eq.(3.36) in [13])

$$
\begin{equation*}
\phi_{\mathcal{D} n}^{\mathrm{gen}}(x ; \boldsymbol{\lambda})=\frac{(-1)^{M} \kappa^{\frac{1}{4} M(M-1)}}{\sqrt{\Xi_{\overline{\mathcal{D}}}(1 ; \boldsymbol{\lambda})}} \frac{\mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda})}{\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})} \sqrt{\prod_{j=1}^{M} \alpha(\boldsymbol{\lambda}) B^{\prime}(0 ; \boldsymbol{\lambda}+(j-1) \tilde{\boldsymbol{\delta}})} \times \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda}), \tag{3.1}
\end{equation*}
$$

where $\alpha(\boldsymbol{\lambda})$ and $B^{\prime}(x ; \boldsymbol{\lambda})$ are given in (A.9) and (A.8), respectively. Let us denote $\phi_{n}^{[s]}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=}$ $\phi_{d_{1} \ldots d_{s} n}^{\text {gen }}(x ; \boldsymbol{\lambda})$. Starting from the original eigenvectors $\phi_{n}^{[0]}(x)=\phi_{n}(x)$, the multi-step Darboux transformations give the eigenvectors of the deformed systems,

$$
\begin{equation*}
\phi_{n}^{[s]}(x)=\hat{\mathcal{A}}_{d_{1} \ldots d_{s}} \phi_{n}^{[s-1]}(x) \quad(s \geq 1) \tag{3.2}
\end{equation*}
$$

where the matrix $\hat{\mathcal{A}}_{d_{1} \ldots d_{s}}$ is $\hat{\mathcal{A}}_{d_{1} \ldots d_{s}}=\sqrt{\hat{B}_{d_{1} \ldots d_{s}}(x)}-e^{\partial} \sqrt{\hat{D}_{d_{1} \ldots d_{s}}(x)}$. Potential functions $\hat{B}_{d_{1} \ldots d_{s}}(x)$ and $\hat{D}_{d_{1} \ldots d_{s}}(x)$ are given in (A.20).

First we note that the matrix $\hat{\mathcal{A}}=\sqrt{\hat{B}(x)}-e^{\partial} \sqrt{\hat{D}(x)}$ with $\hat{D}\left(x_{\max }+1\right)=0$ acts on a vector $\psi(x)$ defined by the product of two vectors $\psi(x)=f(x) \phi(x)$ as

$$
\hat{\mathcal{A}}(f(x) \phi(x))=\sqrt{\hat{B}(x)} f(x) \phi(x)-\sqrt{\hat{D}(x+1)} f(x+1) \phi(x+1)
$$

$$
\begin{align*}
& =f(x)(\sqrt{\hat{B}(x)} \phi(x)-\sqrt{\hat{D}(x+1)} \phi(x+1))+(f(x)-f(x+1)) \sqrt{\hat{D}(x+1)} \phi(x+1) \\
& =f(x) \hat{\mathcal{A}} \phi(x)+(f(x)-f(x+1)) \sqrt{\hat{D}(x+1)} \phi(x+1) \tag{3.3}
\end{align*}
$$

Let us define $\check{R}_{n, k}^{[s]}(x)\left(n, k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq-1}\right)$ as follows:

$$
\begin{align*}
& \check{R}_{n, k}^{[s]}(x)=0 \quad(|k|>s+1 \text { or } n+k<0), \quad \check{R}_{n, 0}^{[-1]}(x)=1 \quad(n \geq 0), \\
& \check{R}_{n, k}^{[s]}(x)=A_{n} \check{R}_{n+1, k-1}^{[s-1]}(x)+\left(B_{n}-\eta(x+s)\right) \check{R}_{n, k}^{[s-1]}(x)+C_{n} \check{R}_{n-1, k+1}^{[s-1]}(x) \quad(s \geq 0) . \tag{3.4}
\end{align*}
$$

Here $A_{n}, B_{n}$ and $C_{n}$ are the coefficients of the three term recurrence relations (2.7) with $A_{-1}=0$ and we regard $A_{-1} \times(\cdots)=0$, which implies that $A_{n}(n<-1), B_{n}(n<0)$ and $C_{n}(n<0)$ do not appear. For example, non-trivial $\check{R}_{n, k}^{[s]}(x)(n+k \geq 0)$ for $s=0,1$ are

$$
\begin{array}{ll}
s=0: & \check{R}_{n, 1}^{[0]}(x)=A_{n}, \quad \check{R}_{n, 0}^{[0]}(x)=B_{n}-\eta(x), \quad \check{R}_{n,-1}^{[0]}(x)=C_{n} \\
s=1: & \check{R}_{n, 2}^{[1]}(x)=A_{n} A_{n+1}, \quad \check{R}_{n, 1}^{[1]}(x)=A_{n}\left(B_{n}+B_{n+1}-\eta(x)-\eta(x+1)\right) \\
& \check{R}_{n, 0}^{[1]}(x)=A_{n} C_{n+1}+A_{n-1} C_{n}+\left(B_{n}-\eta(x)\right)\left(B_{n}-\eta(x+1)\right) \\
& \check{R}_{n,-2}^{[1]}(x)=C_{n} C_{n-1}, \quad \check{R}_{n,-1}^{[1]}(x)=C_{n}\left(B_{n}+B_{n-1}-\eta(x)-\eta(x+1)\right) .
\end{array}
$$

Note that $\check{R}_{n, \pm(s+1)}^{[s]}(x)$ are $x$-independent. By induction in $s$, we can show that

$$
\begin{equation*}
\check{R}_{n, k}^{[s]}(x+1)-\check{R}_{n, k}^{[s]}(x)=(\eta(x)-\eta(x+s+1)) \check{R}_{n, k}^{[s-1]}(x+1) \quad(s \geq 0) \tag{3.5}
\end{equation*}
$$

We will show the following proposition, the $3+2 s$ term recurrence relations of $\phi_{n}^{[s]}(x)$.

## Proposition 1

$$
\begin{equation*}
\sum_{k=-s-1}^{s+1} \check{R}_{n, k}^{[s]}(x) \phi_{n+k}^{[s]}(x)=0 \quad(s \geq 0 ; n \in \mathbb{Z}) \tag{3.6}
\end{equation*}
$$

Proof: We prove this proposition by induction in $s$.
first step : For $s=0$, (3.6) is

$$
A_{n} \phi_{n+1}(x)+\left(B_{n}-\eta(x)\right) \phi_{n}(x)+C_{n} \phi_{n-1}(x)=0
$$

which is the three term recurrence relation itself (2.14). Therefore $s=0$ case holds. second step : Assume that (3.6) holds till $s(s \geq 0)$, we will show that it also holds for $s+1$. By applying $\hat{\mathcal{A}}_{d_{1} \ldots d_{s+1}}$ to (3.6) and using (3.2) and (3.3), we have

$$
0=\sum_{k=-s-1}^{s+1} \check{R}_{n, k}^{[s]}(x) \phi_{n+k}^{[s+1]}(x)+\sum_{k=-s-1}^{s+1}\left(\check{R}_{n, k}^{[s]}(x)-\check{R}_{n, k}^{[s]}(x+1)\right) \sqrt{\hat{D}_{d_{1} \ldots d_{s+1}}(x+1)} \phi_{n+k}^{[s]}(x+1) .
$$

By using (3.5) this is rewritten as

$$
\begin{equation*}
\sum_{k=-s-1}^{s+1} \check{R}_{n, k}^{[s]}(x) \phi_{n+k}^{[s+1]}(x)=(\eta(x)-\eta(x+s+1)) G_{n}^{[s+1]}(x), \tag{3.7}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{n}^{[s+1]}(x) \stackrel{\text { def }}{=} \sqrt{\hat{D}_{d_{1} \ldots d_{s+1}}(x+1)} \sum_{k=-s}^{s} \check{R}_{n, k}^{[s-1]}(x+1) \phi_{n+k}^{[s]}(x+1) . \tag{3.8}
\end{equation*}
$$

We remark that when there is a factor $\check{R}_{n, k}^{[s]}$ in a sum $\sum_{k=-s-1}^{s+1}$, the range of the sum can be extended to $\sum_{k \in \mathbb{Z}}$ due to the definition $\check{R}_{n, k}^{[s]}(x)=0(|k|>s+1)$, which will be abbreviated as $\sum_{k}$. Then we have

$$
\begin{aligned}
& \quad A_{n} G_{n+1}^{[s+1]}(x)+\left(B_{n}-\eta(x+s)\right) G_{n}^{[s+1]}(x)+C_{n} G_{n-1}^{[s+1]}(x) \\
& \\
& \quad+\left(B_{n}-\eta(x+s)\right) \sum_{k} \check{D}_{d_{1} \ldots d_{s+1}}(x+1)\left(A_{n} \sum_{k, k} \check{R}_{n+1, k}^{[s-1]}(x+1) \phi_{n+1+k}^{[s]}(x+1)\right. \\
& \\
& \stackrel{(\mathrm{ii})}{=} \sqrt{\hat{D}_{d_{1} \ldots d_{s+1}}(x+1)} \sum_{n+k}\left(A_{n} \check{R}_{n+1, k-1}^{[s-1]}(x+1)+C_{n} \sum_{k} \check{R}_{n-1, k}^{[s-1]}(x+1) \phi_{n-1+k}^{[s]}(x+1)\right) \\
& \\
& \left.\quad+\left(B_{n}-\eta(x+s)\right) \check{R}_{n, k}^{[s-1]}(x+1)+C_{n} \check{R}_{n-1, k+1}^{[s-1]}(x+1)\right) \phi_{n+k}^{[s]}(x+1) \\
& \stackrel{\text { (iii) }}{=} \\
& \hat{D}_{d_{1} \ldots d_{s+1}}(x+1) \\
& \sum_{k}\left(\check{R}_{n, k}^{[s]}(x+1)+(\eta(x+s+1)-\eta(x+s)) \check{R}_{n, k}^{[s-1]}(x+1)\right) \phi_{n+k}^{[s]}(x+1) \\
& \stackrel{\text { (iv) }}{=} \\
& (\eta(x+s+1)-\eta(x+s)) \sqrt{\hat{D}_{d_{1} \ldots d_{s+1}}(x+1)} \sum_{k} \check{R}_{n, k}^{[s-1]}(x+1) \phi_{n+k}^{[s]}(x+1) \\
& \stackrel{(\mathrm{v})}{=} \\
& (\eta(x+s+1)-\eta(x+s)) G_{n}^{[s+1]}(x),
\end{aligned}
$$

((i): (3.8), (ii): shift of $k$, (iii): (3.4), (iv): assumption (3.6), (v): (3.8) are used), namely,

$$
\begin{equation*}
A_{n} G_{n+1}^{[s+1]}(x)+\left(B_{n}-\eta(x+s+1)\right) G_{n}^{[s+1]}(x)+C_{n} G_{n-1}^{[s+1]}(x)=0 \tag{3.9}
\end{equation*}
$$

From (3.7) and (3.9) we obtain

$$
\begin{aligned}
0= & A_{n} \sum_{k} \check{R}_{n+1, k}^{[s]}(x) \phi_{n+1+k}^{[s+1]}(x)+\left(B_{n}-\eta(x+s+1)\right) \sum_{k} \check{R}_{n, k}^{[s]}(x) \phi_{n+k}^{[s+1]}(x) \\
& +C_{n} \sum_{k} \check{R}_{n-1, k}^{[s]}(x) \phi_{n-1+k}^{[s+1]}(x) \\
= & \sum_{k}\left(A_{n} \check{R}_{n+1, k-1}^{[s]}(x)+\left(B_{n}-\eta(x+s+1)\right) \check{R}_{n, k}^{[s]}(x)+C_{n} \check{R}_{n-1, k+1}^{[s]}(x)\right) \phi_{n+k}^{[s+1]}(x)
\end{aligned}
$$

$$
=\sum_{k} \check{R}_{n, k}^{[s+1]}(x) \phi_{n+k}^{[s+1]}(x)=\sum_{k=-s-2}^{s+2} \check{R}_{n, k}^{[s+1]}(x) \phi_{n+k}^{[s+1]}(x),
$$

which shows (3.6) with $s \rightarrow s+1$. This concludes the induction proof of (3.6).
Note that $\check{R}_{n, k}^{[s]}(x)$ does not depend on the specific values of $d_{j}$ 's.
Since $\phi_{n}^{[s]}(x)$ has the form (3.1), the recurrence relations of $\phi_{n}^{[s]}(x)$ (3.6) imply those of the multi-indexed orthogonal polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ for $s=M$,

$$
\begin{equation*}
\sum_{k=-M-1}^{M+1} \mathcal{C}_{\mathcal{D}, n+k}(\boldsymbol{\lambda}) \check{R}_{n, k}^{[M]}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda})=0 \tag{3.10}
\end{equation*}
$$

where $\mathcal{C}_{\mathcal{D}, n+k}(\boldsymbol{\lambda})$ depends on the specific values of $d_{j}$ 's. More explicitly, by dividing them by $\mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda})$, the recurrence relations become

$$
\begin{gather*}
\sum_{k=-M-1}^{M+1} \prod_{j=1}^{M} \frac{\mathcal{E}_{n+k}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})} \cdot \check{R}_{n, k}^{[M]}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda})=0,  \tag{3.11}\\
\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})= \begin{cases}(n+\mathrm{v}+c)(n-\mathrm{v}+\tilde{d}-c) & : \mathrm{R} \\
q^{-n}\left(1-c q^{n+\mathrm{v}}\right)\left(1-c^{-1} \tilde{d} q^{n-\mathrm{v}}\right) & : q \mathrm{R}\end{cases} \tag{3.12}
\end{gather*}
$$

We can check that $\check{R}_{n, k}^{[s]}(x ; \boldsymbol{\lambda})$ are symmetric polynomials in $\eta(x ; \boldsymbol{\lambda}), \eta(x+1 ; \boldsymbol{\lambda}), \ldots, \eta(x+$ $s ; \boldsymbol{\lambda})$. The elementary symmetric polynomials $e_{k}$ in $\eta(x ; \boldsymbol{\lambda}), \eta(x+1 ; \boldsymbol{\lambda}), \ldots, \eta(x+s ; \boldsymbol{\lambda})$ are polynomials in $\eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})$, because

$$
\begin{aligned}
& \sum_{k=0}^{s+1}(-1)^{k} e_{k} t^{s+1-k}=\prod_{j=0}^{s}(t-\eta(x+j ; \boldsymbol{\lambda})), \quad A=\left\{\begin{array}{ll}
t-\eta\left(x+\frac{s}{2} ; \boldsymbol{\lambda}\right) & : s \text { even } \\
1 & : s \text { odd }
\end{array},\right. \\
= & A \times \prod_{j=0}^{\left[\frac{s-1}{2}\right]}(t-\eta(x+j ; \boldsymbol{\lambda}))(t-\eta(x+s-j ; \boldsymbol{\lambda})) \\
= & A \times \prod_{j=0}^{\left[\frac{s-1}{2}\right]}\left(t^{2}-(\eta(x+j ; \boldsymbol{\lambda})+\eta(x+s-j ; \boldsymbol{\lambda})) t+\eta(x+j ; \boldsymbol{\lambda}) \eta(x+s-j ; \boldsymbol{\lambda})\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \eta\left(x+\frac{s}{2} ; \boldsymbol{\lambda}\right)=B \eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})+\eta\left(\frac{s}{2} ; \boldsymbol{\lambda}\right), \\
& \eta(x+j ; \boldsymbol{\lambda})+\eta(x+s-j ; \boldsymbol{\lambda})=B^{\prime} \eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})+\eta(j ; \boldsymbol{\lambda})+\eta(s-j ; \boldsymbol{\lambda}), \\
& \eta(x+j ; \boldsymbol{\lambda}) \eta(x+s-j ; \boldsymbol{\lambda})=B^{\prime \prime} \eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})^{2}+B^{\prime \prime \prime} \eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})+\eta(j ; \boldsymbol{\lambda}) \eta(s-j ; \boldsymbol{\lambda}), \\
& B=\left\{\begin{array}{ll}
1 & : \mathrm{R} \\
q^{-\frac{s}{2}} & : q \mathrm{R}
\end{array}, \quad B^{\prime}=\left\{\begin{array}{ll}
2 & : \mathrm{R} \\
q^{-j}+q^{j-s} & : q \mathrm{R}
\end{array}, \quad B^{\prime \prime}=\left\{\begin{array}{ll}
1 & \mathrm{R} \\
q^{-s} & : q \mathrm{R}
\end{array},\right.\right.\right.
\end{aligned}
$$

$$
B^{\prime \prime \prime}=\left\{\begin{array}{ll}
2 j(s-j)+s d & : \mathrm{R} \\
q^{-s}\left(2\left(1+d q^{s}\right)-\left(q^{j}+q^{s-j}\right)(1+d)\right) & : q \mathrm{R}
\end{array} .\right.
$$

This implies that any symmetric polynomial in $\eta(x ; \boldsymbol{\lambda}), \eta(x+1 ; \boldsymbol{\lambda}), \ldots, \eta(x+s ; \boldsymbol{\lambda})$ is expressed as a polynomial in $\eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta})$. Therefore we obtain

$$
\begin{align*}
& \check{R}_{n, k}^{[s]}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} R_{n, k}^{[s]}(\eta(x ; \boldsymbol{\lambda}+s \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \quad(|k| \leq s+1) \\
& R_{n, k}^{[s]}(\eta ; \boldsymbol{\lambda}): \text { a polynomial of degree } s+1-|k| \text { in } \eta . \tag{3.13}
\end{align*}
$$

By rewriting the recurrence relations (3.10) and (3.11), we obtain the following theorem.

