# Exactly Solvable Discrete Quantum Mechanical Systems and Multi-indexed Orthogonal Polynomials of the Continuous Hahn and Meixner-Pollaczek Types 

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#### Abstract

We present new exactly solvable systems of the discrete quantum mechanics with pure imaginary shifts, whose physical range of the coordinate is the whole real line. These systems are shape invariant and their eigenfunctions are described by the multiindexed continuous Hahn and Meixner-Pollaczek orthogonal polynomials. The set of degrees of these multi-indexed polynomials are $\left\{\ell_{\mathcal{D}}, \ell_{\mathcal{D}}+1, \ell_{\mathcal{D}}+2, \ldots\right\}$, where $\ell_{\mathcal{D}}$ is an even positive integer ( $\mathcal{D}$ : a multi-index set), but they form a complete set of orthogonal basis in the weighted Hilbert space.


## 1 Introduction

Exactly solvable quantum mechanical systems in one dimension are closely related to the orthogonal polynomials. In ordinary quantum mechanics (oQM), whose Schrödinger equation is the second order differential equation, the Hermite, Laguerre and Jacobi polynomials appear in the harmonica oscillator, the radial oscillator and the Darboux-Pöschl-Teller potential, respectively. In discrete quantum mechanics (dQM) [1, 2, 3], whose Schrödinger equation is the second order difference equation, the Askey-Wilson, $q$-Racah polynomials etc. appear. Orthogonal polynomials satisfying second order differential or difference equations are severely restricted by the Bochner's theorem and its generalizations, and they are summarized as the Askey scheme of the (basic) hypergeometric orthogonal polynomials [4, 5]. We have two types of dQM: dQM with pure imaginary shifts (idQM) and dQM with real shifts (rdQM). The coordinate of idQM is continuous and that of rdQM is discrete.

Recent developments in the theory of orthogonal polynomials and exactly solvable quantum mechanical systems are based on the discovery of new types of orthogonal polynomials: exceptional and multi-indexed polynomials $\left\{P_{\mathcal{D}, n}(\eta) \mid n \in \mathbb{Z}_{\geq 0}\right\}$ [6]-[20]. These polynomials satisfy second order differential or difference equations and form a complete set of orthogonal basis in an appropriate Hilbert space in spite of missing degrees. We distinguish the following two cases; the set of missing degrees $\mathcal{I}=\mathbb{Z}_{\geq 0} \backslash\left\{\operatorname{deg} P_{\mathcal{D}, n}(\eta) \mid n \in \mathbb{Z}_{\geq 0}\right\}$ is case-(1): $\mathcal{I}=\{0,1, \ldots, \ell-1\}$, or case- $(2): \mathcal{I} \neq\{0,1, \ldots, \ell-1\}$, where $\ell$ is a positive integer. The situation of case-(1) is called stable in [11]. Our approach to orthogonal polynomials is based on the quantum mechanical formulation. We deform exactly solvable quantum mechanical systems by multi-step Darboux transformations and obtain multi-indexed polynomials as eigenfunctions of the deformed systems. In the quantum mechanical formulation, the multiindexed orthogonal polynomials appear as polynomials in the sinusoidal coordinate $\eta(x)$ [21, 22], $P_{\mathcal{D}, n}(\eta(x))$, where $x$ is the coordinate of the quantum system.

The range of the coordinate $x$ of oQM is a finite interval ( $0, \frac{1}{2} \pi$ ) for the Darboux-PöschlTeller potential (Jacobi polynomial), the half real line $(0, \infty)$ for the radial oscillator (Laguerre polynomial) and the whole real line $(-\infty, \infty)$ for the harmonic oscillator (Hermite polynomial). The counterparts of idQM to the Jacobi, Laguerre and Hermite polynomials of oQM are Askey-Wilson, Wilson and continuous Hahn polynomials, respectively. Their physical range of the coordinate $x$ is a finite interval $(0, \pi)$, the half real line $(0, \infty)$ and the whole real line $(-\infty, \infty)$, respectively [5]. The situation of the multi-indexed polynomials for oQM and idQM constructed so far is given in Table 1. The case-(2) multi-indexed polynomials are obtained by taking the eigenfunctions as seed solutions of the Darboux transformations, and some eigenvalues are deleted from the original spectrum [23]-[27]. The Darboux transformations with the pseudo virtual state wavefunctions as seed solutions also give the case-(2) multi-indexed polynomials, and some eigenvalues are added to the original spectrum [28]-31]. The case-(1) multi-indexed polynomials are obtained by taking virtual state wavefunctions as seed solutions of the Darboux transformations, and the deformed systems are isospectral to the original systems [13, 15]. For rdQM systems, for example, see [32] for case-(2) and [17, 20] for case-(1).

The purpose of the present paper is to study the case-(1) multi-indexed polynomials of idQM systems on the whole real line, namely, the case-(1) multi-indexed polynomials of the continuous Hahn and Meixner-Pollaczek types, which have not been studied except for

|  | (physical) | (typical) | multi-indexed polynomial |  |
| :---: | :---: | :---: | :---: | :---: |
|  | range of $x$ | orthogonal polynomial | case-(1) | case-(2) |
|  | $\left(0, \frac{1}{2} \pi\right)$ | Jacobi | $\bigcirc$ | $\bigcirc$ |
|  | $(0, \infty)$ | Laguerre | $\bigcirc$ | $\bigcirc$ |
|  | $(-\infty, \infty)$ | Hermite | $\times$ | $\bigcirc$ |
| idQM | $(0, \pi)$ | Askey-Wilson | $\bigcirc$ | $\bigcirc$ |
|  | $(0, \infty)$ | Wilson | $\bigcirc$ | $\bigcirc$ |
|  | $(-\infty, \infty)$ | continuous Hahn | $?$ | $\bigcirc$ |

$\bigcirc$ : possible and constructed, $\times$ : impossible, ?: not yet studied
Table 1: multi-indexed polynomials in oQM and idQM
the 1-indexed Meixner-Pollaczek polynomial with special parameter [3]. The case-(1) multiindexed polynomials are obtained by the Darboux transformations with the virtual state wavefunctions as seed solutions. For oQM on the whole real line, there is no virtual state in the harmonic oscillator and it is impossible to construct the case-(1) multi-indexed Hermite polynomials. For the (Askey-)Wilson cases studied in [15], not only the final deformed Hamiltonian but also the intermediate deformed Hamiltonians are hermitian. For the continuous Hahn and Meixner-Pollaczek cases, however, the intermediate deformed Hamiltonians may be singular. This difference comes from the simple fact that an odd degree polynomial with real coefficients has at least one zero on the whole real line. We ignore the hermiticity of the intermediate deformed Hamiltonians and require the hermiticity of the final deformed Hamiltonian only.

This paper is organized as follows. In section 2 the discrete quantum mechanics with pure imaginary shifts is recapitulated and the data of the continuous Hahn system is presented. In section 3 we deform the continuous Hahn idQM system and obtain new exactly solvable idQM systems and the case-(1) multi-indexed continuous Hahn polynomials. In section 4 we present the case-(1) multi-indexed Meixner-Pollaczek polynomials and new exactly solvable idQM systems. Section 5 is for a summary and comments. In Appendices A and B some properties of the multi-indexed continuous Hahn and Meixner-Pollaczek polynomials are presented, respectively.

## 2 Original Continuous Hahn System

After recapitulating the discrete quantum mechanics with pure imaginary shifts, we present the data of the continuous Hahn system.

### 2.1 Discrete quantum mechanics with pure imaginary shifts

Let us recapitulate the discrete quantum mechanics with pure imaginary shifts (idQM) [2, 3].
The dynamical variables of idQM are the real coordinate $x\left(x_{1} \leq x \leq x_{2}\right)$ and the conjugate momentum $p=-i \partial_{x}$, which are governed by the following factorized positive semi-definite Hamiltonian:

$$
\begin{align*}
& \mathcal{H} \stackrel{\text { def }}{=} \sqrt{V(x)} e^{\gamma p} \sqrt{V^{*}(x)}+\sqrt{V^{*}(x)} e^{-\gamma p} \sqrt{V(x)}-V(x)-V^{*}(x)=\mathcal{A}^{\dagger} \mathcal{A}  \tag{2.1}\\
& \mathcal{A} \stackrel{\text { def }}{=} i\left(e^{\frac{\gamma}{2} p} \sqrt{V^{*}(x)}-e^{-\frac{\gamma}{2} p} \sqrt{V(x)}\right), \quad \mathcal{A}^{\dagger} \stackrel{\text { def }}{=}-i\left(\sqrt{V(x)} e^{\frac{\gamma}{2} p}-\sqrt{V^{*}(x)} e^{-\frac{\gamma}{2} p}\right) \tag{2.2}
\end{align*}
$$

Here the potential function $V(x)$ is an analytic function of $x$ and $\gamma$ is a real constant. The $*-$ operation on an analytic function $f(x)=\sum_{n} a_{n} x^{n}\left(a_{n} \in \mathbb{C}\right)$ is defined by $f^{*}(x)=\sum_{n} a_{n}^{*} x^{n}$, in which $a_{n}^{*}$ is the complex conjugation of $a_{n}$. Since the momentum operator appears in exponentiated forms, the Schrödinger equation

$$
\begin{equation*}
\mathcal{H} \phi_{n}(x)=\mathcal{E}_{n} \phi_{n}(x) \quad(n=0,1,2, \ldots) \tag{2.3}
\end{equation*}
$$

is an analytic difference equation with pure imaginary shifts instead of a differential equation. Throughout this paper we consider those systems which have a square-integrable groundstate together with an infinite number of discrete energy levels: $0=\mathcal{E}_{0}<\mathcal{E}_{1}<\mathcal{E}_{2}<\cdots$. The orthogonality relation reads

$$
\begin{equation*}
\left(\phi_{n}, \phi_{m}\right) \stackrel{\text { def }}{=} \int_{x_{1}}^{x_{2}} d x \phi_{n}^{*}(x) \phi_{m}(x)=h_{n} \delta_{n m} \quad(n, m=0,1,2, \ldots), \quad 0<h_{n}<\infty \tag{2.4}
\end{equation*}
$$

The eigenfunctions $\phi_{n}(x)$ can be chosen 'real', $\phi_{n}^{*}(x)=\phi_{n}(x)$, and the groundstate wavefunction $\phi_{0}(x)$ is determined as the zero mode of the operator $\mathcal{A}, \mathcal{A} \phi_{0}(x)=0$. The norm of a function $f(x)$ is $\|f\| \xlongequal{=}(f, f)^{\frac{1}{2}}$.

The Hamiltonian $\mathcal{H}$ should be hermitian. From its form $\mathcal{H}=\mathcal{A}^{\dagger} \mathcal{A}$, it is formally hermitian, $\mathcal{H}^{\dagger}=\left(\mathcal{A}^{\dagger} \mathcal{A}\right)^{\dagger}=(\mathcal{A})^{\dagger}\left(\mathcal{A}^{\dagger}\right)^{\dagger}=\mathcal{A}^{\dagger} \mathcal{A}=\mathcal{H}$. However, the true hermiticity is defined in terms of the inner product, $\left(f_{1}, \mathcal{H} f_{2}\right)=\left(\mathcal{H} f_{1}, f_{2}\right)$ [2, 22, 15]. To show the hermiticity of
$\mathcal{H}$, singularities of some functions in the rectangular domain $D_{\gamma}$ are important. Here $D_{\gamma}$ is defined by [15]

$$
\begin{equation*}
D_{\gamma} \stackrel{\text { def }}{=}\left\{\left.x \in \mathbb{C}\left|x_{1} \leq \operatorname{Re} x \leq x_{2},|\operatorname{Im} x| \leq \frac{1}{2}\right| \gamma \right\rvert\,\right\} \tag{2.5}
\end{equation*}
$$

In the following, we assume that the eigenfunctions $\phi_{n}(x)(2.3)$ have the following form:

$$
\begin{equation*}
\phi_{n}(x)=\phi_{0}(x) \check{P}_{n}(x), \quad \check{P}_{n}(x) \stackrel{\text { def }}{=} P_{n}(\eta(x)) \quad(n=0,1,2, \ldots), \tag{2.6}
\end{equation*}
$$

where $\eta(x)$ is a sinusoidal coordinate [21, 22] and $P_{n}(\eta)$ is a orthogonal polynomial of degree $n$ in $\eta$. As a polynomial $P_{n}(\eta)$, we consider the Askey-Wilson, Wilson, continuous Hahn polynomials etc., which are members of the Askey-scheme of hypergeometric orthogonal polynomials [5]. We call the idQM system by the name of the orthogonal polynomial: Askey-Wilson system, Wilson system, continuous Hahn system etc. These idQM systems have the property of shape invariance, which is a sufficient condition for exact solvability. Concrete idQM systems have a set of parameters $\boldsymbol{\lambda}=\left(\lambda_{1}, \lambda_{2}, \ldots\right)$. Various quantities depend on them and their dependence is expressed like, $f=f(\boldsymbol{\lambda}), f(x)=f(x ; \boldsymbol{\lambda})$. (We sometimes omit writing $\boldsymbol{\lambda}$-dependence, when it does not cause confusion.)

