# Dual Polynomials of the Multi-Indexed ( $q$-)Racah Orthogonal Polynomials 

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#### Abstract

We consider dual polynomials of the multi-indexed ( $q-$ )Racah orthogonal polynomials. The $M$-indexed ( $q$-)Racah polynomials satisfy the second order difference equations and various $1+2 L(L \geq M+1)$ term recurrence relations with constant coefficients. Therefore their dual polynomials satisfy the three term recurrence relations and various $2 L$-th order difference equations. This means that the dual multi-indexed $(q-)$ Racah polynomials are ordinary orthogonal polynomials and the Krall-type. We obtain new exactly solvable discrete quantum mechanics with real shifts, whose eigenvectors are described by the dual multi-indexed ( $q-$ )Racah polynomials. These quantum systems satisfy the closure relations, from which the creation/annihilation operators are obtained, but they are not shape invariant.


## 1 Introduction

Ordinary orthogonal polynomials in one variable satisfying second order differential equations are severely restricted by Bochner's theorem [1, 2]. Allowed polynomials are the Hermite, Laguerre, Jacobi and Bessel polynomials, but the weight function of the Bessel polynomial is not positive definite. Various attempts to avoid this no-go theorem have been carried out and there are three directions.

The first direction (i) is to change the second order to higher orders. This direction was initiated by Krall [3] and he classified the orthogonal polynomials satisfying fourth order differential equations [4]. Based on the Laguerre and Jacobi polynomials, by adding the Dirac delta functions to the weight functions, orthogonal polynomials satisfying higher order differential equations are obtained [5]-[9]. Such polynomials are called the Krall polynomials. The
second direction (ii) is to replace a differential equation with a difference equation. Studies in this direction were summarized as the Askey scheme of (basic-)hypergeometric orthogonal polynomials and various generalizations of the Bochner's theorem were proposed [10, 2]. By combining (i) and (ii), orthogonal polynomials satisfying higher order difference equations were also studied. We call such polynomials the Krall-type polynomials. Some of them have weight functions with delta functions [11] and some others have those without delta functions [12]-14]. The third direction (iii) is to allow missing degrees. This means the following situation: polynomials $\left\{\mathcal{P}_{n}\right\}\left(n \in \mathbb{Z}_{\geq 0}\right)$ are orthogonal and satisfy second order differential equation and form a complete set, but there are missing degrees, $\left\{\operatorname{deg} \mathcal{P}_{n} \mid n \in \mathbb{Z}_{\geq 0}\right\} \subsetneq \mathbb{Z}_{\geq 0}$ [15, 16]. By combining with (ii), we can also consider the situation in which second order differential equation is replaced with second order difference equation. These polynomials are called exceptional or multi-indexed orthogonal polynomials and various examples have been obtained for classical orthogonal polynomials [15]-[33]. We distinguish the following two cases; the set of missing degrees $\mathcal{I}=\mathbb{Z}_{\geq 0} \backslash\left\{\operatorname{deg} \mathcal{P}_{n} \mid n \in \mathbb{Z}_{\geq 0}\right\}$ is case-(1): $\mathcal{I}=\{0,1, \ldots, \ell-1\}$, or case-(2): $\mathcal{I} \neq\{0,1, \ldots, \ell-1\}$, where $\ell$ is a positive integer. The situation of case-(1) is called stable in [20]. In the case of finite systems such as $(q-)$ Racah polynomials, the index set $\mathbb{Z}_{\geq 0}$ is replaced by $\{0,1, \ldots, N\}$. It is also possible to combine three directions (i), (ii) and (iii), but such examples are not yet known.

Quantum mechanical formulation is useful for studying orthogonal polynomials. We consider three kinds of quantum mechanical systems: ordinary quantum mechanics (oQM), discrete quantum mechanics with pure imaginary shifts (idQM) [34- [37] and discrete quantum mechanics with real shifts (rdQM) [38]-[40]. Their features are the following:

|  | Schrödinger eq. | variable $x$ | examples of orthogonal polynomials |
| :---: | :---: | :---: | :--- |
| oQM | differential eq. | continuous | Hermite, Laguerre, Jacobi |
| idQM | difference eq. | continuous | continuous Hahn, (Askey-)Wilson |
| rdQM | difference eq. | discrete | Hahn, $(q-)$ Racah |

In our previous works we have taken second order differential or difference operators as Hamiltonians, but it is also allowed to take higher order operators as Hamiltonians. Exceptional and multi-indexed polynomials are obtained by applying the Darboux transformations with appropriate seed solutions to the exactly solvable quantum mechanical systems described by the classical orthogonal polynomials in the Askey scheme. When the virtual state wavefunctions are used as seed solutions, the case-(1) multi-indexed polynomials are obtained [21, 23, 25]. When the eigenstate and/or pseudo virtual state wavefunctions are used as seed
solutions, the case-(2) multi-indexed polynomials are obtained [31]-[33]. Another method to obtain exceptional and multi-indexed polynomials is to use the Krall-type polynomials [28]-30].

In this paper we discuss dual polynomials of the case-(1) multi-indexed ( $q$ - )Racah polynomials. Dual polynomials are introduced naturally for orthogonal polynomials of a discrete variable [2] and they are treated in the framework of rdQM [38]-40]. The polynomial $\mathcal{P}_{n}(\eta(x))=\check{\mathcal{P}}_{n}(x)$ and its dual polynomial $\mathcal{Q}_{x}\left(\mathcal{E}_{n}\right)=\check{\mathcal{Q}}_{x}(n)$ are related as $\check{\mathcal{P}}_{n}(x) \propto \check{\mathcal{Q}}_{x}(n)$. The roles of the variable and the label ( $=$ degree for ordinary orthogonal polynomials) are interchanged, and we have the following correspondence:
difference equation (w.r.t $x$ ) for $\check{\mathcal{P}}_{n}(x) \leftrightarrow$ recurrence relation (w.r.t $x$ ) for $\check{\mathcal{Q}}_{x}(n)$,
recurrence relation (w.r.t $n$ ) for $\check{\mathcal{P}}_{n}(x) \leftrightarrow$ difference equation (w.r.t $n$ ) for $\check{\mathcal{Q}}_{x}(n)$.
The multi-indexed $(q-)$ Racah polynomials satisfy the second order difference equations [25]. On the other hand, the multi-indexed polynomials do not satisfy the three term recurrence relations, which characterize the ordinary orthogonal polynomials [2], because they are not the ordinary orthogonal polynomials. They satisfy recurrence relations with more terms [41][48], and such recurrence relations for the multi-indexed ( $q$-) Racah polynomials are studied recently [49]. It is shown that the $M$-indexed ( $q-$ )Racah polynomials satisfy various $1+2 L$ ( $L \geq M+1$ ) term recurrence relations with constant coefficients. Therefore dual polynomials of the multi-indexed $(q$-)Racah polynomials satisfy the three term recurrence relations and various $2 L$-th order difference equations, namely they are ordinary orthogonal polynomials and the Krall-type. The weight functions do not contain delta functions (Kronecker deltas). By using these dual multi-indexed ( $q-$ )Racah polynomials, we construct new exactly solvable rdQM systems, whose Hamiltonians are not tridiagonal but " $(1+2 L)$-diagonal". These quantum systems satisfy the closure relations [50, 38], from which the creation and annihilation operators are obtained, but they are not shape invariant.

This paper is organized as follows. In section 2 the essence of the multi-indexed ( $q$-) Racah polynomials are recapitulated. In section 3 we define the dual polynomials of the multiindexed $(q-)$ Racah polynomials and present their properties. In section 4 we construct new exactly solvable rdQM systems described by the dual multi-indexed ( $q$ - Racah polynomials. The closure relations and the creation and annihilation operators are presented in $\S 4.1$ and the shape invariance is discussed in $\S 4.2$. Some examples are given in $\S 4.3$. Section 5 is for a summary and comments. In Appendix A some basic data of the multi-indexed ( $q$-)Racah
polynomials are summarized for readers' convenience.

## 2 Multi-indexed (q-)Racah Orthogonal Polynomials

In this section we recapitulate the properties of the case-(1) multi-indexed Racah (R) and $q$-Racah ( $q \mathrm{R}$ ) orthogonal polynomials [25, 49]. We follow the notation of [25, 49]. Various quantities depend on a set of parameters $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}, \ldots\right)$ and their dependence is expressed like, $f=f(\boldsymbol{\lambda}), f(x)=f(x ; \boldsymbol{\lambda})$. The parameter $q$ is $0<q<1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{\left(\lambda_{1}, \lambda_{2}, \ldots\right)}=$ $\left(q^{\lambda_{1}}, q^{\lambda_{2}}, \ldots\right)$. See Appendix A for the explicit forms of various quantities $\left(\check{\Xi}_{\mathcal{D}}(x), \Xi_{\mathcal{D}}(\eta)\right.$, $\left.\check{P}_{\mathcal{D}, n}(x), P_{\mathcal{D}, n}(\eta), \check{P}_{n}(x), P_{n}(\eta), B(x), D(x), \tilde{\boldsymbol{\delta}}, \phi_{0}(x), d_{\mathcal{D}, n}, A_{n}, C_{n}, \tilde{\mathcal{E}}_{\mathrm{v}}, I_{\boldsymbol{\lambda}}\right)$.

The set of parameters $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right)$, its shift $\boldsymbol{\delta}$ and $\kappa$ are

$$
\begin{array}{rll}
\mathrm{R}: & \boldsymbol{\lambda}=(a, b, c, d), & \boldsymbol{\delta}=(1,1,1,1),  \tag{2.1}\\
q \mathrm{R}: & q^{\boldsymbol{\lambda}}=(a, b, c, d), & \boldsymbol{\delta}=(1,1,1,1), \\
\kappa=q^{-1}
\end{array}
$$

For $N \in \mathbb{Z}_{>0}$, we take $n_{\max }=x_{\text {max }}=N$ and

$$
\begin{equation*}
\mathrm{R}: a=-N, \quad q \mathrm{R}: a=q^{-N} \tag{2.2}
\end{equation*}
$$

and assume the following parameter ranges:

$$
\begin{align*}
\mathrm{R}: & 0<d<a+b, \quad 0<c<1+d, \quad d+\max (\mathcal{D})+1<a+b \\
q \mathrm{R}: & 0<a b<d<1, \quad q d<c<1, \quad a b<d q^{\max (\mathcal{D})+1} . \tag{2.3}
\end{align*}
$$

Here $\mathcal{D}=\left\{d_{1}, d_{2}, \ldots, d_{M}\right\}\left(d_{1}<d_{2}<\cdots<d_{M}, d_{j} \in \mathbb{Z}_{\geq 1}\right)$ is the multi-index set. The denominator polynomials $\Xi_{\mathcal{D}}(\eta)$ and the multi-indexed $(q-)$ Racah polynomials $P_{\mathcal{D}, n}(\eta)(n=$ $0,1, \ldots, n_{\max }$ ) are polynomials in the sinusoidal coordinate $\eta$,

$$
\begin{align*}
\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \Xi_{\mathcal{D}}(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}) ; \boldsymbol{\lambda}), \quad \operatorname{deg} \Xi_{\mathcal{D}}(\eta)=\ell_{\mathcal{D}},  \tag{2.4}\\
\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} P_{\mathcal{D}, n}(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) ; \boldsymbol{\lambda}), \quad \operatorname{deg} P_{\mathcal{D}, n}(\eta)=\ell_{\mathcal{D}}+n, \tag{2.5}
\end{align*}
$$

where $\ell_{\mathcal{D}}$ is

$$
\begin{equation*}
\ell_{\mathcal{D}} \stackrel{\text { def }}{=} \sum_{j=1}^{M} d_{j}-\frac{1}{2} M(M-1) . \tag{2.6}
\end{equation*}
$$

The sinusoidal coordinates $\eta(x ; \boldsymbol{\lambda})$ are

$$
\eta(x ; \boldsymbol{\lambda})= \begin{cases}x(x+d) & : \mathrm{R}  \tag{2.7}\\ \left(q^{-x}-1\right)\left(1-d q^{x}\right) & : q \mathrm{R}\end{cases}
$$

and the energy eigenvalues $\mathcal{E}_{n}(\boldsymbol{\lambda})$ are

$$
\mathcal{E}_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
n(n+\tilde{d}) & : \mathrm{R}  \tag{2.8}\\
\left(q^{-n}-1\right)\left(1-\tilde{d} q^{n}\right) & : q \mathrm{R}
\end{array}, \quad \tilde{d} \stackrel{\text { def }}{=}\left\{\begin{array}{ll}
a+b+c-d-1 & : \mathrm{R} \\
a b c d^{-1} q^{-1} & : q \mathrm{R}
\end{array} .\right.\right.
$$

The normalization of these quantities is

$$
\begin{equation*}
\eta(0 ; \boldsymbol{\lambda})=\mathcal{E}_{0}(\boldsymbol{\lambda})=0, \quad \check{\Xi}_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=\Xi_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=1, \quad \check{P}_{\mathcal{D}, n}(0 ; \boldsymbol{\lambda})=P_{\mathcal{D}, n}(0 ; \boldsymbol{\lambda})=1 \tag{2.9}
\end{equation*}
$$

In the sequence $\check{P}_{\mathcal{D}, n}(0 ; \boldsymbol{\lambda}), \check{P}_{\mathcal{D}, n}(1 ; \boldsymbol{\lambda}), \ldots, \check{P}_{\mathcal{D}, n}\left(x_{\max } ; \boldsymbol{\lambda}\right)$, the sign changes $n$ times. Note that

$$
\begin{equation*}
\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})=\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \tag{2.10}
\end{equation*}
$$

and $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ is positive for $x=0,1, \ldots, x_{\text {max }}$. We set

$$
\begin{equation*}
P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} 0 \quad(n<0) . \tag{2.11}
\end{equation*}
$$

The original $(q-)$ Racah polynomials $P_{n}(\eta)$ correspond to the " $M=0$ " $(\mathcal{D}=\emptyset)$ case, $P_{n}(\eta)=$ $P_{\emptyset, n}(\eta)$. We remark that if we treat the parameter $a$ as an indeterminate, $\check{P}_{\mathcal{D}, n}(x)$ are defined for $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{C}$. However, for the choice (2.2)) (we take the limit from an indeterminate $a$ to $a$ in (2.21)), $\check{P}_{\mathcal{D}, n}(x)$ are well-defined for $n \in\left\{0,1, \ldots, n_{\max }\right\}$ and $x \in \mathbb{C}$, or $n \in \mathbb{Z}_{>n_{\max }}$ and $x \in\left\{0,1, \ldots, x_{\max }\right\}$.