Theorem 1 The multi-indexed (q-)Racah polynomials satisfy the $3+2 M$ term recurrence relations with variable dependent coefficients:

$$
\begin{gather*}
\sum_{k=-M-1}^{M+1} \mathcal{C}_{\mathcal{D}, n+k}(\boldsymbol{\lambda}) R_{n, k}^{[M]}(\eta ; \boldsymbol{\lambda}) P_{\mathcal{D}, n+k}(\eta ; \boldsymbol{\lambda})=0,  \tag{3.14}\\
\text { or } \quad \sum_{k=-M-1}^{M+1} \prod_{j=1}^{M} \frac{\mathcal{E}_{n+k}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})} \cdot R_{n, k}^{[M]}(\eta ; \boldsymbol{\lambda}) P_{\mathcal{D}, n+k}(\eta ; \boldsymbol{\lambda})=0 . \tag{3.15}
\end{gather*}
$$

Remark We have assumed that $N$ is sufficiently large $(N \gg n)$, or the parameter $a$ is treated as an indeterminate.

According to the same argument for the multi-indexed Laguerre, Jacobi, Wilson and Askey-Wilson polynomials [19], the multi-indexed $(q-)$ Racah polynomials $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})(n \geq$ $M+1$ ) are determined by the $3+2 M$ term recurrence relations (3.14) with $M+1$ "initial data"

$$
\begin{equation*}
P_{\mathcal{D}, 0}(\eta ; \boldsymbol{\lambda}), P_{\mathcal{D}, 1}(\eta ; \boldsymbol{\lambda}), \ldots, P_{\mathcal{D}, M}(\eta ; \boldsymbol{\lambda}) . \tag{3.16}
\end{equation*}
$$

After calculating the initial data (3.16) by (2.21), we can obtain $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ through the $3+2 M$ term recurrence relations (3.14). The calculation cost of this method is much less than the original determinant expression (2.21) for large $M$.

## 4 Recurrence Relations with Constant Coefficients

In this section we present $1+2 L$ term recurrence relations with constant coefficients. Depending on whether the parameter $a$ is an indeterminate or (2.2), we have two different kinds of recurrence relations.

We want to find $X(\eta)=X(\eta ; \boldsymbol{\lambda})$, which is a polynomial of degree $L$ in $\eta$ and gives the following expansion:

$$
X(\eta) P_{\mathcal{D}, n}(\eta)=\sum_{k=-n}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \quad \text { or } \quad \check{X}(x) \check{P}_{\mathcal{D}, n}(x)=\sum_{k=-n}^{L} r_{n, k}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+k}(x)
$$

where $r_{n, k}^{X, \mathcal{D}}$ 's are constants and $\check{X}(x)=\check{X}(x ; \boldsymbol{\lambda})$ is defined by

$$
\begin{equation*}
\check{X}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} X(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \tag{4.1}
\end{equation*}
$$

Since the multi-indexed polynomials $P_{\mathcal{D}, n}(\eta)$ are orthogonal polynomials, the above recurrence relations with constant coefficients are expressed as (see Lemma 1 in [22])

$$
\begin{equation*}
X(\eta) P_{\mathcal{D}, n}(\eta)=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \tag{4.2}
\end{equation*}
$$

under the convention (2.32). Here we have assumed $N$ is sufficiently large. Unlike the multiindexed polynomials in [22], there is a maximal value of $n$ for $P_{\mathcal{D}, n}(\eta), n_{\max }=N$. So (4.2) is expected to be modified as

$$
\begin{equation*}
\check{X}(x) \check{P}_{\mathcal{D}, n}(x)=\sum_{k=-\min (L, n)}^{\min (L, N-n)} r_{n, k}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+k}(x) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} \tag{4.3}
\end{equation*}
$$

where $\check{P}_{\mathcal{D}, n}(x)$ with $n<0$ or $n>n_{\text {max }}$ does not appear. The overall normalization and the constant term of $X(\eta)$ are not important, because the change of the former induces that of the overall normalization of $r_{n, k}^{X, \mathcal{D}}$ and the shift of the latter induces that of $r_{n, 0}^{X, \mathcal{D}}$. Without loss of generality, we take the constant term of $X(\eta)$ as $X(0)=0$.

### 4.1 Parameter $a$ : indeterminate

In this subsection we assume that the parameter $a$ is an indeterminate, $x$ and $\eta$ are continuous variables $(x, \eta \in \mathbb{C})$ and other parameters $\boldsymbol{\lambda}(b, c, d)$ are generic.

### 4.1.1 step 0

The sinusoidal coordinates $\eta(x ; \boldsymbol{\lambda})(2.5)$ have the following property [37, 22]

$$
\begin{equation*}
\frac{\eta(x ; \boldsymbol{\lambda})^{n+1}-\eta(x-1 ; \boldsymbol{\lambda})^{n+1}}{\eta(x ; \boldsymbol{\lambda})-\eta(x-1 ; \boldsymbol{\lambda})}=\sum_{k=0}^{n} g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \eta(x ; \boldsymbol{\lambda}-\boldsymbol{\delta})^{n-k} \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{4.4}
\end{equation*}
$$

where $g_{n}^{\prime(k)}$ is given by

$$
\begin{align*}
& \mathrm{R}: g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sum_{r=0}^{k} \sum_{l=0}^{k-r}\binom{n+1}{r}\binom{n-r-l}{n-k}(-1)^{r+l}\left(\frac{d}{2}\right)^{2 r}\left(\frac{d-1}{2}\right)^{2(k-r-l)} g_{n-r}^{\prime(l) \mathrm{W}}  \tag{4.5}\\
& q \mathrm{R}: g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sum_{r=0}^{k} \sum_{l=0}^{k-r}\binom{n+1}{r}\binom{n-r-l}{n-k}(-1)^{r}\left(2 d^{\frac{1}{2}}\right)^{l} q^{\frac{1}{2}(n-r-l)}(1+d)^{r}\left(1+d q^{-1}\right)^{k-r-l} \\
& \times g_{n-r}^{\prime(l) \text { AW }} \tag{4.6}
\end{align*}
$$

Here $g_{n}^{\prime(k) \mathrm{W}}$ and $g_{n}^{\prime(k) \mathrm{AW}}$ are given by [38, 22]

$$
\begin{aligned}
& g_{n}^{\prime(k) \mathrm{W}} \stackrel{\text { def }}{=} \frac{(-1)^{k}}{2^{2 k+1}}\binom{2 n+2}{2 k+1}, \\
& g_{n}^{\prime(k) \mathrm{AW}} \stackrel{\text { def }}{=} \theta(k: \text { even }) \frac{(n+1)!}{2^{k}} \sum_{r=0}^{\frac{k}{2}}\binom{n-k+r}{r} \frac{(-1)^{r} q^{-\frac{1}{2}(n-k+2 r)}}{\left(\frac{k}{2}-r\right)!\left(n-\frac{k}{2}+1+r\right)!} \frac{1-q^{n-k+1+2 r}}{1-q},
\end{aligned}
$$

and $\theta(P)$ is a step function (an indicator function) for a proposition $P, \theta(P)=1$ ( $P$ : true), 0 ( $P$ : false).

For a polynomial $p(\eta)$ in $\eta$, let us define a polynomial in $\eta, I_{\lambda}[p](\eta)$, as follows:

$$
\begin{equation*}
p(\eta)=\sum_{k=0}^{n} a_{k} \eta^{k} \mapsto I_{\boldsymbol{\lambda}}[p](\eta) \stackrel{\text { def }}{=} \sum_{k=0}^{n+1} b_{k} \eta^{k}, \tag{4.7}
\end{equation*}
$$

where $b_{k}$ 's are defined by

$$
\begin{equation*}
b_{k+1}=\frac{1}{g_{k}^{\prime(0)}(\boldsymbol{\lambda})}\left(a_{k}-\sum_{j=k+1}^{n} g_{j}^{\prime(j-k)}(\boldsymbol{\lambda}) b_{j+1}\right) \quad(k=n, n-1, \ldots, 1,0), \quad b_{0}=0 . \tag{4.8}
\end{equation*}
$$

The constant term of $I_{\boldsymbol{\lambda}}[p](\eta)$ is chosen to be zero. Note that $a_{k}=\sum_{j=k}^{n} g_{j}^{\prime(j-k)}(\boldsymbol{\lambda}) b_{j+1}$. It is easy to show that this polynomial $I_{\boldsymbol{\lambda}}[p](\eta)=P(\eta)$ satisfies

$$
\begin{equation*}
\frac{P(\eta(x ; \boldsymbol{\lambda}))-P(\eta(x-1 ; \boldsymbol{\lambda}))}{\eta(x ; \boldsymbol{\lambda})-\eta(x-1 ; \boldsymbol{\lambda})}=p(\eta(x ; \boldsymbol{\lambda}-\boldsymbol{\delta})) \tag{4.9}
\end{equation*}
$$

So we can call $P(\eta)$ the 'primitive polynomial' of $p(\eta)$. The above equations are valid for $x, \eta \in \mathbb{C}$, but (4.9) with $P(0)=0$ gives the following expression:

$$
\begin{equation*}
P(\eta(x ; \boldsymbol{\lambda}))=\sum_{j=1}^{x}(\eta(j ; \boldsymbol{\lambda})-\eta(j-1 ; \boldsymbol{\lambda})) p(\eta(j ; \boldsymbol{\lambda}-\boldsymbol{\delta})) \quad\left(x \in \mathbb{Z}_{\geq 0}\right) \tag{4.10}
\end{equation*}
$$

It is nontrivial to show directly that the RHS is a polynomial in $\eta(x ; \boldsymbol{\lambda})$, but it is so by construction. Note that the maps $I_{\boldsymbol{\lambda}}$ depend on parameters $\boldsymbol{\lambda}$ only through $d$ because coefficients $g_{n}^{\prime(k)}(\boldsymbol{\lambda})$ depend on $d$ only.

### 4.1.2 step 1

Let us define the set of finite linear combinations of $P_{\mathcal{D}, n}(\eta), \mathcal{U}_{\mathcal{D}} \subset \mathbb{C}[\eta]$, by

$$
\begin{equation*}
\mathcal{U}_{\mathcal{D}} \stackrel{\text { def }}{=} \operatorname{Span}\left\{P_{\mathcal{D}, n}(\eta) \mid n \in \mathbb{Z}_{\geq 0}\right\} \tag{4.11}
\end{equation*}
$$

Since the degree of $P_{\mathcal{D}, n}(\eta)$ is $\ell_{\mathcal{D}}+n$, it is trivial that $p(\eta) \in \mathcal{U}_{\mathcal{D}} \Rightarrow \operatorname{deg} p \geq \ell_{\mathcal{D}}$, except for $p(\eta)=0$. Corresponding to (2.31), the multi-indexed ( $q-$ )Racah polynomials $\check{P}_{\mathcal{D}, n}(x)$ with $x \in \mathbb{C}$ satisfy the second order difference equations,

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{4.12}
\end{equation*}
$$

where $\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})$ is obtained from $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})$ (2.30) by replacing the matrices $e^{ \pm \partial}$ with the shift operators $e^{ \pm \frac{d}{d x}}$,

$$
\begin{align*}
\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})= & B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{\frac{d}{d x}}\right) \\
& +D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{-\frac{d}{d x}}\right) . \tag{4.13}
\end{align*}
$$

For $p(\eta) \in \mathbb{C}[\eta], \widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})$ acts on $\check{p}(x) \stackrel{\text { def }}{=} p(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))$ as

$$
\begin{align*}
\tilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{p}(x)= & B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} \check{p}(x)-\check{p}(x+1)\right) \\
& +D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} \check{p}(x)-\check{p}(x-1)\right) . \tag{4.14}
\end{align*}
$$

Let zeros of $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}+\boldsymbol{\delta})$ be $\beta_{j}^{(\eta)}$ and $\beta_{j}^{\prime(\eta)}\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right)$, respectively, which are simple for generic parameters (This property can be verified by numerical calculation but we do not have its analytical proof. We assume this property in the following.). We define $\beta_{j}$ and $\beta_{j}^{\prime}$ as $\beta_{j}^{(\eta)}=\eta\left(\beta_{j} ; \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)$ and $\beta_{j}^{\prime(\eta)}=\eta\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)$. (For $x \in \mathbb{C}, \eta=\eta(x ; \boldsymbol{\lambda})$ are not one-to-one functions $\left(\eta(x ; \boldsymbol{\lambda})=\eta(-x-d ; \boldsymbol{\lambda})\right.$ for $\mathrm{R}, \eta(x ; \boldsymbol{\lambda})=\eta\left(-x-\lambda_{4} ; \boldsymbol{\lambda}\right)=\eta\left(x+\frac{2 \pi i}{\log q} ; \boldsymbol{\lambda}\right)$ for $q \mathrm{R}$ ), but it does not cause any problems in the following argument, because we need (4.16) and it is determined by the values of $\eta\left(\beta_{j}: \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)$ and $\eta\left(\beta_{j}-1: \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)$ instead of multi-valued $\beta_{j}$.)

Let us consider the condition such that $\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{p}(x)$ (4.14) is a polynomial in $\eta(x ; \boldsymbol{\lambda}+$ $M \boldsymbol{\delta})$. The poles at $x=\beta_{j}^{\prime}, \beta_{j}, \beta_{j}-1$ in (4.14) should be canceled. First we consider $x=\beta_{j}^{\prime}$. Since $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})(n>0)$ do not have common roots for generic
parameters (This property can be verified by numerical calculation but we do not have its analytical proof. We assume this property in the following.), (4.14) with $\check{p}(x)=\check{P}_{\mathcal{D}, n}(x)$ implies that the poles at $x=\beta_{j}^{\prime}$ are canceled, namely,

$$
\begin{align*}
& B\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+1 ; \boldsymbol{\lambda}\right)} \check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right) \\
+ & D\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+1 ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}\right)} \check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)=0 \quad\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right) \tag{4.15}
\end{align*}
$$

This relation implies that we do not need bother the poles at $x=\beta_{j}^{\prime}$ in (4.14) for general $p(\eta)$. Next we consider $x=\beta_{j}, \beta_{j}-1$. For generic parameters, $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})$ do not have common roots, and the numerators of $B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})$ and $D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})$ do not cancel the poles coming from $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})$, and zeros of the denominators of $B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})$ and $D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})$ do not coincide with $\beta_{j}$ and $\beta_{j}-1$. The residue of the first term of (4.14) at $x=\beta_{j}-1$ is

$$
B\left(\beta_{j}-1 ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-1 ; \boldsymbol{\lambda}\right)}{\left.\frac{d}{d x} \check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})\right|_{x=\beta_{j}-1}}\left(\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}-1\right)-\check{p}\left(\beta_{j}\right)\right),
$$

and that of the second term of (4.14) at $x=\beta_{j}$ is

$$
D\left(\beta_{j} ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}+1 ; \boldsymbol{\lambda}\right)}{\left.\frac{d}{d x} \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})\right|_{x=\beta_{j}}}\left(\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}\right)-\check{p}\left(\beta_{j}-1\right)\right) .
$$

These residues should be vanished. So we obtain the conditions:

$$
\begin{equation*}
\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}-1\right)=\check{p}\left(\beta_{j}\right) \quad\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right) \tag{4.16}
\end{equation*}
$$

Let us assume $\operatorname{deg} p(\eta)<\ell_{\mathcal{D}}$. Without loss of generality, we take $p(\eta)$ as a monic polynomial. Then the number of adjustable coefficients of $p(\eta)$ is $\operatorname{deg} p(\eta)$. On the other hand, the number of conditions (4.16) is $\ell_{\mathcal{D}}$. Therefore the conditions (4.16) can not be satisfied for generic parameters, except for $p(\eta)=0$.