The shape invariant condition is the following [2, 22, 3]:

$$
\begin{equation*}
\mathcal{A}(\boldsymbol{\lambda}) \mathcal{A}(\boldsymbol{\lambda})^{\dagger}=\kappa \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^{\dagger} \mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_{1}(\boldsymbol{\lambda}) \tag{2.7}
\end{equation*}
$$

where $\kappa$ is a real positive constant and $\boldsymbol{\delta}$ is the shift of the parameters. This condition combined with the Crum's theorem allows the wavefunction $\phi_{n}(x)$ and energy eigenvalue $\mathcal{E}_{n}$ of the excited states to be expressed in terms of the ground state wavefunction $\phi_{0}(x)$ and the first excited state energy eigenvalue $\mathcal{E}_{1}$ with shifted parameters. As a consequence of the shape invariance, we have

$$
\begin{equation*}
\mathcal{A}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda})=f_{n}(\boldsymbol{\lambda}) \phi_{n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{A}(\boldsymbol{\lambda})^{\dagger} \phi_{n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=b_{n-1}(\boldsymbol{\lambda}) \phi_{n}(x ; \boldsymbol{\lambda}), \tag{2.8}
\end{equation*}
$$

where $f_{n}(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are some constants satisfying $f_{n}(\boldsymbol{\lambda}) b_{n-1}(\boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda})$. These relations can be rewritten as

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda})=f_{n}(\boldsymbol{\lambda}) \check{P}_{n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{B}(\boldsymbol{\lambda}) \check{P}_{n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=b_{n-1}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}) \tag{2.9}
\end{equation*}
$$

Here the forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ are defined by

$$
\begin{equation*}
\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_{0}(x ; \boldsymbol{\lambda})=i \varphi(x)^{-1}\left(e^{\frac{\gamma}{2} p}-e^{-\frac{\gamma}{2} p}\right), \tag{2.10}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^{\dagger} \circ \phi_{0}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=-i\left(V(x ; \boldsymbol{\lambda}) e^{\frac{\gamma}{2} p}-V^{*}(x ; \boldsymbol{\lambda}) e^{-\frac{\gamma}{2} p}\right) \varphi(x) \tag{2.11}
\end{equation*}
$$

where $\varphi(x)$ is an auxiliary function $\left(\varphi(x) \propto \eta\left(x-i \frac{\gamma}{2}\right)-\eta\left(x+i \frac{\gamma}{2}\right)\right)$. The difference operator $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ acting on the polynomial eigenfunctions is square root free:

$$
\begin{align*}
& \widetilde{\mathcal{H}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}(\boldsymbol{\lambda}) \circ \phi_{0}(x ; \boldsymbol{\lambda})=\mathcal{B}(\boldsymbol{\lambda}) \mathcal{F}(\boldsymbol{\lambda}) \\
& \quad=V(x ; \boldsymbol{\lambda})\left(e^{\gamma p}-1\right)+V^{*}(x ; \boldsymbol{\lambda})\left(e^{-\gamma p}-1\right)  \tag{2.12}\\
& \widetilde{\mathcal{H}}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}) \tag{2.13}
\end{align*}
$$

### 2.2 Continuous Hahn system

Let us consider the continuous Hahn system. The lower bound $x_{1}$, upper bound $x_{2}$ and the parameter $\gamma$ are

$$
\begin{equation*}
x_{1}=-\infty, \quad x_{2}=\infty, \quad \gamma=1 \tag{2.14}
\end{equation*}
$$

Namely, the physical range of the coordinate $x$ is a whole real line. A set of parameters $\boldsymbol{\lambda}$ is

$$
\begin{equation*}
\boldsymbol{\lambda}=\left(a_{1}, a_{2}\right), \quad a_{i} \in \mathbb{C}, \quad \operatorname{Re} a_{i}>0 \tag{2.15}
\end{equation*}
$$

Here are the fundamental data [2]:

$$
\begin{align*}
& V(x ; \boldsymbol{\lambda})=\left(a_{1}+i x\right)\left(a_{2}+i x\right) \quad\left(\Rightarrow V^{*}(x ; \boldsymbol{\lambda})=\left(a_{1}^{*}-i x\right)\left(a_{2}^{*}-i x\right)\right),  \tag{2.16}\\
& \eta(x)=x, \quad \varphi(x)=1, \quad \mathcal{E}_{n}(\boldsymbol{\lambda})=n\left(n+b_{1}-1\right), \quad b_{1} \stackrel{\text { def }}{=} a_{1}+a_{2}+a_{1}^{*}+a_{2}^{*},  \tag{2.17}\\
& \phi_{n}(x ; \boldsymbol{\lambda})=\phi_{0}(x ; \boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}),  \tag{2.18}\\
& \phi_{0}(x ; \boldsymbol{\lambda})=\sqrt{\Gamma\left(a_{1}+i x\right) \Gamma\left(a_{2}+i x\right) \Gamma\left(a_{1}^{*}-i x\right) \Gamma\left(a_{2}^{*}-i x\right)},  \tag{2.19}\\
& \check{P}_{n}(x ; \boldsymbol{\lambda})=P_{n}(\eta(x) ; \boldsymbol{\lambda})=p_{n}\left(\eta(x) ; a_{1}, a_{2}, a_{1}^{*}, a_{2}^{*}\right)  \tag{2.20}\\
& =i^{n} \frac{\left(a_{1}+a_{1}^{*}, a_{1}+a_{2}^{*}\right)_{n}}{n!}{ }_{3} F_{2}\left(\begin{array}{c|c}
-n, n+b_{1}-1, a_{1}+i x & 1) \\
a_{1}+a_{1}^{*}, a_{1}+a_{2}^{*} & 1) ~
\end{array}\right.  \tag{2.21}\\
& =c_{n}(\boldsymbol{\lambda}) \eta(x)^{n}+(\text { lower order terms }), \quad c_{n}(\boldsymbol{\lambda})=\frac{\left(n+b_{1}-1\right)_{n}}{n!},  \tag{2.22}\\
& h_{n}(\boldsymbol{\lambda})=2 \pi \frac{\prod_{j, k=1}^{2} \Gamma\left(n+a_{j}+a_{k}^{*}\right)}{n!\left(2 n+b_{1}-1\right) \Gamma\left(n+b_{1}-1\right)},  \tag{2.23}\\
& \boldsymbol{\delta}=\left(\frac{1}{2}, \frac{1}{2}\right), \quad \kappa=1, \quad f_{n}(\boldsymbol{\lambda})=n+b_{1}-1, \quad b_{n-1}(\boldsymbol{\lambda})=n . \tag{2.24}
\end{align*}
$$

(Although the notation $b_{1}$ conflicts with $b_{n-1}(\boldsymbol{\lambda})$, we think this does not cause any confusion.) Here $p_{n}\left(\eta ; a_{1}, a_{2}, a_{3}, a_{4}\right)$ in (2.20) is the continuous Hahn polynomial of degree $n$ in $\eta$ [5], and the symbol $(a)_{n}$ is the shifted factorial. Note that $\phi_{0}^{*}(x ; \boldsymbol{\lambda})=\phi_{0}(x ; \boldsymbol{\lambda})$ and $\check{P}_{n}^{*}(x ; \boldsymbol{\lambda})=$ $\check{P}_{n}(x ; \boldsymbol{\lambda})$. It is not necessary to distinguish $\check{P}_{n}$ and $P_{n}$ since $\eta(x)=x$, but we will use both notations to compare with other cases in [15].

## 3 New Exactly Solvable idQM Systems and Multi-indexed Continuous Hahn Polynomials

In this section we deform the continuous Hahn system by applying the multi-step Darboux transformations with the virtual state wavefunctions as seed solutions. The eigenfunctions of the deformed systems are described by the case-(1) multi-indexed continuous Hahn polynomials.

### 3.1 Virtual state wavefunctions

Let us introduce two types of twist operations $\mathfrak{t}$ and constants $\tilde{\boldsymbol{\delta}}$ :

$$
\begin{array}{ll}
\text { type I : } \quad \mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(1-a_{1}^{*}, a_{2}\right), & \tilde{\boldsymbol{\delta}}^{\mathrm{I}} \stackrel{\text { def }}{=}\left(-\frac{1}{2}, \frac{1}{2}\right), \\
\text { type II : } \mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(a_{1}, 1-a_{2}^{*}\right), & \tilde{\boldsymbol{\delta}}^{\mathrm{II}} \stackrel{\text { def }}{=}\left(\frac{1}{2},-\frac{1}{2}\right) . \tag{3.1}
\end{array}
$$

Each twist operation is an involution $\mathfrak{t}^{2}=\mathrm{id}$, and satisfies $\mathfrak{t}(\boldsymbol{\lambda}+\beta \boldsymbol{\delta})=\mathfrak{t}(\boldsymbol{\lambda})+\beta \tilde{\boldsymbol{\delta}}(\beta \in \mathbb{R})$. Their composition $\mathfrak{t}^{\mathrm{III}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}\left(\mathfrak{t}^{\mathrm{I}} \circ \mathfrak{t}^{\mathrm{II}}\right)(\boldsymbol{\lambda})=\left(1-a_{1}^{*}, 1-a_{2}^{*}\right)$ was used to construct the pseudo virtual state wavefunctions [31]. For each of $\mathfrak{t}^{\mathrm{I}}$ and $\mathfrak{t}^{\mathrm{II}}$, the potential function $V(x ; \boldsymbol{\lambda})$ satisfies

$$
\begin{align*}
& V(x ; \boldsymbol{\lambda}) V^{*}(x-i \gamma ; \boldsymbol{\lambda})=\alpha(\boldsymbol{\lambda})^{2} V(x ; \mathfrak{t}(\boldsymbol{\lambda})) V^{*}(x-i \gamma ; \mathfrak{t}(\boldsymbol{\lambda})), \\
& V(x ; \boldsymbol{\lambda})+V^{*}(x ; \boldsymbol{\lambda})=\alpha(\boldsymbol{\lambda})\left(V(x ; \mathfrak{t}(\boldsymbol{\lambda}))+V^{*}(x ; \mathfrak{t}(\boldsymbol{\lambda}))\right)-\alpha^{\prime}(\boldsymbol{\lambda}) \tag{3.2}
\end{align*}
$$

where $\alpha(\boldsymbol{\lambda})$ and $\alpha^{\prime}(\boldsymbol{\lambda})$ are

$$
\left\{\begin{array}{r}
\alpha^{\mathrm{I}}(\boldsymbol{\lambda})=1  \tag{3.3}\\
\alpha^{\mathrm{II}}(\boldsymbol{\lambda})=1
\end{array}, \quad\left\{\begin{aligned}
& \alpha^{\prime \mathrm{I}}(\boldsymbol{\lambda})=-\left(a_{1}+a_{1}^{*}-1\right)\left(a_{2}+a_{2}^{*}\right) \\
& \alpha^{\mathrm{II}}(\boldsymbol{\lambda})=-\left(a_{2}+a_{2}^{*}-1\right)\left(a_{1}+a_{1}^{*}\right)
\end{aligned}\right. \text {. }\right.
$$

In the following, we assume $\operatorname{Re} a_{i}>\frac{1}{2}(i=1,2)$, which gives $\alpha^{\prime}(\boldsymbol{\lambda})<0$. The relations (3.2) imply a linear relation between two Hamiltonians [15]:

$$
\begin{equation*}
\mathcal{H}(\boldsymbol{\lambda})=\alpha(\boldsymbol{\lambda}) \mathcal{H}(\mathfrak{t}(\boldsymbol{\lambda}))+\alpha^{\prime}(\boldsymbol{\lambda}) \tag{3.4}
\end{equation*}
$$

Therefore $\phi_{n}(x ; \mathfrak{t}(\boldsymbol{\lambda}))$ satisfies the Schrödinger equation $\mathcal{H}(\boldsymbol{\lambda}) \phi_{n}(x ; \mathfrak{t}(\boldsymbol{\lambda}))=\tilde{\mathcal{E}}_{n}(\boldsymbol{\lambda}) \phi_{n}(x ; \mathfrak{t}(\boldsymbol{\lambda}))$ with $\tilde{\mathcal{E}}_{n}(\boldsymbol{\lambda})=\alpha(\boldsymbol{\lambda}) \mathcal{E}_{n}(\mathfrak{t}(\boldsymbol{\lambda}))+\alpha^{\prime}(\boldsymbol{\lambda})$. Two types of virtual state wavefunctions $\tilde{\phi}_{\mathrm{v}}(x ; \boldsymbol{\lambda})$ $\left(\mathrm{v} \in \mathcal{V} \subset \mathbb{Z}_{\geq 0}\right)$ are defined by
type I: $\quad \tilde{\phi}_{\mathrm{v}}^{\mathrm{I}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{\mathrm{v}}\left(x ; \mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda})\right)=\tilde{\phi}_{0}^{\mathrm{I}}(x ; \boldsymbol{\lambda}) \check{\xi}_{\mathrm{v}}^{\mathrm{I}}(x ; \boldsymbol{\lambda}), \quad \tilde{\phi}_{0}^{\mathrm{I}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}\left(x ; \mathrm{t}^{\mathrm{I}}(\boldsymbol{\lambda})\right)$,

$$
\begin{equation*}
\check{\xi}_{\mathrm{v}}^{\mathrm{I}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \xi_{\mathrm{v}}^{\mathrm{I}}(\eta(x) ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{P}_{\mathrm{v}}\left(x ; \mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda})\right)=P_{\mathrm{v}}\left(\eta(x) ; \mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda})\right) \quad\left(\mathrm{v} \in \mathcal{V}^{\mathrm{I}}\right), \tag{3.5}
\end{equation*}
$$

type II $\tilde{\phi}_{\mathrm{v}}^{\mathrm{II}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{\mathrm{v}}\left(x ; \mathrm{t}^{\mathrm{II}}(\boldsymbol{\lambda})\right)=\tilde{\phi}_{0}^{\mathrm{II}}(x ; \boldsymbol{\lambda}) \check{\xi}_{\mathrm{v}}^{\mathrm{II}}(x ; \boldsymbol{\lambda}), \quad \tilde{\phi}_{0}^{\mathrm{II}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}\left(x ; \mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda})\right)$,

$$
\begin{equation*}
\check{\xi}_{\mathrm{v}}^{\mathrm{II}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \xi_{\mathrm{v}}^{\mathrm{II}}(\eta(x) ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{P}_{\mathrm{v}}\left(x ; \mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda})\right)=P_{\mathrm{v}}\left(\eta(x) ; \mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda})\right) \quad\left(\mathrm{v} \in \mathcal{V}^{\mathrm{II}}\right) \tag{3.6}
\end{equation*}
$$