The Hamiltonian of the deformed system $\mathcal{H}_{\mathcal{D}}=\left(\mathcal{H}_{\mathcal{D} ; x, y}\right)_{0 \leq x, y \leq x_{\max }}$ is a real symmetric matrix (a tridiagonal matrix in this case),

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}}=-\sqrt{B_{\mathcal{D}}(x)} e^{\partial} \sqrt{D_{\mathcal{D}}(x)}-\sqrt{D_{\mathcal{D}}(x)} e^{-\partial} \sqrt{B_{\mathcal{D}}(x)}+B_{\mathcal{D}}(x)+D_{\mathcal{D}}(x) \tag{2.12}
\end{equation*}
$$

where matrices $e^{ \pm \partial}$ are $\left(e^{ \pm \partial}\right)_{x, y}=\delta_{x \pm 1, y}$ and the unit matrix $\mathbf{1}=\left(\delta_{x, y}\right)$ are suppressed. The notation $f(x) A g(x)$ (or $f(x) \circ A \circ g(x)$ ), where $f(x)$ and $g(x)$ are functions of $x$ and $A$ is a matrix $A=\left(A_{x, y}\right)$, stands for a matrix whose $(x, y)$-element is $f(x) A_{x, y} g(y)$. Namely, it is a matrix product $\operatorname{diag}\left(f(0), f(1), \ldots f\left(x_{\max }\right)\right) A \operatorname{diag}\left(g(0), g(1), \ldots, g\left(x_{\max }\right)\right)$. The notation $A f(x)$ stands for a vector whose $x$-th component is $\sum_{y=0}^{x_{\text {max }}} A_{x, y} f(y)$. Note that the matrices $e^{\partial}$ and $e^{-\partial}$ are not inverse to each other. The potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ are

$$
\begin{align*}
& B_{\mathcal{D}}(x ; \boldsymbol{\lambda})=B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\ddot{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})} \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})},  \tag{2.13}\\
& D_{\mathcal{D}}(x ; \boldsymbol{\lambda})=D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}  \tag{2.14}\\
& \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})
\end{align*}
$$

which satisfy the boundary conditions

$$
\begin{equation*}
D_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=0, \quad B_{\mathcal{D}}\left(x_{\max } ; \boldsymbol{\lambda}\right)=0 . \tag{2.15}
\end{equation*}
$$

The eigenvectors of the Hamiltonian are

$$
\begin{align*}
& \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})=\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}), \quad \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})=\sqrt{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \frac{\phi_{0}(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}},  \tag{2.16}\\
& \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right) \tag{2.17}
\end{align*}
$$

and the normalization of $\psi_{\mathcal{D}}$ and $\phi_{\mathcal{D} n}$ is $\psi_{\mathcal{D}}(0 ; \boldsymbol{\lambda})=\phi_{\mathcal{D} n}(0 ; \boldsymbol{\lambda})=1$. Namely the multi-indexed $(q-)$ Racah polynomials satisfy the second order difference equations

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right) \tag{2.18}
\end{equation*}
$$

where the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})=\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ is

$$
\begin{align*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})= & B(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{\partial}\right) \\
& +D(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x+1 ; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}\left(\frac{\check{\Xi}_{\mathcal{D}}(x-1 ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}-e^{-\partial}\right) . \tag{2.19}
\end{align*}
$$

The orthogonality relations of the multi-indexed $(q-)$ Racah polynomials are

$$
\begin{equation*}
\sum_{x=0}^{x_{\max }} \frac{\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{2}}{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, m}(x ; \boldsymbol{\lambda})=\frac{\delta_{n m}}{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}} \quad\left(n, m=0,1, \ldots, n_{\max }\right) \tag{2.20}
\end{equation*}
$$

We remark that

$$
\begin{equation*}
\frac{d_{\mathcal{D}, n}(\boldsymbol{\lambda})}{d_{\mathcal{D}, 0}(\boldsymbol{\lambda})}=\left(\prod_{m=0}^{n-1} \frac{A_{m}(\boldsymbol{\lambda})}{C_{m+1}(\boldsymbol{\lambda})} \cdot \prod_{j=1}^{M} \frac{\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}\right)^{\frac{1}{2}} \tag{2.21}
\end{equation*}
$$

where $A_{n}$ and $C_{n}$ are the coefficients of the three term recurrence relations for the original $(q-)$ Racah polynomials $P_{n}(\eta)$ and $\tilde{\mathcal{E}}_{\mathrm{v}}$ is the virtual state energy.

The multi-indexed $(q-)$ Racah polynomials satisfy the recurrence relations with constant coefficients [49]. We have the following results:

Theorem 1 49] For any polynomial $Y(\eta)(\neq 0)$, we take $X(\eta)=X(\eta ; \boldsymbol{\lambda})=X^{\mathcal{D}, Y}(\eta ; \boldsymbol{\lambda})$ as

$$
\begin{equation*}
X(\eta)=I_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}\left[\Xi_{\mathcal{D}} Y\right](\eta), \quad \operatorname{deg} X(\eta)=L=\ell_{\mathcal{D}}+\operatorname{deg} Y(\eta)+1 \tag{2.22}
\end{equation*}
$$

where $\Xi_{\mathcal{D}} Y$ means a polynomial $\left(\Xi_{\mathcal{D}} Y\right)(\eta)=\Xi_{\mathcal{D}}(\eta) Y(\eta)$, and define $\check{X}(x)=\check{X}(x ; \boldsymbol{\lambda})$ by

$$
\begin{equation*}
\check{X}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} X(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \quad(x \in \mathbb{C}) \tag{2.23}
\end{equation*}
$$

$$
\begin{align*}
=\sum_{j=1}^{x} & (\eta(j ; \boldsymbol{\lambda}+M \boldsymbol{\delta})-\eta(j-1 ; \boldsymbol{\lambda}+M \boldsymbol{\delta})) \\
& \times \check{\Xi}_{\mathcal{D}}(j ; \boldsymbol{\lambda}) Y(\eta(j ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta})) \quad\left(x \in \mathbb{Z}_{\geq 0}\right) \tag{2.24}
\end{align*}
$$

Then the multi-indexed ( $q$-)Racah polynomials $P_{\mathcal{D}, n}(\eta)$ satisfy $1+2 L$ term recurrence relations with constant coefficients:

$$
\begin{equation*}
\check{X}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\sum_{k=-\min (L, n)}^{\min (L, N-n)} r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n+k}(x ; \boldsymbol{\lambda}) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} . \tag{2.25}
\end{equation*}
$$

Remark For $L>\frac{1}{2} N$, number of terms is not $1+2 L$ but $N+1$. Unless $N-L+1 \leq n \leq N$, (2.25) is an equation as a polynomial, namely it holds for $x \in \mathbb{C}$. On the other hand, for $N-L+1 \leq n \leq N$, (2.25) holds only for $x=0,1, \ldots, x_{\max }$.

We note that the overall normalization and the constant term of $X(\eta)$ are not important, because the change of the former induces that of the overall normalization of $r_{n, k}^{X, \mathcal{D}}$ and the shift of the latter induces that of $r_{n, 0}^{X, \mathcal{D}}$. The constant term of $X(\eta)$ is chosen as $X(0)=0$. There are the following relations among the coefficients $r_{n, k}^{X, \mathcal{D}}$ [49]

$$
\begin{equation*}
r_{n+k,-k}^{X, \mathcal{D}}(\boldsymbol{\lambda})=\frac{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}}{d_{\mathcal{D}, n+k}(\boldsymbol{\lambda})^{2}} r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \quad\left(1 \leq k \leq L ; n+k \leq n_{\max }\right) . \tag{2.26}
\end{equation*}
$$

Direct verification of this theorem is rather straightforward for lower $M$ and smaller $d_{j}, n$, $\operatorname{deg} Y$ and $N$, by a computer algebra system, e.g. Mathematica. The coefficients $r_{n, k}^{X, \mathcal{D}}$ are explicitly obtained for small $d_{j}$ and $n$. However, to obtain the closed expression of $r_{n, k}^{X, \mathcal{D}}$ for general $n$ is not an easy task even for small $d_{j}$, and it is a different kind of problem. Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. Although not all of them are independent, the relations among them are unclear. Note that $L \geq M+1$ because of $\ell_{\mathcal{D}} \geq M$. The minimal degree one, which corresponds to $Y(\eta)=1$, is $X_{\min }(\eta)=I_{\boldsymbol{\lambda}+M \boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta)$, $\operatorname{deg} X_{\text {min }}(\eta)=\ell_{\mathcal{D}}+1$.

## 3 Dual Polynomials of the Multi-Indexed ( $q$-)Racah Polynomials

For ordinary orthogonal polynomials, a discrete orthogonality relation of a system of polynomials induces an orthogonality relation for the dual system where the role of the variable and the degree are interchanged [2] (see also [38]-40]). In this section we consider dual
polynomials of the multi-indexed $(q-)$ Racah polynomials, where the degree is replaced by the number of sign changes.

Corresponding to the multi-indexed $(q-)$ Racah polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$, let us define $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$ by

$$
\begin{equation*}
\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})}{\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})}\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} . \tag{3.1}
\end{equation*}
$$

(Remark that if the parameter $a$ is treated as an indeterminate, $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$ is defined for $n, x \in \mathbb{Z}_{\geq 0}$.) Then we have

$$
\begin{equation*}
\check{Q}_{\mathcal{D}, x}(0 ; \boldsymbol{\lambda})=1, \quad \check{Q}_{\mathcal{D}, 0}(n ; \boldsymbol{\lambda})=1 . \tag{3.2}
\end{equation*}
$$

Orthogonality relations (2.20) are rewritten as

$$
\begin{equation*}
\sum_{x=0}^{x_{\max }} \hat{\phi}_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \hat{\phi}_{\mathcal{D} m}(x ; \boldsymbol{\lambda})=\delta_{n m} \quad\left(n, m=0,1, \ldots, n_{\max }\right) \tag{3.3}
\end{equation*}
$$

where the normalized eigenvectors $\hat{\phi}_{\mathcal{D}}{ }_{n}(x ; \boldsymbol{\lambda})$ are

$$
\begin{equation*}
\hat{\phi}_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{d_{\mathcal{D}, n}(\boldsymbol{\lambda})}{\sqrt{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})}} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\frac{\phi_{\mathcal{D} 0}(x ; \boldsymbol{\lambda})}{\sqrt{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})}} d_{\mathcal{D}, n}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda}) . \tag{3.4}
\end{equation*}
$$

Since the matrix size is finite $n_{\max }=x_{\max }=N$, (3.3) implies

$$
\begin{equation*}
\sum_{n=0}^{n_{\max }} \hat{\phi}_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) \hat{\phi}_{\mathcal{D} n}(y ; \boldsymbol{\lambda})=\delta_{x y} \quad\left(x, y=0,1, \ldots, x_{\max }\right) \tag{3.5}
\end{equation*}
$$

namely dual orthogonality relations

$$
\begin{equation*}
\sum_{n=0}^{n_{\max }} \frac{d_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}}{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda})} \check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, y}(n ; \boldsymbol{\lambda})=\frac{\delta_{x y}}{\phi_{\mathcal{D} 0}(x ; \boldsymbol{\lambda})^{2}} \quad\left(x, y=0,1, \ldots, x_{\max }\right) \tag{3.6}
\end{equation*}
$$

The second order difference equations for $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})(2.18)$ are rewritten as the three term recurrence relations for $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$,

$$
\begin{array}{r}
\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})=A_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x+1}(n ; \boldsymbol{\lambda})+B_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})+C_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x-1}(n ; \boldsymbol{\lambda}) \\
\left(n=0,1, \ldots, n_{\max } ; x=0,1, \ldots, x_{\max }\right) \tag{3.7}
\end{array}
$$

where we have used (2.10). Here $A_{\mathcal{D}, x}^{\text {dual }}, B_{\mathcal{D}, x}^{\text {dual }}$ and $C_{\mathcal{D}, x}^{\text {dual }}$ are

$$
\begin{equation*}
A_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}-B_{\mathcal{D}}(x ; \boldsymbol{\lambda}), \quad C_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}-D_{\mathcal{D}}(x ; \boldsymbol{\lambda}), \quad B_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}-A_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda})-C_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) \tag{3.8}
\end{equation*}
$$

and satisfy the boundary conditions

$$
\begin{equation*}
C_{\mathcal{D}, 0}^{\text {dual }}(\boldsymbol{\lambda})=0, \quad A_{\mathcal{D}, x_{\max }}^{\text {dual }}(\boldsymbol{\lambda})=0 \tag{3.9}
\end{equation*}
$$

This means that $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$ are generated by the three term recurrence relations (3.7) with the initial conditions

$$
\begin{equation*}
\check{Q}_{\mathcal{D}, 0}(n ; \boldsymbol{\lambda})=1, \quad \check{Q}_{\mathcal{D},-1}(n ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} 0 . \tag{3.10}
\end{equation*}
$$

Therefore $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$ are polynomials in $\mathcal{E}_{n}(\boldsymbol{\lambda})$,

$$
\begin{equation*}
\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} Q_{\mathcal{D}, x}\left(\mathcal{E}_{n}(\boldsymbol{\lambda}) ; \boldsymbol{\lambda}\right), \quad \operatorname{deg} Q_{\mathcal{D}, x}(\mathcal{E})=x \tag{3.11}
\end{equation*}
$$