Since any polynomial $p(\eta)$ is expanded as

$$
p(\eta)=\sum_{n=0}^{\operatorname{deg} p-\ell_{\mathcal{D}}} a_{n} P_{\mathcal{D}, n}(\eta)+r(\eta), \quad \operatorname{deg} r(\eta)<\ell_{\mathcal{D}} \quad\left(p(\eta)=r(\eta) \text { for } \operatorname{deg} p<\ell_{\mathcal{D}}\right)
$$

we have

$$
\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{p}(x): \text { a polynomial in } \eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})
$$

$$
\Leftrightarrow \widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{r}(x): \text { a polynomial in } \eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) \Leftrightarrow r(\eta)=0 .
$$

Therefore we obtain the following proposition:
Proposition 2 For $p(\eta) \in \mathbb{C}[\eta]$, the following holds:

$$
\begin{equation*}
p(\eta) \in \mathcal{U}_{\mathcal{D}} \Leftrightarrow \widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{p}(x): \text { a polynomial in } \eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) . \tag{4.17}
\end{equation*}
$$

### 4.1.3 step 2

Let us consider a polynomial $X(\eta)$ satisfying (4.2). From Proposition2, $X(\eta)$ in (4.2) should satisfy

$$
\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right) \text { : a polynomial in } \eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) .
$$

Action of $\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})$ on $\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ is

$$
\begin{aligned}
& \tilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right) \\
&=\check{X}(x) \widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})-B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}(\check{X}(x+1)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x+1 ; \boldsymbol{\lambda}) \\
& \quad-D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}(\check{X}(x-1)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x-1 ; \boldsymbol{\lambda}),
\end{aligned}
$$

namely,

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right)=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})+F(x) . \tag{4.18}
\end{equation*}
$$

Here $F(x)$ is

$$
\begin{align*}
F(x)= & -B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}(\check{X}(x+1)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x+1 ; \boldsymbol{\lambda}) \\
& -D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}(\check{X}(x-1)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x-1 ; \boldsymbol{\lambda}) . \tag{4.19}
\end{align*}
$$

Equations (4.4) and (4.1) imply

$$
\begin{aligned}
\check{X}(x)-\check{X}(x-1)= & (\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
& \times(\text { a polynomial in } \eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})) .
\end{aligned}
$$

In order to cancel the zeros of $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})=\Xi_{\mathcal{D}}(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}) ; \boldsymbol{\lambda})$ in (4.19), the polynomial appeared in the above expression should have the following form,

$$
\begin{equation*}
\check{X}(x)-\check{X}(x-1)=(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) Y(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})), \tag{4.20}
\end{equation*}
$$

where $Y(\eta)$ is an arbitrary polynomial in $\eta$. Note that this $X(\eta)$ can be expressed in terms of the map $I_{\lambda}$ (4.7) by (4.9),

$$
\begin{equation*}
X(\eta)=I_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}\left[\Xi_{\mathcal{D}} Y\right](\eta) \tag{4.21}
\end{equation*}
$$

Then $F(x)$ (4.19) becomes

$$
\begin{align*}
F(x)=- & B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
& \times \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) Y(\eta(x+1 ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})) \check{P}_{\mathcal{D}, n}(x+1 ; \boldsymbol{\lambda}) \\
- & D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))  \tag{4.22}\\
& \times \check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}) Y(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})) \check{P}_{\mathcal{D}, n}(x-1 ; \boldsymbol{\lambda}) .
\end{align*}
$$

From the explicit forms of $B(x ; \boldsymbol{\lambda})$ and $D(x ; \boldsymbol{\lambda})$ A.1), we have

$$
\begin{align*}
& B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
= & \begin{cases}-\frac{(x+a)(x+b)(x+M+c)(x+M+d)}{2 x+M+d} & : \mathrm{R} \\
-(1-q) \frac{\left(1-a q^{x}\right)\left(1-b q^{x}\right)\left(1-c q^{x+M}\right)\left(1-d q^{x+M}\right)}{q^{x+1}\left(1-d q^{2 x+M}\right)} & : q \mathrm{R}\end{cases}  \tag{4.23}\\
= & \begin{cases}\frac{(x+M+d}{}(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
(1-q) \frac{a b c}{d} \frac{\left(1-a^{-1} d q^{x+M}\right)\left(1-b^{-1} d q^{x+M}\right)\left(1-c^{-1} d q^{x}\right)\left(1-q^{x}\right)}{q^{x+1}\left(1-d q^{2 x+M}\right)} & : q \mathrm{R}\end{cases}
\end{align*}
$$

These denominators vanish at $x=-\frac{1}{2}\left(M+\lambda_{4}\right) \stackrel{\text { def }}{=} x_{0}$ and their residues are related as

$$
\begin{aligned}
& \operatorname{Res}_{x=x_{0}}(B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))) \\
= & -\operatorname{Res}_{x=x_{0}}(D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))) .
\end{aligned}
$$

(For $q \mathrm{R}$, (4.23)-(4.24) are rational functions of $z=q^{x}$, and it is better to consider the residue with respect to $z$ at $z= \pm\left(d q^{M}\right)^{-\frac{1}{2}}$.) At $x=x_{0}$, we have
$\eta\left(x_{0} ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}\right)=\eta\left(x_{0}+1 ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}\right), \quad \eta\left(x_{0}+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)=\eta\left(x_{0}-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta}\right)$.
Combining these and (4.22), we obtain

$$
\operatorname{Res}_{x=x_{0}} F(x)=0
$$

Therefore $F(x)$ (4.22) is a (Laurent) polynomial in $x\left(q^{x}\right)$. For the involution $\mathcal{I}_{\boldsymbol{\lambda}}$ (2.15), we have

$$
\begin{aligned}
& \mathcal{I}_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}(B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))) \\
= & D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})(\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})), \\
& \mathcal{I}_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}))=\eta(x+1 ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}), \\
& \mathcal{I}_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}(\eta(x+1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta}))=\eta(x-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) .
\end{aligned}
$$

Hence $F(x)$ (4.22) satisfies $\mathcal{I}_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}(F(x))=F(x)$. By Lemma回, $F(x)$ is a polynomial in $\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})$. Therefore, from (4.18), we have shown that $\widetilde{\mathcal{H}}_{\mathcal{D}}^{\text {cont }}(\boldsymbol{\lambda})\left(\check{X}^{(x)} \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right)$ is a polynomial in $\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta})$.

### 4.1.4 step 3

Let us summarize the result. For the denominator polynomial $\Xi_{\mathcal{D}}(\eta)=\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and a polynomial in $\eta, Y(\eta)(\neq 0)$, we set $X(\eta)=X(\eta ; \boldsymbol{\lambda})=X^{\mathcal{D}, Y}(\eta ; \boldsymbol{\lambda})$ as

$$
\begin{equation*}
X(\eta)=I_{\lambda+M \boldsymbol{\delta}}\left[\Xi_{\mathcal{D}} Y\right](\eta), \quad \operatorname{deg} X(\eta)=L=\ell_{\mathcal{D}}+\operatorname{deg} Y(\eta)+1 \tag{4.25}
\end{equation*}
$$

where $\Xi_{\mathcal{D}} Y$ means a polynomial $\left(\Xi_{\mathcal{D}} Y\right)(\eta)=\Xi_{\mathcal{D}}(\eta) Y(\eta)$. Note that $L \geq M+1$ because of $\ell_{\mathcal{D}} \geq M$. The minimal degree one, which corresponds to $Y(\eta)=1$, is

$$
\begin{equation*}
X_{\min }(\eta)=I_{\lambda+M \delta}\left[\Xi_{\mathcal{D}}\right](\eta), \quad \operatorname{deg} X_{\min }(\eta)=\ell_{\mathcal{D}}+1 \tag{4.26}
\end{equation*}
$$

Then we have the following theorem.
Theorem 2 Let the parameter a be an indeterminate. For any polynomial $Y(\eta)(\neq 0)$, we take $X(\eta)=X^{\mathcal{D}, Y}(\eta)$ as (4.25). Then the multi-indexed ( $q$-)Racah polynomials $P_{\mathcal{D}, n}(\eta)$ satisfy $1+2 L$ term recurrence relations with constant coefficients:

$$
\begin{align*}
X(\eta) P_{\mathcal{D}, n}(\eta) & =\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \quad\left(n \in \mathbb{Z}_{\geq 0} ; \eta \in \mathbb{C}\right),  \tag{4.27}\\
\text { or } \quad \check{X}(x) \check{P}_{\mathcal{D}, n}(x) & =\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+k}(x) \quad\left(n \in \mathbb{Z}_{\geq 0} ; x \in \mathbb{C}\right) . \tag{4.28}
\end{align*}
$$

Remark 1 We have assumed the convention (2.32). If we replace $\sum_{k=-L}^{L}$ with $\sum_{k=-\min (L, n)}^{L}$, it is unnecessary.

Remark 2 As shown near (4.20), any polynomial $X(\eta)$ giving the recurrence relations with constant coefficients must have the form (4.25).

Remark 3 Direct verification of this theorem is rather straightforward for lower $M$ and smaller $d_{j}, n$ and deg $Y$, by a computer algebra system, e.g. Mathematica. The coefficients $r_{n, k}^{X, \mathcal{D}}$ are explicitly obtained for small $d_{j}$ and $n$. However, to obtain the closed expression of $r_{n, k}^{X, \mathcal{D}}$ for general $n$ is not an easy task even for small $d_{j}$, and it is a different kind of problem. We present some examples in $\S 4.3$.
Remark 4 Explicit examples (see $\S(4.3)$ suggest that, for $1 \leq k \leq L$, the coefficients $r_{n, k}^{X, \mathcal{D}}$ have the factor $\left\{\begin{array}{ll}(a+n)_{k} & : \mathrm{R} \\ \left(a q^{n} ; q\right)_{k} & : q \mathrm{R}\end{array}\right.$ and $r_{n,-k}^{X, \mathcal{D}}$ have the factor $\left\{\begin{array}{ll}(n-k+1)_{k} & : \mathrm{R} \\ \left(q^{n-k+1} ; q\right)_{k} & : q \mathrm{R}\end{array}\right.$.
Remark 5 Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. However not all of them are independent. The relations among them are unclear. For ' $M=0$ case' (namely, ordinary orthogonal polynomials), it is trivial that recurrence relations obtained from arbitrary $Y(\eta)(\operatorname{deg} Y \geq 1)$ are derived by the three term recurrence relations.

### 4.2 Parameter $a$ : (2.2)

In this subsection we assume that the parameter $a$ is given by (2.2). We write $a, \check{\Xi}_{\mathcal{D}}(x)$, $\check{P}_{\mathcal{D}, n}(x), \check{X}(x), Y(\eta), r_{n, k}^{X, \mathcal{D}}$ etc. in $\S 4.1$ as those with bar: $\bar{a}, \check{\Xi}_{\mathcal{D}}(x), \check{\bar{P}}_{\mathcal{D}, n}(x), \check{X}(\eta), \bar{Y}(\eta)$, $\bar{r}_{n, k}^{\bar{X}, \mathcal{D}}$ etc. This notation is used in this subsection only. In the limit $\bar{a} \rightarrow a$, the quantities with bar reduce to the quantities without bar, if they exist. As remarked in the end of $\S 2$, the $\bar{a} \rightarrow a$ limit of $\tilde{\bar{P}}_{\mathcal{D}, n}(x)$ exists for $n \in\left\{0,1, \ldots, n_{\max }\right\}$ and $x \in \mathbb{C}$, or $n \in \mathbb{Z}_{>n_{\max }}$ and $x \in\left\{0,1, \ldots, x_{\max }\right\}$, but does not exist for $n \in \mathbb{Z}_{>n_{\max }}$ and $x \in \mathbb{C} \backslash\left\{0,1, \ldots, x_{\max }\right\}$, for which $\check{\bar{P}}_{\mathcal{D}, n}(x)$ behaves as $\sim \frac{1}{\bar{a}-a}$. Recall the relation (2.24). We take $\bar{Y}(\eta)$ such that its $\bar{a} \rightarrow a$ limit exists, $\lim _{\bar{a} \rightarrow a} \bar{Y}(\eta)=Y(\eta)$. Then $\overline{\bar{X}}(x)$ also has a finite limit, $\lim _{\bar{a} \rightarrow a} \bar{X}(x)=\check{X}(x)$. In (4.28) with $x \in\left\{0,1, \ldots, x_{\max }\right\}, \check{\bar{X}}(x), \check{\bar{P}}_{\mathcal{D}, n}(x)$ and $\overline{\bar{P}}_{\mathcal{D}, n+k}(x)$ have finite $\bar{a} \rightarrow a$ limits. So the coefficients $\bar{r}_{n, k}^{\bar{X}, \mathcal{D}}$ also have finite limits, $\lim _{\bar{a} \rightarrow a} \bar{r}_{n, k}^{\bar{X}, \mathcal{D}}=r_{n, k}^{X, \mathcal{D}}$.

For $n \in\left\{0,1, \ldots, n_{\max }\right\}$, the $\bar{a} \rightarrow a$ limit of (4.28) gives

$$
\begin{equation*}
\check{X}(x) \check{P}_{\mathcal{D}, n}(x)=\sum_{k=-\min (L, n)}^{\min (L, N-n)} r_{n, k}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+k}(x)+\lim _{\bar{a} \rightarrow a} \sum_{k=\min (L, N-n)+1}^{L} \bar{r}_{n, k}^{\bar{X}, \mathcal{D}} \check{\bar{P}}_{\mathcal{D}, n+k}(x) \quad(x \in \mathbb{C}), \tag{4.29}
\end{equation*}
$$

where the second sum is zero unless $N-L+1 \leq n \leq N$. So, unless $N-L+1 \leq n \leq N$, we have obtained the recurrence relations for $a(2.2)$. Let us consider the case $N-L+1 \leq n \leq N$.

This relation implies

$$
\lim _{\bar{a} \rightarrow a} \sum_{k=\min (L, N-n)+1}^{L} \bar{r}_{n, k}^{\bar{X}, \mathcal{D}} \tilde{\bar{P}}_{\mathcal{D}, n+k}(x): \text { finite } \quad(x \in \mathbb{C})
$$

Since this is a polynomial, this means

$$
\lim _{\bar{a} \rightarrow a} \bar{r}_{n, k}^{\bar{X}, \mathcal{D}} \check{\bar{P}}_{\mathcal{D}, n+k}(x): \text { finite }(\min (L, N-n)+1 \leq k \leq L ; x \in \mathbb{C})
$$

Note that $\min (L, N-n)+1 \leq k \leq L \Leftrightarrow N-n+1 \leq k \leq L$. By setting $x \in \mathbb{C} \backslash\left\{0,1, \ldots, x_{\max }\right\}$, for which $\check{\bar{P}}_{\mathcal{D}, n+k}(x)$ diverge in the $\bar{a} \rightarrow a$ limit, we obtain

$$
\begin{equation*}
r_{n, k}^{X, \mathcal{D}}=\lim _{\bar{a} \rightarrow a} \bar{r}_{n, k}^{\bar{x}, \mathcal{D}}=0 \quad(N-n+1 \leq k \leq L) . \tag{4.30}
\end{equation*}
$$

Note that Remark 4 below Theorem2 is consistent with this. The relation (4.29) with $x \in$ $\left\{0,1, \ldots, x_{\max }\right\}$, for which $\overline{\bar{P}}_{\mathcal{D}, n+k}(x)$ have finite $\bar{a} \rightarrow a$ limits, gives the following theorem.

Theorem 3 Let the parameter a be (2.21). For any polynomial $Y(\eta)(\neq 0)$, we take $X(\eta)=$ $X^{\mathcal{D}, Y}(\eta)$ as (4.25). Then the multi-indexed ( $q$-)Racah polynomials $\check{P}_{\mathcal{D}, n}(x)$ satisfy $1+2 L$ term recurrence relations with constant coefficients:

$$
\begin{equation*}
\check{X}(x) \check{P}_{\mathcal{D}, n}(x)=\sum_{k=-\min (L, n)}^{\min (L, N-n)} r_{n, k}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+k}(x) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} . \tag{4.31}
\end{equation*}
$$

Remark 1 For $L>\frac{1}{2} N$, the number of terms is not $1+2 L$ but $N+1$.
Remark 2 Unless $N-L+1 \leq n \leq N$, (4.31) is an equation as a polynomial, namely it holds for $x \in \mathbb{C}$. On the other hand, for $N-L+1 \leq n \leq N$, (4.31) holds only for $x=0,1, \ldots, x_{\text {max }}$.
Remark 3 If we set $r_{n, k}^{X, \mathcal{D}}=0$ unless $0 \leq n+k \leq n_{\text {max }}$ (see Remark 3 below Theorem(2), the sum $\sum_{k=-\min (L, n)}^{\min (L, N-n)}$ in (4.31) can be rewritten as $\sum_{k=-L}^{L}$,

$$
\begin{equation*}
\check{X}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda}) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} . \tag{4.32}
\end{equation*}
$$

Remark 4 By (4.10), $\check{X}(x ; \boldsymbol{\lambda})$ is expressed as

$$
\begin{align*}
\check{X}(x ; \boldsymbol{\lambda})=\sum_{j=1}^{x} & (\eta(j ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(j-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
& \times \check{\Xi}_{\mathcal{D}}(j ; \boldsymbol{\lambda}) Y(\eta(j ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})) \quad\left(x \in \mathbb{Z}_{\geq 0}\right) . \tag{4.33}
\end{align*}
$$

For later use, we provide a conjecture about $r_{n, k}^{X, \mathcal{D}}$.