The virtual state polynomials $\xi_{\mathrm{v}}(\eta ; \boldsymbol{\lambda})$ are polynomials of degree v in $\eta$. They are chosen ${ }^{\prime}$ real,' $\tilde{\phi}_{0}^{*}(x ; \boldsymbol{\lambda})=\tilde{\phi}_{0}(x ; \boldsymbol{\lambda}), \check{\xi}_{\mathrm{v}}^{*}(x ; \boldsymbol{\lambda})=\check{\xi}_{\mathrm{v}}(x ; \boldsymbol{\lambda})$ and the virtual energies $\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})$ are

$$
\left\{\begin{array}{c}
\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{I}}(\boldsymbol{\lambda})=-\left(a_{1}+a_{1}^{*}-\mathrm{v}-1\right)\left(a_{2}+a_{2}^{*}+\mathrm{v}\right)  \tag{3.7}\\
\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{II}}(\boldsymbol{\lambda})=-\left(a_{2}+a_{2}^{*}-\mathrm{v}-1\right)\left(a_{1}+a_{1}^{*}+\mathrm{v}\right)
\end{array} .\right.
$$

Note that $\alpha^{\prime}(\boldsymbol{\lambda})=\tilde{\mathcal{E}}_{0}(\boldsymbol{\lambda})<0$ and

$$
\begin{equation*}
\tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{I}}(\boldsymbol{\lambda})<0 \Leftrightarrow a_{1}+a_{1}^{*}>\mathrm{v}+1, \quad \tilde{\mathcal{E}}_{\mathrm{v}}^{\mathrm{II}}(\boldsymbol{\lambda})<0 \Leftrightarrow a_{2}+a_{2}^{*}>\mathrm{v}+1, \tag{3.8}
\end{equation*}
$$

for $\mathrm{v} \geq 0$. We choose $\mathcal{V}^{\mathrm{I}}$ and $\mathcal{V}^{\mathrm{II}}$ as

$$
\begin{equation*}
\mathcal{V}^{\mathrm{I}}=\left\{0,1,2, \ldots,\left[a_{1}+a_{1}^{*}-1\right]^{\prime}\right\}, \quad \mathcal{V}^{\mathrm{II}}=\left\{0,1,2, \ldots,\left[a_{2}+a_{2}^{*}-1\right]^{\prime}\right\} \tag{3.9}
\end{equation*}
$$

where $[x]^{\prime}$ denotes the greatest integer not equal or exceeding $x$. Although we have included 0 in $\mathcal{V}$, the Darboux transformations with the label 0 virtual state do not give essentially new systems, see the end of $\S 3.3$.

### 3.2 New exactly solvable systems

By applying multi-step Darboux transformations to the continuous Hahn system in §2.2, we can deform it and obtain new exactly solvable idQM systems. The virtual state wavefunctions in $\S 3.1$ are used as seed solutions, and new systems are isospectral to the original one.

The deformed systems are labeled by $\mathcal{D}=\left\{d_{1}, \ldots, d_{M}\right\}=\left\{d_{1}^{\mathrm{I}}, \ldots, d_{M_{\mathrm{I}}}^{\mathrm{I}}, d_{1}^{\mathrm{II}}, \ldots, d_{M_{\mathrm{II}}}^{\mathrm{II}}\right\}$ ( $M=M_{\mathrm{I}}+M_{\mathrm{II}}, d_{j}^{\mathrm{I}} \in \mathcal{V}^{\mathrm{I}}:$ mutually distinct, $d_{j}^{\mathrm{II}} \in \mathcal{V}^{\mathrm{II}}:$ mutually distinct), which are the degrees and types of the virtual state wavefunctions used in $M$-step Darboux transformations. The Hamiltonian is deformed as $\mathcal{H} \rightarrow \mathcal{H}_{d_{1}} \rightarrow \mathcal{H}_{d_{1} d_{2}} \rightarrow \cdots \rightarrow \mathcal{H}_{d_{1} \ldots d_{s}} \rightarrow \cdots \rightarrow \mathcal{H}_{d_{1} \ldots d_{M}}=\mathcal{H}_{\mathcal{D}}$ by $M$-step Darboux transformations. Exactly speaking, $\mathcal{D}$ is an ordered set. Various quantities of the deformed systems are denoted as $\mathcal{H}_{\mathcal{D}}, \phi_{\mathcal{D} n}, \mathcal{A}_{\mathcal{D}}$, etc. The general formula is as follows [15]:

$$
\begin{align*}
& \mathcal{H}_{\mathcal{D}} \phi_{\mathcal{D} n}(x)=\mathcal{E}_{n} \phi_{\mathcal{D} n}(x) \quad(n=0,1,2, \ldots),  \tag{3.10}\\
& \mathcal{H}_{\mathcal{D}}=\mathcal{A}_{\mathcal{D}}^{\dagger} \mathcal{A}_{\mathcal{D}}  \tag{3.11}\\
& \mathcal{A}_{\mathcal{D}}=i\left(e^{\frac{\gamma}{2} p} \sqrt{V_{\mathcal{D}}^{*}(x)}-e^{-\frac{\gamma}{2} p} \sqrt{V_{\mathcal{D}}(x)}\right), \quad \mathcal{A}_{\mathcal{D}}^{\dagger}=-i\left(\sqrt{V_{\mathcal{D}}(x)} e^{\frac{\gamma}{2} p}-\sqrt{V_{\mathcal{D}}^{*}(x)} e^{-\frac{\gamma}{2} p}\right), \tag{3.12}
\end{align*}
$$

$$
\begin{align*}
V_{\mathcal{D}}(x)= & \sqrt{V\left(x-i \frac{M}{2} \gamma\right) V^{*}\left(x-i \frac{M+2}{2} \gamma\right)} \\
& \times \frac{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right]\left(x+i \frac{\gamma}{2}\right)}{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right]\left(x-i \frac{\gamma}{2}\right)} \frac{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{0}\right](x-i \gamma)}{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{0}\right](x)}  \tag{3.13}\\
\phi_{\mathcal{D} n}(x)= & \mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{n}\right](x) \\
& \times\left(\frac{\sqrt{\prod_{j=0}^{M-1} V\left(x+i\left(\frac{M}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{M}{2}-j\right) \gamma\right)}}{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right]\left(x+i \frac{\gamma}{2}\right)}\right)^{\frac{1}{2}} \tag{3.14}
\end{align*}
$$

where $\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right]$ is the Casorati determinant of a set of $n$ functions $\left\{f_{j}(x)\right\}$,

$$
\begin{equation*}
\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} i^{\frac{1}{2} n n(n-1)} \operatorname{det}\left(f_{k}\left(x_{j}^{(n)}\right)\right)_{1 \leq j, k \leq n}, \quad x_{j}^{(n)} \stackrel{\text { def }}{=} x+i\left(\frac{n+1}{2}-j\right) \gamma \tag{3.15}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}_{\gamma}[\cdot](x)=1$ ). These properties of the Darboux transformation are proved algebraically, and their analytical properties are not considered. Therefore, the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ may be singular. In order to obtain well-defined deformed systems, we have to check the regularity and hermiticity of $\mathcal{H}_{\mathcal{D}}$. It is also necessary to check the square integrability of $\phi_{\mathcal{D} n}(x)$.

To obtain the concrete forms of $V_{\mathcal{D}}(x)$ and $\phi_{\mathcal{D} n}(x)$, we have to evaluate the Casoratians. Let us define the following functions:

$$
\begin{align*}
& \nu^{\mathrm{I}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\phi_{0}(x ; \boldsymbol{\lambda})}{\tilde{\phi}_{0}^{\mathrm{I}}(x ; \boldsymbol{\lambda})}, \quad r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \stackrel{\text { def }}{=} \frac{\nu^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right)}{\nu^{\mathrm{I}}\left(x ; \boldsymbol{\lambda}+(M-1) \tilde{\boldsymbol{\delta}}^{\mathrm{I}}\right)}(j=1,2, \ldots, M), \\
& \nu^{\mathrm{II}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\phi_{0}(x ; \boldsymbol{\lambda})}{\tilde{\phi}_{0}^{\mathrm{II}}(x ; \boldsymbol{\lambda})}, \quad r_{j}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \stackrel{\text { def }}{=} \frac{\nu^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right)}{\nu^{\mathrm{II}}\left(x ; \boldsymbol{\lambda}+(M-1) \tilde{\boldsymbol{\delta}}^{\mathrm{II}}\right)}(j=1,2, \ldots, M), \tag{3.16}
\end{align*}
$$

whose explicit forms are

$$
\begin{align*}
r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) & =(-1)^{j-1} i^{1-M}\left(a_{1}-\frac{M-1}{2}+i x\right)_{j-1}\left(a_{1}^{*}-\frac{M-1}{2}-i x\right)_{M-j}, \\
r_{j}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) & =(-1)^{j-1} i^{1-M}\left(a_{2}-\frac{M-1}{2}+i x\right)_{j-1}\left(a_{2}^{*}-\frac{M-1}{2}-i x\right)_{M-j} . \tag{3.17}
\end{align*}
$$

Furthermore, let us define $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ as follows:

$$
\begin{align*}
& i^{\frac{1}{2} M(M-1)}\left|\begin{array}{llllll}
\vec{X}_{d_{1}^{\mathrm{I}}}^{(M)} & \ldots & \vec{X}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{(M)} & \vec{Y}_{d_{1}^{\mathrm{I}}}^{(M)} & \ldots & \vec{Y}_{d_{M_{\mathrm{II}}}^{(M)}}^{(M)}
\end{array}\right|=\varphi_{M}(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \times A,  \tag{3.18}\\
& i^{\frac{1}{2} M(M+1)}\left|\begin{array}{llllllll}
\vec{X}_{d_{1}^{\mathrm{I}}}^{(M+1)} & \ldots & \vec{X}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{(M+1)} & \vec{Y}_{d_{1}^{\mathrm{I}}}^{(M+1)} & \ldots & \vec{Y}_{d_{M_{\mathrm{II}}}^{(M+1)}}^{(M+1)} & \vec{Z}_{n}^{(M+1)}
\end{array}\right| \\
& =\varphi_{M+1}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \times B, \tag{3.19}
\end{align*}
$$

where $A$ and $B$ are

$$
\begin{align*}
& A=\prod_{j=1}^{M_{\mathrm{I}}-1}\left(a_{2}-\frac{M-1}{2}+i x, a_{2}^{*}-\frac{M-1}{2}-i x\right)_{j} \cdot \prod_{j=1}^{M_{\mathrm{II}}-1}\left(a_{1}-\frac{M-1}{2}+i x, a_{1}^{*}-\frac{M-1}{2}-i x\right)_{j},  \tag{3.20}\\
& B=\prod_{j=1}^{M_{\mathrm{I}}}\left(a_{2}-\frac{M}{2}+i x, a_{2}^{*}-\frac{M}{2}-i x\right)_{j} \cdot \prod_{j=1}^{M_{\mathrm{II}}}\left(a_{1}-\frac{M}{2}+i x, a_{1}^{*}-\frac{M}{2}-i x\right)_{j} \tag{3.21}
\end{align*}
$$

and $\vec{X}_{\mathrm{v}}^{(M)}, \vec{Y}_{\mathrm{v}}^{(M)}$ and $\vec{Z}_{\mathrm{v}}^{(M)}$ are

$$
\begin{align*}
& \left(\vec{X}_{\mathrm{v}}^{(M)}\right)_{j}=r_{j}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \check{\mathrm{y}}_{\mathrm{v}}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right), \quad(j=1,2, \ldots, M), \\
& \left(\vec{Y}_{\mathrm{v}}^{(M)}\right)_{j}=r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \check{\xi}_{\mathrm{v}}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right), \\
& \left(\vec{Z}_{n}^{(M)}\right)_{j}=r_{j}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \check{P}_{n}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right) . \tag{3.22}
\end{align*}
$$