These polynomials $Q_{\mathcal{D}, x}(\mathcal{E})$ are ordinary orthogonal polynomials. So, the sign changes $x$ times in the sequence $\check{Q}_{\mathcal{D}, x}(0 ; \boldsymbol{\lambda}), \check{Q}_{\mathcal{D}, x}(1 ; \boldsymbol{\lambda}), \ldots, \check{Q}_{\mathcal{D}, x}\left(n_{\max } ; \boldsymbol{\lambda}\right)$. We call $Q_{\mathcal{D}, x}(\mathcal{E})$ as the dual multi-indexed $(q-)$ Racah polynomials. Correspondence of this duality is

$$
\begin{equation*}
x \leftrightarrow n, \quad \eta(x) \leftrightarrow \mathcal{E}_{n}, \quad P_{\mathcal{D}, n}(\eta) \leftrightarrow Q_{\mathcal{D}, x}(\mathcal{E}), \quad \frac{\phi_{\mathcal{D} 0}(x)}{\phi_{\mathcal{D} 0}(0)} \leftrightarrow \frac{d_{\mathcal{D}, n}}{d_{\mathcal{D}, 0}} . \tag{3.12}
\end{equation*}
$$

The multi-indexed ( $q-$ )Racah polynomials $P_{\mathcal{D}, n}(\eta)$ and their dual polynomials $Q_{\mathcal{D}, x}(\mathcal{E})$ are different polynomials. This contrasts with the original $(q-)$ Racah cases. The ( $q-$ ) Racah polynomials and their dual polynomials are same polynomials with the parameter correspondence $(a, b, c, d) \leftrightarrow(a, b, c, \tilde{d})$. We remark that if we treat the parameter $a$ as andeterminate, the dual multi-indexed $(q-)$ Racah polynomials $Q_{\mathcal{D}, x}(\mathcal{E})$ are defined for $x \in \mathbb{Z}_{\geq 0}$ and $\mathcal{E} \in \mathbb{C}$ by the three term recurrence relations

$$
\begin{equation*}
\mathcal{E} Q_{\mathcal{D}, x}(\mathcal{E} ; \boldsymbol{\lambda})=A_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) Q_{\mathcal{D}, x+1}(\mathcal{E} ; \boldsymbol{\lambda})+B_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) Q_{\mathcal{D}, x}(\mathcal{E} ; \boldsymbol{\lambda})+C_{\mathcal{D}, x}^{\text {dual }}(\boldsymbol{\lambda}) Q_{\mathcal{D}, x-1}(\mathcal{E} ; \boldsymbol{\lambda}) \tag{3.13}
\end{equation*}
$$

with the initial condition $Q_{\mathcal{D}, 0}(\mathcal{E})=1$ and $Q_{\mathcal{D},-1}(\mathcal{E}) \stackrel{\text { def }}{=} 0$. For $(2.2)$, we have $A_{\mathcal{D}, N}^{\text {dual }}(\boldsymbol{\lambda})=0$.
The $1+2 L$ term recurrence relations with constant coefficients for $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})(2.25)$ are rewritten as the $2 L$-th order difference equations for $\check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})$,

$$
\begin{equation*}
\sum_{k=-\min (L, n)}^{\min (L, N-n)} r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x}(n+k ; \boldsymbol{\lambda})=\check{X}(x ; \boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, x}(n ; \boldsymbol{\lambda})\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }} \tag{3.14}
\end{equation*}
$$

(Remark: For $L>\frac{1}{2} N$, the order is not $2 L$ but $N$.) Therefore the dual multi-indexed $\left(q\right.$-)Racah polynomials $Q_{\mathcal{D}, x}(\mathcal{E})$ are the Krall-type polynomials. Since we can take various $X=X^{\mathcal{D}, Y}$ for a multi-indexed set $\mathcal{D}$, these Krall-type polynomials $Q_{\mathcal{D}, x}(\mathcal{E})$ satisfy various difference equations of order $2 L \geq 2 M+2$.

In the $q \rightarrow 1$ limit, the $q$-Racah polynomial reduced to the Racah polynomial [10]: $\lim _{q \rightarrow 1} \check{P}_{n}^{q \mathrm{R}}(x ; \boldsymbol{\lambda})=\check{P}_{n}^{\mathrm{R}}(x ; \boldsymbol{\lambda})$. Similarly, the (dual) multi-indexed $q$-Racah polynomials reduce to the (dual) multi-indexed Racah polynomials: $\lim _{q \rightarrow 1} \check{P}_{\mathcal{D}, n}^{q \mathrm{R}}(x ; \boldsymbol{\lambda})=\check{P}_{\mathcal{D}, n}^{\mathrm{R}}(x ; \boldsymbol{\lambda})$ and $\lim _{q \rightarrow 1} \check{Q}_{\mathcal{D}, x}^{q \mathrm{R}}(n ; \boldsymbol{\lambda})=$ $\check{Q}_{\mathcal{D}, x}^{\mathrm{R}}(n ; \boldsymbol{\lambda})$.

## 4 New Exactly Solvable rdQM Systems

In this section we present new exactly solvable rdQM systems, whose eigenvectors are described by the dual multi-indexed ( $q$-)Racah polynomials. Unlike the previous section, we assume that the coordinate is $x$ and the label of states is $n$ as usual. Although tridiagonal matrices have been considered in our previous papers [38, 39, 40, 33], the Hamiltonian of rdQM is not restricted to tridiagonal matrices and any real symmetric matrix is allowed.

Let us fix the multi-index set $\mathcal{D}=\left\{d_{1}, d_{2}, \ldots, d_{M}\right\}\left(d_{1}<d_{2}<\cdots<d_{M}, d_{j} \in \mathbb{Z}_{\geq 1}\right)$. We take a polynomial $X(\eta)=X^{\mathcal{D}, Y}(\eta)(2.22)$ and assume that $Y(\eta)(\neq 0)$ is a polynomial with real non-negative coefficients. For each $X(\eta)$, we define the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})=$ $\left(\mathcal{H}_{\mathcal{D} ; x, y}^{X \text { dual }}(\boldsymbol{\lambda})\right)_{0 \leq x, y \leq x_{\text {max }}}$ by

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}}^{X} \text { dual }(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sum_{k=-\min (L, x)}^{\min (L, N-x)} r_{x, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \frac{d_{\mathcal{D}, x}(\boldsymbol{\lambda})}{d_{\mathcal{D}, x+k}(\boldsymbol{\lambda})} e^{k \partial} \tag{4.1}
\end{equation*}
$$

where matrices $e^{k \partial}(k \in \mathbb{Z})$ are defined by

$$
e^{k \partial} \stackrel{\text { def }}{=}\left\{\begin{array}{ll}
\left(e^{\partial}\right)^{k} & (k \geq 0)  \tag{4.2}\\
\left(e^{-\partial}\right)^{-k} & (k<0)
\end{array}, \text { namely } \quad\left(e^{k \partial}\right)_{x, y}=\delta_{x+k, y}\right.
$$

This Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}$ dual is a band matrix with lower and upper bandwidth $L$, namely a " $(1+2 L)$-diagonal" matrix, e.g. $L=1(\Leftrightarrow M=0, \mathcal{D}=\emptyset)$ : tridiagonal matrix, $L=2$ : pentadiagonal matrix, etc. (For $L>N$, it is " $(1+2 N)$-diagonal".) By using (2.21) (with $n \rightarrow x$ ), explicit forms of $d_{\mathcal{D}, x} / d_{\mathcal{D}, x+k}$ with $0 \leq x+k \leq N$ are (convention: $\prod_{i=n}^{n-1} *=1$ )

$$
\frac{d_{\mathcal{D}, x}(\boldsymbol{\lambda})}{d_{\mathcal{D}, x+k}(\boldsymbol{\lambda})}=\left(\prod_{j=1}^{M} \frac{\mathcal{E}_{x}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{\mathcal{E}_{x+k}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}\right)^{\frac{1}{2}} \times \begin{cases}\left(\prod_{i=1}^{k} \frac{C_{x+i}(\boldsymbol{\lambda})}{A_{x+i-1}(\boldsymbol{\lambda})}\right)^{\frac{1}{2}} & (0 \leq k \leq L)  \tag{4.3}\\ \left(\prod_{i=1}^{-k} \frac{A_{x-i}(\boldsymbol{\lambda})}{C_{x+1-i}(\boldsymbol{\lambda})}\right)^{\frac{1}{2}} & (-L \leq k \leq-1)\end{cases}
$$

This Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}$ is real symmetric because of (2.26) (with $\left.n \rightarrow x\right)$.

Multiplying (3.14) (with replacement $x \leftrightarrow n$ ) by $d_{\mathcal{D}, x}(\boldsymbol{\lambda}) / d_{\mathcal{D}, 0}(\boldsymbol{\lambda})$, we have

$$
\begin{gather*}
\sum_{k=-\min (L, x)}^{\min (L, N-x)} r_{x, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) \frac{d_{\mathcal{D}, x}(\boldsymbol{\lambda})}{d_{\mathcal{D}, x+k}(\boldsymbol{\lambda})} \frac{d_{\mathcal{D}, x+k}(\boldsymbol{\lambda})}{d_{\mathcal{D}, 0}(\boldsymbol{\lambda})} \check{Q}_{\mathcal{D}, n}(x+k ; \boldsymbol{\lambda})=\check{X}(n ; \boldsymbol{\lambda}) \frac{d_{\mathcal{D}, x}(\boldsymbol{\lambda})}{d_{\mathcal{D}, 0}(\boldsymbol{\lambda})} \check{Q}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \\
\left(n=0,1, \ldots, n_{\max } ; x=0,1, \ldots, x_{\max }\right) . \tag{4.4}
\end{gather*}
$$

Therefore eigenvectors of the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})$,

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda})=\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right), \tag{4.5}
\end{equation*}
$$

are given by

$$
\begin{align*}
& \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{d_{\mathcal{D}, x}(\boldsymbol{\lambda})}{d_{\mathcal{D}, 0}(\boldsymbol{\lambda})} \check{Q}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\binom{n=0,1, \ldots, n_{\max }}{x=0,1, \ldots, x_{\max }},  \tag{4.6}\\
& \mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{X}(n ; \boldsymbol{\lambda}), \quad \check{Q}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=Q_{\mathcal{D}, n}\left(\mathcal{E}_{x}(\boldsymbol{\lambda}) ; \boldsymbol{\lambda}\right) . \tag{4.7}
\end{align*}
$$

Their orthogonality relations are obtained from (3.6):

$$
\begin{equation*}
\sum_{x=0}^{x_{\text {max }}} \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda}) \phi_{\mathcal{D} m}^{\text {dual }}(x ; \boldsymbol{\lambda})=\frac{\check{\Xi}_{\mathcal{D}}(1 ; \boldsymbol{\lambda}) \delta_{n m}}{d_{\mathcal{D}, 0}(\boldsymbol{\lambda})^{2} \phi_{\mathcal{D} 0}(n ; \boldsymbol{\lambda})^{2}} \quad\left(n, m=0,1, \ldots, n_{\max }\right) \tag{4.8}
\end{equation*}
$$

Since each term of the sum in (2.24) is positive for $x=1,2, \ldots, x_{\max }$, we have

$$
\begin{equation*}
0=\check{X}(0 ; \boldsymbol{\lambda})<\check{X}(1 ; \boldsymbol{\lambda})<\cdots<\check{X}\left(x_{\max } ; \boldsymbol{\lambda}\right) . \tag{4.9}
\end{equation*}
$$

Therefore the energy eigenvalues $\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}(\boldsymbol{\lambda})$ satisfy

$$
\begin{equation*}
0=\mathcal{E}_{\mathcal{D}, 0}^{X \text { dual }}(\boldsymbol{\lambda})<\mathcal{E}_{\mathcal{D}, 1}^{X \text { dual }}(\boldsymbol{\lambda})<\cdots<\mathcal{E}_{\mathcal{D}, n_{\max }}^{X \text { dual }}(\boldsymbol{\lambda}) \tag{4.10}
\end{equation*}
$$

and the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}$ is positive semi-definite.
For a multi-index set $\mathcal{D}$, we can take various $X=X^{\mathcal{D}, Y}$, because a polynomial $Y$ with real non-negative coefficients is arbitrary. The Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}$ dual and energy eigenvalues $\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}$ depend on $X$, but the eigenvectors $\phi_{\mathcal{D} n}^{\text {dual }}(x)$ do not. Hence $\phi_{\mathcal{D} n}^{\text {dual }}(x)$ are simultaneous eigenvectors of various $\mathcal{H}_{\mathcal{D}}^{X}$ dual . In other words, the Hamiltonians associated with $X_{1}=X^{\mathcal{D}, Y_{1}}$ and $X_{2}=X^{\mathcal{D}, Y_{2}}$ commute with each other,

$$
\begin{equation*}
\left[\mathcal{H}_{\mathcal{D}}^{X_{1} \text { dual }}, \mathcal{H}_{\mathcal{D}}^{X_{2} \text { dual }}\right]=0 \tag{4.11}
\end{equation*}
$$

By the similarity transformation in terms of the ground state eigenvector $\phi_{\mathcal{D} 0}^{\text {dual }}(x)$, the Schrödinger equation (4.5) is rewritten as

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}(\boldsymbol{\lambda}) \check{Q}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \quad\left(n=0,1, \ldots, n_{\max }\right), \tag{4.12}
\end{equation*}
$$

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})=\phi_{\mathcal{D} 0}^{\text {dual }}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda}) \circ \phi_{\mathcal{D} 0}^{\text {dual }}(x ; \boldsymbol{\lambda})=\sum_{k=-\min (L, x)}^{\min (L, N-x)} r_{x, k}^{X, \mathcal{D}}(\boldsymbol{\lambda}) e^{k \partial} \tag{4.13}
\end{equation*}
$$