Conjecture 1 The coefficients $r_{n, k}^{X, \mathcal{D}}$ are rational functions of $n(f o r ~ R)$ or $q^{n}$ (for $q R$ ). They satisfy

$$
\begin{equation*}
\mathrm{R}:\left.r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})\right|_{n \rightarrow-n-\tilde{d}}=r_{n,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda}), \quad q \mathrm{R}:\left.r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})\right|_{q^{n} \rightarrow q^{-n} \tilde{d}^{-1}}=r_{n,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \tag{4.34}
\end{equation*}
$$

for $1 \leq k \leq L$. Therefore $r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})+r_{n,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda})(1 \leq k \leq L)$ is a rational function of $\mathcal{E}_{n}(\boldsymbol{\lambda})$ and let this rational function be $I_{k}(z)=I_{k}(z ; \boldsymbol{\lambda})$, namely $I_{k}\left(\mathcal{E}_{n}(\boldsymbol{\lambda}) ; \boldsymbol{\lambda}\right)=r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})+r_{n,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda})$. The following function $I(z)=I(z ; \boldsymbol{\lambda})$,

$$
\begin{equation*}
I(z) \stackrel{\text { def }}{=} \prod_{j=1}^{L} \alpha_{j}(z) \alpha_{2 L+1-j}(z) \times \sum_{k=1}^{L} I_{k}(z), \tag{4.35}
\end{equation*}
$$

is a polynomial of degree $2 L$ in $z$. Here $\alpha_{j}(z) \alpha_{2 L+1-j}(z)$ will be given in (5.10).
The recurrence relations (4.27) or (4.31) with $\eta=0$ or $x=0$ and the normalization (2.22) give

$$
\begin{equation*}
r_{n, 0}^{X, \mathcal{D}}=-\sum_{k=1}^{L}\left(r_{n, k}^{X, \mathcal{D}}+r_{n,-k}^{X, \mathcal{D}}\right) . \tag{4.36}
\end{equation*}
$$

Hence (4.34) also holds for $k=0$ and the second factor of $I(z)$ (4.35) corresponds to $-r_{n, 0}^{X, \mathcal{D}}$. By the recurrence relations (4.31) and orthogonality relations (2.29) (with an appropriate range of parameters (2.19)), we have

$$
\begin{aligned}
& \sum_{x=0}^{x_{\text {max }}} \frac{\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{2}}{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})}\left(\check{X}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda})=\frac{r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})}{d_{\mathcal{D}, n+k}(\boldsymbol{\lambda})^{2}} \\
= & \sum_{x=0}^{x_{\max }} \frac{\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{2}}{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\left(\check{X}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda})\right)=\frac{r_{n+k,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda})}{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}},
\end{aligned}
$$

for $1 \leq k \leq L$ and $n+k \leq n_{\max }$. So we obtain the relations among the coefficients $r_{n, k}^{X, \mathcal{D}}$,

$$
\begin{equation*}
r_{n+k,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda})=\frac{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}}{d_{\mathcal{D}, n+k}(\boldsymbol{\lambda})^{2}} r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \quad\left(1 \leq k \leq L ; n+k \leq n_{\max }\right) \tag{4.37}
\end{equation*}
$$

which are valid for any parameter ranges (except for the zeros of the denominators). Therefore it is sufficient to find $r_{n, k}^{X, \mathcal{D}}(1 \leq k \leq L)$. For sufficiently large $N$ (or treating $a$ as an indeterminate), the top coefficient $r_{n, L}^{X, \mathcal{D}}$ is easily obtained by comparing the highest degree terms,

$$
\begin{equation*}
r_{n, L}^{X, \mathcal{D}}=\frac{c^{X} c_{\mathcal{D}, n}^{P}}{c_{\mathcal{D}, n+L}^{P}}, \tag{4.38}
\end{equation*}
$$

where $c^{X}$ is the coefficient of the highest term of $X(\eta)=c^{X} \eta^{L}+$ (lower order terms) and $c_{\mathcal{D}, n}^{P}$ is given by (A.17).

### 4.3 Examples

For illustration, we present some examples of the coefficients $r_{n, k}^{X, \mathcal{D}}$ of the recurrence relations (4.2) for $X(\eta)=X_{\min }(\eta)$ and small $d_{j}$. The parameter $a$ is treated as an indeterminate. Since the overall normalization of $X(\eta)$ is not important, we multiply $X(\eta)$ (4.25) by an appropriate factor.

### 4.3.1 multi-indexed Racah polynomials

We set $\sigma_{1}=a+b, \sigma_{2}=a b, \sigma_{1}^{\prime}=c+d$ and $\sigma_{2}^{\prime}=c d$.
$\underline{\text { Ex. } 1} \mathcal{D}=\{1\}, Y(\eta)=1\left(\Rightarrow X(\eta)=X_{\min }(\eta)\right): 5$-term recurrence relations

$$
\begin{aligned}
X(\eta)= & 2 c(d-a+1)(d-b+1) I_{\lambda+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
= & \eta\left(\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) \eta-\sigma_{1}\left(2 c+d+2 \sigma_{2}^{\prime}\right)+2 \sigma_{2} c+2 \sigma_{1}^{\prime}+\sigma_{2}^{\prime}(5+2 d)+d^{2}\right), \\
r_{n, 2}^{X, \mathcal{D}}= & \frac{\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right)(c+n)(c+n+3)(a+n, b+n, \tilde{d}+n)_{2}}{(\tilde{d}+2 n)_{4}}, \\
r_{n,-2}^{X, \mathcal{D}}= & \frac{\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right)(\tilde{d}-c+n-3)(\tilde{d}-c+n)(\tilde{d}-a+n-1, \tilde{d}-b+n-1, n-1)_{2}}{(\tilde{d}+2 n-3)_{4}}, \\
r_{n, 1}^{X, \mathcal{D}}= & \frac{2(a+n)(b+n)(c+n)(c+n+2)(\tilde{d}-c+n)(\tilde{d}+n)}{(\tilde{d}+2 n+3)(\tilde{d}+2 n-1)_{3}} \\
& \times\left(-2\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) n(n+\tilde{d}+1)+2(1-\tilde{d})\left(1+c-\sigma_{2}\right)+d\left(1-\tilde{d}^{2}\right)\right), \\
r_{n,-1}^{X, \mathcal{D}=}= & \frac{2 n(\tilde{d}-a+n)(\tilde{d}-b+n)(c+n)(\tilde{d}-c+n-2)(\tilde{d}-c+n)}{(\tilde{d}+2 n-3)(\tilde{d}+2 n-1)_{3}} \\
& \times\left(-2\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) n(n+\tilde{d}-1)+2\left(1+c-\sigma_{2}\right)+2\left(\sigma_{2}+c-\tilde{d}\right) \tilde{d}+d\left(1-\tilde{d}^{2}\right)\right), \\
r_{n, 0}^{X, \mathcal{D}}= & -\sum_{k=1}^{2}\left(r_{n,-k}^{X, \mathcal{D}}+r_{n, k}^{X, \mathcal{D}}\right) .
\end{aligned}
$$

Direct calculation shows that $I(z)$ (4.35) is a polynomial of degree 4 in $z$. Its explicit form is somewhat lengthy and we omit it.

We have also obtained 7 -term recurrence relations for $\mathcal{D}=\{2\},\{1,2\}$ with $X(\eta)=$ $X_{\min }(\eta)$ and $\mathcal{D}=\{1\}$ with non-minimal $X(\eta)(Y(\eta)=\eta)$. Since the explicit forms of $r_{n, k}^{X, \mathcal{D}}$ are somewhat lengthy, we do not write down them here.

### 4.3.2 multi-indexed $q$-Racah polynomials

We set $\sigma_{1}=a+b, \sigma_{2}=a b, \sigma_{1}^{\prime}=c+d$ and $\sigma_{2}^{\prime}=c d$.
$\underline{\text { Ex. } 1} \mathcal{D}=\{1\}, Y(\eta)=1\left(\Rightarrow X(\eta)=X_{\min }(\eta)\right): 5$-term recurrence relations

$$
\begin{align*}
& X(\eta)=(1+q)(1-c)\left(1-a^{-1} d q\right)\left(1-b^{-1} d q\right) I_{\boldsymbol{\lambda}+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
&= \eta\left(\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right) \eta+\sigma_{2}^{-1} q^{2}(1+q-2 c q) d^{2}\right. \\
&\left.\quad-\sigma_{2}^{-1}\left(\sigma_{1} q(1+q)(1-c)+(1-q)\left(\sigma_{2}+c q^{2}\right)\right) d+2-c(1+q)\right), \\
& r_{n, 2}^{X, \mathcal{D}}= \frac{\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right)\left(1-c q^{n}\right)\left(1-c q^{n+3}\right)\left(a q^{n}, b q^{n}, \tilde{d} q^{n} ; q\right)_{2}}{\left(\tilde{d} q^{2 n} ; q\right)_{4}}, \\
& r_{n,-2}^{X, \mathcal{D}=}= \frac{d^{2} q^{2}\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right)\left(1-c^{-1} \tilde{d} q^{n-3}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(a^{-1} \tilde{d} q^{n-1}, b^{-1} \tilde{d} q^{n-1}, q^{n-1} ; q\right)_{2}}{\left(\tilde{d} q^{2 n-3} ; q\right)_{4}}, \\
& r_{n, 1}^{X, \mathcal{D}=}= \frac{(1+q)\left(1-a q^{n}\right)\left(1-b q^{n}\right)\left(1-c q^{n}\right)\left(1-c q^{n+2}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(1-\tilde{d} q^{n}\right)}{\sigma_{2} d\left(1-\tilde{d} q^{2 n+3}\right)\left(\tilde{d} q^{2 n-1} ; q\right)_{3}} \\
& \times\left(-\left(\sigma_{2} \sigma_{1}^{\prime}+\sigma_{1}(1-c) d q-\sigma_{1}^{\prime} d q^{2}\right)\left(\sigma_{2} c q^{2 n}+d\right)\right. \\
&\left.+\left(q+q^{-1}\right) d\left(\sigma_{1} \sigma_{2} c+\sigma_{2}(1-c) \sigma_{1}^{\prime} q-\sigma_{1} \sigma_{2}^{\prime} q^{2}\right) q^{n}\right),  \tag{4.40}\\
& \sigma_{2}\left(1-\tilde{d} q^{2 n-3}\right)\left(\tilde{d} q^{2 n-1} ; q\right)_{3} \\
& r_{n,-1}^{X, \mathcal{D}=}= \frac{(1+q)\left(1-q^{n}\right)\left(1-a^{-1} \tilde{d} q^{n}\right)\left(1-b^{-1} \tilde{d} q^{n}\right)\left(1-c q^{n}\right)\left(1-c^{-1} \tilde{d} q^{n-2}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)}{\sigma_{2}} \\
& \times\left(-\left(\sigma_{2} \sigma_{1}^{\prime}+\sigma_{1}(1-c) d q-\sigma_{1}^{\prime} d q^{2}\right)\left(\sigma_{2} c q^{2 n-1}+d q\right)\right. \\
&\left.+\left(q+q^{-1}\right) d\left(\sigma_{1} \sigma_{2} c+\sigma_{2}(1-c) \sigma_{1}^{\prime} q-\sigma_{1} \sigma_{2}^{\prime} q^{2}\right) q^{n}\right), \\
& r_{n, 0}^{X, \mathcal{D}=}= \sum_{k=1}^{2}\left(r_{n,-k}^{X, \mathcal{D}}+r_{n, k}^{X, \mathcal{D}) .}\right.
\end{align*}
$$

Direct calculation shows that $I(z)$ (4.35) is a polynomial of degree 4 in $z$. Its explicit form is somewhat lengthy and we omit it.

We have also obtained 7-term recurrence relations for $\mathcal{D}=\{2\}$ with $X(\eta)=X_{\min }(\eta)$. Since the explicit forms of $r_{n, k}^{X, \mathcal{D}}$ are somewhat lengthy, we do not write down them here.

## 5 Generalized Closure Relations and Creation and Annihilation Operators

In this section we discuss the generalized closure relations and the creation and annihilation operators of the multi-indexed $(q-)$ Racah rdQM systems described by $\mathcal{H}_{\mathcal{D}}$ (2.27).

First let us recapitulate the essence of the (generalized) closure relation [25]. The closure relation of order $K$ is an algebraic relation between a Hamiltonian $\mathcal{H}$ and some operator $X$ $(=X(\eta(x))=\check{X}(x))$ [25]:

$$
\begin{equation*}
(\operatorname{ad} \mathcal{H})^{K} X=\sum_{i=0}^{K-1}(\operatorname{ad} \mathcal{H})^{i} X \cdot R_{i}(\mathcal{H})+R_{-1}(\mathcal{H}) \tag{5.1}
\end{equation*}
$$

where $(\operatorname{ad} \mathcal{H}) X=[\mathcal{H}, X],(\operatorname{ad} \mathcal{H})^{0} X=X$ and $R_{i}(z)=R_{i}^{X}(z)$ is a polynomial in $z$. The original closure relation [39, 30] corresponds to $K=2$. Since the closure relation of order $K$ implies that of order $K^{\prime}>K$, we are interested in the smallest integer $K$ satisfying (5.1). We assume that the matrix $A=\left(a_{i j}\right)_{1 \leq i, j \leq K}\left(a_{i+1, i}=1(1 \leq i \leq K-1), a_{i+1, K}=R_{i}(z)\right.$ $(0 \leq i \leq K-1), a_{i j}=0$ (others)) has $K$ distinct real non-vanishing eigenvalues $\alpha_{i}=\alpha_{i}(z)$ for $z \geq 0$, which are indexed in decreasing order $\alpha_{1}(z)>\alpha_{2}(z)>\cdots>\alpha_{K}(z)$. Then we obtain the exact Heisenberg solution of $X$,

$$
\begin{equation*}
X_{\mathrm{H}}(t) \stackrel{\text { def }}{=} e^{i \mathcal{H} t} X e^{-i \mathcal{H} t}=\sum_{n=0}^{\infty} \frac{(i t)^{n}}{n!}(\operatorname{ad} \mathcal{H})^{n} X=\sum_{j=1}^{K} a^{(j)} e^{i \alpha_{j}(\mathcal{H}) t}-R_{-1}(\mathcal{H}) R_{0}(\mathcal{H})^{-1} \tag{5.2}
\end{equation*}
$$

Here $a^{(j)}=a^{(j)}(\mathcal{H}, X)(1 \leq j \leq K)$ are creation or annihilation operators,

$$
\begin{equation*}
a^{(j)}=\left(\sum_{i=1}^{K}(\operatorname{ad} \mathcal{H})^{i-1} X \cdot p_{i j}(\mathcal{H})+R_{-1}(\mathcal{H}) \alpha_{j}(\mathcal{H})^{-1}\right) \prod_{\substack{k=1 \\ k \neq j}}^{K}\left(\alpha_{j}(\mathcal{H})-\alpha_{k}(\mathcal{H})\right)^{-1} \tag{5.3}
\end{equation*}
$$

where $p_{i j}(z)(1 \leq i, j \leq K)$ are

$$
\begin{equation*}
p_{i j}(z)=\alpha_{j}(z)^{K-i}-\sum_{k=1}^{K-i} R_{K-k}(z) \alpha_{j}(z)^{K-i-k} \tag{5.4}
\end{equation*}
$$

Let us consider the rdQM systems described by the multi-indexed ( $q$-) Racah polynomials. The Hamiltonian is $\mathcal{H}_{\mathcal{D}}(2.27)$ and a candidate of the operator $X$ is a polynomial $X(\eta(x))=$ $\check{X}(x)$ discussed in $\S 4$. The closure relation (5.1) is now

$$
\begin{equation*}
\left(\operatorname{ad} \mathcal{H}_{\mathcal{D}}\right)^{K} \underline{X}=\sum_{i=0}^{K-1}\left(\operatorname{ad} \mathcal{H}_{\mathcal{D}}\right)^{i} \underline{X} \cdot R_{i}\left(\mathcal{H}_{\mathcal{D}}\right)+R_{-1}\left(\mathcal{H}_{\mathcal{D}}\right) \tag{5.5}
\end{equation*}
$$

where $\underline{X}$ is a diagonal matrix $\underline{X}=\left(\check{X}(x) \delta_{x, y}\right)_{0 \leq x, y \leq x_{\max }}$. (In the notation used in (2.10), this matrix $\underline{X}$ is expressed as $\check{X}(x) \mathbf{1}$ or simply $\check{X}(x)$.) From the tridiagonal form of $\mathcal{H}_{\mathcal{D}}$ and by assuming that $N$ is sufficiently large, polynomials $R_{i}(z)=R_{i}^{X}(z)$ have the following degrees,

$$
\begin{equation*}
R_{i}(z)=\sum_{j=0}^{K-i} r_{i}^{(j)} z^{j} \quad(0 \leq i \leq K-1), \quad R_{-1}(z)=\sum_{j=0}^{K} r_{-1}^{(j)} z^{j} \tag{5.6}
\end{equation*}
$$

where $r_{i}^{(j)}=r_{i}^{X(j)}$ are coefficients.
The method of [25] is (i) Find $X$ and $R_{i}(z)$ satisfying (5.1), (ii) Calculate the eigenvalues $\alpha_{j}(z)$, (iii) Heisenberg solution $X_{\mathrm{H}}(t)$ is obtained, (iv) Creation/annihilation operators $a^{(j)}$ are obtained. Here we reverse a part of the logic, namely exchange (i) and (ii). First we define functions $\alpha_{j}(z)$ by guess work. Next, polynomials $R_{i}(z)$ are defined by using $\alpha_{j}(z)$ and Conjecture 1. Then we check the closure relation (5.5) for these $R_{i}(z)$ and $\underline{X}$ with $\check{X}(x)$ given in $\S$.