The auxiliary function $\varphi_{M}(x)$ introduced in [27] is $\varphi_{M}(x)=1$ in the present case. These $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are 'real', $\check{\Xi}_{\mathcal{D}}^{*}(x ; \boldsymbol{\lambda})=\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}^{*}(x ; \boldsymbol{\lambda})=\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$. They are polynomials in the sinusoidal coordinate $\eta(x)$ :

$$
\begin{equation*}
\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \Xi_{\mathcal{D}}(\eta(x) ; \boldsymbol{\lambda}), \quad \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} P_{\mathcal{D}, n}(\eta(x) ; \boldsymbol{\lambda}) \tag{3.23}
\end{equation*}
$$

We call $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})$ the denominator polynomial and $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ the multi-indexed polynomial. Their degrees are $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}}+n$, respectively (we assume $c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \neq 0$ and $c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \neq 0$, see (A.1) $-($ A.2) $)$. Here $\ell_{\mathcal{D}}$ is

$$
\begin{equation*}
\ell_{\mathcal{D}} \stackrel{\text { def }}{=} \sum_{j=1}^{M} d_{j}-\frac{1}{2} M(M-1)+2 M_{\mathrm{I}} M_{\mathrm{II}} . \tag{3.24}
\end{equation*}
$$

Then, the Casoratians $\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right](x)$ and $\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{n}\right](x)$ are expressed as

$$
\begin{align*}
& \mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right](x ; \boldsymbol{\lambda}) \\
= & \prod_{j=1}^{M} \phi_{0}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right) \cdot \mathrm{W}_{\gamma}\left[\frac{1}{\nu^{\mathrm{I}}} \check{\xi}_{d_{\mathrm{I}}^{\mathrm{I}}}^{\mathrm{I}}, \ldots, \frac{1}{\nu^{\mathrm{I}}} \check{\xi}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{\mathrm{I}}, \frac{1}{\nu^{\mathrm{II}}} \check{\xi}_{d_{\mathrm{I}}^{\mathrm{II}}}^{\mathrm{I}}, \ldots, \frac{1}{\nu^{\mathrm{II}}} \check{\xi}_{d_{M_{\mathrm{II}}}^{\mathrm{II}}}^{\mathrm{I}}\right](x ; \boldsymbol{\lambda}) \\
= & \prod_{j=1}^{M} \phi_{0}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right) \cdot \nu^{\mathrm{I}}\left(x ; \boldsymbol{\lambda}+(M-1) \tilde{\boldsymbol{\delta}}^{\mathrm{I}}\right)^{-M_{\mathrm{I}}} \nu^{\mathrm{II}}\left(x ; \boldsymbol{\lambda}+(M-1) \tilde{\boldsymbol{\delta}}^{\mathrm{II}}\right)^{-M_{\mathrm{II}}} \\
& \times \prod_{j=1}^{M+1} r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right)^{-1} r_{j}^{\mathrm{II}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right)^{-1} \times \varphi_{M}(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) A,  \tag{3.25}\\
& \mathrm{~W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{n}\right](x ; \boldsymbol{\lambda})
\end{align*}
$$

$$
\begin{align*}
= & \prod_{j=1}^{M+1} \phi_{0}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}\right) \cdot \mathrm{W}_{\gamma}\left[\frac{1}{\nu^{\mathrm{I}}} \check{\xi}_{d_{1}^{\mathrm{I}}}^{\mathrm{I}}, \ldots, \frac{1}{\nu^{\mathrm{I}}} \check{g}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{\mathrm{I}}, \frac{1}{\nu^{\mathrm{II}}} \check{\xi}_{d_{1}^{\mathrm{II}}}^{\mathrm{II}}, \ldots, \frac{1}{\nu^{\mathrm{II}}} \check{\xi}_{d_{M_{\mathrm{II}}}^{\mathrm{II}}}^{\mathrm{II}}, \check{P}_{n}\right](x ; \boldsymbol{\lambda}) \\
= & \prod_{j=1}^{M+1} \phi_{0}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}\right) \cdot \nu^{\mathrm{I}}\left(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}^{\mathrm{I}}\right)^{-M_{\mathrm{I}}} \nu^{\mathrm{II}}\left(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}^{\mathrm{II}}\right)^{-M_{\mathrm{II}}} \\
& \times \prod_{j=1}^{M+1} r_{j}^{\mathrm{I}}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}, M+1\right)^{-1} r_{j}^{\mathrm{II}}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}, M+1\right)^{-1} \times \varphi_{M+1}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) B, \tag{3.26}
\end{align*}
$$

where $A$ and $B$ are given in (3.20) and (3.21) respectively. After some calculation, the eigenfunction (3.14) is rewritten as

$$
\begin{align*}
& \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})=\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}),  \tag{3.27}\\
& \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\phi_{0}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}]}\right]}\right.}{\sqrt{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) \check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}}, \quad \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]} \stackrel{\text { def }}{=} \boldsymbol{\lambda}+M_{\mathrm{I}} \tilde{\boldsymbol{\delta}}^{\mathrm{I}}+M_{\mathrm{II}} \tilde{\boldsymbol{\delta}}^{\mathrm{II}} . \tag{3.28}
\end{align*}
$$

The ground state wavefunction $\phi_{\mathcal{D} 0}$ is annihilated by $\mathcal{A}_{\mathcal{D}}, \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D} 0}(x ; \boldsymbol{\lambda})=0$. The lowest degree multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ is proportional to $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})$, see (A.3). The potential function $V_{\mathcal{D}}(x)(3.13)$ is expressed neatly in terms of the denominator polynomial:

$$
\begin{equation*}
V_{\mathcal{D}}(x ; \boldsymbol{\lambda})=V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)} \frac{\check{\Xi}_{\mathcal{D}}(x-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} . \tag{3.29}
\end{equation*}
$$

Since the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is expressed in terms of the potential function $V_{\mathcal{D}}(x ; \boldsymbol{\lambda}), \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is determined by the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$, whose normalization is irrelevant. Under the permutation of $d_{j}$ 's, the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ is invariant, but the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ and the multi-indexed polynomials $\check{P}_{\mathcal{D}, n}(x)$ may change their signs.

As mentioned before, we have to check the regularity and hermiticity of $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$. Let us consider the function $g(x)$,

$$
\begin{align*}
g(x) \stackrel{\text { def }}{=} & V\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \phi_{0}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right)^{2} \\
= & \Gamma\left(a_{1}-\frac{1}{2} M^{\prime}+\frac{1}{2}+i x\right) \Gamma\left(a_{2}+\frac{1}{2} M^{\prime}+\frac{1}{2}+i x\right) \\
& \times \Gamma\left(a_{1}^{*}-\frac{1}{2} M^{\prime}+\frac{1}{2}-i x\right) \Gamma\left(a_{2}^{*}+\frac{1}{2} M^{\prime}+\frac{1}{2}-i x\right), \tag{3.30}
\end{align*}
$$

where $M^{\prime}=M_{\mathrm{I}}-M_{\mathrm{II}}$. Asymptotic behavior of $g(x)$ at $x \sim \pm \infty$ is $g(x) \sim 4 \pi^{2} e^{\mp \pi \operatorname{Im}\left(a_{1}+a_{2}\right)}$ $|x|^{2 \operatorname{Re}\left(a_{1}+a_{2}\right)} e^{-2 \pi|x|}($ for $x \in \mathbb{R})$, where we have used the asymptotic formula of the gamma function $|\Gamma(x+i y)|^{2} \sim 2 \pi|y|^{2 x-1} e^{-\pi|y|}(x, y \in \mathbb{R}, x$ :fixed, $y \sim \pm \infty)$. The necessary and
sufficient condition for $g(x)$ to have no poles in the rectangular domain $D_{\gamma}$ is $\operatorname{Re} a_{1}-\frac{1}{2} M^{\prime}>$ 0 and $\operatorname{Re} a_{2}+\frac{1}{2} M^{\prime}>0$. This condition is automatically satisfied, because (3.9) implies $M_{\mathrm{I}}-1 \leq \max _{j}\left\{d_{j}^{\mathrm{I}}\right\}<2 \operatorname{Re} a_{1}-1$ and $M_{\mathrm{II}}-1 \leq \max _{j}\left\{d_{j}^{\mathrm{II}}\right\}<2 \operatorname{Re} a_{2}-1$, for $M_{\mathrm{I}}, M_{\mathrm{II}}>0$. (For $M_{\mathrm{I}}=0$ or $M_{\mathrm{II}}=0$, it is trivial.) By the same argument as ref. 15] ( $\S 2.2$ and $\S 3.4$ ), the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is well-defined and hermitian, if the following condition is satisfied:

$$
\begin{equation*}
\text { The denominator polynomial } \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \text { has no zero in } D_{\gamma}(2.5) \text {. } \tag{3.31}
\end{equation*}
$$

This is a sufficient condition for the hermiticity. For the Wilson and Askey-Wilson cases [15], not only the final $M$-step Hamiltonian $\mathcal{H}_{\mathcal{D}}$ but also the intermediate $s$-step Hamiltonians $\mathcal{H}_{d_{1} \ldots d_{s}}$ are well-defined and hermitian. In the present case, however, the intermediate $s$ step Hamiltonians $\mathcal{H}_{d_{1} \ldots d_{s}}$ may be singular. This situation is similar to the $M$-step Darboux transformations with the eigenfunctions as seed solutions [27], in which the intermediate $s$-step Hamiltonians may be singular but the final $M$-step Hamiltonian is well-defined and hermitian if the Krein-Adler condition is satisfied: $\prod_{j=1}^{M}\left(n-d_{j}\right) \geq 0\left(\forall n \in \mathbb{Z}_{\geq 0}\right)$. To satisfy the condition (3.31), the degree of $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}), \ell_{\mathcal{D}}$, should be even, because the rectangular domain $D_{\gamma}$ contains the real axis. Although we have no analytical proof that there exists a range of parameters $\boldsymbol{\lambda}$ satisfying the condition (3.31), we can verify that there exists such a range of $\boldsymbol{\lambda}$ by numerical calculation (for small $M$ and $d_{j}$ ). We have observed various sufficient conditions for the parameter range satisfying (3.31), with $M_{\mathrm{I}} M_{\mathrm{II}} \neq 0$ or $=0$ and $\operatorname{Im} a_{i} \neq 0$ or $=0$. For example, for $M_{\text {II }}=0$ case, the following parameter ranges seem to be sufficient conditions:

- $M_{\mathrm{I}}=1,0 \leq d_{1}<2 \operatorname{Re} a_{1}-1, d_{1}:$ even, $\operatorname{Re} a_{1}-\operatorname{Re} a_{2}<\frac{1}{2}\left(d_{1}+1\right)$,
- $M_{\mathrm{I}}=2,0 \leq d_{1}<d_{2}<2 \operatorname{Re} a_{1}-1, d_{1}:$ even, $d_{2}:$ odd, $\operatorname{Re} a_{1}-\operatorname{Re} a_{2}<\frac{1}{2}\left(d_{1}+1\right)$,
- $M_{\mathrm{I}}=2,0 \leq d_{1}<d_{2}<2 \operatorname{Re} a_{1}-1, d_{1}:$ even, $d_{2}=d_{1}+1, \operatorname{Re} a_{1}-\operatorname{Re} a_{2}>d_{2}+2$,
$\cdot 0 \leq d_{1}<d_{2}<\cdots<d_{M}<2 \operatorname{Re} a_{1}-1,(-1)^{d_{j}}=(-1)^{j-1}(1 \leq j \leq M), \operatorname{Re} a_{1} \ll \operatorname{Re} a_{2}$.
In the following we assume that the condition (3.31) is satisfied.
If the deformed systems is well-defined, the general formula gives the orthogonality of the eigenfunctions [15]:

$$
\begin{equation*}
\left(\phi_{\mathcal{D} n}, \phi_{\mathcal{D} m}\right)=\prod_{j=1}^{M}\left(\mathcal{E}_{n}-\tilde{\mathcal{E}}_{d_{j}}\right) \cdot h_{n} \delta_{n m} \quad(n, m=0,1,2, \ldots) \tag{3.32}
\end{equation*}
$$

Namely, the orthogonality relations of the multi-indexed polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are

$$
\begin{align*}
& \int_{x_{1}}^{x_{2}} d x \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{2} \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \check{P}_{\mathcal{D}, m}(x ; \boldsymbol{\lambda})=h_{\mathcal{D}, n}(\boldsymbol{\lambda}) \delta_{n m} \quad(n, m=0,1,2, \ldots),  \tag{3.33}\\
& h_{\mathcal{D}, n}(\boldsymbol{\lambda})=h_{n}(\boldsymbol{\lambda}) \prod_{j=1}^{M_{\mathrm{I}}}\left(\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}^{\mathrm{I}}}^{\mathrm{I}}(\boldsymbol{\lambda})\right) \cdot \prod_{j=1}^{M_{\mathrm{II}}}\left(\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}^{\mathrm{II}}}^{\mathrm{II}}(\boldsymbol{\lambda})\right) . \tag{3.34}
\end{align*}
$$

The multi-indexed orthogonal polynomial $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ has $n$ zeros in the physical region $\eta \in \mathbb{R}$ ( $\Leftrightarrow \eta\left(x_{1}\right)<\eta<\eta\left(x_{2}\right)$ ), which interlace the $n+1$ zeros of $P_{\mathcal{D}, n+1}(\eta ; \boldsymbol{\lambda})$ in the physical region, and $\ell_{\mathcal{D}}$ zeros in the unphysical region $\eta \in \mathbb{C} \backslash \mathbb{R}$. These properties and (3.33) can be verified by numerical calculation.