Multiplying (3.7) $(x \leftrightarrow n)$ by $d_{\mathcal{D}, x}(\boldsymbol{\lambda}) / d_{\mathcal{D}, 0}(\boldsymbol{\lambda})$, we obtain the three term recurrence relations for the eigenvectors $\phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda})$,

$$
\begin{align*}
\mathcal{E}_{x}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda})=A_{\mathcal{D}, n}^{\text {dual }}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n+1}^{\text {dual }}(x ; \boldsymbol{\lambda})+B_{\mathcal{D}, n}^{\text {dual }}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}^{\text {dual }}(x ; \boldsymbol{\lambda})+C_{\mathcal{D}, n}^{\text {dual }}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n-1}^{\text {dual }}(x ; \boldsymbol{\lambda}) \\
\left(n=0,1, \ldots, n_{\max } ; x=0,1, \ldots, x_{\max }\right) \tag{4.14}
\end{align*}
$$

### 4.1 Closure relation

Corresponding to the three term recurrence relations (4.14), the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}$ dual and the sinusoidal coordinate $\mathcal{E}_{x}$ are expected to satisfy the ordinary closure relation [50, 38] (closure relation of order 2 [48]),

$$
\begin{equation*}
\left[\mathcal{H}_{\mathcal{D}}^{X} \text { dual },\left[\mathcal{H}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right]\right]=\underline{\mathcal{E}} R_{0}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)+\left[\mathcal{H}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right] R_{1}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)+R_{-1}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right), \tag{4.15}
\end{equation*}
$$

where $\underline{\mathcal{E}}$ is a diagonal matrix $\underline{\mathcal{E}}=\left(\mathcal{E}_{x} \delta_{x, y}\right)_{0 \leq x, y \leq x_{\max }}$ and $R_{i}(z)$ 's are polynomials in $z$. (In the notation used in (2.12), this matrix $\underline{\mathcal{E}}$ is expressed as $\mathcal{E}_{x} 1$ or simply $\mathcal{E}_{x}$.) For the original $(q$ - $)$ Racah systems, the degrees of $R_{i}(z)$ are $\left(\operatorname{deg} R_{0}, \operatorname{deg} R_{1}, \operatorname{deg} R_{-1}\right)=(2,1,2)$. For the dual multi-indexed ( $q-$ )Racah systems, the degrees of $R_{i}(x)$ need to be much higher. This is because the expression of $\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}$ is more complicated than that of $\mathcal{E}_{n}$. We can find $R_{i}(z)$ satisfying (4.15), whose degrees are ( $\left.\operatorname{deg} R_{0}, \operatorname{deg} R_{1}, \operatorname{deg} R_{-1}\right)=(N, N-2, N),(N, N, N-$ 2), $(N-1, N-1, N)$, etc. In order to construct the creation/annihilation operators, however, these minimal choices (the number of coefficients of $R_{i}(z)$ 's is $3 N+1$ ) are not appropriate, because the relations (4.20)-(4.22) are not satisfied, which are the desired properties (4.28). For this purpose, we take $\left(\operatorname{deg} R_{0}, \operatorname{deg} R_{1}, \operatorname{deg} R_{-1}\right)=(N, N, N)$ (the number of coefficients of $R_{i}(z)$ 's is $\left.3 N+3\right)$.

The method of closure relation [50, 38] is the following: (i) Find $R_{i}(z)$ satisfying (4.15), (ii) Calculate $\alpha_{ \pm}(z)$ from $R_{i}(z)$, (iii) Heisenberg solution $\underline{\mathcal{E}}_{\mathrm{H}}(t)$ is obtained, (iv) Creation/ annihilation operators $a^{( \pm)}$are obtained. Here we change a part of the logic, namely mix (i) and (ii) by using some consequence of (iv). We define functions $\alpha_{ \pm}(z)$ and polynomials $R_{i}(z)$ by guess work, which is expected from some consequence of (iv). Then we check the closure relation (4.15) for these $R_{i}(z)$.

Let us define $R_{0}(z), R_{1}(z), R_{-1}(z)$ and $\alpha_{ \pm}(z)$ by

$$
\begin{align*}
& R_{0}(z) \stackrel{\text { def }}{=} \sum_{j=0}^{N} r_{0}^{(j)} z^{j}, \quad R_{1}(z) \stackrel{\text { def }}{=} \sum_{j=0}^{N} r_{1}^{(j)} z^{j}, \quad R_{-1}(z) \stackrel{\text { def }}{=} \sum_{j=0}^{N} r_{-1}^{(j)} z^{j},  \tag{4.16}\\
& \alpha_{ \pm}(z) \stackrel{\text { def }}{=} \frac{1}{2}\left(R_{1}(z) \pm \sqrt{R_{1}(z)^{2}+4 R_{0}(z)}\right) \Leftrightarrow\left\{\begin{array}{l}
R_{0}(z)=-\alpha_{+}(z) \alpha_{-}(z) \\
R_{1}(z)=\alpha_{+}(z)+\alpha_{-}(z)
\end{array}\right. \tag{4.17}
\end{align*}
$$

where coefficients $r_{0}^{(j)}$ and $r_{1}^{(j)}$ are determined by the condition

$$
\begin{equation*}
\alpha_{ \pm}(\check{X}(j))=\check{X}(j \pm 1)-\check{X}(j) \quad(j=0,1, \ldots, N), \tag{4.18}
\end{equation*}
$$

and coefficients $r_{-1}^{(j)}$ are determined by the condition

$$
\begin{equation*}
R_{-1}(\check{X}(j))=-(\check{X}(j+1)-\check{X}(j))(\check{X}(j)-\check{X}(j-1)) B_{\mathcal{D}, j}^{\text {dual }} \quad(j=0,1, \ldots, N) \tag{4.19}
\end{equation*}
$$

The condition (4.18) is rewritten as

$$
\begin{align*}
& R_{0}(\check{X}(j))=(\check{X}(j+1)-\check{X}(j))(\check{X}(j)-\check{X}(j-1)) \quad(j=0,1, \ldots, N),  \tag{4.20}\\
& R_{1}(\check{X}(j))=\check{X}(j+1)-2 \check{X}(j)+\check{X}(j-1) \quad(j=0,1, \ldots, N) \tag{4.21}
\end{align*}
$$

and the condition (4.19) is rewritten as

$$
\begin{equation*}
R_{-1}(\check{X}(j))=-R_{0}(\check{X}(j)) B_{\mathcal{D}, j}^{\text {dual }} \quad(j=0,1, \ldots, N) . \tag{4.22}
\end{equation*}
$$

Note that

$$
\begin{equation*}
R_{1}(\check{X}(j))^{2}+4 R_{0}(\check{X}(j))=(\check{X}(j+1)-\check{X}(j-1))^{2} \quad(j=0,1, \ldots, N) . \tag{4.23}
\end{equation*}
$$

These systems of linear equations (4.20), (4.21) and (4.19) are solved by using Cramer's rule. By using $\check{X}(0)=0$, we obtain

$$
\begin{align*}
R_{i}(z)= & \prod_{j=1}^{N} \check{X}(j)^{-1} \cdot \prod_{1 \leq k<j \leq N}(\check{X}(j)-\check{X}(k))^{-1} \\
& \times\left|\begin{array}{ccccc}
\check{X}(1) & \check{X}(1)^{2} & \ldots & \check{X}(1)^{N} & \beta_{i}^{(0)}-\beta_{i}^{(1)} \\
\check{X}(2) & \check{X}(2)^{2} & \cdots & \check{X}(2)^{N} & \beta_{i}^{(0)}-\beta_{i}^{(2)} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\check{X}(N) & \check{X}(N)^{2} & \cdots & \check{X}(N)^{N} & \beta_{i}^{(0)}-\beta_{i}^{(N)} \\
z & z^{2} & \cdots & z^{N} & \beta_{i}^{(0)}
\end{array}\right|, \tag{4.24}
\end{align*}
$$

where $\beta_{i}^{(j)}(i=0,1,-1 ; j=0,1, \ldots, N)$ are

$$
\left\{\begin{array}{l}
\beta_{0}^{(j)}=(\check{X}(j+1)-\check{X}(j))(\check{X}(j)-\check{X}(j-1))  \tag{4.25}\\
\beta_{1}^{(j)}=\check{X}(j+1)-2 \check{X}(j)+\check{X}(j-1) \\
\beta_{-1}^{(j)}=-(\check{X}(j+1)-\check{X}(j))(\check{X}(j)-\check{X}(j-1)) B_{\mathcal{D}, j}^{\text {dual }}
\end{array}\right.
$$

We remark that conditions (4.20), (4.21) and (4.19) are $N+1$ equations for $N+1$ unknown coefficients $r_{i}^{(j)}$ and those relations do not hold for $j \neq 0,1, \ldots, N$. This contrasts with the original ( $q$-) Racah systems, in which the degrees of $R_{i}(z)$ are 2 or 1 and similar relations hold for any $j$ [38].

Then we have the following conjecture.
Conjecture 1 The closure relation (4.15) holds for $R_{i}(z)$ (4.16), (4.24).
From the closure relation (4.15), the exact Heisenberg operator solution of the sinusoidal coordinate $\underline{\mathcal{E}}$ can be obtained [50, 38,

$$
\begin{align*}
\underline{\mathcal{E}}_{\mathrm{H}}(t) & \stackrel{\text { def }}{=} e^{i \mathcal{H}_{\mathcal{D}}^{X} \text { dual }} \underline{\underline{E}} e^{-i \mathcal{H}_{\mathcal{D}}^{X} \text { dual } t} \\
& =a^{(+)} e^{i \alpha_{+}\left(\mathcal{H}_{\mathcal{D}}^{X} \text { dual }\right) t}+a^{(-)} e^{i \alpha_{-}\left(\mathcal{H}_{\mathcal{D}}^{X} \text { dual }\right) t}-R_{-1}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right) R_{0}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)^{-1} \tag{4.26}
\end{align*}
$$

where $a^{( \pm)}=a^{( \pm)}\left(\mathcal{H}_{\mathcal{D}}^{X}\right.$ dual,$\left.\underline{\mathcal{E}}\right)$ are

$$
\begin{align*}
a^{( \pm)} \stackrel{\text { def }}{=} & \pm\left(\left[\mathcal{H}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right]-\left(\underline{\mathcal{E}}+R_{-1}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right) R_{0}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)^{-1}\right) \alpha_{\mp}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)\right) \\
& \times\left(\alpha_{+}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)-\alpha_{-}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}\right)\right)^{-1} . \tag{4.27}
\end{align*}
$$

Note that square roots in $\alpha_{ \pm}\left(\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}\right)$ are well-defined, because the matrix $R_{1}\left(\mathcal{H}_{\mathcal{D}}^{X} \text { dual }\right)^{2}+$ $4 R_{0}\left(\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}\right)$ is positive semi-definite, see (4.23). Action of (4.26) on $\phi_{\mathcal{D}}^{n}$ dual $(x)$ is

$$
\begin{aligned}
\underline{\mathcal{E}}_{\mathrm{H}}(t) \phi_{\mathcal{D} n}^{\text {dual }}(x)= & e^{i \alpha_{+}\left(\mathcal{E}_{\mathcal{D}, n}^{X} \text { dual }\right) t} a^{(+)} \phi_{\mathcal{D} n}^{\text {dual }}(x)+e^{i \alpha_{-}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right) t} a^{(-)} \phi_{\mathcal{D} n}^{\text {dual }}(x) \\
& -R_{-1}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right) R_{0}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right)^{-1} \phi_{\mathcal{D} n}^{\text {dual }}(x) .
\end{aligned}
$$

On the other hand it turns out to be

$$
\begin{aligned}
& \mathcal{E}_{\mathrm{H}}(t) \phi_{\mathcal{D} n}^{\text {dual }}(x)=e^{i \mathcal{H}_{\mathcal{D}}^{X} \text { dual }} \underline{\mathcal{E}} e^{-i \mathcal{H}_{\mathcal{D}}^{X} \text { dual }} \phi_{\mathcal{D} n}^{\text {dual }}(x)=e^{i \mathcal{H} \mathcal{D}_{\mathcal{D}}^{X} \text { dual }} \underline{\mathcal{E}} e^{-i \mathcal{E}_{\mathcal{D}, n}^{X} \text { dual } t} \phi_{\mathcal{D} n}^{\text {dual }}(x) \\
= & e^{-i \mathcal{E}_{\mathcal{D}, n}^{X \text { dual }} t} e^{i \mathcal{H}_{\mathcal{D}}^{X \text { dual }} t}\left(A_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n+1}^{\text {dual }}(x)+B_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n}^{\text {dual }}(x)+C_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n-1}^{\text {dual }}(x)\right) \\
= & e^{i\left(\mathcal{E}_{\mathcal{D}, n+1}^{X}-\mathcal{E}_{\mathcal{D}, n}^{X},{ }^{\text {dual }}\right) t} A_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n+1}^{\text {dual }}(x)+B_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n}^{\text {dual }}(x)+e^{i\left(\mathcal{E}_{\mathcal{D}, n-1}^{X \text { dual }}-\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right) t} C_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n-1}^{\text {dual }}(x),
\end{aligned}
$$

where we have used (4.14). Comparing these $t$-dependence, we obtain

$$
\begin{align*}
& \alpha_{ \pm}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right)=\mathcal{E}_{\mathcal{D}, n \pm 1}^{X \text { dual }}-\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}, \quad-R_{-1}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right) R_{0}\left(\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\right)^{-1}=B_{\mathcal{D}, n}^{\text {dual }}  \tag{4.28}\\
& a^{(+)} \phi_{\mathcal{D} n}^{\text {dual }}(x)=A_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n+1}^{\text {dual }}(x), \quad a^{(-)} \phi_{\mathcal{D} n}^{\text {dual }}(x)=C_{\mathcal{D}, n}^{\text {dual }} \phi_{\mathcal{D} n-1}^{\text {dual }}(x) \tag{4.29}
\end{align*}
$$

Therefore $a^{(+)}$and $a^{(-)}$are creation and annihilation operators, respectively. The relations (4.28) correspond to (4.20)-(4.22).