Let us define $\alpha_{j}(z)(1 \leq j \leq 2 L)$ as follows:

$$
\begin{align*}
& \mathrm{R}: \quad \alpha_{j}(z)= \begin{cases}(L+1-j)^{2}+(L+1-j) \sqrt{4 z+\tilde{d}^{2}} & (1 \leq j \leq L) \\
(j-L)^{2}-(j-L) \sqrt{4 z+\tilde{d}^{2}} & (L+1 \leq j \leq 2 L)\end{cases}  \tag{5.7}\\
& q \mathrm{R}: \quad \alpha_{j}(z)=\left\{\begin{array}{c}
\frac{1}{2}\left(\left(q^{-\frac{1}{2}(L+1-j)}-q^{\frac{1}{2}(L+1-j)}\right)^{2}(z+1+\tilde{d})\right. \\
\left.+\left(q^{-(L+1-j)}-q^{L+1-j}\right) \sqrt{(z+1+\tilde{d})^{2}-4 \tilde{d}}\right) \quad(1 \leq j \leq L) \\
\frac{1}{2}\left(\left(q^{-\frac{1}{2}(j-L)}-q^{\frac{1}{2}(j-L)}\right)^{2}(z+1+\tilde{d})\right. \\
\left.-\left(q^{-(j-L)}-q^{j-L}\right) \sqrt{(z+1+\tilde{d})^{2}-4 \tilde{d}}\right) \quad(L+1 \leq j \leq 2 L)
\end{array}\right. \tag{5.8}
\end{align*}
$$

The pair of $\alpha_{j}(z)$ and $\alpha_{2 L+1-j}(z)(1 \leq j \leq L)$ satisfies

$$
\begin{align*}
& \alpha_{j}(z)+\alpha_{2 L+1-j}(z)=\left\{\begin{array}{ll}
2(L+1-j)^{2} & : \mathrm{R} \\
\left(q^{-\frac{1}{2}(L+1-j)}-q^{\frac{1}{2}(L+1-j)}\right)^{2}(z+1+\tilde{d}) & : q \mathrm{R}
\end{array},\right.  \tag{5.9}\\
& \alpha_{j}(z) \alpha_{2 L+1-j}(z)=\left\{\begin{array}{ll}
(L+1-j)^{2}\left((L+1-j)^{2}-4 z-\tilde{d}^{2}\right) & : \mathrm{R} \\
\left(q^{-\frac{1}{2}(L+1-j)}-q^{\frac{1}{2}(L+1-j)}\right)^{2} & \\
\times\left(\left(q^{-\frac{1}{2}(L+1-j)}+q^{\frac{1}{2}(L+1-j)}\right)^{2} \tilde{d}-(z+1+\tilde{d})^{2}\right) & : q \mathrm{R}
\end{array} .\right. \tag{5.10}
\end{align*}
$$

These $\alpha_{j}(z)$ satisfy

$$
\begin{equation*}
\alpha_{1}(z)>\alpha_{2}(z)>\cdots>\alpha_{L}(z)>0>\alpha_{L+1}(z)>\alpha_{L+2}(z)>\cdots>\alpha_{2 L}(z) \quad(z \geq 0) \tag{5.11}
\end{equation*}
$$

for $\tilde{d}>2 L-1(\mathrm{R})$ and $\tilde{d}<q^{2 L-1}(q \mathrm{R})$. We remark that $\alpha_{j}\left(\mathcal{E}_{n}\right)$ is square root free, $\sqrt{4 \mathcal{E}_{n}+\tilde{d}^{2}}=2 n+\tilde{d}$ for R and $\sqrt{\left(\mathcal{E}_{n}+1+\tilde{d}\right)^{2}-4 \tilde{d}^{2}}=q^{-n}-\tilde{d} q^{n}$ for $q \mathrm{R}$. It is easy to show the following:

$$
\alpha_{j}\left(\mathcal{E}_{n}\right)= \begin{cases}\mathcal{E}_{n+L+1-j}-\mathcal{E}_{n}>0 & (1 \leq j \leq L)  \tag{5.12}\\ \mathcal{E}_{n-(j-L)}-\mathcal{E}_{n}<0 & (L+1 \leq j \leq 2 L)\end{cases}
$$

Like the Wilson and Askey-Wilson cases [22], we conjecture the following.

Conjecture 2 Take $X(\eta)$ as Theorem 3 and take $R_{i}(z)(-1 \leq i \leq 2 L-1)$ as follows:

$$
\begin{align*}
R_{i}(z) & =(-1)^{i+1} \sum_{1 \leq j_{1}<j_{2}<\cdots<j_{2 L-i} \leq 2 L} \alpha_{j_{1}}(z) \alpha_{j_{2}}(z) \cdots \alpha_{j_{2 L-i}}(z) \quad(0 \leq i \leq 2 L-1)  \tag{5.13}\\
R_{-1}(z) & =-I(z) \tag{5.14}
\end{align*}
$$

where $I(z)$ is given by (4.35). Then the closure relation of order $K=2 L$ (5.5) holds.
We remark that $R_{i}(z)$ in (5.13) are indeed polynomials in $z$, because RHS of (5.13) are symmetric under the exchange of $\alpha_{j}$ and $\alpha_{2 L+1-j}$ and their sum and product are polynomials in $z$, (5.9) -(5.10). Since $R_{i}(z)(0 \leq i \leq 2 L-1)$ are expressed in terms of $\alpha_{j}(z)$, they do not depend on $\mathcal{D}$ and $X$ (except for $\operatorname{deg} X=L$ ). Only $R_{-1}(z)$ depends on $\mathcal{D}$ and $X$. For ' $L=1$ case', namely the original system $\left(\mathcal{D}=\emptyset, \ell_{\mathcal{D}}=0, \Xi_{\mathcal{D}}(\eta)=1, X(\eta)=X_{\min }(\eta)=\eta\right)$, this generalized closure relation reduces to the original closure relation [30]. Direct verification of this conjecture is straightforward for lower $M$ and smaller $d_{j}$, $\operatorname{deg} Y$ and $N$, by a computer algebra system.

Let us assume $\tilde{d}>2 L-1$ for R and $\tilde{d}<q^{2 L-1}$ for $q \mathrm{R}$. If Conjecture 2 is true, we have the exact Heisenberg operator solution $X_{\mathrm{H}}(t)$ (5.2) and the creation/annihilation operators $a^{(j)}=a^{\mathcal{D}, X(j)}$ (5.3). Action of (5.2) on $\phi_{\mathcal{D} n}(x)$ (2.25) is

$$
e^{i \mathcal{H}_{\mathcal{D}} t} \underline{X} e^{-i \mathcal{H}_{\mathcal{D}} t} \phi_{\mathcal{D} n}(x)=\sum_{j=1}^{2 L} e^{i \alpha_{j}\left(\mathcal{E}_{n}\right) t} a^{(j)} \phi_{\mathcal{D} n}(x)-R_{-1}\left(\mathcal{E}_{n}\right) R_{0}\left(\mathcal{E}_{n}\right)^{-1} \phi_{\mathcal{D} n}(x) .
$$

On the other hand the LHS turns out to be

$$
\begin{aligned}
e^{i \mathcal{H}_{\mathcal{D}} t} \underline{X} e^{-i \mathcal{H}_{\mathcal{D}} t} \phi_{\mathcal{D} n}(x) & =e^{i \mathcal{H}_{\mathcal{D}} t} \underline{X} e^{-i \mathcal{E}_{n} t} \phi_{\mathcal{D} n}(x)=e^{-i \mathcal{E}_{n} t} e^{i \mathcal{H}_{\mathcal{D}} t} \sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} \phi_{\mathcal{D} n+k}(x) \\
& =\sum_{k=-L}^{L} e^{i\left(\mathcal{E}_{n+k}-\mathcal{E}_{n}\right) t} r_{n, k}^{X, \mathcal{D}} \phi_{\mathcal{D} n+k}(x)
\end{aligned}
$$

where we have used (4.32). Comparing these $t$-dependence, we obtain (5.12) and

$$
\begin{align*}
& a^{(j)} \phi_{\mathcal{D} n}(x)= \begin{cases}r_{n, L+1-j}^{X, \mathcal{D}} \phi_{\mathcal{D} n+L+1-j}(x) & (1 \leq j \leq L) \\
r_{n,-(j-L)}^{X, \mathcal{D}} \phi_{\mathcal{D} n-(j-L)}(x) & (L+1 \leq j \leq 2 L)\end{cases}  \tag{5.15}\\
& -R_{-1}\left(\mathcal{E}_{n}\right) R_{0}\left(\mathcal{E}_{n}\right)^{-1}=r_{n, 0}^{X, \mathcal{D}} \tag{5.16}
\end{align*}
$$

where $r_{n, k}^{X, \mathcal{D}}=0$ unless $0 \leq n+k \leq n_{\max }$. Note that (5.16) is consistent with Conjecture 1 and (4.36). Therefore $a^{(j)}(1 \leq j \leq L)$ and $a^{(j)}(L+1 \leq j \leq 2 L)$ are creation and annihilation operators, respectively. Among them, $a^{(L)}$ and $a^{(L+1)}$ are fundamental, $a^{(L)} \phi_{\mathcal{D}, n}(x) \propto \phi_{\mathcal{D} n+1}(x)$ and $a^{(L+1)} \phi_{\mathcal{D}, n}(x) \propto \phi_{\mathcal{D} n-1}(x)$. Furthermore, $X=X_{\text {min }}$ case is the most basic.

By the similarity transformation (see (2.30)), the closure relation (5.5) becomes

$$
\begin{equation*}
\left(\operatorname{ad} \widetilde{\mathcal{H}}_{\mathcal{D}}\right)^{K} \underline{X}=\sum_{i=0}^{K-1}\left(\operatorname{ad} \widetilde{\mathcal{H}}_{\mathcal{D}}\right)^{i} \underline{X} \cdot R_{i}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}\right)+R_{-1}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}\right) \tag{5.17}
\end{equation*}
$$

and the creation/annihilation operators for eigenpolynomials can be obtained,

$$
\begin{align*}
& \tilde{a}^{(j)} \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x)^{-1} \circ a^{(j)}\left(\mathcal{H}_{\mathcal{D}}, \underline{X}\right) \circ \psi_{\mathcal{D}}(x)=a^{(j)}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}, \underline{X}\right),  \tag{5.18}\\
& \tilde{a}^{(j)} \check{P}_{\mathcal{D}, n}(x)=\left\{\begin{array}{ll}
r_{n, L+1-j}^{X, \mathcal{D}} \check{P}_{\mathcal{D}, n+L+1-j}(x) & (1 \leq j \leq L) \\
r_{n,-(j-L)}^{X, \mathcal{D}} & \check{P}_{\mathcal{D}, n-(j-L)}(x)
\end{array}(L+1 \leq j \leq 2 L)\right. \tag{5.19}
\end{align*} .
$$

## 6 Summary and Comments

Following the preceding papers on the multi-indexed Laguerre and Jacobi polynomials in oQM [19, 22, 24] and the multi-indexed Wilson and Askey-Wilson polynomials in idQM [19, 22], we have discussed the recurrence relations for the multi-indexed Racah and $q$ Racah polynomials in rdQM. The $3+2 M$ term recurrence relations with variable dependent coefficients (3.14) are derived (Theorem(1). They provide an efficient method to calculate the multi-indexed $(q-)$ Racah polynomials. Two different kinds of the $1+2 L$ term $(L \geq M+1)$ recurrence relations with constant coefficients (4.27) and (4.31) are derived (Theorem2, (3), and their examples are presented. Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. Not all of them are independent, but the relations among them are unclear. To clarify their relations is an important problem. The most basic one is the minimal degree one $X_{\text {min }}(\eta)$ (4.26), which gives $3+2 \ell_{\mathcal{D}}$ term recurrence relations. Corresponding to the recurrence relations with constant coefficients, the rdQM systems described by the multi-indexed ( $q-$ )Racah polynomials satisfy the generalized closure relations, from which the creation and annihilation operators are obtained. There are many creation and annihilation operators and it is an interesting problem to study their relations. A proof and some data of the recurrence relations with constant coefficients for the multi-indexed Wilson and AskeyWilson polynomials are presented in Appendix B and C , respectively.

In rdQM, dual polynomials are introduced naturally [2, 30, 31]. The polynomial $\mathcal{P}_{n}(\eta(x))$ and its dual polynomial $\mathcal{Q}_{x}\left(\mathcal{E}_{n}\right)$ are related as $\mathcal{P}_{n}(\eta(x))=\mathcal{Q}_{x}\left(\mathcal{E}_{n}\right)$, where the roles of the variable and the 'degree' (the number of zeros) are interchanged. The multi-indexed ( $q$-)Racah polynomials $P_{\mathcal{D}, n}(\eta(x))$ satisfy the second order difference equations [13] and the $1+2 L$ term recurrence relations with constant coefficients derived in this paper. Let us introduce
dual polynomial $Q_{\mathcal{D}, x}\left(\mathcal{E}_{n}\right)$ as $P_{\mathcal{D}, n}(\eta(x)) \propto Q_{\mathcal{D}, x}\left(\mathcal{E}_{n}\right)$. Then dual polynomials $Q_{\mathcal{D}, x}\left(\mathcal{E}_{n}\right)$ satisfy the three term recurrence relations and various $2 L$-th order difference equations which depend on the choice of $Y(\eta)$. Therefore dual polynomials $Q_{\mathcal{D}, x}\left(\mathcal{E}_{n}\right)$ are ordinary orthogonal polynomials and they are the Krall-type. It is an interesting problem to study these dual polynomials in detail. We will report on this topic elsewhere 40.

The $(q-)$ Racah polynomial $P_{n}^{(q) \mathrm{R}}(\eta)$ and the (Askey-)Wilson polynomial $P_{n}^{(\mathrm{A}) \mathrm{W}}(\eta)$ are the 'same' polynomials [33]. The replacement rule of this correspondence is

$$
\begin{equation*}
i x^{(\mathrm{A}) \mathrm{W}}=\gamma\left(x^{(q) \mathrm{R}}+\frac{1}{2} \lambda_{4}^{(q) \mathrm{R}}\right), \quad \boldsymbol{\lambda}^{(\mathrm{A}) \mathrm{W}}=\boldsymbol{\lambda}^{(q) \mathrm{R}}-\frac{1}{2} \lambda_{4}^{(q) \mathrm{R}} \boldsymbol{\delta}^{(q) \mathrm{R}}, \tag{6.1}
\end{equation*}
$$

which gives

$$
\begin{align*}
P_{n}^{\mathrm{R}}\left(\eta ; \boldsymbol{\lambda}^{\mathrm{R}}\right) & =(a, b, c)_{n}^{-1} P_{n}^{\mathrm{W}}\left(-\eta-\frac{1}{4} d^{2} ; \boldsymbol{\lambda}^{\mathrm{W}}\right),  \tag{6.2}\\
P_{n}^{q \mathrm{R}}\left(\eta ; \boldsymbol{\lambda}^{q \mathrm{R}}\right) & =d^{\frac{n}{2}}(a, b, c ; q)_{n}^{-1} P_{n}^{\mathrm{AW}}\left(\frac{1}{2} d^{-\frac{1}{2}}(\eta+1+d) ; \boldsymbol{\lambda}^{\mathrm{AW}}\right) \tag{6.3}
\end{align*}
$$

(The relation between $q \mathrm{R}$ and AW is given in [36]. See [36] for notation.) This property is inherited to the multi-indexed polynomials. The multi-indexed ( $q$-)Racah polynomial $P_{\mathcal{D}, n}^{(q) \mathrm{R}}(\eta)$ and the multi-indexed (Askey-)Wilson polynomial $P_{\mathcal{D}, n}^{(\mathrm{A}) \mathrm{W}}(\eta)$ with all type I indices are the 'same' polynomials. The replacement rule of this correspondence is

$$
\begin{equation*}
i x^{(\mathrm{A}) \mathrm{W}}=\gamma\left(x^{(q) \mathrm{R}}+\frac{1}{2} M+\frac{1}{2} \lambda_{4}^{(q) \mathrm{R}}\right), \quad \boldsymbol{\lambda}^{(\mathrm{A}) \mathrm{W}}=\boldsymbol{\lambda}^{(q) \mathrm{R}}-\frac{1}{2} \lambda_{4}^{(q) \mathrm{R}} \boldsymbol{\delta}^{(q) \mathrm{R}} \tag{6.4}
\end{equation*}
$$

which gives

$$
\begin{align*}
& P_{\mathcal{D}, n}^{\mathrm{R}}\left(\eta ; \boldsymbol{\lambda}^{\mathrm{R}}\right)=(-1)^{\ell_{\mathcal{D}}+n} \frac{c_{\mathcal{D}, n}^{P \mathrm{R}}\left(\boldsymbol{\lambda}^{\mathrm{R}}\right)}{c_{\mathcal{D}, n}^{P \mathrm{~W}}\left(\boldsymbol{\lambda}^{\mathrm{W}}\right)} P_{\mathcal{D}, n}^{\mathrm{W}}\left(-\eta-\frac{1}{4}(d+M)^{2} ; \boldsymbol{\lambda}^{\mathrm{W}}\right),  \tag{6.5}\\
& P_{\mathcal{D}, n}^{q \mathrm{R}}\left(\eta ; \boldsymbol{\lambda}^{q \mathrm{R}}\right)=\left(2 d^{\frac{1}{2}} q^{\frac{1}{2} M}\right)^{\ell_{\mathcal{D}}+n} \frac{c_{\mathcal{D}, n}^{P q \mathrm{R}}\left(\boldsymbol{\lambda}^{q \mathrm{R}}\right)}{c_{\mathcal{D}, n}^{P \mathrm{AW}}\left(\boldsymbol{\lambda}^{\mathrm{AW}}\right)} P_{\mathcal{D}, n}^{\mathrm{AW}}\left(\frac{1}{2} d^{-\frac{1}{2}} q^{-\frac{1}{2} M}\left(\eta+1+d q^{M}\right) ; \boldsymbol{\lambda}^{\mathrm{AW}}\right) \tag{6.6}
\end{align*}
$$

Here $c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda})$ is the coefficient of the highest degree term of $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ and they are given by (A.17) (eq.(3.59) in [13]) and eq.(A.7) in [11]. Therefore the recurrence relations of the multiindexed (Askey-)Wilson polynomials give those of the multi-indexed ( $q$-)Racah polynomials. Conversely, the recurrence relations of the multi-indexed ( $q$ - $)$ Racah polynomials give those of the multi-indexed (Askey-)Wilson polynomials with all type I indices.