For the cases of type I only $\left(M_{\mathrm{I}}=M, M_{\mathrm{II}}=0, \mathcal{D}=\left\{d_{1}, \ldots, d_{M}\right\}\right)$, the expressions (3.18) and (3.19) are slightly simplified,

$$
\begin{align*}
& \mathrm{W}_{\gamma}\left[\check{\xi}_{d_{1}}^{\mathrm{I}}, \ldots, \check{\xi}_{d_{M}}^{\mathrm{I}}\right](x ; \boldsymbol{\lambda})=\varphi_{M}(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}),  \tag{3.35}\\
& \nu^{\mathrm{I}}\left(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}^{\mathrm{I}}\right)^{-1} \mathrm{~W}_{\gamma}\left[\check{\xi}_{d_{1}}^{\mathrm{I}}, \ldots, \check{\xi}_{d_{M}}^{\mathrm{I}}, \nu^{\mathrm{I}} \check{P}_{n}\right](x ; \boldsymbol{\lambda})=\varphi_{M+1}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \\
& =i^{\frac{1}{2} M(M+1)}\left|\begin{array}{cccc}
\check{\xi}_{d_{1}}^{\mathrm{I}}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) & \ldots & \check{\xi}_{d_{M}}^{\mathrm{I}}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) & r_{1}^{\mathrm{I}}\left(x_{1}^{(M+1)}\right) \check{P}_{n}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) \\
\check{\xi}_{d_{1}}^{\mathrm{I}}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda}\right) & \cdots & \check{\xi}_{d_{M}}^{\mathrm{I}}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda}\right) & r_{2}^{\mathrm{I}}\left(x_{2}^{(M+1)}\right) \check{P}_{n}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda}\right) \\
\vdots & \ldots & \vdots & \vdots \\
\check{\xi}_{d_{1}}^{\mathrm{I}}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda}\right) & \ldots & \check{\xi}_{d_{M}}^{\mathrm{I}}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda )}\right. & r_{M+1}^{\mathrm{I}}\left(x_{M+1}^{(M+1)}\right) \check{P}_{n}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda}\right)
\end{array}\right|, \tag{3.36}
\end{align*}
$$

where $r_{j}^{\mathrm{I}}(x)=r_{j}^{\mathrm{I}}(x ; \boldsymbol{\lambda}, M+1)$. The cases of type II only $\left(M_{\mathrm{I}}=0, M_{\mathrm{II}}=M\right)$ are similar.

### 3.3 Shape invariance

The shape invariance of the original system is inherited by the deformed systems. By the same argument of [15], the Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is shape invariant:

$$
\begin{equation*}
\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}=\kappa \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}+\boldsymbol{\delta})^{\dagger} \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_{1}(\boldsymbol{\lambda}) \tag{3.37}
\end{equation*}
$$

As a consequence of the shape invariance, the actions of $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}$ on the eigenfunctions $\phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})$ are

$$
\begin{align*}
& \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})=\kappa^{\frac{M}{2}} f_{n}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \\
& \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger} \phi_{\mathcal{D} n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=\kappa^{-\frac{M}{2}} b_{n-1}(\boldsymbol{\lambda}) \phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda}) . \tag{3.38}
\end{align*}
$$

The forward and backward shift operators are defined by

$$
\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})
$$

$$
\begin{align*}
&= \frac{i}{\varphi(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})}\left(\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right) e^{\frac{\gamma}{2} p}-\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}+\boldsymbol{\delta}\right) e^{-\frac{\gamma}{2} p}\right),  \tag{3.39}\\
& \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger} \circ \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}) \\
&=\frac{-i}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}\left(V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) e^{\frac{\gamma}{2} p}\right. \\
&\left.\quad-V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) e^{-\frac{\gamma}{2} p}\right) \varphi(x) \tag{3.40}
\end{align*}
$$

and their actions on $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are

$$
\begin{align*}
& \mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=f_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \\
& \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n-1}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})=b_{n-1}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) . \tag{3.41}
\end{align*}
$$

The similarity transformed Hamiltonian is square root free:

$$
\begin{align*}
& \widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=} \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})=\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) \mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \\
&= V\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\Xi_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(e^{\gamma p}-\frac{\stackrel{\Xi}{\Xi}_{\mathcal{D}}(x-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}\right) \\
&+V^{*}\left(x ; \boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]}\right) \frac{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\Xi_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}\left(e^{-\gamma p}-\frac{\check{\Xi}_{\mathcal{D}}(x+i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\Xi_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})}\right), \tag{3.42}
\end{align*}
$$

and the multi-indexed orthogonal polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are its eigenpolynomials:

$$
\begin{equation*}
\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})=\mathcal{E}_{n}(\boldsymbol{\lambda}) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \tag{3.43}
\end{equation*}
$$

The properties (A.4)-(A.5) are also the consequences of the shape invariance. By (A.3), the similar property holds for the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$. By the remark below (3.29), the $M$-step deformed system labeled by $\mathcal{D}$ with 0 is equivalent to the ( $M-1$ )-step deformed system labeled by $\mathcal{D}^{\prime}$ with shifted parameters $\boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}$.

### 3.4 Limit from the Wilson system

The continuous Hahn polynomial can be obtained from the Wilson polynomial [5]. The potential function $V(x ; \boldsymbol{\lambda})$, the energy eigenvalue $\mathcal{E}_{n}(\boldsymbol{\lambda})$, the sinusoidal coordinate $\eta(x)$ and the eigenfunctions $\phi_{n}(x ; \boldsymbol{\lambda})$ of the Wilson system are [15]

$$
\begin{align*}
& \boldsymbol{\lambda}=\left(a_{1}, a_{2}, a_{3}, a_{4}\right), \quad \operatorname{Re} a_{i}>0, \quad\left\{a_{1}^{*}, a_{2}^{*}, a_{3}^{*}, a_{4}^{*}\right\}=\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}(\text { as a set }), \quad b_{1}=\sum_{j=1}^{4} a_{j}, \\
& V(x ; \boldsymbol{\lambda})=\frac{\prod_{j=1}^{4}\left(a_{j}+i x\right)}{2 i x(2 i x+1)}, \quad \mathcal{E}_{n}(\boldsymbol{\lambda})=n\left(n+b_{1}-1\right), \quad \eta(x)=x^{2} \tag{3.44}
\end{align*}
$$

$$
\begin{align*}
\phi_{n}(x ; \boldsymbol{\lambda}) & =\phi_{0}(x ; \boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}), \quad \phi_{0}(x ; \boldsymbol{\lambda})=\sqrt{\frac{\prod_{j=1}^{4} \Gamma\left(a_{j}+i x\right) \Gamma\left(a_{j}-i x\right)}{\Gamma(2 i x) \Gamma(-2 i x)}}  \tag{3.45}\\
\check{P}_{n}(x ; \boldsymbol{\lambda}) & =P_{n}(\eta(x) ; \boldsymbol{\lambda})=W_{n}\left(\eta(x) ; a_{1}, a_{2}, a_{3}, a_{4}\right) \\
& =\left(a_{1}+a_{2}, a_{1}+a_{3}, a_{1}+a_{4}\right)_{n} \cdot{ }_{4} F_{3}\left(\begin{array}{c}
-n, n+b_{1}-1, a_{1}+i x, a_{1}-i x \\
a_{1}+a_{2}, a_{1}+a_{3}, a_{1}+a_{4}
\end{array} 1\right), \tag{3.46}
\end{align*}
$$

where $W_{n}\left(\eta ; a_{1}, a_{2}, a_{3}, a_{4}\right)$ is the Wilson polynomial of degree $n$ in $\eta$ [5].
Let us consider the following limit:

$$
\begin{equation*}
x^{\mathrm{W}}=x+t, \quad \lambda^{\mathrm{W}}=\left(a_{1}-i t, a_{1}^{*}+i t, a_{2}-i t, a_{2}^{*}+i t\right), \quad t \rightarrow \infty . \tag{3.47}
\end{equation*}
$$

Here the superscript W indicates the quantities of the Wilson system, and $x, a_{1}$ and $a_{2}$ are quantities of the continuous Hahn system. The physical range of $x^{\mathrm{W}}\left(0 \leq x^{\mathrm{W}}<\infty\right)$ gives the physical range of $x(-\infty<x<\infty)$. The continuous Hahn polynomial is obtained from the Wilson polynomial [5],

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{(-2 t)^{n} n!} \check{P}_{n}^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right)=\check{P}_{n}(x ; \boldsymbol{\lambda}) \tag{3.48}
\end{equation*}
$$

Note that $\check{P}_{n}^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right)$ is a polynomial in $\eta^{\mathrm{W}}\left(x^{\mathrm{W}}\right)=\left(x^{\mathrm{W}}\right)^{2}$ and $\check{P}_{n}(x ; \boldsymbol{\lambda})$ is a polynomial in $\eta(x)=x$. Other quantities are also obtained:

$$
\begin{align*}
& \lim _{t \rightarrow \infty} V^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right)=V(x ; \boldsymbol{\lambda}), \quad \lim _{t \rightarrow \infty} \mathcal{E}_{n}^{\mathrm{W}}\left(\boldsymbol{\lambda}^{\mathrm{W}}\right)=\mathcal{E}_{n}(\boldsymbol{\lambda}), \\
& \lim _{t \rightarrow \infty} \frac{e^{\frac{\pi}{2}\left(\operatorname{Im}\left(a_{1}+a_{2}\right)+2 t\right)}}{\sqrt{2 \pi(2 t)^{b_{1}-1}}} \phi_{0}^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right)=\phi_{0}(x ; \boldsymbol{\lambda}) \tag{3.49}
\end{align*}
$$

The continuous Hahn system is obtained from the Wilson system by the limit (3.47).
Next let us consider the deformed case. We can show that the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and the multi-indexed polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ of the continuous Hahn type are obtained from those of the Wilson type by the same limit (3.47):

$$
\begin{align*}
\lim _{t \rightarrow \infty} \frac{(-1)^{\frac{1}{2} M(M-1)}}{(-2 t)^{\ell} \prod_{j=1}^{M} d_{j}!} \check{\Xi}_{\mathcal{D}}^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right) & =\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}),  \tag{3.50}\\
\lim _{t \rightarrow \infty} \frac{(-1)^{\frac{1}{2} M(M-1)}}{(-2 t)^{\ell_{\mathcal{D}}+n} n!\prod_{j=1}^{M} d_{j}!} \check{P}_{\mathcal{D}, n}^{\mathrm{W}}\left(x^{\mathrm{W}} ; \boldsymbol{\lambda}^{\mathrm{W}}\right) & =\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}), \tag{3.51}
\end{align*}
$$

where explicit forms of $\check{\Xi}_{\mathcal{D}}^{W}$ and $\check{P}_{\mathcal{D}, n}^{\mathrm{W}}$ are found in [15]. We remark that the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ is obtained as (3.50) algebraically, but the condition (3.31) is not inherited from that of the Wilson type in general. Therefore the limit (3.47) of the deformed Wilson systems do not give the deformed continuous Hahn systems in general.

## 4 New Exactly Solvable idQM Systems and Multi-indexed Meixner-Pollaczek Polynomials

In this section we deform the Meixner-Pollaczek system. Since the method is the same as in §3, we present results briefly. The eigenfunctions of the deformed systems are described by the case-(1) multi-indexed Meixner-Pollaczek polynomials.

### 4.1 Original Meixner-Pollaczek system

The Meixner-Pollaczek system is the idQM system with (2.14) and a set of parameters $\boldsymbol{\lambda}$ is

$$
\begin{equation*}
\boldsymbol{\lambda}=(a, \phi), \quad a>0, \quad 0<\phi<\pi . \tag{4.1}
\end{equation*}
$$

The fundamental data are the following [2]:

$$
\begin{align*}
& V(x ; \boldsymbol{\lambda})=e^{i\left(\frac{\pi}{2}-\phi\right)}(a+i x), \quad \eta(x)=x, \quad \varphi(x)=1, \quad \mathcal{E}_{n}(\boldsymbol{\lambda})=2 n \sin \phi,  \tag{4.2}\\
& \phi_{n}(x ; \boldsymbol{\lambda})=\phi_{0}(x ; \boldsymbol{\lambda}) \check{P}_{n}(x ; \boldsymbol{\lambda}), \quad \phi_{0}(x ; \boldsymbol{\lambda})=e^{\left(\phi-\frac{\pi}{2}\right) x} \sqrt{\Gamma(a+i x) \Gamma(a-i x)},  \tag{4.3}\\
& \check{P}_{n}(x ; \boldsymbol{\lambda})=P_{n}(\eta(x) ; \boldsymbol{\lambda})=P_{n}^{(a)}(\eta(x) ; \phi)=\frac{(2 a)_{n}}{n!} e^{i n \phi}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-n, a+i x \\
2 a
\end{array} \right\rvert\, 1-e^{-2 i \phi}\right),  \tag{4.4}\\
& h_{n}(\boldsymbol{\lambda})=2 \pi \frac{\Gamma(n+2 a)}{n!(2 \sin \phi)^{2 a}}, \quad c_{n}(\boldsymbol{\lambda})=\frac{(2 \sin \phi)^{n}}{n!},  \tag{4.5}\\
& \boldsymbol{\delta}=\left(\frac{1}{2}, 0\right), \quad \kappa=1, \quad f_{n}(\boldsymbol{\lambda})=2 \sin \phi, \quad b_{n-1}(\boldsymbol{\lambda})=n . \tag{4.6}
\end{align*}
$$

Here $P_{n}^{(a)}(\eta ; \phi)$ in (4.4) is the Meixner-Pollaczek polynomial of degree $n$ in $\eta$. 5 .