Remark: The value of function $\check{X}(x)(2.23)$ at $x=-1$ is

$$
\check{X}(-1)=\left\{\begin{array}{ll}
-(d+M-1) Y(0) & : \mathrm{R}  \tag{4.30}\\
-(1-q)\left(1-d q^{M-1}\right) Y(0) & : q \mathrm{R}
\end{array} .\right.
$$

For $Y(0)=0$, we have $\check{X}(-1)=0$. By (4.20), this and $\check{X}(0)=0$ give $r_{0}^{(0)}=0$, namely $R_{0}(0)=0$. Therefore action of $a^{( \pm)}$on $\phi_{\mathcal{D} 0}^{\text {dual }}(x)$ is not well-defined for this case. Although the coefficient $r_{-1}^{(0)}$ also vanishes by (4.19), namely $R_{-1}(0)=0$, and the limit $\lim _{z \rightarrow 0} R_{-1}(z) R_{0}(z)^{-1}$ exists, it does not coincide with $-B_{\mathcal{D}, 0}^{\text {dual }}$.

By the similarity transformation, the closure relation (4.15) is rewritten for the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}_{\mathcal{D}}^{X}$ dual $(\boldsymbol{\lambda})$ (4.13),

$$
\begin{equation*}
\left[\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }},\left[\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right]\right]=\underline{\mathcal{E}} R_{0}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}\right)+\left[\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right] R_{1}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}\right)+R_{-1}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}\right) \tag{4.31}
\end{equation*}
$$

From the creation and annihilation operators $a^{( \pm)}=a^{( \pm)}\left(\mathcal{H}_{\mathcal{D}}^{X}\right.$ dual,$\underline{\mathcal{E}}$ ) (4.27), we obtain the creation and annihilation operators for polynomial eigenvectors,

$$
\begin{align*}
& \tilde{a}^{( \pm)} \stackrel{\text { def }}{=} \phi_{\mathcal{D} 0}^{\text {dual }}(x)^{-1} \circ a^{( \pm)}\left(\mathcal{H}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right) \circ \phi_{\mathcal{D} 0}^{\text {dual }}(x)=a^{( \pm)}\left(\widetilde{\mathcal{H}}_{\mathcal{D}}^{X \text { dual }}, \underline{\mathcal{E}}\right)  \tag{4.32}\\
& \tilde{a}^{(+)} \check{Q}_{\mathcal{D}, n}(x)=A_{\mathcal{D}, n}^{\text {dual }} \check{Q}_{\mathcal{D}, n+1}(x), \quad \tilde{a}^{(-)} \check{Q}_{\mathcal{D}, n}(x)=C_{\mathcal{D}, n}^{\text {dual }} \check{Q}_{\mathcal{D}, n-1}(x) \tag{4.33}
\end{align*}
$$

### 4.2 No shape invariance

We will show that the rdQM system described by $\mathcal{H}_{\mathcal{D}}^{X}$ dual $(\boldsymbol{\lambda})$ is not shape invariant.
First let us factorize the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})$ (4.1). Since it is positive semi-definite, it can be factorized as

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}}^{X} \text { dual }(\boldsymbol{\lambda})=\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})^{\dagger} \mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda}) \tag{4.34}
\end{equation*}
$$

where $\mathcal{A}_{\mathcal{D}}^{X}$ dual $(\boldsymbol{\lambda})$ is an upper triangular matrix (with upper bandwidth $L$ for $L \leq \frac{1}{2} N$ ). By imposing the condition $\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})_{x, x} \geq 0\left(x=0,1, \ldots, x_{\text {max }}\right)$, this upper triangular matrix
$\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})=\left(a_{x, y}\right)_{0 \leq x, y \leq x_{\text {max }}}$ is given by

$$
a_{x, y}=\left\{\begin{array}{ll}
0 & (0 \leq y \leq x-1)  \tag{4.35}\\
\sqrt{h_{x, x}^{\prime}} & (y=x) \\
\frac{h_{x, y}^{\prime}}{\sqrt{h_{x, x}^{\prime}}} & \left(x+1 \leq y \leq x_{\max }\right)
\end{array} \quad\left(0 \leq x \leq x_{\max }\right)\right.
$$

Here $h_{x, y}^{\prime}$ are defined by

$$
\begin{equation*}
h_{x, y}^{\prime}=h_{x, y}-\sum_{z=0}^{x-1} \frac{h_{z, x}^{\prime} h_{z, y}^{\prime}}{\sqrt{h_{z, z}^{\prime}}} \quad\left(0 \leq x \leq y \leq x_{\max }\right) \tag{4.36}
\end{equation*}
$$

where $h_{x, y}=\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})_{x, y}$ and the convention $\sum_{i=n}^{n-1} *=0$ is assumed. Note that the zero eigenvalue of $\mathcal{H}_{\mathcal{D}}^{X}$ dual $(\boldsymbol{\lambda})$ implies $\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})_{x_{\max }, x_{\max }}=0$. So, the last row of $\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})$ is zero.

Next we recall the general theory of the shape invariance for finite rdQM systems ( $x_{\max }=$ $\left.n_{\max }=N\right)$ 38. The Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})=\left(\mathcal{H}(\boldsymbol{\lambda})_{x, y}\right)_{0 \leq x, y \leq x_{\max }}$ is positive semi-definite, whose eigenvalues are $0=\mathcal{E}_{0}(\boldsymbol{\lambda})<\mathcal{E}_{1}(\boldsymbol{\lambda})<\cdots<\mathcal{E}_{n_{\max }}(\boldsymbol{\lambda})$ and corresponding eigenvectors are $\phi_{n}(x ; \boldsymbol{\lambda})$, and factorized as $\mathcal{H}(\boldsymbol{\lambda})=\mathcal{A}(\boldsymbol{\lambda})^{\dagger} \mathcal{A}(\boldsymbol{\lambda})$, where $\mathcal{A}(\boldsymbol{\lambda})$ is upper triangular and $\mathcal{A}(\boldsymbol{\lambda})_{x_{\max }, x_{\max }}=0$. Since the last row of $\mathcal{A}(\boldsymbol{\lambda})$ is zero, we have

$$
\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}=\left(\begin{array}{cc}
B & \overrightarrow{0} \\
t \overrightarrow{0} & 0
\end{array}\right), \quad \mathcal{A}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})=\binom{\vec{b}}{0} .
$$

Let us write these $N \times N$ matrix $B$ and $N$ component vector $\vec{b}$ as

$$
\begin{equation*}
B=\left(\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\right)^{[N \times N]}, \quad \vec{b}=\left(\mathcal{A}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})\right)^{[N]} \tag{4.37}
\end{equation*}
$$

(In our previous studies [38]-[40], we treat tridiagonal Hamiltonians and $\mathcal{A}$ is an upper triangular matrix with upper bandwidth 1. Here this is not assumed.) Shape invariance is a relation between the system with parameters $\boldsymbol{\lambda}$ and that with $\boldsymbol{\lambda}^{\prime}$. Usually, appropriate choice of parameters allow us to express $\boldsymbol{\lambda}^{\prime}$ as shifts of parameters $\boldsymbol{\lambda}^{\prime}=\boldsymbol{\lambda}+\boldsymbol{\delta}$, but here we do not assume this. The number $N$, which corresponds to the size of the Hamiltonian, is one element of $\boldsymbol{\lambda}$ and it changes to $N-1$ in $\boldsymbol{\lambda}^{\prime}$. Then the shape invariant condition is

$$
\begin{equation*}
\left(\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\right)^{[N \times N]}=\kappa \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)^{\dagger} \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)+\mathcal{E}_{1}(\boldsymbol{\lambda}) \tag{4.38}
\end{equation*}
$$

where $\kappa$ is a positive constant and $\mathcal{E}_{1}(\boldsymbol{\lambda})$ is the abbreviation for $\mathcal{E}_{1}(\boldsymbol{\lambda}) \mathbf{1}_{N}$. The Darboux transformation is defined by

$$
\begin{equation*}
\mathcal{H}^{\text {new }}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)^{\dagger} \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)=\mathcal{H}\left(\boldsymbol{\lambda}^{\prime}\right) \tag{4.39}
\end{equation*}
$$

$$
\begin{equation*}
\phi_{n}^{\text {new }}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(\mathcal{A}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]} \quad(n=0,1, \ldots, N-1) \tag{4.40}
\end{equation*}
$$

The shape invariant condition (4.38) gives

$$
\begin{aligned}
& \left(\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\right)^{[N \times N]} \phi_{n}^{\mathrm{new}}(x ; \boldsymbol{\lambda})=\left(\kappa \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)^{\dagger} \mathcal{A}\left(\boldsymbol{\lambda}^{\prime}\right)+\mathcal{E}_{1}(\boldsymbol{\lambda})\right) \phi_{n}^{\mathrm{new}}(x ; \boldsymbol{\lambda}) \\
= & \left(\left(\begin{array}{cc}
\left(\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\right)^{[N \times N]} & \overrightarrow{0} \\
{ }^{t} \overrightarrow{0} & 0
\end{array}\right)\binom{\left(\mathcal{A}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]}}{0}\right)^{[N]} \\
= & \left(\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger} \cdot \mathcal{A}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]} \\
= & \left(\mathcal{A}(\boldsymbol{\lambda}) \cdot \mathcal{A}(\boldsymbol{\lambda})^{\dagger} \mathcal{A}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]}=\left(\mathcal{A}(\boldsymbol{\lambda}) \cdot \mathcal{H}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]} \\
= & \left(\mathcal{E}_{n+1}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda}) \phi_{n+1}(x ; \boldsymbol{\lambda})\right)^{[N]}=\mathcal{E}_{n+1}(\boldsymbol{\lambda}) \phi_{n}^{\mathrm{new}}(x ; \boldsymbol{\lambda}),
\end{aligned}
$$

namely,

$$
\begin{equation*}
\mathcal{H}^{\text {new }}(\boldsymbol{\lambda}) \phi_{n}^{\text {new }}(x ; \boldsymbol{\lambda})=\frac{1}{\kappa}\left(\mathcal{E}_{n+1}(\boldsymbol{\lambda})-\mathcal{E}_{1}(\boldsymbol{\lambda})\right) \phi_{n}^{\text {new }}(x ; \boldsymbol{\lambda}) \tag{4.41}
\end{equation*}
$$

From the relation $\mathcal{H}^{\text {new }}(\boldsymbol{\lambda})=\mathcal{H}\left(\boldsymbol{\lambda}^{\prime}\right)$, we obtain

$$
\begin{equation*}
\mathcal{E}_{n}\left(\boldsymbol{\lambda}^{\prime}\right)=\frac{1}{\kappa}\left(\mathcal{E}_{n+1}(\boldsymbol{\lambda})-\mathcal{E}_{1}(\boldsymbol{\lambda})\right) \quad(n=0,1, \ldots, N-1) \tag{4.42}
\end{equation*}
$$

This relation implies that the energy eigenvalues $\mathcal{E}_{n}(\boldsymbol{\lambda})$ are determined by the information of the first excited state energy $\mathcal{E}_{1}(\boldsymbol{\lambda})$. For example, the energy eigenvalues $\mathcal{E}_{n}(\boldsymbol{\lambda})$ for the original $(q-)$ Racah systems are given by (2.8), and new set of parameters $\boldsymbol{\lambda}^{\prime}$ is $\boldsymbol{\lambda}+\boldsymbol{\delta}$. It is easy to check (4.42) and $\mathcal{E}_{n}(\boldsymbol{\lambda})=\sum_{s=0}^{n-1} \kappa^{s} \mathcal{E}_{1}(\boldsymbol{\lambda}+s \boldsymbol{\delta})(n=0,1, \ldots, N)$ for these cases.

Now let us consider the shape invariance for the new exactly solvable rdQM $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})$. Assume that this system is shape invariant,

$$
\left(\mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda}) \mathcal{A}_{\mathcal{D}}^{X \text { dual }}(\boldsymbol{\lambda})^{\dagger}\right)^{[N \times N]}=\kappa^{\text {dual }} \mathcal{A}_{\mathcal{D}}^{X \text { dual }}\left(\boldsymbol{\lambda}^{\prime}\right)^{\dagger} \mathcal{A}_{\mathcal{D}}^{X \text { dual }}\left(\boldsymbol{\lambda}^{\prime}\right)+\mathcal{E}_{\mathcal{D}, 1}^{X \text { dual }}(\boldsymbol{\lambda})
$$

Then (4.42) gives

$$
\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}\left(\boldsymbol{\lambda}^{\prime}\right)=\frac{1}{\kappa^{\text {dual }}}\left(\mathcal{E}_{\mathcal{D}, n+1}^{X \text { dual }}(\boldsymbol{\lambda})-\mathcal{E}_{\mathcal{D}, 1}^{X \text { dual }}(\boldsymbol{\lambda})\right) \quad(n=0,1, \ldots, N-1)
$$

namely,

$$
\check{X}\left(x ; \boldsymbol{\lambda}^{\prime}\right)=\frac{1}{\kappa^{\text {dual }}}(\check{X}(x+1 ; \boldsymbol{\lambda})-\check{X}(1 ; \boldsymbol{\lambda})) \quad(x=0,1, \ldots, N-1) .
$$

However, these relations do not hold in general, because the concrete expression of $\mathcal{E}_{\mathcal{D}, n}^{X}$ dual $(\boldsymbol{\lambda})=$ $\check{X}(n ; \boldsymbol{\lambda})$ is much more complicated than $\mathcal{E}_{n}(\boldsymbol{\lambda})(2.8)$. We will convince this by calculating small $M$ examples, see $\S 4.3$. Therefore the new exactly solvable $\operatorname{rdQM} \mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})$ is not shape invariant.