## Acknowledgments

I thank R. Sasaki for discussion and useful comments on the manuscript.

## A Data for Multi-indexed ( $q$-)Racah Polynomials

In this appendix we present some data for the multi-indexed ( $q-$ )Racah polynomials [30, 13], which are not presented in the main text.

- potential functions:

$$
\begin{align*}
& B(x ; \boldsymbol{\lambda})= \begin{cases}-\frac{(x+a)(x+b)(x+c)(x+d)}{(2 x+d)(2 x+1+d)} & : \mathrm{R} \\
-\frac{\left(1-a q^{x}\right)\left(1-b q^{x}\right)\left(1-c q^{x}\right)\left(1-d q^{x}\right)}{\left(1-d q^{2 x}\right)\left(1-d q^{2 x+1}\right)} & : q \mathrm{R}\end{cases} \\
& D(x ; \boldsymbol{\lambda})= \begin{cases}-\frac{(x+d-a)(x+d-b)(x+d-c) x}{(2 x-1+d)(2 x+d)} & : \mathrm{R} \\
-\tilde{d} \frac{\left(1-a^{-1} d q^{x}\right)\left(1-b^{-1} d q^{x}\right)\left(1-c^{-1} d q^{x}\right)\left(1-q^{x}\right)}{\left(1-d q^{2 x-1}\right)\left(1-d q^{2 x}\right)} & : q \mathrm{R}\end{cases} \tag{A.1}
\end{align*}
$$

- coefficients of the three term recurrence relations: $\left(A_{-1}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} 0\right)$

$$
\begin{align*}
& B_{n}(\boldsymbol{\lambda})=-A_{n}(\boldsymbol{\lambda})-C_{n}(\boldsymbol{\lambda}), \\
& A_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
\frac{(n+a)(n+b)(n+c)(n+\tilde{d})}{(2 n+\tilde{d})(2 n+1+\tilde{d})} & : \mathrm{R} \\
\frac{\left(1-a q^{n}\right)\left(1-b q^{n}\right)\left(1-c q^{n}\right)\left(1-\tilde{d} q^{n}\right)}{\left(1-\tilde{d} q^{2 n}\right)\left(1-\tilde{d} q^{2 n+1}\right)} & : q \mathrm{R}
\end{array},\right.  \tag{A.2}\\
& C_{n}(\boldsymbol{\lambda})= \begin{cases}\frac{(n+\tilde{d}-a)(n+\tilde{d}-b)(n+\tilde{d}-c) n}{(2 n-1+\tilde{d})(2 n+\tilde{d})} & : \mathrm{R} \\
d \frac{\left(1-a^{-1} \tilde{d} q^{n}\right)\left(1-b^{-1} \tilde{d} q^{n}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(1-q^{n}\right)}{\left(1-\tilde{d} q^{2 n-1}\right)\left(1-\tilde{d} q^{2 n}\right)} & : q \mathrm{R}\end{cases}
\end{align*}
$$

- ground state eigenvector: $\phi_{0}(x ; \boldsymbol{\lambda})>0$

$$
\phi_{0}(x ; \boldsymbol{\lambda})^{2}= \begin{cases}\frac{(a, b, c, d)_{x}}{(d-a+1, d-b+1, d-c+1,1)_{x}} \frac{2 x+d}{d} & : \mathrm{R}  \tag{A.3}\\ \frac{(a, b, c, d ; q)_{x}}{\left(a^{-1} d q, b^{-1} d q, c^{-1} d q, q ; q\right)_{x} \tilde{d}^{x}} \frac{1-d q^{2 x}}{1-d} & : q \mathrm{R}\end{cases}
$$

- normalization constant: $d_{n}(\boldsymbol{\lambda})>0$

$$
d_{n}(\boldsymbol{\lambda})^{2}=\left\{\begin{array}{ll}
\frac{(a, b, c, \tilde{d})_{n}}{\frac{(\tilde{d}-a+1, \tilde{d}-b+1, \tilde{d}-c+1,1)_{n}}{} \frac{2 n+\tilde{d}}{\tilde{d}}}  \tag{A.4}\\
\times \frac{(-1)^{N}(d-a+1, d-b+1, d-c+1)_{N}}{(\tilde{d}+1)_{N}(d+1)_{2 N}} & : \mathrm{R} \\
\frac{(a, b, c, \tilde{d} ; q)_{n}}{\left(a^{-1} \tilde{d} q, b^{-1} \tilde{d} q, c^{-1} \tilde{d} q, q ; q\right)_{n} d^{n}} \frac{1-\tilde{d} q^{2 n}}{1-\tilde{d}} \\
\times \frac{(-1)^{N}\left(a^{-1} d q, b^{-1} d q, c^{-1} d q ; q\right)_{N} \tilde{d^{N}} q^{\frac{1}{2} N(N+1)}}{(\tilde{d} q ; q)_{N}(d q ; q)_{2 N}} & : q \mathrm{R}
\end{array} .\right.
$$

- energy eigenvalue:

$$
\mathcal{E}_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
n(n+\tilde{d}) & : \mathrm{R}  \tag{A.5}\\
\left(q^{-n}-1\right)\left(1-\tilde{d} q^{n}\right) & : q \mathrm{R}
\end{array} .\right.
$$

- auxiliary functions: (convention: $\prod_{1 \leq j<k \leq M} *=1$ for $M=0,1$ )

$$
\begin{align*}
& \varphi(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\eta(x+1 ; \boldsymbol{\lambda})-\eta(x ; \boldsymbol{\lambda})}{\eta(1 ; \boldsymbol{\lambda})}=\left\{\begin{array}{ll}
\frac{2 x+d+1}{d+1} & : \mathrm{R} \\
\frac{q^{-x}-d q^{x+1}}{1-d q} & : q \mathrm{R}
\end{array},\right.  \tag{A.6}\\
& \varphi_{M}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \prod_{1 \leq j<k \leq M} \frac{\eta(x+k-1 ; \boldsymbol{\lambda})-\eta(x+j-1 ; \boldsymbol{\lambda})}{\eta(k-j ; \boldsymbol{\lambda})} \quad\left(\varphi_{0}(x)=\varphi_{1}(x)=1\right) \\
&=\prod_{1 \leq j<k \leq M} \varphi(x+j-1 ; \boldsymbol{\lambda}+(k-j-1) \boldsymbol{\delta}) . \tag{A.7}
\end{align*}
$$

- potential functions $B^{\prime}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} B(x ; \mathfrak{t}(\boldsymbol{\lambda})), D^{\prime}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} D(x ; \mathfrak{t}(\boldsymbol{\lambda}))$ :

$$
\begin{align*}
& B^{\prime}(x ; \boldsymbol{\lambda})=\left\{\begin{array}{ll}
-\frac{(x+d-a+1)(x+d-b+1)(x+c)(x+d)}{(2 x+d)(2 x+1+d)} & : \mathrm{R} \\
-\frac{\left(1-a^{-1} d q^{x+1}\right)\left(1-b^{-1} d q^{x+1}\right)\left(1-c q^{x}\right)\left(1-d q^{x}\right)}{\left(1-d q^{2 x}\right)\left(1-d q^{2 x+1}\right)} & : q \mathrm{R}
\end{array},\right. \\
& D^{\prime}(x ; \boldsymbol{\lambda})= \begin{cases}-\frac{(x+a-1)(x+b-1)(x+d-c) x}{(2 x-1+d)(2 x+d)} & : \mathrm{R} \\
-\frac{c d q}{a b} \frac{\left(1-a q^{x-1}\right)\left(1-b q^{x-1}\right)\left(1-c^{-1} d q^{x}\right)\left(1-q^{x}\right)}{\left(1-d q^{2 x-1}\right)\left(1-d q^{2 x}\right)} & : q \mathrm{R}\end{cases} \tag{A.8}
\end{align*}
$$

- $\alpha(\boldsymbol{\lambda})$ and virtual state energy $\tilde{\mathcal{E}}_{\mathrm{v}}$ :

$$
\alpha(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
1 & : \mathrm{R}  \tag{A.9}\\
a b d^{-1} q^{-1} & : q \mathrm{R}
\end{array}, \quad \tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
-(c+\mathrm{v})(\tilde{d}-c-\mathrm{v}) & : \mathrm{R} \\
-\left(1-c q^{\mathrm{v}}\right)\left(1-c^{-1} \tilde{d} q^{-\mathrm{v}}\right) & : q \mathrm{R}
\end{array} .\right.\right.
$$

- $r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)(1 \leq j \leq M+1):\left(x_{j} \stackrel{\text { def }}{=} x+j-1\right)$

$$
r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)= \begin{cases}\frac{(x+a, x+b)_{j-1}(x+d-a+j, x+d-b+j)_{M+1-j}}{(d-a+1, d-b+1)_{M}} & : \mathrm{R}  \tag{A.10}\\ \frac{\left(a q^{x}, b q^{x} ; q\right)_{j-1}\left(a^{-1} d q^{x+j}, b^{-1} d q^{x+j} ; q\right)_{M+1-j}}{\left(a b d^{-1} q^{-1}\right)^{j-1} q^{M x}\left(a^{-1} d q, b^{-1} d q ; q\right)_{M}} & : q \mathrm{R}\end{cases}
$$

- normalization constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}), \mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda}), \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})>0$ and $d_{\mathcal{D}, n}(\boldsymbol{\lambda})>0$ :

$$
\begin{align*}
\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) & =\frac{1}{\varphi_{M}(0 ; \boldsymbol{\lambda})} \prod_{1 \leq j<k \leq M} \frac{\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{k}}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda}) B^{\prime}(j-1 ; \boldsymbol{\lambda})}  \tag{A.11}\\
\mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda}) & =(-1)^{M} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2} \tag{A.12}
\end{align*}
$$

$$
\begin{align*}
\tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2} & =\frac{\varphi_{M}(0 ; \boldsymbol{\lambda})}{\varphi_{M+1}(0 ; \boldsymbol{\lambda})} \prod_{j=1}^{M} \frac{\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda}) B^{\prime}(j-1 ; \boldsymbol{\lambda})}  \tag{A.13}\\
d_{\mathcal{D}, n}(\boldsymbol{\lambda}) & =d_{n}(\boldsymbol{\lambda}) \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda}) \tag{A.14}
\end{align*}
$$

- coefficients of the highest degree term:
$P_{n}(\eta ; \boldsymbol{\lambda})=c_{n}(\boldsymbol{\lambda}) \eta^{n}+$ (lower order terms), $P_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})=c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}+n}+$ (lower order terms), $\xi_{\mathrm{v}}(\eta ; \boldsymbol{\lambda})=\tilde{c}_{\mathrm{v}}(\boldsymbol{\lambda}) \eta^{\mathrm{v}}+($ lower order terms $), \quad \Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}}+($ lower order terms $)$,

$$
\begin{align*}
& c_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{lll}
\frac{(\tilde{d}+n)_{n}}{(a, b, c)_{n}} & : \mathrm{R} \\
\frac{\left(\tilde{d} q^{n} ; q\right)_{n}}{(a, b, c ; q)_{n}} & : q \mathrm{R}
\end{array}, \quad \tilde{c}_{\mathrm{v}}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
\frac{(c+d-a-b+\mathrm{v}+1)_{\mathrm{v}}}{(d-a+1, d-b+1, c)_{\mathrm{v}}} & : \mathrm{R} \\
\frac{\left(a^{-1} b^{-1} c d q^{\mathrm{v}+1} ; q\right)_{\mathrm{v}}}{\left(a^{-1} d q, b^{-1} d q, c ; q\right)_{\mathrm{v}}} & : q \mathrm{R}
\end{array},\right.\right.  \tag{A.15}\\
& c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})=\prod_{j=1}^{M} \tilde{c}_{d_{j}}(\boldsymbol{\lambda}) \times \begin{cases}\frac{\prod_{j=1}^{M}(d-a+1, d-b+1, c)_{j-1}}{\prod_{1 \leq j<k \leq M}\left(c+d-a-b+d_{j}+d_{k}+1\right)} & : \mathrm{R} \\
\frac{\prod_{j=1}^{M}\left(a^{-1} d q, b^{-1} d q, c ; q\right)_{j-1}}{\prod_{1 \leq j<k \leq M}\left(1-a^{-1} b^{-1} c d q^{d_{j}+d_{k}+1}\right)}\end{cases}  \tag{A.16}\\
& c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda})=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) c_{n}(\boldsymbol{\lambda}) \times \begin{cases}\prod_{j=1}^{M} \frac{c+j-1}{c+d_{j}+n} & : \mathrm{R} \\
\prod_{j=1}^{M} \frac{1-c q^{j-1}}{1-c q^{d_{j}+n}} & : q \mathrm{R}\end{cases} \tag{A.17}
\end{align*}
$$

- potential functions $B_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $D_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ :

$$
\begin{align*}
& B_{\mathcal{D}}(x ; \boldsymbol{\lambda})=B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\frac{\Xi_{\mathcal{\Xi}}(x+1 ; \boldsymbol{\lambda})}{} \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}} \\
& D_{\mathcal{D}}(x ; \boldsymbol{\lambda})=D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}  \tag{A.18}\\
& \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})
\end{align*} \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{} .
$$

- Casorati determinant (Casoratian) of a set of $n$ functions $\left\{f_{j}(x)\right\}$ :

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} \operatorname{det}\left(f_{k}(x+j-1)\right)_{1 \leq j, k \leq n} \tag{A.19}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}_{\mathrm{C}}[\cdot](x)=1$ ).

- potential functions $\hat{B}_{d_{1} \ldots d_{s}}(x ; \boldsymbol{\lambda})$ and $\hat{D}_{d_{1} \ldots d_{s}}(x ; \boldsymbol{\lambda})$ :

$$
\begin{align*}
& \hat{B}_{d_{1} \ldots d_{s}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \alpha B^{\prime}(x+s-1 ; \boldsymbol{\lambda}) \frac{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s-1}}\right](x ; \boldsymbol{\lambda})}{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s-1}}\right](x+1 ; \boldsymbol{\lambda})} \frac{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s}}\right](x+1 ; \boldsymbol{\lambda})}{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s}}\right](x ; \boldsymbol{\lambda})} \\
& \hat{D}_{d_{1} \ldots d_{s}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \alpha D^{\prime}(x ; \boldsymbol{\lambda}) \frac{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s-1}}\right](x+1 ; \boldsymbol{\lambda})}{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s-1}}\right](x ; \boldsymbol{\lambda})} \frac{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s}}\right](x-1 ; \boldsymbol{\lambda})}{\mathrm{W}_{\mathrm{C}}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{s}}\right](x ; \boldsymbol{\lambda})} \tag{A.20}
\end{align*}
$$

## B Proof of Conjecture 2 in Ref. [22]

In [22] we discussed the recurrence relations with constant coefficients for the multi-indexed Wilson (W) and Askey-Wilson (AW) polynomials and presented Conjecture 2,

Conjecture 2 in [22] For any polynomial $Y(\eta)$, we take $X(\eta)$ as

$$
X(\eta)=I\left[\Xi_{\mathcal{D}} Y\right](\eta), \quad \operatorname{deg} X(\eta)=L=\ell_{\mathcal{D}}+\operatorname{deg} Y(\eta)+1
$$

Then the multi-indexed Wilson and Askey-Wilson polynomials $P_{\mathcal{D}, n}(\eta)$ satisfy $1+2 L$ term recurrence relations with constant coefficients:

$$
X(\eta) P_{\mathcal{D}, n}(\eta)=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \quad\left(\forall n \in \mathbb{Z}_{\geq 0}\right)
$$

Here we prove this conjecture by the same method used in $\S$ 4.1. The 'step 0 ' was given in [22]. We follow the notation of [11, 22]. (Many same symbols are used for ( $q$ - $) \mathrm{R}$ and (A)W cases, but all the quantities used in this appendix correspond to (A)W cases.)