### 4.2 Virtual state wavefunctions

Let us introduce a twist operation $\mathfrak{t}$,

$$
\begin{equation*}
\mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text { def }}{=}(1-a, \phi), \quad \tilde{\boldsymbol{\delta}} \stackrel{\text { def }}{=}\left(-\frac{1}{2}, 0\right), \tag{4.7}
\end{equation*}
$$

which is an involution $\mathfrak{t}^{2}=$ id and satisfies $\mathfrak{t}(\boldsymbol{\lambda}+\beta \boldsymbol{\delta})=\mathfrak{t}(\boldsymbol{\lambda})+\beta \tilde{\boldsymbol{\delta}}(\beta \in \mathbb{R})$. The potential function $V(x ; \boldsymbol{\lambda})$ satisfies (3.2) with

$$
\begin{equation*}
\alpha(\boldsymbol{\lambda})=1, \quad \alpha^{\prime}(\boldsymbol{\lambda})=2(1-2 a) \sin \phi \tag{4.8}
\end{equation*}
$$

In the following, we assume $a>\frac{1}{2}$, which gives $\alpha^{\prime}(\boldsymbol{\lambda})<0$. The virtual state wavefunctions $\tilde{\phi}_{\mathrm{v}}(x ; \boldsymbol{\lambda})\left(\mathrm{v} \in \mathcal{V} \subset \mathbb{Z}_{\geq 0}\right)$ are defined by

$$
\tilde{\phi}_{\mathrm{v}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{\mathrm{v}}(x ; \mathfrak{t}(\boldsymbol{\lambda}))=\tilde{\phi}_{0}(x ; \boldsymbol{\lambda}) \check{\xi}_{\mathrm{v}}(x ; \boldsymbol{\lambda}), \quad \tilde{\phi}_{0}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \phi_{0}(x ; \mathfrak{t}(\boldsymbol{\lambda})),
$$

$$
\begin{equation*}
\check{\xi}_{\mathrm{v}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \xi_{\mathrm{v}}(\eta(x) ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \check{P}_{\mathrm{v}}(x ; \mathfrak{t}(\boldsymbol{\lambda}))=P_{\mathrm{v}}(\eta(x) ; \mathfrak{t}(\boldsymbol{\lambda})) \quad(\mathrm{v} \in \mathcal{V}) \tag{4.9}
\end{equation*}
$$

which satisfy the Schrödinger equation $\mathcal{H}(\boldsymbol{\lambda}) \tilde{\phi}_{\mathrm{v}}(x ; \boldsymbol{\lambda})=\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda}) \tilde{\phi}_{\mathrm{v}}(x ; \boldsymbol{\lambda})$. The virtual state polynomial $\xi_{\mathrm{v}}(\eta ; \boldsymbol{\lambda})$ is a polynomial of degree v in $\eta$, and the virtual energy $\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})$ is

$$
\begin{equation*}
\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda})=2(\mathrm{v}+1-2 a) \sin \phi, \tag{4.10}
\end{equation*}
$$

which is negative for $2 a>\mathrm{v}+1$. We choose $\mathcal{V}$ as

$$
\begin{equation*}
\mathcal{V}=\left\{0,1,2, \ldots,[2 a-1]^{\prime}\right\} \tag{4.11}
\end{equation*}
$$

### 4.3 New exactly solvable systems

Isospectral deformations of the Meixner-Pollaczek system are obtained by the multi-step Darboux transformations with the virtual state wavefunctions as seed solutions. The deformed systems are labeled by $\mathcal{D}=\left\{d_{1}, \ldots, d_{M}\right\}\left(d_{j} \in \mathcal{V}\right.$ : mutually distinct $)$.

Let us define the following functions:

$$
\begin{equation*}
\nu(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\phi_{0}(x ; \boldsymbol{\lambda})}{\tilde{\phi}_{0}(x ; \boldsymbol{\lambda})}, \quad r_{j}^{\mathrm{I}}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right) \stackrel{\text { def }}{=} \frac{\nu\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right)}{\nu(x ; \boldsymbol{\lambda}+(M-1) \tilde{\boldsymbol{\delta}})}(j=1,2, \ldots, M), \tag{4.12}
\end{equation*}
$$

whose explicit form is

$$
\begin{equation*}
r_{j}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}, M\right)=(-1)^{j-1} i^{1-M}\left(a-\frac{M-1}{2}+i x\right)_{j-1}\left(a-\frac{M-1}{2}-i x\right)_{M-j} . \tag{4.13}
\end{equation*}
$$

The denominator polynomial and the multi-indexed polynomial are defined by (3.23) and

$$
\begin{align*}
& \mathrm{W}_{\gamma}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{M}}\right](x ; \boldsymbol{\lambda})=\varphi_{M}(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}),  \tag{4.14}\\
& \nu(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})^{-1} \mathrm{~W}_{\gamma}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{M}}, \nu \check{P}_{n}\right](x ; \boldsymbol{\lambda})=\varphi_{M+1}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) \\
& =i^{\frac{1}{2} M(M+1)}\left|\begin{array}{cccc}
\check{\xi}_{d_{1}}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) & \ldots & \check{\xi}_{d_{M}}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) & r_{1}\left(x_{1}^{(M+1)}\right) \check{P}_{n}\left(x_{1}^{(M+1)} ; \boldsymbol{\lambda}\right) \\
\check{\xi}_{d_{1}}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda )}\right. & \cdots & \check{\xi}_{d_{M}}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda}\right) & r_{2}\left(x_{2}^{(M+1)}\right) \check{P}_{n}\left(x_{2}^{(M+1)} ; \boldsymbol{\lambda}\right) \\
\vdots & \cdots & \vdots & \vdots \\
\check{\xi}_{d_{1}}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda )}\right. & \cdots & \check{\xi}_{d_{M}}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda )}\right. & r_{M+1}\left(x_{M+1}^{(M+1)}\right) \check{P}_{n}\left(x_{M+1}^{(M+1)} ; \boldsymbol{\lambda )}\right.
\end{array}\right|, \tag{4.15}
\end{align*}
$$

where $r_{j}(x)=r_{j}(x ; \boldsymbol{\lambda}, M+1)$ and $\varphi_{M}(x)=1$. Their degrees are $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}}+n$, respectively (we assume $c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \neq 0$ and $c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \neq 0$, see $(\overline{\mathrm{B} .1})-(\overline{\mathrm{B} .2})$ ). Here $\ell_{\mathcal{D}}$ is (3.24) with $M_{\mathrm{II}}=0$. These $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are 'real', $\check{\Xi}_{\mathcal{D}}^{*}(x ; \boldsymbol{\lambda})=\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D}, n}^{*}(x ; \boldsymbol{\lambda})=\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$. Then, the Casoratians $\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right](x)$ and $\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{n}\right](x)$ are expressed as

$$
\mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}\right](x ; \boldsymbol{\lambda})=\prod_{j=1}^{M} \phi_{0}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right) \cdot \mathrm{W}_{\gamma}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{M}}\right](x ; \boldsymbol{\lambda})
$$

$$
\begin{align*}
= & \prod_{j=1}^{M} \phi_{0}\left(x_{j}^{(M)} ; \boldsymbol{\lambda}\right) \times \varphi_{M}(x) \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}),  \tag{4.16}\\
& \mathrm{W}_{\gamma}\left[\tilde{\phi}_{d_{1}}, \ldots, \tilde{\phi}_{d_{M}}, \phi_{n}\right](x ; \boldsymbol{\lambda})=\prod_{j=1}^{M+1} \phi_{0}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}\right) \cdot \mathrm{W}_{\gamma}\left[\check{\xi}_{d_{1}}, \ldots, \check{\xi}_{d_{M}}, \nu \check{P}_{n}\right](x ; \boldsymbol{\lambda}) \\
= & \prod_{j=1}^{M+1} \phi_{0}\left(x_{j}^{(M+1)} ; \boldsymbol{\lambda}\right) \times \nu(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \varphi_{M+1}(x) \check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) . \tag{4.17}
\end{align*}
$$

The eigenfunction $\phi_{\mathcal{D} n}(x ; \boldsymbol{\lambda})$ (3.14) is rewritten as (3.27) and $\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ is

$$
\begin{equation*}
\psi_{\mathcal{D}}(x ; \boldsymbol{\lambda}) \stackrel{\text { def }}{=} \frac{\phi_{0}(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right) \check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}} \tag{4.18}
\end{equation*}
$$

The ground state wavefunction $\phi_{\mathcal{D} 0}$ is annihilated by $\mathcal{A}_{\mathcal{D}}, \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D} 0}(x ; \boldsymbol{\lambda})=0$. The lowest degree multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ is proportional to $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})$, see (B.3). The potential function $V_{\mathcal{D}}(x)$ (3.13) is expressed as

$$
\begin{equation*}
V_{\mathcal{D}}(x ; \boldsymbol{\lambda})=V(x ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)}{\check{\Xi}_{\mathcal{D}}\left(x-i \frac{\gamma}{2} ; \boldsymbol{\lambda}\right)} \frac{\check{\Xi}_{\mathcal{D}}(x-i \gamma ; \boldsymbol{\lambda}+\boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta})} . \tag{4.19}
\end{equation*}
$$

To check the regularity and hermiticity of $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$, let us consider the function $g(x)$,

$$
\begin{align*}
g(x) & \stackrel{\text { def }}{=} V\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right) \phi_{0}\left(x+i \frac{\gamma}{2} ; \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}\right)^{2} \\
& =e^{2\left(\phi-\frac{\pi}{2}\right) x} \Gamma\left(a-\frac{1}{2} M+\frac{1}{2}+i x\right) \Gamma\left(a-\frac{1}{2} M+\frac{1}{2}-i x\right) \tag{4.20}
\end{align*}
$$

Asymptotic behavior of $g(x)$ at $x \sim \pm \infty$ is $g(x) \sim 2 \pi|x|^{2 a-M} e^{2\left(\phi-\frac{\pi}{2}\right) x-\pi|x|}$ (for $x \in \mathbb{R}$ ). The necessary and sufficient condition for $g(x)$ to have no poles in the rectangular domain $D_{\gamma}$ is $a-\frac{1}{2} M>0$. This condition is automatically satisfied because of (4.11). By the same argument as $\S$ 3.2, the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is well-defined and hermitian, if the condition (3.31) is satisfied. To satisfy the condition (3.31), the degree of $\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}), \ell_{\mathcal{D}}$, should be even. Although we have no analytical proof that there exists a range of parameters $\boldsymbol{\lambda}$ satisfying the condition (3.31), numerical calculation (for small $M$ and $d_{j}$ ) suggests the following conjecture.

Conjecture 1 Let $d_{j}$ 's be $0 \leq d_{1}<d_{2}<\cdots<d_{M}<2 a-1$. Then, the necessary and sufficient condition for the condition (3.31) is $(-1)^{d_{j}}=(-1)^{j-1}(j=1,2, \ldots, M)$.

In the following we assume that the condition (3.31) is satisfied.

If the deformed systems is well-defined, the eigenfunctions are orthogonal. Namely, the orthogonality relations of the multi-indexed polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ are (3.33) with

$$
\begin{equation*}
h_{\mathcal{D}, n}(\boldsymbol{\lambda})=h_{n}(\boldsymbol{\lambda}) \prod_{j=1}^{M}\left(\mathcal{E}_{n}(\boldsymbol{\lambda})-\tilde{\mathcal{E}}_{d_{j}}(\boldsymbol{\lambda})\right) \tag{4.21}
\end{equation*}
$$

The multi-indexed orthogonal polynomial $P_{\mathcal{D}, n}(\eta ; \boldsymbol{\lambda})$ has $n$ zeros in the physical region $\eta \in \mathbb{R}$ ( $\Leftrightarrow \eta\left(x_{1}\right)<\eta<\eta\left(x_{2}\right)$ ), which interlace the $n+1$ zeros of $P_{\mathcal{D}, n+1}(\eta ; \boldsymbol{\lambda})$ in the physical region, and $\ell_{\mathcal{D}}$ zeros in the unphysical region $\eta \in \mathbb{C} \backslash \mathbb{R}$. These properties and (3.33) can be verified by numerical calculation.