### 4.3 Examples

To write down the Hamiltonian $\mathcal{H}_{\mathcal{D}}^{X}{ }^{\text {dual }}(\boldsymbol{\lambda})$ (4.1) or similarity transformed one $\widetilde{\mathcal{H}}_{\mathcal{D}}^{X}$ dual $(\boldsymbol{\lambda})$ (4.13), we need explicit form of the coefficients $r_{n, k}^{X, \mathcal{D}}(\boldsymbol{\lambda})(2.25)$. We can calculate $r_{n, k}^{X, \mathcal{D}} \operatorname{explic-}$ itly for small $M, d_{j}, \operatorname{deg} Y$ and $n$, and check various properties of the system for small $N$ : the Schrödinger equation (4.5) (or (4.12)), commutativity (4.11), the closure relation (4.15) (or (4.31)), action of the creation/annihilation operators (4.29) (or (4.33)), etc. Monotonically increasing property of the eigenvalues (4.10) can be also checked for small $N$ by assigning various numerical values to $q, b, c, d$ satisfying (2.3). However, to find the closed expression of $r_{n, k}^{X, \mathcal{D}}$ for general $n$ is very difficult. We have obtained such general $n$ expression of $r_{n, k}^{X, \mathcal{D}}$ for

$$
\begin{aligned}
\mathrm{R}: \mathcal{D} & =\{1\}, Y=1 ; \mathcal{D}=\{2\}, Y=1 ; \mathcal{D}=\{1,2\}, Y=1 ; \mathcal{D}=\{1\}, Y(\eta)=\eta, \\
q \mathrm{R}: \mathcal{D} & =\{1\}, Y=1 ; \mathcal{D}=\{2\}, Y=1,
\end{aligned}
$$

but their explicit forms are somewhat lengthy. Here we write down $r_{n, k}^{X, \mathcal{D}}$ for $\mathcal{D}=\{1\}$ and $Y=1$ [49]. For other cases, we present $X(\eta)$ only. From $X(\eta)=X(\eta ; \boldsymbol{\lambda})$ and (2.23), the energy eigenvalues $\mathcal{E}_{\mathcal{D}, n}^{X \text { dual }}(\boldsymbol{\lambda})$ (4.7) are obtained. Since the overall normalization of $X(\eta)$ is not important, we multiply $X(\eta)(2.22)$ by an appropriate positive factor.

### 4.3.1 dual multi-indexed Racah systems

We set $\sigma_{1}=a+b, \sigma_{2}=a b, \sigma_{1}^{\prime}=c+d$ and $\sigma_{2}^{\prime}=c d$.
$\underline{\text { Ex. } 1} \mathcal{D}=\{1\}, Y(\eta)=1\left(\Rightarrow L=2, X=X_{\min }\right)$

$$
\begin{align*}
X(\eta)= & -2 c(d-a+1)(d-b+1) I_{\lambda+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
= & -\eta\left(\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) \eta-\sigma_{1}\left(2 c+d+2 \sigma_{2}^{\prime}\right)+2 \sigma_{2} c+2 \sigma_{1}^{\prime}+\sigma_{2}^{\prime}(5+2 d)+d^{2}\right),  \tag{4.43}\\
r_{n, 2}^{X, \mathcal{D}}= & -\frac{\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right)(c+n)(c+n+3)(a+n, b+n, \tilde{d}+n)_{2}}{(\tilde{d}+2 n)_{4}}, \\
r_{n,-2}^{X, \mathcal{D}}= & -\frac{\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right)(\tilde{d}-c+n-3)(\tilde{d}-c+n)(\tilde{d}-a+n-1, \tilde{d}-b+n-1, n-1)_{2}}{(\tilde{d}+2 n-3)_{4}}, \\
r_{n, 1}^{X, \mathcal{D}}= & -\frac{2(a+n)(b+n)(c+n)(c+n+2)(\tilde{d}-c+n)(\tilde{d}+n)}{(\tilde{d}+2 n+3)(\tilde{d}+2 n-1)_{3}} \\
& \times\left(-2\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) n(n+\tilde{d}+1)+2(1-\tilde{d})\left(1+c-\sigma_{2}\right)+d\left(1-\tilde{d}^{2}\right)\right),  \tag{4.44}\\
r_{n,-1}^{X, \mathcal{D}}= & -\frac{2 n(\tilde{d}-a+n)(\tilde{d}-b+n)(c+n)(\tilde{d}-c+n-2)(\tilde{d}-c+n)}{(\tilde{d}+2 n-3)(\tilde{d}+2 n-1)_{3}}
\end{align*}
$$

$$
\begin{aligned}
& \times\left(-2\left(2-\sigma_{1}+\sigma_{1}^{\prime}\right) n(n+\tilde{d}-1)+2\left(1+c-\sigma_{2}\right)+2\left(\sigma_{2}+c-\tilde{d}\right) \tilde{d}+d\left(1-\tilde{d}^{2}\right)\right) \\
r_{n, 0}^{X, \mathcal{D}}= & -\sum_{k=1}^{2}\left(r_{n,-k}^{X, \mathcal{D}}+r_{n, k}^{X, \mathcal{D}}\right)
\end{aligned}
$$

$\underline{\text { Ex. } 2} \mathcal{D}=\{2\}, Y(\eta)=1\left(\Rightarrow L=3, X=X_{\min }\right)$

$$
\begin{align*}
X(\eta)=3 c & (1+c)(d-a+1)(d-a+2)(d-b+1)(d-b+2) I_{\lambda+\delta}\left[\Xi_{\mathcal{D}}\right](\eta) \\
=\eta & \left(\left(\sigma_{1}-\sigma_{1}^{\prime}-4\right)\left(\sigma_{1}-\sigma_{1}^{\prime}-3\right) \eta^{2}\right. \\
& -\left(\sigma_{1}-\sigma_{1}^{\prime}-3\right)\left(3(1+c)(d-a)(d-b)+2(5+6 c) d-2 \sigma_{1}(2+3 c)+4+10 c\right) \eta \\
& +\left(3 c^{2}+6 c+2\right) d^{2}\left(d-\sigma_{1}\right)^{2}+\left(12+40 c+21 c^{2}\right) d^{3}  \tag{4.45}\\
& +\left(22+88 c+50 c^{2}-\sigma_{1}\left(16+55 c+30 c^{2}\right)+\sigma_{2}\left(3+9 c+6 c^{2}\right)\right) d^{2} \\
& +\left(12+70 c+46 c^{2}-\sigma_{1}\left(16+71 c+45 c^{2}\right)+\sigma_{1}^{2}\left(4+15 c+9 c^{2}\right)+3 \sigma_{2}\left(3+10 c+7 c^{2}\right)\right. \\
& \left.\left.-3 \sigma_{1} \sigma_{2}(1+c)(1+2 c)\right) d+3(a-2)(a-1)(b-2)(b-1) c(c+1)\right) .
\end{align*}
$$

$\underline{\text { Ex. } 3} \mathcal{D}=\{1,2\}, Y(\eta)=1\left(\Rightarrow L=3, X=X_{\text {min }}\right)$

$$
\begin{align*}
& X(\eta)=3 c(c+1)(d-a+1)(d-a+2)(d-b+1)(d-b+2) I_{\boldsymbol{\lambda}+2 \boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
&=\eta\left(\left(\sigma_{1}-\sigma_{1}^{\prime}-3\right)\left(\sigma_{1}-\sigma_{1}^{\prime}-2\right) \eta^{2}\right. \\
&-\left(\sigma_{1}-\sigma_{1}^{\prime}-3\right)\left(3(1+c) d\left(d-\sigma_{1}\right)+(7+9 c) d+2+4 c-\sigma_{1}-3 c\left(\sigma_{1}-\sigma_{2}\right)\right) \eta \\
&+\left(2+6 c+3 c^{2}\right) d^{2}\left(d-\sigma_{1}\right)^{2}+\left(12+40 c+21 c^{2}\right) d^{3}  \tag{4.46}\\
&+\left(22+89 c+50 c^{2}-\sigma_{1}\left(14+55 c+30 c^{2}\right)+3 \sigma_{2} c(3+2 c)\right) d^{2} \\
&+\left(12+76 c+47 c^{2}-\sigma_{1}\left(10+73 c+45 c^{2}\right)+\sigma_{1}^{2}\left(2+15 c+9 c^{2}\right)-3 \sigma_{1} \sigma_{2} c(3+2 c)\right. \\
&\left.\left.+3 \sigma_{2} c(10+7 c)\right) d-3\left(\sigma_{1}-\sigma_{2}-1\right) c\left(7+5 c-\sigma_{1}(3+2 c)+\sigma_{2}(1+c)\right)\right) .
\end{align*}
$$

$\underline{\text { Ex. } 4} \mathcal{D}=\{1\}, Y(\eta)=\eta(\Rightarrow L=3)$

$$
\begin{array}{rl}
X(\eta)=-6 & c(d-a+1)(d-b+1) I_{\boldsymbol{\lambda}+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}} Y\right](\eta) \\
=-\eta & \left(2\left(\sigma_{1}^{\prime}-\sigma_{1}+2\right) \eta^{2}\right. \\
& +\left(3 d(1+c)\left(d-\sigma_{1}\right)+(5+9 c) d-2+2 c+\sigma_{1}+3 c\left(\sigma_{2}-\sigma_{1}\right)\right) \eta  \tag{4.47}\\
& \left.+d\left(d(1+3 c)\left(d-\sigma_{1}\right)+(1+7 c) d-2+2 c+\sigma_{1}+3 c\left(\sigma_{2}-\sigma_{1}\right)\right)\right)
\end{array}
$$

### 4.3.2 dual multi-indexed $q$-Racah systems

We set $\sigma_{1}=a+b, \sigma_{2}=a b, \sigma_{1}^{\prime}=c+d$ and $\sigma_{2}^{\prime}=c d$.
Ex. $1 \mathcal{D}=\{1\}, Y(\eta)=1\left(\Rightarrow L=2, X=X_{\text {min }}\right)$

$$
\begin{aligned}
X(\eta)= & -(1+q)(1-c)\left(1-a^{-1} d q\right)\left(1-b^{-1} d q\right) I_{\boldsymbol{\lambda}+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
= & -\eta\left(\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right) \eta+\sigma_{2}^{-1} q^{2}(1+q-2 c q) d^{2}\right. \\
& \left.\quad-\sigma_{2}^{-1}\left(\sigma_{1} q(1+q)(1-c)+(1-q)\left(\sigma_{2}+c q^{2}\right)\right) d+2-c(1+q)\right), \\
r_{n, 2}^{X, \mathcal{D}}= & -\frac{\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right)\left(1-c q^{n}\right)\left(1-c q^{n+3}\right)\left(a q^{n}, b q^{n}, \tilde{d} q^{n} ; q\right)_{2}}{\left(\tilde{d} q^{2 n} ; q\right)_{4}}, \\
r_{n,-2}^{X, \mathcal{D}}= & -\frac{d^{2} q^{2}\left(1-\sigma_{2}^{-1} \sigma_{2}^{\prime} q^{2}\right)\left(1-c^{-1} \tilde{d} q^{n-3}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(a^{-1} \tilde{d} q^{n-1}, b^{-1} \tilde{d} q^{n-1}, q^{n-1} ; q\right)_{2}}{\left(\tilde{d} q^{2 n-3} ; q\right)_{4}}, \\
r_{n, 1}^{X, \mathcal{D}=}= & -\frac{(1+q)\left(1-a q^{n}\right)\left(1-b q^{n}\right)\left(1-c q^{n}\right)\left(1-c q^{n+2}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(1-\tilde{d} q^{n}\right)}{\sigma_{2} d\left(1-\tilde{d} q^{2 n+3}\right)\left(\tilde{d} q^{2 n-1} ; q\right)_{3}} \\
& \times\left(-\left(\sigma_{2} \sigma_{1}^{\prime}+\sigma_{1}(1-c) d q-\sigma_{1}^{\prime} d q^{2}\right)\left(\sigma_{2} c q^{2 n}+d\right)\right. \\
& \left.+\left(q+q^{-1}\right) d\left(\sigma_{1} \sigma_{2} c+\sigma_{2}(1-c) \sigma_{1}^{\prime} q-\sigma_{1} \sigma_{2}^{\prime} q^{2}\right) q^{n}\right), \\
r_{n,-1}^{X, \mathcal{D}=}= & \frac{(1+q)\left(1-q^{n}\right)\left(1-a^{-1} \tilde{d} q^{n}\right)\left(1-b^{-1} \tilde{d} q^{n}\right)\left(1-c q^{n}\right)\left(1-c^{-1} \tilde{d} q^{n-2}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)}{\sigma_{2}\left(1-\tilde{d} q^{2 n-3}\right)\left(\tilde{d} q^{2 n-1} ; q\right)_{3}} \\
\times & \left(-\left(\sigma_{2} \sigma_{1}^{\prime}+\sigma_{1}(1-c) d q-\sigma_{1}^{\prime} d q^{2}\right)\left(\sigma_{2} c q^{2 n-1}+d q\right)\right. \\
& \left.+\left(q+q^{-1}\right) d\left(\sigma_{1} \sigma_{2} c+\sigma_{2}(1-c) \sigma_{1}^{\prime} q-\sigma_{1} \sigma_{2}^{\prime} q^{2}\right) q^{n}\right), \\
r_{n, 0}^{X, \mathcal{D}=}= & \sum_{k=1}^{2}\left(r_{n,-k}^{X, \mathcal{D}}+r_{n, k}^{X, \mathcal{D}) .}\right.
\end{aligned}
$$