## B. 1 Step 1

The sinusoidal coordinates are $\eta(x)=x^{2}(\mathrm{~W})$ and $\eta(x)=\cos x$ (AW), and the parameters $a_{i}(i=1,2,3,4)$ satisfy $\left\{a_{1}^{*}, a_{2}^{*}\right\}=\left\{a_{1}, a_{2}\right\}$ (as a set) and $\left\{a_{3}^{*}, a_{4}^{*}\right\}=\left\{a_{3}, a_{4}\right\}$ (as a set). The denominator polynomials $\Xi_{\mathcal{D}}(\eta)$ and the multi-indexed (Askey-)Wilson polynomials $P_{\mathcal{D}, n}(\eta)$ ( $n \in \mathbb{Z}_{\geq 0}$ ) are

$$
\begin{array}{rlrl}
\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \Xi_{\mathcal{D}}(\eta(x) ; \boldsymbol{\lambda}), & \operatorname{deg} \Xi_{\mathcal{D}}(\eta)=\ell_{\mathcal{D}}, & \Xi_{\mathcal{D}}^{*}(\eta)=\Xi_{\mathcal{D}}(\eta) \\
\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} P_{\mathcal{D}, n}(\eta(x) ; \boldsymbol{\lambda}), & \operatorname{deg} P_{\mathcal{D}, n}(\eta)=\ell_{\mathcal{D}}+n, \quad P_{\mathcal{D}, n}^{*}(\eta ; \boldsymbol{\lambda})=P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda}), \tag{B.2}
\end{array}
$$

and $P_{\mathcal{D}, 0}(\eta ; \boldsymbol{\lambda}) \propto \Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}+\boldsymbol{\delta})$. Let us define the set of finite linear combinations of $P_{\mathcal{D}, n}(\eta)$, $\mathcal{U}_{\mathcal{D}} \subset \mathbb{C}[\eta]$, by

$$
\begin{equation*}
\mathcal{U}_{\mathcal{D}} \stackrel{\text { def }}{=} \operatorname{Span}\left\{P_{\mathcal{D}, n}(\eta) \mid n \in \mathbb{Z}_{\geq 0}\right\} \tag{B.3}
\end{equation*}
$$

Since the degree of $P_{\mathcal{D}, n}(\eta)$ is $\ell_{\mathcal{D}}+n$, it is trivial that $p(\eta) \in \mathcal{U}_{\mathcal{D}} \Rightarrow \operatorname{deg} p \geq \ell_{\mathcal{D}}$, except for $p(\eta)=0$. The multi-indexed (Askey-)Wilson polynomials $\check{P}_{\mathcal{D}, n}(x)$ with $x \in \mathbb{C}$ satisfy second order difference equations,

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{B.4}
\end{equation*}
$$

where $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})$ is

$$
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})=V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(e^{\gamma p}-\frac{\check{\Xi}_{\mathcal{D}}(x-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}\right)
$$

$$
\begin{equation*}
+V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(e^{-\gamma p}-\frac{\check{\Xi}_{\mathcal{D}}(x+i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}\right) \tag{B.5}
\end{equation*}
$$

For $p(\eta) \in \mathbb{C}[\eta], \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})$ acts on $\check{p}(x) \stackrel{\text { def }}{=} p(\eta(x))$ as

$$
\begin{align*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{p}(x)= & V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}]}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(\check{p}(x-i \gamma)-\frac{\check{\Xi}_{\mathcal{D}}(x-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} \check{p}(x)\right) \\
& +V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(\check{p}(x+i \gamma)-\frac{\check{\Xi}_{\mathcal{D}}(x+i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} \check{p}(x)\right) . \tag{B.6}
\end{align*}
$$

Let zeros of $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}+\boldsymbol{\delta})$ be $\beta_{j}^{(\eta)}$ and $\beta_{j}^{\prime(\eta)}\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right)$, respectively, which are simple for generic parameters. We define $\beta_{j}$ and $\beta_{j}^{\prime}$ as $\beta_{j}^{(\eta)}=\eta\left(\beta_{j}\right)$ and $\beta_{j}^{\prime(\eta)}=\eta\left(\beta_{j}^{\prime}\right)$. (For $x \in \mathbb{C}, \eta=\eta(x)$ are not one-to-one functions, but it does not cause any problems in the following argument.)

Let us consider the condition such that $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{p}(x)(\overline{\mathrm{B} .6)}$ is a polynomial in $\eta(x)$. The poles at $x=\beta_{j}^{\prime}, \beta_{j} \pm i \frac{\gamma}{2}$ in (B.6) should be canceled. First we consider $x=\beta_{j}^{\prime}$. Since $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}) \propto \check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})(n>0)$ do not have common roots for generic parameters, (B.6) with $\check{p}(x)=\check{P}_{\mathcal{D}, n}(x)$ implies that the poles at $x=\beta_{j}^{\prime}$ are canceled, namely,

$$
\begin{align*}
& V\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\ddot{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)} \check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right) \\
+ & V^{*}\left(\beta_{j}^{\prime} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)} \check{\Xi}_{\mathcal{D}}\left(\beta_{j}^{\prime}+i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)=0 \quad\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right) . \tag{B.7}
\end{align*}
$$

This relation implies that we do not need bother the poles at $x=\beta_{j}^{\prime}$ in (B.6) for general $p(\eta)$. Next we consider $x=\beta_{j} \pm i \frac{\gamma}{2}$. For generic parameters, $\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)$ and $\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)$ do not common roots, and the numerators of $V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)$ and $V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)$ do not cancel the poles coming from $\check{\Xi}_{\mathcal{D}}\left(x \pm i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)$, and zeros of the denominators of $V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)$ and $V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)$ do not coincide with $\beta_{j} \pm i \frac{\gamma}{2}$. The residue of the first term of (B.6) at $x=\beta_{j}+i \frac{\gamma}{2}$ is

$$
V\left(\beta_{j}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}+i \gamma ; \boldsymbol{\lambda}\right)}{\left.\frac{d}{d x} \check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)\right|_{x=\beta_{j}+i \frac{\gamma}{2}}}\left(\check{p}\left(\beta_{j}-i \frac{\gamma}{2}\right)-\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}+i \frac{\gamma}{2}\right)\right),
$$

and that of the second term of $(\overline{\mathrm{B} .6})$ at $x=\beta_{j}-i \frac{\gamma}{2}$ is
$V^{*}\left(\beta_{j}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-i \gamma ; \boldsymbol{\lambda}\right)}{\left.\frac{d}{d x} \check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)\right|_{x=\beta_{j}-i \frac{\gamma}{2}}}\left(\check{p}\left(\beta_{j}+i \frac{\gamma}{2}\right)-\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}-i \frac{\gamma}{2}\right)\right)$.

These residues should be vanished. So we obtain the conditions:

$$
\begin{equation*}
\frac{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}-i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)}{\check{\Xi}_{\mathcal{D}}\left(\beta_{j}+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right)} \check{p}\left(\beta_{j}+i \frac{\gamma}{2}\right)=\check{p}\left(\beta_{j}-i \frac{\gamma}{2}\right) \quad\left(j=1,2, \ldots, \ell_{\mathcal{D}}\right) . \tag{B.8}
\end{equation*}
$$

Let us assume $\operatorname{deg} p(\eta)<\ell_{\mathcal{D}}$. Without loss of generality, we take $p(\eta)$ is a monic polynomial. Then the number of adjustable coefficients of $p(\eta)$ is $\operatorname{deg} p(\eta)$. On the other hand, the number of conditions (B.8) is $\ell_{\mathcal{D}}$. Therefore the conditions (B.8) can not be satisfied for generic parameters, except for $p(\eta)=0$.

Since any polynomial $p(\eta)$ is expanded as

$$
p(\eta)=\sum_{n=0}^{\operatorname{deg} p-\ell_{\mathcal{D}}} a_{n} P_{\mathcal{D}, n}(\eta)+r(\eta), \quad \operatorname{deg} r(\eta)<\ell_{\mathcal{D}} \quad\left(p(\eta)=r(\eta) \text { for } \operatorname{deg} p<\ell_{\mathcal{D}}\right)
$$

we have

$$
\begin{aligned}
& \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{p}(x): \text { a polynomial in } \eta(x) \\
\Leftrightarrow & \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{r}(x): \text { a polynomial in } \eta(x) \Leftrightarrow r(\eta)=0
\end{aligned}
$$

Therefore we obtain the following proposition:

Proposition 3 For $p(\eta) \in \mathbb{C}[\eta]$, the following holds:

$$
\begin{equation*}
p(\eta) \in \mathcal{U}_{\mathcal{D}} \Leftrightarrow \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{p}(x): \text { a polynomial in } \eta(x) \tag{B.9}
\end{equation*}
$$

## B. 2 Step 2

Let us consider a polynomial $X(\eta)$ giving the following recurrence relations with constant coefficients,

$$
\begin{equation*}
X(\eta) P_{\mathcal{D}, n}(\eta)=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \quad\left(\forall n \in \mathbb{Z}_{\geq 0}\right) \tag{B.10}
\end{equation*}
$$

where $P_{\mathcal{D}, n}(\eta)=0(n<0)$. For $X(\eta), \check{X}(x)$ is defined by

$$
\begin{equation*}
\check{X}(x) \stackrel{\text { def }}{=} X(\eta(x)) . \tag{B.11}
\end{equation*}
$$

From Proposition $3, X(\eta)$ in (B.10) should satisfy

$$
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right): \text { a polynomial in } \eta(x)
$$

Action of $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})$ on $\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ is

$$
\begin{aligned}
& \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right) \\
& =\check{X}(x) \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \\
& \quad+V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}(\check{X}(x-i \gamma)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x-i \gamma ; \boldsymbol{\lambda}) \\
& \quad+V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}]}\right]} \frac{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}(\check{X}(x+i \gamma)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x+i \gamma ; \boldsymbol{\lambda}),\right.
\end{aligned}
$$

namely,

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right)=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})+F(x) . \tag{B.12}
\end{equation*}
$$

Here $F(x)$ is

$$
\begin{align*}
F(x)= & V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}(\check{X}(x-i \gamma)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x-i \gamma ; \boldsymbol{\lambda}) \\
& +V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}(\check{X}(x+i \gamma)-\check{X}(x)) \check{P}_{\mathcal{D}, n}(x+i \gamma ; \boldsymbol{\lambda}) . \tag{B.13}
\end{align*}
$$

Equations (3.6) in [22] and (B.11) imply

$$
\check{X}(x-i \gamma)-\check{X}(x)=(\eta(x-i \gamma)-\eta(x)) \times\left(\text { a polynomial in } \eta\left(x-i \frac{\gamma}{2}\right)\right) .
$$

In order to cancel the zeros of $\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)=\Xi_{\mathcal{D}}\left(\eta\left(x-i \frac{\gamma}{2}\right) ; \boldsymbol{\lambda}\right)$ in (B.13), the polynomial appeared in the above expression should have the following form,

$$
\begin{equation*}
\check{X}(x-i \gamma)-\check{X}(x)=(\eta(x-i \gamma)-\eta(x)) \check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) Y\left(\eta\left(x-i \frac{\gamma}{2}\right)\right) \tag{B.14}
\end{equation*}
$$

where $Y(\eta)$ is an arbitrary polynomial in $\eta$. Note that this $X(\eta)$ can be expressed in terms of the map $I$ eq.(3.10) in [22] by eq.(3.12) in [22],

$$
\begin{equation*}
X(\eta)=I\left[\Xi_{\mathcal{D}} Y\right](\eta) \tag{B.15}
\end{equation*}
$$

Then $F(x)(\bar{B} .13)$ becomes

$$
\begin{align*}
F(x)= & V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x-i \gamma)-\eta(x)) \check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) Y\left(\eta\left(x-i \frac{\gamma}{2}\right)\right) \check{P}_{\mathcal{D}, n}(x-i \gamma ; \boldsymbol{\lambda}) \quad(\mathrm{B}  \tag{B.16}\\
& +V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x+i \gamma)-\eta(x)) \check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) Y\left(\eta\left(x+i \frac{\gamma}{2}\right)\right) \check{P}_{\mathcal{D}, n}(x+i \gamma ; \boldsymbol{\lambda}) .
\end{align*}
$$

From the explicit forms of $V(x ; \boldsymbol{\lambda})$, we have

$$
V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x-i \gamma)-\eta(x))
$$

$$
\begin{align*}
& = \begin{cases}-\frac{\prod_{j=1}^{2}\left(a_{j}-\frac{1}{2} M^{\prime}+i x\right) \cdot \prod_{j=3}^{4}\left(a_{j}+\frac{1}{2} M^{\prime}+i x\right)}{2 i x} & : \mathrm{W} \\
-\frac{\left(q^{-1}-1\right) \prod_{j=1}^{2}\left(1-a_{j} q^{-\frac{1}{2} M^{\prime}} e^{i x}\right) \cdot \prod_{j=3}^{4}\left(1-a_{j} q^{\frac{1}{2} M^{\prime}} e^{i x}\right)}{2 e^{2 i x}\left(e^{i x}-e^{-i x}\right)} & : \mathrm{AW}\end{cases}  \tag{B.17}\\
& = \begin{cases}\frac{\prod_{j=1}^{2}\left(a_{j}-\frac{1}{2} M^{\prime}-i x\right) \cdot \prod_{j=3}^{4}\left(a_{j}+\frac{1}{2} M^{\prime}-i x\right)}{2 i x} & : \mathrm{W} \\
\frac{\left(q^{-1}-1\right) \prod_{j=1}^{2}\left(1-a_{j} q^{-\frac{1}{2} M^{\prime}} e^{-i x}\right) \cdot \prod_{j=3}^{4}\left(1-a_{j} q^{\frac{1}{2} M^{\prime}} e^{-i x}\right)}{2 e^{-2 i x}\left(e^{i x}-e^{-i x}\right)} & : \mathrm{AW}\end{cases}
\end{align*}
$$

where $M^{\prime}=M_{\mathrm{I}}-M_{\mathrm{II}}$. For AW case, they are rational functions of $z=e^{i x}$. Residues of (B.17) (B.18) at $x=0(\mathrm{~W})$ or $z= \pm 1$ (AW) are related as

$$
\begin{aligned}
\mathrm{W}: & \operatorname{Res}_{x=0}\left(V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x-i \gamma)-\eta(x))\right) \\
& =-\operatorname{Res}_{x=0}\left(V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x+i \gamma)-\eta(x))\right) \\
& \begin{aligned}
\mathrm{AW}: & \operatorname{Res}_{z= \pm 1}\left(V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x-i \gamma)-\eta(x))\right) \\
& =-\operatorname{Res}_{z= \pm 1}\left(V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x+i \gamma)-\eta(x))\right) .
\end{aligned}
\end{aligned}
$$

At $x=0$ or $z= \pm 1$, we have

$$
\begin{aligned}
\mathrm{W}:\left.\quad \eta\left(x-i \frac{\gamma}{2}\right)\right|_{x=0}=\left.\eta\left(x+i \frac{\gamma}{2}\right)\right|_{x=0},\left.\quad \eta(x-i \gamma)\right|_{x=0}=\left.\eta(x+i \gamma)\right|_{x=0} \\
\mathrm{AW}:\left.\eta\left(x-i \frac{\gamma}{2}\right)\right|_{z= \pm 1}=\left.\eta\left(x+i \frac{\gamma}{2}\right)\right|_{z= \pm 1},\left.\quad \eta(x-i \gamma)\right|_{z= \pm 1}=\left.\eta(x+i \gamma)\right|_{z= \pm 1}
\end{aligned}
$$

Combining these and (B.16), we obtain

$$
\mathrm{W}: \operatorname{Res}_{x=0} F(x)=0, \quad \mathrm{AW}: \operatorname{Res}_{z= \pm 1} F(x)=0 .
$$

Therefore $F(x)(\overline{\mathrm{B} .16})$ is a polynomial in $x$ for W , a Laurent polynomial in $z$ for AW. By introducing an involution $\mathcal{I}: x \rightarrow-x\left(\Rightarrow z \rightarrow z^{-1}\right)$, we have

$$
\begin{aligned}
& \mathcal{I}\left(V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x-i \gamma)-\eta(x))\right)=V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)(\eta(x+i \gamma)-\eta(x)), \\
& \mathcal{I}\left(\eta\left(x-i \frac{\gamma}{2}\right)\right)=\eta\left(x+i \frac{\gamma}{2}\right), \quad \mathcal{I}(\eta(x-i \gamma))=\eta(x+i \gamma) .
\end{aligned}
$$

Hence $F(x)(\overline{\mathrm{B} .16)}$ satisfies $\mathcal{I}(F(x))=F(x)$, which implies that $F(x)$ is a polynomial in $\eta(x)$. Therefore, from (B.12), we have shown that $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\left(\check{X}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right)$ is a polynomial in $\eta(x)$.

## B. 3 Step 3

Let us summarize the result. For the denominator polynomial $\Xi_{\mathcal{D}}(\eta)=\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and a polynomial in $\eta, Y(\eta)(\neq 0)$, we set $X(\eta)=X(\eta ; \boldsymbol{\lambda})=X^{\mathcal{D}, Y}(\eta ; \boldsymbol{\lambda})$ as

$$
\begin{equation*}
X(\eta)=I\left[\Xi_{\mathcal{D}} Y\right](\eta), \quad \operatorname{deg} X(\eta)=L=\ell_{\mathcal{D}}+\operatorname{deg} Y(\eta)+1 \tag{B.19}
\end{equation*}
$$

where $\Xi_{\mathcal{D}} Y$ means a polynomial $\left(\Xi_{\mathcal{D}} Y\right)(\eta)=\Xi_{\mathcal{D}}(\eta) Y(\eta)$. Note that $L \geq M+1$ because of $\ell_{\mathcal{D}} \geq M$. The minimal degree one, which corresponds to $Y(\eta)=1$, is

$$
\begin{equation*}
X_{\min }(\eta)=I\left[\Xi_{\mathcal{D}}\right](\eta), \quad \operatorname{deg} X_{\min }(\eta)=\ell_{\mathcal{D}}+1 \tag{B.20}
\end{equation*}
$$

Then we have the following theorem.
Theorem 4 For any polynomial $Y(\eta)(\neq 0)$, we take $X(\eta)=X^{\mathcal{D}, Y}(\eta)$ as (B.19). Then the multi-indexed (Askey-)Wilson polynomials $P_{\mathcal{D}, n}(\eta)$ satisfy $1+2 L$ term recurrence relations with constant coefficients:

$$
\begin{equation*}
X(\eta) P_{\mathcal{D}, n}(\eta)=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}(\eta) \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{B.21}
\end{equation*}
$$

Remark 1 We have assumed the convention $P_{\mathcal{D}, n}(\eta)=0(n<0)$. If we replace $\sum_{k=-L}^{L}$ with $\sum_{k=-\min (L, n)}^{L}$, it is unnecessary.
Remark 2 As shown near ( $\overline{\text { B.14 }})$, any polynomial $X(\eta)$ giving the recurrence relations with constant coefficients must have the form ( $\overline{\mathrm{B} .19}$ ).
Remark 3 If $Y(\eta)$ satisfies $Y^{*}(\eta)=Y(\eta)$, we have $X^{*}(\eta)=X(\eta)$ and $r_{n, k}^{X, \mathcal{D} *}=r_{n, k}^{X, \mathcal{D}}$.