The shape invariance of the original system is inherited by the deformed systems. The properties (3.37)-(3.43) (with the replacement $\boldsymbol{\lambda}^{\left[M_{\mathrm{I}}, M_{\mathrm{II}}\right]} \rightarrow \boldsymbol{\lambda}+M \tilde{\boldsymbol{\delta}}$ ) hold. The property (B.4) implies that the $M$-step deformed system labeled by $\mathcal{D}$ with 0 is equivalent to the ( $M-1$ )-step deformed system labeled by $\mathcal{D}^{\prime}$ with shifted parameters $\boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}$.

### 4.4 Limit from the continuous Hahn system

The Meixner-Pollaczek polynomial can be obtained from the continuous Hahn polynomial [5]. Let us consider the following limit:

$$
\begin{equation*}
x^{\mathrm{cH}}=x+\frac{t}{\tan \phi}, \quad \lambda^{\mathrm{cH}}=\left(a-i \frac{t}{\tan \phi}, t\right), \quad t \rightarrow \infty . \tag{4.22}
\end{equation*}
$$

Here the superscript cH indicates the quantities of the continuous Hahn system.
Under this limit (4.22), the Meixner-Pollaczek polynomial is obtained from the continuous Hahn polynomial [5,

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left(\frac{\sin \phi}{t}\right)^{n} \check{P}_{n}^{\mathrm{cH}}\left(x^{\mathrm{cH}} ; \boldsymbol{\lambda}^{\mathrm{cH}}\right)=\check{P}_{n}(x ; \boldsymbol{\lambda}) . \tag{4.23}
\end{equation*}
$$

Other quantities are also obtained:

$$
\begin{align*}
& \lim _{t \rightarrow \infty} \frac{\sin \phi}{t} V^{\mathrm{cH}}\left(x^{\mathrm{cH}} ; \boldsymbol{\lambda}^{\mathrm{cH}}\right)=V(x ; \boldsymbol{\lambda}), \quad \lim _{t \rightarrow \infty} \frac{\sin \phi}{t} \mathcal{E}_{n}^{\mathrm{cH}}\left(\boldsymbol{\lambda}^{\mathrm{cH}}\right)=\mathcal{E}_{n}(\boldsymbol{\lambda}), \\
& \lim _{t \rightarrow \infty}\left(\frac{\sin \phi}{t}\right)^{t-\frac{1}{2}} \frac{e^{\frac{t}{\tan \phi}\left(\frac{\pi}{2}-\phi\right)+t}}{\sqrt{2 \pi}} \phi_{0}^{\mathrm{cH}}\left(x^{\mathrm{cH}} ; \boldsymbol{\lambda}^{\mathrm{cH}}\right)=\phi_{0}(x ; \boldsymbol{\lambda}) . \tag{4.24}
\end{align*}
$$

The Meixner-Pollaczek system is obtained from the continuous Hahn system by the limit (4.22).

Next let us consider the deformed case. Under the limit (4.22), the type I twist operation of the continuous Hahn system $\mathfrak{t}^{\mathrm{cHI}}$ reduces to the twist operation of the Meixner-Pollaczek
system $\mathfrak{t}$, but type II $\mathfrak{t}^{\mathrm{cHII}}$ does not have a good limit. Hence we consider the deformed continuous Hahn systems with $M_{\text {II }}=0$. We can show that the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ and the multi-indexed polynomials $\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})$ of the Meixner-Pollaczek type are obtained from those of the continuous Hahn type (with type I only) by the limit (4.22)

$$
\begin{align*}
\lim _{t \rightarrow \infty}\left(\frac{\sin \phi}{t}\right)^{\sum_{j=1}^{M} d_{j}} \check{\Xi}_{\mathcal{D}}^{\mathrm{cH}}\left(x^{\mathrm{cH}} ; \boldsymbol{\lambda}^{\mathrm{cH}}\right) & =\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}),  \tag{4.25}\\
\lim _{t \rightarrow \infty}\left(\frac{\sin \phi}{t}\right)^{\sum_{j=1}^{M} d_{j}+n} \check{P}_{\mathcal{D}, n}^{\mathrm{cH}}\left(x^{\mathrm{cH}} ; \boldsymbol{\lambda}^{\mathrm{cH}}\right) & =\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda}) . \tag{4.26}
\end{align*}
$$

We conjecture that the condition (3.31) of the Meixner-Pollaczek system is obtained from that of the continuous Hahn system. If this is true, the deformed Meixner-Pollaczek systems are obtained from the deformed continuous Hahn systems (with type I only) by the limit (4.22).

### 4.5 Limit to the harmonic oscillator

The harmonic oscillator is an ordinary quantum mechanical system and its eigenfunctions are the following:

$$
\begin{align*}
& H=p^{2}+x^{2}-1, \quad \mathcal{E}_{n}=2 n, \quad \eta(x)=x, \quad-\infty<x<\infty  \tag{4.27}\\
& \phi_{n}(x)=\phi_{0}(x) \check{P}_{n}(x), \quad \phi_{0}(x)=e^{-\frac{1}{2} x^{2}}  \tag{4.28}\\
& \check{P}_{n}(x)=P_{n}(\eta(x))=H_{n}(\eta(x))=(2 x)^{n} \cdot{ }_{2} F_{0}\left(\left.\begin{array}{c}
-\frac{n}{2},-\frac{n-1}{2} \\
-
\end{array} \right\rvert\,-\frac{1}{x^{2}}\right) \tag{4.29}
\end{align*}
$$

where $H_{n}(\eta)$ is the Hermite polynomial of degree $n$ in $\eta$ [5].
Let us consider the following limit of the Meixner-Pollaczek system:

$$
\begin{equation*}
x^{\mathrm{MP}}=\sqrt{t} x, \quad \boldsymbol{\lambda}^{\mathrm{MP}}=\left(t, \frac{\pi}{2}\right), \quad t \rightarrow \infty \tag{4.30}
\end{equation*}
$$

Under this limit, the Meixner-Pollaczek system reduces to the harmonic oscillator:

$$
\begin{align*}
& \lim _{t \rightarrow \infty} H^{\mathrm{MP}}\left(\boldsymbol{\lambda}^{\mathrm{MP}}\right)=H, \quad \lim _{t \rightarrow \infty} \mathcal{E}_{n}^{\mathrm{MP}}\left(\boldsymbol{\lambda}^{\mathrm{MP}}\right)=\mathcal{E}_{n}  \tag{4.31}\\
& \lim _{t \rightarrow \infty} \frac{e^{t}}{\sqrt{2 \pi} t^{t-\frac{1}{2}}} \phi_{0}^{\mathrm{MP}}\left(x^{\mathrm{MP}} ; \boldsymbol{\lambda}^{\mathrm{MP}}\right)=\phi_{0}(x), \quad \lim _{t \rightarrow \infty} \frac{n!}{t^{\frac{n}{2}}} \check{P}_{n}^{\mathrm{MP}}\left(x^{\mathrm{MP}} ; \boldsymbol{\lambda}^{\mathrm{MP}}\right)=\check{P}_{n}(x) \tag{4.32}
\end{align*}
$$

We remark that the Meixner-Pollaczek polynomial $P_{n}^{(a)}(x ; \phi)$ with any $\phi$ reduces to the Hermite polynomial as [5]

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{n!}{t^{\frac{n}{2}}} P_{n}^{(t)}\left(\frac{\sqrt{t} x-t \cos \phi}{\sin \phi} ; \phi\right)=H_{n}(x) \tag{4.33}
\end{equation*}
$$

but this limit does not lead to a good limit of the quantum system.
Next let us consider the deformed case. There is no virtual state in the harmonic oscillator [29]. Hence the limit (4.30) of the virtual state wavefunction of the Meixner-Pollaczek system can not be a virtual state wavefunction. In fact, the limit of $\tilde{\phi}_{\mathrm{v}}^{\mathrm{MP}}$ is

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{t^{t-\frac{1}{2}} e^{-t}}{\sqrt{2 \pi}} \tilde{\phi}_{0}^{\mathrm{MP}}\left(x^{\mathrm{MP}} ; \boldsymbol{\lambda}^{\mathrm{MP}}\right)=e^{\frac{1}{2} x^{2}}, \quad \lim _{t \rightarrow \infty} \frac{\mathrm{v}!}{t^{\frac{\mathrm{v}}{2}}} \check{\mathrm{v}}_{\mathrm{v}}^{\mathrm{MP}}\left(x^{\mathrm{MP}} ; \boldsymbol{\lambda}^{\mathrm{MP}}\right)=i^{-\mathrm{v}} H_{\mathrm{v}}(i x) \tag{4.34}
\end{equation*}
$$

This is the pseudo virtual state wavefunction of the harmonic oscillator $\tilde{\phi}_{\mathrm{v}}(x)=i^{-\mathrm{v}} \phi_{\mathrm{v}}(i x)$ [29]. The deformed harmonic oscillator system, which is obtained by the Darboux transformations with $\tilde{\phi}_{\mathrm{v}}(\mathrm{v} \in \mathcal{D})$ as seed solutions, has energy eigenvalues $\mathcal{E}_{n}(n=0,1, \ldots)$ and $\tilde{\mathcal{E}}_{d_{j}}$ $(j=1, \ldots, M)$. The eigenfunctions with $\mathcal{E}_{n}$ are obtained as the limit of $\phi_{\mathcal{D} n}^{\mathrm{MP}}$, but those with $\tilde{\mathcal{E}}_{d_{j}}$ can not be obtained from the eigenfunctions of the deformed Meixner-Pollaczek system. In this sense the limit (4.30) of the deformed Meixner-Pollaczek system is not a good limit.

## 5 Summary and Comments

The continuous Hahn and Meixner-Pollaczek idQM systems are exactly solvable and their physical range of the coordinate is the whole real line. We deform them by the multistep Darboux transformations with the virtual state wavefunctions as seed solutions, and obtain new exactly solvable idQM systems and the case-(1) multi-indexed continuous Hahn and Meixner-Pollaczek polynomials. By this result, the construction of the multi-indexed polynomials in idQM is essentially completed. The remaining task is to study the properties of various multi-indexed polynomials and to use them to investigate quantum mechanical systems.

The deformed quantum system labeled by an index set $\mathcal{D}$ may be equivalent to another labeled by a different index set $\mathcal{D}^{\prime}$ with shifted parameters, which means that the corresponding two multi-indexed orthogonal polynomials labeled by $\mathcal{D}$ and $\mathcal{D}^{\prime}$ with shifted parameters are proportional. Such equivalence is studied for the case-(2) multi-indexed polynomials of Hermite, Laguerre, Jacobi, Wilson and Askey-Wilson types [29, 31] and for the case-(1) multi-indexed polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types [33] (see also [34]). The case-(1) multi-indexed continuous Hahn polynomials obtained in this paper have equivalence in the same form as the (Askey-)Wilson cases [33], which is derived from the properties (A.4)-(A.5). The case-(1) multi-indexed Meixner-Pollaczek polynomials also have similar equivalence derived from the property (B.4).

The multi-indexed orthogonal polynomials do not satisfy the three term recurrence relations, which are characterizations of the ordinary orthogonal polynomials [4], because they are not ordinary orthogonal polynomials. Instead, they satisfy the recurrence relations with more terms [35]-43]. The case-(1) multi-indexed continuous Hahn and Meixner-Pollaczek polynomials satisfy such recurrence relations. The recurrence relations with constant coefficients are related to the generalized closure relations [42], which give the creation and annihilation operators of the deformed quantum systems. We will report these topics elsewhere [44].

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## A Some Properties of the Multi-indexed Continuous Hahn Polynomials

We present some properties of the multi-indexed continuous Hahn polynomials.