$\underline{\text { Ex. } 2} \mathcal{D}=\{2\}, Y(\eta)=1\left(\Rightarrow L=3, X=X_{\min }\right)$

$$
\begin{align*}
X(\eta)= & \left(1+q+q^{2}\right)(1-c)(1-c q)(a-d q)\left(a-d q^{2}\right)(b-d q)\left(b-d q^{2}\right) I_{\boldsymbol{\lambda}+\boldsymbol{\delta}}\left[\Xi_{\mathcal{D}}\right](\eta) \\
= & \eta\left(\left(\sigma_{2}-c d q^{3}\right)\left(\sigma_{2}-c d q^{4}\right) \eta^{2}+\left(\sigma_{2}-c d q^{3}\right)\left(\left(1+q+q^{2}\right)\left(q^{3} d^{2}-q(1-c q) \sigma_{1} d-c \sigma_{2}\right)\right.\right. \\
& \left.\quad-3 c q^{5} d^{2}-(1-q)^{2}\left(\sigma_{2}-c q^{3}\right) d+3 \sigma_{2}\right) \eta \\
& +\left(1+q+q^{2}\right)\left(q^{6}(1-2 c q) d^{4}-q^{3}\left(q(1-c q)(1+q-2 c q) \sigma_{1}+(1-q)\left(\sigma_{2}+c q^{3}\right)\right) d^{3}\right. \\
& +q\left(q^{2}(1-c)(1-c q) \sigma_{1}^{2}+(1-q)(1-c q)\left(\sigma_{2}+c q^{3}\right) \sigma_{1}+q(1+q)\left(1+c^{2} q^{2}\right) \sigma_{2}\right) d^{2} \\
& \left.\quad-\left(q(1-c q)(2-(1+q) c) \sigma_{1}-(1-q)\left(\sigma_{2}+q^{3} c\right) c\right) \sigma_{2} d-(2-c q) c \sigma_{2}^{2}\right)  \tag{4.50}\\
+ & 3 q^{9} c^{2} d^{4}+q^{4}(1-q)\left((2+q) \sigma_{2}+q^{2}\left(1+2 q^{2}\right) c\right) c d^{3} \\
& -q\left((1-q)^{2}\left(\sigma_{2}^{2}+q^{5} c^{2}\right)+q(1+q)\left(1+4 q^{2}+q^{4}\right) c \sigma_{2}\right) d^{2}
\end{align*}
$$

$$
\left.-\sigma_{2}(1-q)\left(\left(2+q^{2}\right) \sigma_{2}+q^{3}(1+2 q) c\right) d+3 \sigma_{2}^{2}\right)
$$

## 5 Summary and Comments

The case-(1) multi-indexed ( $M$-indexed) ( $q$-)Racah orthogonal polynomials $\check{P}_{\mathcal{D}, n}(x)$ satisfy the second order difference equations (2.18) [25] and various $1+2 L(L \geq M+1)$ term recurrence relations with constant coefficients (2.25) [49]. Corresponding to these properties, their dual polynomials $\check{Q}_{\mathcal{D}, x}(n)$ (3.1) satisfy the three term recurrence relations (3.7) and various $2 L$-th order difference equations (3.14). That is, the dual multi-indexed ( $q$-)Racah polynomials are ordinary orthogonal polynomials and the Krall-type. Their weight functions do not contain delta functions (Kronecker deltas).

We construct new exactly solvable rdQM systems, whose eigenvectors are described by the dual multi-indexed $(q-)$ Racah polynomials. Their Hamiltonians (4.1) are not tridiagonal but " $(1+2 L)$-diagonal". These quantum systems satisfy the closure relations (4.15), from which the creation/annihilation operators (4.29) are obtained, but they are not shape invariant. As a sufficient condition for exact solvability, we know two conditions: the closure relation and the shape invariance. Concerning the exactly solvable models we have studied, we observe that when they satisfy the (generalized) closure relation, they are also shape invariant. The new exactly solvable rdQM systems (4.1) give counterexamples to this observation.

Finally we list some problems related to the dual multi-indexed ( $q$-) Racah polynomials.

1. The commutativity (4.11) originates from the non-uniqueness of $X$ giving the recurrence relations with constant coefficients (2.25). The relations among the recurrence relations for various $X(Y)$ are unclear. It is an important problem to clarify them.
2. Orthogonal polynomials of a discrete variable in the Askey-scheme can be obtained as certain limits of the ( $q$-) Racah polynomials [10]. It is an interesting problem to study various limits of the (dual) multi-indexed ( $q-$ )Racah polynomials. We remark that the (dual) multi-indexed ( $q-$ )Racah polynomials may not reduce to good polynomials in the same limits used for the ( $q$ - $)$ Racah polynomials. For example, the case-(1) multiindexed polynomials are not allowed for some reduced polynomials. See 51] for similar situation in the Krall-type case.
3. For each (exactly solvable) rdQM system, we can construct the (exactly solvable)
birth and death process [52], which is a stationary Markov chain. The new exactly solvable systems described by the dual multi-indexed ( $q-$ )Racah polynomials provide new exactly solvable birth and death processes.
4. We may be able to deform the new exactly solvable systems (4.1) by multi-step Darboux transformations with appropriate seed solutions. At least it is possible to take eigenvectors as seed solutions. However, some formulas in [39] may be modified, because the Hamiltonians (4.1) are not tridiagonal. It is an interesting problem to study such deformation and to clarify whether virtual state vectors exist or not. This will give examples of the combined three directions (i)-(iii) in $\S$ [
5. The case-(1) multi-indexed polynomials of the Laguerre, Jacobi, Wilson and AskeyWilson types satisfy the second order difference equations [21, 23] and various $1+2 L$ term recurrence relations with constant coefficients [45, 47]. But their variable $x$ is continuous and dual polynomials are not defined naturally. It is a challenging problem to construct the Krall-type polynomials related to these multi-indexed polynomials.

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## A Data for Multi-indexed ( $q-$ )Racah Polynomials

For readers' convenience, we present some data for the multi-indexed ( $q$-)Racah polynomials [38, 25, 49], which are not presented in the main text.

- $(q-)$ Racah polynomials $P_{n}(\eta ; \boldsymbol{\lambda})$ :

$$
\begin{align*}
\check{P}_{n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} P_{n}(\eta(x ; \boldsymbol{\lambda}) ; \boldsymbol{\lambda}) & =\left\{\begin{array}{cl}
{ }_{4} F_{3}\left(\left.\begin{array}{c}
-n, n+\tilde{d},-x, x+d \\
a, b, c
\end{array} \right\rvert\, 1\right): \mathrm{R} \\
{ }_{4} \phi_{3}\left(\left.\begin{array}{c}
q^{-n}, \tilde{d} q^{n}, q^{-x}, d q^{x} \\
a, b, c
\end{array} \right\rvert\, q ; q\right): q \mathrm{R}
\end{array}\right.  \tag{A.1}\\
& = \begin{cases}R_{n}(\eta(x ; \boldsymbol{\lambda}) ; a-1, \tilde{d}-a, c-1, d-c) & \mathrm{R} \\
R_{n}\left(\eta(x ; \boldsymbol{\lambda})+1+d ; a q^{-1}, \tilde{d} a^{-1}, c q^{-1}, d c^{-1} \mid q\right) & : q \mathrm{R}\end{cases}
\end{align*}
$$

where $\eta(x ; \boldsymbol{\lambda})$ is given by (2.7) and $\tilde{d}$ is given by (2.8). Here $R_{n}(x(x+\gamma+\delta+1) ; \alpha, \beta, \gamma, \delta)$ and $R_{n}\left(q^{-x}+\gamma \delta q^{x+1} ; \alpha, \beta, \gamma, \delta \mid q\right)$ are the Racah and $q$-Racah polynomials in the conventional
parametrization [10], respectively. Our parametrization respects the correspondence between the ( $q$-)Racah and (Askey-)Wilson polynomials, and symmetries in ( $a, b, c, d$ ) are transparent. - potential functions:

$$
\begin{align*}
& B(x ; \boldsymbol{\lambda})=\left\{\begin{array}{ll}
-\frac{(x+a)(x+b)(x+c)(x+d)}{(2 x+d)(2 x+1+d)} & : \mathrm{R} \\
-\frac{\left(1-a q^{x}\right)\left(1-b q^{x}\right)\left(1-c q^{x}\right)\left(1-d q^{x}\right)}{\left(1-d q^{2 x}\right)\left(1-d q^{2 x+1}\right)} & : q \mathrm{R}
\end{array},\right. \\
& D(x ; \boldsymbol{\lambda})=\left\{\begin{array}{ll}
-\frac{(x+d-a)(x+d-b)(x+d-c) x}{(2 x-1+d)(2 x+d)} & : \mathrm{R} \\
-\tilde{d} \frac{\left(1-a^{-1} d q^{x}\right)\left(1-b^{-1} d q^{x}\right)\left(1-c^{-1} d q^{x}\right)\left(1-q^{x}\right)}{\left(1-d q^{2 x-1}\right)\left(1-d q^{2 x}\right)} & : q \mathrm{R}
\end{array} .\right. \tag{A.2}
\end{align*}
$$

- three term recurrence relations: $\left(P_{n}(\eta ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} 0(n<0)\right)$

$$
\begin{equation*}
\eta P_{n}(\eta ; \boldsymbol{\lambda})=A_{n}(\boldsymbol{\lambda}) P_{n+1}(\eta ; \boldsymbol{\lambda})+B_{n}(\boldsymbol{\lambda}) P_{n}(\eta ; \boldsymbol{\lambda})+C_{n}(\boldsymbol{\lambda}) P_{n-1}(\eta ; \boldsymbol{\lambda}) . \tag{A.3}
\end{equation*}
$$

- coefficients of the three term recurrence relations: $\left(A_{-1}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} 0\right)$

$$
\begin{align*}
& B_{n}(\boldsymbol{\lambda})=-A_{n}(\boldsymbol{\lambda})-C_{n}(\boldsymbol{\lambda}), \\
& A_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
\frac{(n+a)(n+b)(n+c)(n+\tilde{d})}{(2 n+\tilde{d})(2 n+1+\tilde{d})} & : \mathrm{R} \\
\frac{\left(1-a q^{n}\right)\left(1-b q^{n}\right)\left(1-c q^{n}\right)\left(1-\tilde{d} q^{n}\right)}{\left(1-\tilde{d} q^{2 n}\right)\left(1-\tilde{d} q^{2 n+1}\right)} & : q \mathrm{R}
\end{array},\right.  \tag{A.4}\\
& C_{n}(\boldsymbol{\lambda})= \begin{cases}\frac{(n+\tilde{d}-a)(n+\tilde{d}-b)(n+\tilde{d}-c) n}{(2 n-1+\tilde{d})(2 n+\tilde{d})} & : \mathrm{R} \\
d \frac{\left(1-a^{-1} \tilde{d} q^{n}\right)\left(1-b^{-1} \tilde{d} q^{n}\right)\left(1-c^{-1} \tilde{d} q^{n}\right)\left(1-q^{n}\right)}{\left(1-\tilde{d} q^{2 n-1}\right)\left(1-\tilde{d} q^{2 n}\right)} & : q \mathrm{R}\end{cases}
\end{align*}
$$

- ground state eigenvector: $\phi_{0}(x ; \boldsymbol{\lambda})>0$

$$
\phi_{0}(x ; \boldsymbol{\lambda})^{2}= \begin{cases}\frac{(a, b, c, d)_{x}}{(d-a+1, d-b+1, d-c+1,1)_{x}} \frac{2 x+d}{d} & : \mathrm{R}  \tag{A.5}\\ \frac{(a, b, c, d ; q)_{x}}{\left(a^{-1} d q, b^{-1} d q, c^{-1} d q, q ; q\right)_{x} \tilde{d}^{x}} \frac{1-d q^{2 x}}{1-d} & : q \mathrm{R}\end{cases}
$$

- normalization constant: $d_{n}(\boldsymbol{\lambda})>0$

$$
d_{n}(\boldsymbol{\lambda})^{2}=\left\{\begin{array}{ll}
\frac{(a, b, c, \tilde{d})_{n}}{(\tilde{d}-a+1, \tilde{d}-b+1, \tilde{d}-c+1,1)_{n}} \frac{2 n+\tilde{d}}{\tilde{d}}  \tag{A.6}\\
\times \frac{(-1)^{N}(d-a+1, d-b+1, d-c+1)_{N}}{(\tilde{d}+1)_{N}(d+1)_{2 N}} & : \mathrm{R} \\
\frac{(a, b, c, \tilde{d} ; q)_{n}}{\left(a^{-1} \tilde{d} q, b^{-1} \tilde{d} q, c^{-1} \tilde{d} q, q ; q\right)_{n} d^{n}} \frac{1-\tilde{d} q^{2 n}}{1-\tilde{d}} \\
\times \frac{(-1)^{N}\left(a^{-1} d q, b^{-1} d q, c^{-1} d q ; q\right)_{N} \tilde{d}^{N} q^{\frac{1}{2} N(N+1)}}{(\tilde{d} q ; q)_{N}(d q ; q)_{2 N}} & : q \mathrm{R}
\end{array} .\right.
$$

- auxiliary functions: (convention: $\prod_{1 \leq j<k \leq M} *=1$ for $M=0,1$ )

$$
\begin{align*}
& \varphi(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\eta(x+1 ; \boldsymbol{\lambda})-\eta(x ; \boldsymbol{\lambda})}{\eta(1 ; \boldsymbol{\lambda})}= \begin{cases}\frac{2 x+d+1}{d+1} & : \mathrm{R} \\
\frac{q^{-x}-d q^{x+1}}{1-d q} & : q \mathrm{R}\end{cases}  \tag{A.7}\\
& \varphi_{M}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \prod_{1 \leq j<k \leq M} \frac{\eta(x+k-1 ; \boldsymbol{\lambda})-\eta(x+j-1 ; \boldsymbol{\lambda})}{\eta(k-j ; \boldsymbol{\lambda})} \quad\left(\varphi_{0}(x)=\varphi_{1}(x)=1\right) \\
&=\prod_{1 \leq j<k \leq M} \varphi(x+j-1 ; \boldsymbol{\lambda}+(k-j-1) \boldsymbol{\delta}) . \tag{A.8}
\end{align*}
$$

- twist operation $\mathfrak{t}$ and twisted shift $\tilde{\boldsymbol{\delta}}$ :

$$
\begin{equation*}
\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(\lambda_{4}-\lambda_{1}+1, \lambda_{4}-\lambda_{2}+1, \lambda_{3}, \lambda_{4}\right), \quad \tilde{\boldsymbol{\delta}} \stackrel{\text { def }}{=}(0,0,1,1) \tag{A.9}
\end{equation*}
$$

Note that $\eta(x ; \mathfrak{t}(\boldsymbol{\lambda}))=\eta(x ; \boldsymbol{\lambda})$ and $\eta(x ; \boldsymbol{\lambda}+\beta \tilde{\boldsymbol{\delta}})=\eta(x ; \boldsymbol{\lambda}+\beta \boldsymbol{\delta})(\beta \in \mathbb{R})$.