## C $\quad r_{n, 0}^{X, \mathcal{D}}$ in (B.11) and (B.12) of Ref. [22]

In [22] we discussed the recurrence relations with constant coefficients for the multi-indexed Wilson (W) and Askey-Wilson (AW) polynomials. As examples, the explicit forms of $r_{n, k}^{X, \mathcal{D}}$ for $\mathcal{D}=\left\{1^{\mathrm{I}}\right\}$ (type I) and $X(\eta)=X_{\min }(\eta)$ are presented in Appendix B. 3 and B. 4 of [22]. However, we did not write down $r_{n, 0}^{X, \mathcal{D}}$ explicitly due to their lengthy expressions. Here we present concise expressions of $r_{n, 0}^{X, \mathcal{D}}$. We follow the notation of [22].

For $\mathrm{W}, r_{n, 0}^{X, \mathcal{D}}$ in eq.(B.11) of [22] is expressed as

$$
r_{n, 0}^{X, \mathcal{D}}=X_{0}-\frac{\sigma_{1}+n}{\sigma_{1}+n-2}\left(\sigma_{1}^{\prime}+n+3\right)\left(\sigma_{1}^{\prime}+n\right) \prod_{j=1}^{2}\left(a_{j}+a_{4}+n\right)_{2} \cdot r_{n, 2}^{X, \mathcal{D}}
$$

$$
\begin{align*}
& -\frac{\sigma_{1}+n-1}{\sigma_{1}+n-2} \frac{\sigma_{1}^{\prime}+n+2}{\sigma_{1}^{\prime}+n+1}\left(\sigma_{1}^{\prime}+n\right) \prod_{j=1}^{2}\left(a_{j}+a_{4}+n\right) \cdot r_{n, 1}^{X, \mathcal{D}} \\
& -\frac{\sigma_{1}+n-3}{\sigma_{1}+n-2} \frac{\sigma_{1}^{\prime}+n}{\sigma_{1}^{\prime}+n+1} \frac{r_{n,-1}^{X, \mathcal{D}}}{\left(\sigma_{1}^{\prime}+n-1\right) \prod_{j=1}^{2}\left(a_{j}+a_{4}+n-1\right)} \\
& -\frac{\sigma_{1}+n-4}{\sigma_{1}+n-2} \frac{r_{n,-2}^{X, \mathcal{D}}}{\left(\sigma_{1}^{\prime}+n+1\right)\left(\sigma_{1}^{\prime}+n-2\right) \prod_{j=1}^{2}\left(a_{j}+a_{4}+n-2\right)_{2}}, \tag{C.1}
\end{align*}
$$

where $X_{0}$ is given by

$$
\begin{align*}
& X_{0} \stackrel{\text { def }}{=} X\left(-\left(a_{4}+\frac{1}{2}\right)^{2}\right) \\
& =\frac{1}{32}\left(1+2 a_{4}\right)^{2}\left(2-7 \sigma_{1}^{\prime}-20 \sigma_{2}^{\prime}-8 a_{4}-4\left(3+\sigma_{1}^{\prime}\right) a_{4}^{2}\right. \\
&  \tag{C.2}\\
& \left.\quad+\left(8\left(\sigma_{1}^{\prime}+\sigma_{2}^{\prime}\right)-1+4 a_{4}\left(1+a_{4}\right)\right) \sigma_{1}-8 \sigma_{1}^{\prime} \sigma_{2}\right) .
\end{align*}
$$

For AW, $r_{n, 0}^{X, \mathcal{D}}$ in eq.(B.12) of [22] is expressed as

$$
\begin{align*}
r_{n, 0}^{X, \mathcal{D}}= & X_{0}-a_{4}^{-2} q^{-2} \frac{1-\sigma_{2} q^{n}}{1-\sigma_{2} q^{n-2}}\left(1-\sigma_{2}^{\prime} q^{n+3}\right)\left(1-\sigma_{2}^{\prime} q^{n}\right) \prod_{j=1}^{2}\left(a_{j} a_{4} q^{n} ; q\right)_{2} \cdot r_{n, 2}^{X, \mathcal{D}} \\
& -a_{4}^{-1} q^{-1} \frac{1-\sigma_{2} q^{n-1}}{1-\sigma_{2} q^{n-2}} \frac{1-\sigma_{2}^{\prime} q^{n+2}}{1-\sigma_{2}^{\prime} q^{n+1}}\left(1-\sigma_{2}^{\prime} q^{n}\right) \prod_{j=1}^{2}\left(1-a_{j} a_{4} q^{n}\right) \cdot r_{n, 1}^{X, \mathcal{D}} \\
& -a_{4} q \frac{1-\sigma_{2} q^{n-3}}{1-\sigma_{2} q^{n-2}} \frac{1-\sigma_{2}^{\prime} q^{n}}{1-\sigma_{2}^{\prime} q^{n+1}} \frac{r_{n,-1}^{X, \mathcal{D}}}{\left(1-\sigma_{2}^{\prime} q^{n-1}\right) \prod_{j=1}^{2}\left(1-a_{j} a_{4} q^{n-1}\right)} \\
& -a_{4}^{2} q^{2} \frac{1-\sigma_{2} q^{n-4}}{1-\sigma_{2} q^{n-2}} \frac{r_{n,-2}^{X, \mathcal{D}}}{\left(1-\sigma_{2}^{\prime} q^{n+1}\right)\left(1-\sigma_{2}^{\prime} q^{n-2}\right) \prod_{j=1}^{2}\left(a_{j} a_{4} q^{n-2} ; q\right)_{2}} \tag{C.3}
\end{align*}
$$

where $X_{0}$ is given by

$$
\begin{align*}
& X_{0} \stackrel{\text { def }}{=} X\left(\frac{1}{2}\left(a_{4} q^{\frac{1}{2}}+a_{4}^{-1} q^{-\frac{1}{2}}\right)\right) \\
& =\frac{1+a_{4}^{2} q}{2 q^{\frac{1}{2}}(1+q) \sigma_{2} a_{4}^{2}}\left(\sigma_{1}^{\prime} q^{3} a_{4}+\left(1-\sigma_{2}^{\prime} q\right) a_{4}^{2} q^{2}-q(1+q)\left(1-\sigma_{2}^{\prime}\right) \sigma_{1} a_{4}\right. \\
&  \tag{C.4}\\
& \left.\quad+\left(1-a_{4}^{2}-(1+q) \sigma_{2}^{\prime}\right) \sigma_{2}\right) .
\end{align*}
$$

These expressions are obtained by the following simple observation. Substituting some specific value $\eta_{0}$ for $\eta$ in the recurrence relations, we have $X\left(\eta_{0}\right) P_{\mathcal{D}, n}\left(\eta_{0}\right)=\sum_{k=-L}^{L} r_{n, k}^{X, \mathcal{D}} P_{\mathcal{D}, n+k}\left(\eta_{0}\right)$, which gives

$$
\begin{equation*}
r_{n, 0}^{X, \mathcal{D}}=X\left(\eta_{0}\right)-\sum_{\substack{k=-L \\ k \neq 0}}^{L} \frac{P_{\mathcal{D}, n+k}\left(\eta_{0}\right)}{P_{\mathcal{D}, n}\left(\eta_{0}\right)} r_{n, k}^{X, \mathcal{D}} . \tag{C.5}
\end{equation*}
$$

Let the index set $\mathcal{D}$ be $\mathcal{D}=\left\{d_{1}, d_{2}, \ldots, d_{M}\right\}=\left\{d_{1}^{\mathrm{I}}, \ldots, d_{M_{\mathrm{I}}}^{\mathrm{I}}, d_{1}^{\mathrm{II}}, \ldots, d_{M_{\mathrm{II}}}^{\mathrm{II}}\right\}\left(0 \leq d_{1}^{\mathrm{I}}<\right.$ $\left.\cdots<d_{M_{\mathrm{I}}}^{\mathrm{I}}, 0 \leq d_{1}^{\mathrm{II}}<\cdots<d_{M_{\mathrm{II}}}^{\mathrm{II}}, M=M_{\mathrm{I}}+M_{\mathrm{II}}\right)$. Let $x_{0}$ and $\eta_{0}$ be

$$
\begin{align*}
& x_{0} \stackrel{\text { def }}{=}-i \gamma\left(\lambda_{4}+\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)\right), \\
& \eta_{0} \stackrel{\text { def }}{=} \eta\left(x_{0}\right)=\left\{\begin{array}{ll}
-\left(a_{4}+\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)\right)^{2} & : \mathrm{W} \\
\frac{1}{2}\left(a_{4} q^{\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)}+a_{4}^{-1} q^{-\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)}\right) & : \mathrm{AW}
\end{array} .\right. \tag{C.6}
\end{align*}
$$

Note that, as coordinates $x$ and $\eta$, these values $x_{0}$ and $\eta_{0}$ are unphysical ( $x_{0}$ is imaginary, $\eta_{0}$ is out of the range of $\eta$ ). The multi-indexed (Askey-)Wilson polynomials $P_{\mathcal{D}, n}(\eta)$ take 'simple' values at these 'unphysical' values $\eta_{0}$ :

$$
\begin{align*}
& P_{\mathcal{D}, n}\left(\eta_{0}\right): W \\
& =(-1)^{\ell_{\mathcal{D}}+n} c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \\
& \times \prod_{j=1}^{M_{\mathrm{I}}} \frac{\left(a_{4}-a_{1}+1, a_{4}-a_{2}+1, a_{3}+a_{4}\right)_{d_{j}^{\mathrm{I}}}}{\left(a_{3}+a_{4}-a_{1}-a_{2}+d_{j}^{\mathrm{I}}+1\right)_{d_{j}^{\mathrm{I}}}} \cdot \frac{\prod_{1 \leq j<k \leq M_{\mathrm{I}}}\left(a_{3}+a_{4}-a_{1}-a_{2}+d_{j}^{\mathrm{I}}+d_{k}^{\mathrm{I}}+1\right)}{\prod_{j=1}^{M_{\mathrm{I}}}\left(a_{4}-a_{1}+1, a_{4}-a_{2}+1, a_{3}+a_{4}\right)_{j-1}} \\
& \times \prod_{j=1}^{M_{\mathrm{II}}} \frac{\left(a_{1}-a_{4}+1, a_{2}-a_{4}+1,1-a_{3}-a_{4}\right)_{d_{j}^{\mathrm{II}}}}{\left(a_{1}+a_{2}-a_{3}-a_{4}+d_{j}^{\mathrm{II}}+1\right)_{d_{j}^{\mathrm{II}}}} \cdot \frac{\prod_{1 \leq j<k \leq M_{\mathrm{II}}}\left(a_{1}+a_{2}-a_{3}-a_{4}+d_{j}^{\mathrm{II}}+d_{k}^{\mathrm{II}}+1\right)}{\prod_{j=1}^{M_{\mathrm{II}}}\left(a_{1}-a_{4}+1, a_{2}-a_{4}+1,1-a_{3}-a_{4}\right)_{j-1}} \\
& \times \prod_{\substack{1 \leq j \leq M_{\mathrm{I}} \\
1 \leq k \leq M_{\mathrm{II}}}} \frac{\left(a_{4}-a_{1}+j-k\right)\left(a_{4}-a_{2}+j-k\right)\left(a_{3}+a_{4}+j-k\right)}{a_{3}+a_{4}-a_{1}-a_{2}+d_{j}^{\mathrm{I}}-d_{k}^{\mathrm{II}}} \\
& \times \frac{\left(a_{1}+a_{4}, a_{2}+a_{4}, a_{3}+a_{4}\right)_{n}}{\left(a_{1}+a_{2}+a_{3}+a_{4}+n-1\right)_{n}} \prod_{j=1}^{M_{\mathrm{I}}} \frac{a_{3}+a_{4}+d_{j}^{\mathrm{I}}+n}{a_{3}+a_{4}+j-1} \cdot \prod_{j=1}^{M_{\mathrm{II}}} \frac{d_{j}^{\mathrm{II}}+1-a_{3}-a_{4}}{d_{j}^{\mathrm{II}}+1-n-a_{3}-a_{4}},  \tag{C.7}\\
& P_{\mathcal{D}, n}\left(\eta_{0}\right): A W \\
& =\left(2 a_{4} q^{\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)}\right)^{-\ell_{\mathcal{D}}-n} c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \\
& \times \prod_{j=1}^{M_{\mathrm{I}}} \frac{\left(a_{1}^{-1} a_{4} q, a_{2}^{-1} a_{4} q, a_{3} a_{4} ; q\right)_{d_{j}^{\mathrm{I}}}}{\left(a_{1}^{-1} a_{2}^{-1} a_{3} a_{4} q^{d_{j}^{\mathrm{I}}+1} ; q\right)_{d_{j}^{\mathrm{I}}}} \cdot \frac{\prod_{1 \leq j<k \leq M_{\mathrm{I}}}\left(1-a_{1}^{-1} a_{2}^{-1} a_{3} a_{4} q^{d_{j}^{\mathrm{I}}+d_{k}^{\mathrm{I}}+1}\right)}{\prod_{j=1}^{M_{\mathrm{I}}}\left(a_{1}^{-1} a_{4} q, a_{2}^{-1} a_{4} q, a_{3} a_{4} ; q\right)_{j-1}} \\
& \times \prod_{j=1}^{M_{\mathrm{II}}} \frac{\left(a_{1} a_{4}^{-1} q, a_{2} a_{4}^{-1} q, a_{3}^{-1} a_{4}^{-1} q ; q\right)_{d_{j}^{\mathrm{II}}}}{\left(a_{1} a_{2} a_{3}^{-1} a_{4}^{-1} q^{d_{j}^{\mathrm{II}}+1} ; q\right)_{d_{j}^{\mathrm{II}}}} \cdot \frac{\prod_{1 \leq j<k \leq M_{\mathrm{II}}}\left(1-a_{1} a_{2} a_{3}^{-1} a_{4}^{-1} q^{d_{j}^{\mathrm{II}}+d_{k}^{\mathrm{II}}+1}\right)}{\prod_{j=1}^{M_{\mathrm{II}}}\left(a_{1} a_{4}^{-1} q, a_{2} a_{4}^{-1} q, a_{3}^{-1} a_{4}^{-1} q ; q\right)_{j-1}} \\
& \times\left(a_{4} q^{\frac{1}{2}\left(M_{\mathrm{I}}-M_{\mathrm{II}}\right)}\right)^{2 \sum_{j=1}^{M_{\mathrm{II}}} d_{j}^{\mathrm{II}}-M_{\mathrm{II}}\left(M_{\mathrm{II}}-1\right)} q^{-M_{\mathrm{I}} M_{\mathrm{II}}\left(M_{\mathrm{II}}-1\right)}
\end{align*}
$$

$$
\begin{align*}
& \times \prod_{\substack{1 \leq j \leq M_{\mathrm{I}} \\
1 \leq k \leq M_{\mathrm{II}}}} \frac{\left(a_{4} q^{j-1}-a_{1} q^{k-1}\right)\left(a_{4} q^{j-1}-a_{2} q^{k-1}\right)\left(a_{3} a_{4} q^{j-1}-q^{k-1}\right)}{a_{3} a_{4} q^{d_{j}^{\mathrm{I}}}-a_{1} a_{2} q^{d_{k}^{\mathrm{II}}}} \\
& \times q^{-M_{\mathrm{II}} n} \frac{\left(a_{1} a_{4}, a_{2} a_{4}, a_{3} a_{4} ; q\right)_{n}}{\left(a_{1} a_{2} a_{3} a_{4} q^{n-1} ; q\right)_{n}} \prod_{j=1}^{M_{\mathrm{I}}} \frac{1-a_{3} a_{4} q^{d_{j}^{\mathrm{I}}+n}}{1-a_{3} a_{4} q^{j-1}} \cdot \prod_{j=1}^{M_{\mathrm{II}}} \frac{1-a_{3}^{-1} a_{4}^{-1} q^{d_{j}^{\mathrm{II}}+1}}{1-a_{3}^{-1} a_{4}^{-1} q^{d_{j}^{\mathrm{I}}+1-n}}, \tag{C.8}
\end{align*}
$$

where $c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda})$ is given by eq.(A.7) in [11] and $\ell_{\mathcal{D}}=\sum_{j=1}^{M} d_{j}-\frac{1}{2} M(M-1)+2 M_{\mathrm{I}} M_{\mathrm{II}}$. For $M_{\text {II }}=0$ (type I only), this is a consequence of (6.5) $-(6.6)$.

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