- coefficients of the highest degree terms:

$$
\begin{align*}
\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})= & c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}}+(\text { lower order terms }), \\
c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})= & \prod_{j=1}^{M_{\mathrm{I}}} c_{d_{j}^{\mathrm{I}}}\left(\mathfrak{t}^{\mathrm{I}}(\boldsymbol{\lambda})\right) \cdot \prod_{j=1}^{M_{\mathrm{II}}} c_{d_{j}^{\mathrm{II}}}\left(\mathfrak{t}^{\mathrm{II}}(\boldsymbol{\lambda})\right) \cdot \prod_{1 \leq j<k \leq M_{\mathrm{I}}}\left(d_{k}^{\mathrm{I}}-d_{j}^{\mathrm{I}}\right) \cdot \prod_{1 \leq j<k \leq M_{\mathrm{II}}}\left(d_{k}^{\mathrm{II}}-d_{j}^{\mathrm{II}}\right) \\
& \times \prod_{j=1}^{M_{\mathrm{I}}} \prod_{k=1}^{M_{\mathrm{II}}}\left(a_{1}+a_{1}^{*}-d_{j}^{\mathrm{I}}-a_{2}-a_{2}^{*}+d_{k}^{\mathrm{II}}\right),  \tag{A.1}\\
P_{\mathcal{D}}(\eta ; \boldsymbol{\lambda})= & c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}+n}+(\text { lower order terms }), \\
c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda})= & c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) c_{n}(\boldsymbol{\lambda}) \prod_{j=1}^{M_{\mathrm{I}}}\left(-a_{1}-a_{1}^{*}-n+d_{j}^{\mathrm{I}}+1\right) \cdot \prod_{j=1}^{M_{\mathrm{II}}}\left(-a_{2}-a_{2}^{*}-n+d_{j}^{\mathrm{II}}+1\right) . \tag{A.2}
\end{align*}
$$

- $\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ vs $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ :

$$
\begin{align*}
& \check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})=A \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \\
& A=\prod_{j=1}^{M_{\mathrm{I}}}\left(-a_{1}-a_{1}^{*}+d_{j}^{\mathrm{I}}+1\right) \cdot \prod_{j=1}^{M_{\mathrm{II}}}\left(-a_{2}-a_{2}^{*}+d_{j}^{\mathrm{II}}+1\right) . \tag{A.3}
\end{align*}
$$

- $d_{j}=0$ case :

$$
\begin{align*}
& \left.\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right|_{d_{M_{\mathrm{I}}}^{\mathrm{I}}=0}=A \check{P}_{\mathcal{D}^{\prime}, n}\left(x ; \boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}^{\mathrm{I}}\right), \\
& \quad \mathcal{D}^{\prime}=\left\{d_{1}^{\mathrm{I}}-1, \ldots, d_{M_{\mathrm{I}}-1}^{\mathrm{I}}-1, d_{1}^{\mathrm{II}}+1, \ldots, d_{M_{\mathrm{II}}}^{\mathrm{II}}+1\right\}, \\
& \quad A=(-1)^{M_{\mathrm{I}}}\left(a_{1}+a_{1}^{*}+n-1\right) \prod_{j=1}^{M_{\mathrm{I}}-1}\left(-a_{1}-a_{1}^{*}+a_{2}+a_{2}^{*}+d_{j}^{\mathrm{I}}+1\right) \cdot \prod_{j=1}^{M_{\mathrm{II}}}\left(d_{j}^{\mathrm{II}}+1\right),  \tag{A.4}\\
& \left.\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right|_{d_{M_{\mathrm{II}}}^{\mathrm{II}}=0}=B \check{P}_{\mathcal{D}^{\prime}, n}\left(x ; \boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}^{\mathrm{II}}\right), \\
& \mathcal{D}^{\prime}=\left\{d_{1}^{\mathrm{I}}+1, \ldots, d_{M_{\mathrm{I}}}^{\mathrm{I}}+1, d_{1}^{\mathrm{II}}-1, \ldots, d_{M_{\mathrm{II}}-1}^{\mathrm{II}}-1\right\}, \\
& B=(-1)^{M}\left(a_{2}+a_{2}^{*}+n-1\right) \prod_{j=1}^{M_{\mathrm{II}}-1}\left(-a_{2}-a_{2}^{*}+a_{1}+a_{1}^{*}+d_{j}^{\mathrm{II}}+1\right) \cdot \prod_{j=1}^{M_{\mathrm{I}}}\left(d_{j}^{\mathrm{I}}+1\right) \tag{A.5}
\end{align*}
$$

## B Some Properties of the Multi-indexed Meixner-Pollaczek Polynomials

We present some properties of the multi-indexed Meixner-Pollaczek polynomials.

- coefficients of the highest degree terms:

$$
\begin{align*}
\Xi_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}) & =c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}}+(\text { lower order terms }), \\
c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) & =\prod_{j=1}^{M} c_{d_{j}}(\mathfrak{t}(\boldsymbol{\lambda})) \cdot \prod_{1 \leq j<k \leq M}\left(d_{k}-d_{j}\right),  \tag{B.1}\\
P_{\mathcal{D}}(\eta ; \boldsymbol{\lambda}) & =c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) \eta^{\ell_{\mathcal{D}}+n}+(\text { lower order terms }), \\
c_{\mathcal{D}, n}^{P}(\boldsymbol{\lambda}) & =c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) c_{n}(\boldsymbol{\lambda}) \prod_{j=1}^{M}\left(-2 a-n+d_{j}+1\right) \tag{B.2}
\end{align*}
$$

- $\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})$ vs $\check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda})$ :

$$
\begin{equation*}
\check{P}_{\mathcal{D}, 0}(x ; \boldsymbol{\lambda})=A \check{\Xi}_{\mathcal{D}}(x ; \boldsymbol{\lambda}+\boldsymbol{\delta}), \quad A=\prod_{j=1}^{M}\left(-2 a+d_{j}+1\right) \tag{B.3}
\end{equation*}
$$

- $d_{j}=0$ case :

$$
\begin{align*}
& \left.\check{P}_{\mathcal{D}, n}(x ; \boldsymbol{\lambda})\right|_{d_{M}=0}=A \check{P}_{\mathcal{D}^{\prime}, n}(x ; \boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}), \quad \mathcal{D}^{\prime}=\left\{d_{1}-1, \ldots, d_{M-1}-1\right\} \\
& \quad A=(-1)^{M}(2 a+n-1)(2 \sin \phi)^{M-1} \tag{B.4}
\end{align*}
$$

## References

[1] S. Odake and R. Sasaki, "Orthogonal Polynomials from Hermitian Matrices," J. Math. Phys. 49 (2008) 053503 (43pp), arXiv:0712.4106[math.CA].
[2] S. Odake and R. Sasaki, "Exactly solvable 'discrete' quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states," Prog. Theor. Phys. 119 (2008) 663-700. arXiv:0802.1075[quant-ph].
[3] S. Odake and R. Sasaki, "Discrete quantum mechanics," (Topical Review) J. Phys. A44 (2011) 353001 (47pp), arXiv:1104.0473[math-ph].
[4] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, vol. 98 of Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge (2005).
[5] R. Koekoek, P.A.Lesky and R.F.Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer-Verlag Berlin-Heidelberg (2010).
[6] D. Gómez-Ullate, N. Kamran and R. Milson, "An extension of Bochner's problem: exceptional invariant subspaces," J. Approx. Theory 162 (2010) 987-1006, arXiv:0805. 3376 [math-ph] ;
[7] D. Gómez-Ullate, N. Kamran and R. Milson, "An extended class of orthogonal polynomials defined by a Sturm-Liouville problem," J. Math. Anal. Appl. 359 (2009) 352-367, arXiv:0807.3939[math-ph].
[8] C. Quesne, "Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry," J. Phys. A41 (2008) 392001 (6pp), arXiv:0807.4087[quant-ph].
[9] S. Odake and R. Sasaki, "Infinitely many shape invariant potentials and new orthogonal polynomials," Phys. Lett. B679 (2009) 414-417, arXiv:0906.0142[math-ph].
[10] S. Odake and R. Sasaki, "Another set of infinitely many exceptional ( $X_{\ell}$ ) Laguerre polynomials," Phys. Lett. B684 (2010) 173-176, arXiv:0911.3442[math-ph].
[11] D. Gómez-Ullate, N. Kamran and R. Milson, "On orthogonal polynomials spanning a non-standard flag," Contemp. Math. 563 (2011) 51-72, arXiv:1101.5584[math-ph].
[12] D. Gómez-Ullate, N. Kamran and R. Milson, "Two-step Darboux transformations and exceptional Laguerre polynomials," J. Math. Anal. Appl. 387 (2012) 410-418, arXiv: 1103.5724 [math-ph].
[13] S. Odake and R.Sasaki, "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. B702 (2011) 164-170, arXiv: 1105.0508 [math-ph].
[14] S. Odake and R. Sasaki, "Infinitely many shape invariant discrete quantum mechanical systems and new exceptional orthogonal polynomials related to the Wilson and AskeyWilson polynomials," Phys. Lett. B682 (2009) 130-136, arXiv:0909.3668[math-ph].
[15] S. Odake and R. Sasaki, "Multi-indexed Wilson and Askey-Wilson polynomials," J. Phys. A46 (2013) 045204 (22pp), arXiv:1207. $5584[m a t h-p h]$.
[16] S. Odake and R. Sasaki, "Exceptional $\left(X_{\ell}\right)(q)$-Racah polynomials," Prog. Theor. Phys. 125 (2011) 851-870, arXiv:1102.0812[math-ph].
[17] S. Odake and R. Sasaki, "Multi-indexed ( $q$-)Racah polynomials," J. Phys. A 45 (2012) 385201 (21pp), arXiv:1203.5868[math-ph].
[18] D. Gómez-Ullate, Y. Grandati and R. Milson, "Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials," J. Phys. A47 (2014) 015203 (27pp), arXiv:1306.5143[math-ph].
[19] A. J. Durán, "Exceptional Meixner and Laguerre orthogonal polynomials," J. Approx. Theory 184 (2014) 176-208, arXiv:1310.4658[math.CA].
[20] S. Odake and R. Sasaki, "Multi-indexed Meixner and Little $q$-Jacobi (Laguerre) Polynomials," J. Phys. A50 (2017) 165204 (23pp), arXiv:1610.09854[math.CA].
[21] S. Odake and R. Sasaki, "Unified theory of annihilation-creation operators for solvable ('discrete’) quantum mechanics," J. Math. Phys. 47 (2006) 102102 (33pp), arXiv: quant-ph/0605215.
[22] S. Odake and R. Sasaki, "Unified theory of exactly and quasi-exactly solvable 'discrete' quantum mechanics: I. Formalism," J. Math. Phys 51 (2010) 083502 (24pp), arXiv: 0903.2604 [math-ph].
[23] M. M. Crum, "Associated Sturm-Liouville systems," Quart. J. Math. Oxford Ser. (2) 6 (1955) 121-127, arXiv:physics/9908019.
[24] M. G. Krein, "On continuous analogue of a formula of Christoffel from the theory of orthogonal polynomials," (Russian) Doklady Acad. Nauk. CCCP, 113 (1957) 970-973.
[25] V.É. Adler, "A modification of Crum's method," Theor. Math. Phys. 101 (1994) 13811386.
[26] S. Odake and R. Sasaki, "Crum's theorem for 'discrete' quantum mechanics," Prog. Theor. Phys. 122 (2009) 1067-1079, arXiv:0902.2593[math-ph].
[27] L. García-Gutiérrez, S. Odake and R. Sasaki, "Modification of Crum's theorem for 'discrete' quantum mechanics," Prog. Theor. Phys. 124 (2010) 1-26, arXiv:1004.0289 [math-ph].
[28] C. Quesne, "Novel enlarged shape invariance property and exactly solvable rational extensions of the Rosen-Morse II and Eckart potentials," SIGMA 8 (2012) 080 (19pp), arXiv:1208.6165[math-ph].
[29] S. Odake and R. Sasaki, "Krein-Adler transformations for shape-invariant potentials and pseudo virtual states," J. Phys. A46 (2013) 245201 (24pp), arXiv:1212.6595[math$\mathrm{ph}]$.
[30] S. Odake and R. Sasaki, "Extensions of solvable potentials with finitely many discrete eigenstates," J. Phys. A46 (2013) 235205 (15pp), arXiv:1301.3980[math-ph].
[31] S. Odake and R. Sasaki, "Casoratian Identities for the Wilson and Askey-Wilson Polynomials," J. Approx. Theory 193 (2015) 184-209, arXiv:1308.4240[math-ph].
[32] S. Odake and R. Sasaki, "Dual Christoffel transformations," Prog. Theor. Phys. 126 (2011) 1-34, arXiv:1101.5468[math-ph].
[33] S. Odake, "Equivalences of the Multi-Indexed Orthogonal Polynomials," J. Math. Phys. 55 (2014) 013502 (17pp), arXiv:1309.2346[math-ph].
[34] K. Takemura, "Multi-indexed Jacobi polynomials and Maya diagrams," J. Math. Phys. 55 (2014) 113501 (10pp), arXiv:1311.3570[math-ph].
[35] R. Sasaki, S. Tsujimoto and A. Zhedanov, "Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations," J. Phys. A43 (2010) 315204, arXiv:1004.4711[math-ph].
[36] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials," J. Math. Phys. 54 (2013) 083506 (18pp), arXiv:1303.5820[math-ph].
[37] A. J. Durán, "Higher order recurrence relation for exceptional Charlier, Meixner, Hermite and Laguerre orthogonal polynomials," Integral Transforms Spec. Funct. 26 (2015) 357-376, arXiv:1409.4697[math.CA].
[38] H. Miki and S. Tsujimoto, "A new recurrence formula for generic exceptional orthogonal polynomials," J. Math. Phys. 56 (2015) 033502 (13pp), arXiv:1410.0183[math.CA].
[39] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : II," J. Math. Phys. 56 (2015) 053506 (18pp), arXiv:1410.8236[math-ph].
[40] D. Gómez-Ullate, A. Kasman, A. B. J. Kuijlaars and R. Milson, "Recurrence Relations for Exceptional Hermite Polynomials," J. Approx. Theory 204 (2016) 1-16, arXiv: 1506.03651 [math. CA].
[41] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials : III," J. Math. Phys. 57 (2016) 023514 (24pp), arXiv:1509.08213[math-ph].
[42] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials IV : closure relations and creation/annihilation operators," J. Math. Phys. 57 (2016) 113503 (22pp), arXiv:1606.0283[math-ph].
[43] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials V : Racah and $q$-Racah types," J. Math. Phys. 60 (2019) 023508 (30pp), arXiv:1804. 10352 [math-ph].
[44] S. Odake, "Recurrence Relations of the Multi-Indexed Orthogonal Polynomials VI : Meixner-Pollaczek and continuous Hahn types," in preparation.