- virtual state polynomial $\xi_{\mathrm{v}}(\eta ; \boldsymbol{\lambda})$ :

$$
\begin{equation*}
\check{\xi}_{\mathrm{v}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \xi_{\mathrm{v}}(\eta(x ; \boldsymbol{\lambda}) ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{P}_{\mathrm{v}}(x ; \mathfrak{t}(\boldsymbol{\lambda}))=P_{\mathrm{v}}(\eta(x ; \boldsymbol{\lambda}) ; \mathfrak{t}(\boldsymbol{\lambda})) . \tag{A.10}
\end{equation*}
$$

- potential functions $B^{\prime}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} B(x ; \mathfrak{t}(\boldsymbol{\lambda})), D^{\prime}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} D(x ; \mathfrak{t}(\boldsymbol{\lambda}))$ :

$$
\begin{align*}
& B^{\prime}(x ; \boldsymbol{\lambda})= \begin{cases}-\frac{(x+d-a+1)(x+d-b+1)(x+c)(x+d)}{(2 x+d)(2 x+1+d)} & : \mathrm{R} \\
-\frac{\left(1-a^{-1} d q^{x+1}\right)\left(1-b^{-1} d q^{x+1}\right)\left(1-c q^{x}\right)\left(1-d q^{x}\right)}{\left(1-d q^{2 x}\right)\left(1-d q^{2 x+1}\right)} & : q \mathrm{R}\end{cases} \\
& D^{\prime}(x ; \boldsymbol{\lambda})= \begin{cases}-\frac{(x+a-1)(x+b-1)(x+d-c) x}{(2 x-1+d)(2 x+d)} & : \mathrm{R} \\
-\frac{c d q}{a b} \frac{\left(1-a q^{x-1}\right)\left(1-b q^{x-1}\right)\left(1-c^{-1} d q^{x}\right)\left(1-q^{x}\right)}{\left(1-d q^{2 x-1}\right)\left(1-d q^{2 x}\right)} & : q \mathrm{R}\end{cases} \tag{A.11}
\end{align*}
$$

- $\alpha(\boldsymbol{\lambda})$ and virtual state energy $\tilde{\mathcal{E}}_{\mathrm{v}}$ :

$$
\alpha(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
1 & : \mathrm{R}  \tag{A.12}\\
a b d^{-1} q^{-1} & : q \mathrm{R}
\end{array}, \quad \tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
-(c+\mathrm{v})(\tilde{d}-c-\mathrm{v}) & : \mathrm{R} \\
-\left(1-c q^{\mathrm{v}}\right)\left(1-c^{-1} \tilde{d} q^{-\mathrm{v}}\right) & : q \mathrm{R}
\end{array} .\right.\right.
$$

- Casorati determinant (Casoratian) of a set of $n$ functions $\left\{f_{j}(x)\right\}$ :

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} \operatorname{det}\left(f_{k}(x+j-1)\right)_{1 \leq j, k \leq n} \tag{A.13}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}_{\mathrm{C}}[\cdot](x)=1$ ).

- $r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)(1 \leq j \leq M+1):\left(x_{j} \stackrel{\text { def }}{=} x+j-1\right)$

$$
r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)= \begin{cases}\frac{(x+a, x+b)_{j-1}(x+d-a+j, x+d-b+j)_{M+1-j}}{(d-a+1, d-b+1)_{M}} & : \mathrm{R}  \tag{A.14}\\ \frac{\left(a q^{x}, b q^{x} ; q\right)_{j-1}\left(a^{-1} d q^{x+j}, b^{-1} d q^{x+j} ; q\right)_{M+1-j}}{\left(a b d^{-1} q^{-1}\right)^{j-1} q^{M x}\left(a^{-1} d q, b^{-1} d q ; q\right)_{M}} & : q \mathrm{R}\end{cases}
$$

- normalization constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}), \mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda}), \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})>0$ and $d_{\mathcal{D}, n}(\boldsymbol{\lambda})>0$ :

$$
\begin{align*}
\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) & =\frac{1}{\varphi_{M}(0 ; \boldsymbol{\lambda})} \prod_{1 \leq j<k \leq M} \frac{\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{k}}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda}) B^{\prime}(j-1 ; \boldsymbol{\lambda})}  \tag{A.15}\\
\mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda}) & =(-1)^{M} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2}  \tag{A.16}\\
\tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{2} & =\frac{\varphi_{M}(0 ; \boldsymbol{\lambda})}{\varphi_{M+1}(0 ; \boldsymbol{\lambda})} \prod_{j=1}^{M} \frac{\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda}) B^{\prime}(j-1 ; \boldsymbol{\lambda )}}  \tag{A.17}\\
d_{\mathcal{D}, n}(\boldsymbol{\lambda}) & =d_{n}(\boldsymbol{\lambda}) \tilde{d}_{\mathcal{D}, n}(\boldsymbol{\lambda}) \tag{A.18}
\end{align*}
$$

- denominator polynomial $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ and multi-indexed $(q-)$ Racah polynomials $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ :

$$
\begin{align*}
\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) & \stackrel{\text { def }}{=} \Xi_{\mathcal{D}}(\eta(x ; \boldsymbol{\lambda}+(M-1) \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \\
& \stackrel{\text { def }}{=} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})^{-1} \varphi_{M}(x ; \boldsymbol{\lambda})^{-1} \operatorname{det}\left(\check{\xi}_{d_{k}}\left(x_{j} ; \boldsymbol{\lambda}\right)\right)_{1 \leq j, k \leq M}  \tag{A.19}\\
\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) & \stackrel{\text { def }}{=} P_{\mathcal{D}, n}(\eta(x ; \boldsymbol{\lambda}+M \boldsymbol{\delta}) ; \boldsymbol{\lambda}) \\
& \stackrel{\text { def }}{=} \mathcal{C}_{\mathcal{D}, n}(\boldsymbol{\lambda})^{-1} \varphi_{M+1}(x ; \boldsymbol{\lambda})^{-1} \\
& \times\left|\begin{array}{cccc}
\check{\xi}_{d_{1}}\left(x_{1} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{1} ; \boldsymbol{\lambda}\right) & r_{1}\left(x_{1}\right) \check{P}_{n}\left(x_{1} ; \boldsymbol{\lambda}\right) \\
\check{\xi}_{d_{1}}\left(x_{2} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{2} ; \boldsymbol{\lambda}\right) & r_{2}\left(x_{2}\right) \check{P}_{n}\left(x_{2} ; \boldsymbol{\lambda}\right) \\
\vdots & \cdots & \vdots & \vdots \\
\check{\xi}_{d_{1}}\left(x_{M+1} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}\left(x_{M+1} ; \boldsymbol{\lambda}\right) & r_{M+1}\left(x_{M+1}\right) \check{P}_{n}\left(x_{M+1} ; \boldsymbol{\lambda}\right)
\end{array}\right| \tag{A.20}
\end{align*}
$$

where $x_{j} \stackrel{\text { def }}{=} x+j-1$ and $r_{j}\left(x_{j}\right)=r_{j}\left(x_{j} ; \boldsymbol{\lambda}, M\right)(1 \leq j \leq M+1)$ are given in (A.14). Other determinant expressions of $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ can be found in [53].

- coefficients of the highest degree term:
$P_{n}(\eta ; \boldsymbol{\lambda})=c_{n}(\boldsymbol{\lambda}) \eta^{n}+($ lower order terms $), P_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})=c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}+n}+$ (lower order terms), $\xi_{\mathrm{v}}(\eta ; \boldsymbol{\lambda})=\tilde{c}_{\mathrm{v}}(\boldsymbol{\lambda}) \eta^{\mathrm{v}}+($ lower order terms $), \quad \Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}}+($ lower order terms $)$,

$$
\begin{align*}
& c_{n}(\boldsymbol{\lambda})=\left\{\begin{array}{ll}
\frac{(\tilde{d}+n)_{n}}{(a, b, c)_{n}} & : \mathrm{R} \\
\frac{\left(\tilde{d} q^{n} ; q\right)_{n}}{(a, b, c ; q)_{n}} & : q \mathrm{R}
\end{array}, \quad \tilde{c}_{\mathrm{v}}(\boldsymbol{\lambda})= \begin{cases}\frac{(c+d-a-b+\mathrm{v}+1)_{\mathrm{v}}}{(d-a+1, d-b+1, c)_{\mathrm{v}}} & : \mathrm{R} \\
\frac{\left(a^{-1} b^{-1} c d q^{\mathrm{v}+1} ; q\right)_{\mathrm{v}}}{\left(a^{-1} d q, b^{-1} d q, c ; q\right)_{\mathrm{v}}} & : q \mathrm{R}\end{cases} \right.  \tag{A.21}\\
& c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})=\prod_{j=1}^{M} \tilde{c}_{d_{j}}(\boldsymbol{\lambda}) \times\left\{\begin{array}{l}
\frac{\prod_{j=1}^{M}(d-a+1, d-b+1, c)_{j-1}}{\prod_{1 \leq j<k \leq M}\left(c+d-a-b+d_{j}+d_{k}+1\right)} \\
\frac{\prod_{j=1}^{M}\left(a^{-1} d q, b^{-1} d q, c ; q\right)_{j-1}}{\prod_{1 \leq j<k \leq M}\left(1-a^{-1} b^{-1} c d q^{d_{j}+d_{k}+1}\right)} \quad: \mathrm{R}
\end{array}\right.  \tag{A.22}\\
& c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda})=c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) c_{n}(\boldsymbol{\lambda}) \times\left\{\begin{array}{l}
\prod_{j=1}^{M} \frac{c+j-1}{c+d_{j}+n}: \mathrm{R} \\
\prod_{j=1}^{M} \frac{1-c q^{j-1}}{1-c q^{d_{j}+n}} \quad: q \mathrm{R}
\end{array}\right. \tag{A.23}
\end{align*}
$$

- coefficients $g_{n}^{\prime(k)}(\boldsymbol{\lambda})$ :

$$
\begin{equation*}
\frac{\eta(x ; \boldsymbol{\lambda})^{n+1}-\eta(x-1 ; \boldsymbol{\lambda})^{n+1}}{\eta(x ; \boldsymbol{\lambda})-\eta(x-1 ; \boldsymbol{\lambda})}=\sum_{k=0}^{n} g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \eta(x ; \boldsymbol{\lambda}-\boldsymbol{\delta})^{n-k} \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{A.24}
\end{equation*}
$$

where $g_{n}^{\prime(k)}(\boldsymbol{\lambda})$ is given by

$$
\begin{align*}
& \mathrm{R}: g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sum_{r=0}^{k} \sum_{l=0}^{k-r}\binom{n+1}{r}\binom{n-r-l}{n-k}(-1)^{r+l}\left(\frac{d}{2}\right)^{2 r}\left(\frac{d-1}{2}\right)^{2(k-r-l)} g_{n-r}^{\prime(l) \mathrm{W}},  \tag{A.25}\\
& q \mathrm{R}: g_{n}^{\prime(k)}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \sum_{r=0}^{k} \sum_{l=0}^{k-r}\binom{n+1}{r}\binom{n-r-l}{n-k}(-1)^{r}\left(2 d^{\frac{1}{2}}\right)^{l} q^{\frac{1}{2}(n-r-l)}(1+d)^{r}\left(1+d q^{-1}\right)^{k-r-l} \\
& \times g_{n-r}^{\prime(l) \mathrm{AW}}, \tag{A.26}
\end{align*}
$$

Here $g_{n}^{\prime(k) \mathrm{W}}$ and $g_{n}^{\prime(k) \mathrm{AW}}$ are 45]

$$
\begin{aligned}
& g_{n}^{\prime(k) \mathrm{W}} \stackrel{\text { def }}{=} \frac{(-1)^{k}}{2^{2 k+1}}\binom{2 n+2}{2 k+1}, \\
& g_{n}^{\prime(k) \mathrm{AW}} \stackrel{\text { def }}{=} \theta(k: \text { even }) \frac{(n+1)!}{2^{k}} \sum_{r=0}^{\frac{k}{2}}\binom{n-k+r}{r} \frac{(-1)^{r} q^{-\frac{1}{2}(n-k+2 r)}}{\left(\frac{k}{2}-r\right)!\left(n-\frac{k}{2}+1+r\right)!} \frac{1-q^{n-k+1+2 r}}{1-q},
\end{aligned}
$$

and $\theta(P)$ is a step function for a proposition $P, \theta(P)=1(P$ : true $), 0(P$ : false $)$.

- map $I_{\boldsymbol{\lambda}}:\{$ polynomial $\} \rightarrow\{$ polynomial $\}:$

$$
\begin{equation*}
p(\eta)=\sum_{k=0}^{n} a_{k} \eta^{k} \mapsto I_{\boldsymbol{\lambda}}[p](\eta) \stackrel{\text { def }}{=} \sum_{k=0}^{n+1} b_{k} \eta^{k} \tag{A.27}
\end{equation*}
$$

where $b_{k}$ 's are defined by

$$
\begin{equation*}
b_{k+1}=\frac{1}{g_{k}^{\prime(0)}(\boldsymbol{\lambda})}\left(a_{k}-\sum_{j=k+1}^{n} g_{j}^{\prime(j-k)}(\boldsymbol{\lambda}) b_{j+1}\right) \quad(k=n, n-1, \ldots, 1,0), \quad b_{0}=0 \tag{A.28}
\end{equation*}
$$

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