Recurrence Relations of the Multi-Indexed Orthogonal Polynomials VI : Meixner-Pollaczek and continuous Hahn types

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Abstract

In previous papers, we discussed the recurrence relations of the multi-indexed orthogonal polynomials of the Laguerre, Jacobi, Wilson, Askey-Wilson, Racah and q-Racah types. In this paper we explore those of the Meixner-Pollaczek and continuous Hahn types. For the M-indexed Meixner-Pollaczek and continuous Hahn polynomials, we present $3 + 2M$ term recurrence relations with variable dependent coefficients and $1 + 2L$ term ($L \geq M + 1$) recurrence relations with constant coefficients. Based on the latter, the generalized closure relations and the creation/annihilation operators of the quantum mechanical systems described by the multi-indexed Meixner-Pollaczek and continuous Hahn polynomials are obtained.

1   Introduction

Orthogonal polynomials in the Askey scheme of the (basic) hypergeometric orthogonal polynomials [1], which satisfy second order differential or difference equations, can be successfully studied in the quantum mechanical formulation: ordinary quantum mechanics (oQM), discrete quantum mechanics with pure imaginary shifts (idQM) [2]–[5] and discrete quantum mechanics with real shifts (rdQM) [6]–[8]. By deforming the exactly solvable quantum systems described by the orthogonal polynomials in the Askey scheme, new types of orthogonal polynomials are obtained [9]–[24]. They are called exceptional or multi-indexed orthogonal polynomials $\{\mathcal{P}_n(\eta)|n \in \mathbb{Z}_{\geq 0}\}$ (the range of n is finite for finite rdQM systems). They satisfy second order differential or difference equations and form a complete set of orthogonal basis in an appropriate Hilbert space in spite of missing degrees. This degree missing

is a characteristic feature of them, and thereby the Bochner's theorem and its generalizations [25, 26] are avoided. We distinguish the following two cases; the set of missing degrees $\mathcal{I} = \mathbb{Z}_{\geq 0} \backslash \{\deg \mathcal{P}_n | n \in \mathbb{Z}_{\geq 0}\}$ is case-(1): $\mathcal{I} = \{0, 1, \ldots, \ell-1\}$, or case-(2): $\mathcal{I} \neq \{0, 1, \ldots, \ell-1\}$, where ℓ is a positive integer. The situation of case-(1) is called stable in [13].

Ordinary orthogonal polynomials in one variable are characterized by the three term recurrence relations [26]. Since the multi-indexed orthogonal polynomials are not ordinary orthogonal polynomials, they do not satisfy the three term recurrence relations. Instead of the three term recurrence relations, they satisfy some recurrence relations with more terms [27]–[35].

In previous papers [28, 31, 33, 34, 35], the recurrence relations for the case-(1) multi-indexed orthogonal polynomials (Laguerre (L) and Jacobi (J) types in oQM [15], Wilson (W) and Askey-Wilson (AW) types in idQM [17], Racah (R) and q-Racah (qR) types in rdQM [19]) were studied. There are two kinds of recurrence relations: with variable dependent coefficients and with constant coefficients.

Recently, based on the idQM systems whose physical range of the coordinate is the whole real line, the case-(1) multi-indexed orthogonal polynomials of Meixner-Pollaczek (MP) and continuous Hahn (cH) types are constructed [36]. We remark that the MP and cH polynomials reduce to the Hermite (H) polynomial in certain limits but there are no case-(1) multi-indexed Hermite orthogonal polynomials.

In this paper we explore the recurrence relations for the case-(1) multi-indexed orthogonal polynomials of MP and cH types. By similar methods used in W and AW cases, we derive two kinds of recurrence relations. The recurrence relations with constant coefficients are closely related to the generalized closure relations [34]. The generalized closure relations provide the exact Heisenberg operator solution of a certain operator, from which the creation and annihilation operators of the system are obtained. We present the creation and annihilation operators of the deformed MP and cH systems.

The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ is determined by the denominator polynomial $\Xi_{\mathcal{D}}(\eta)$, where $\mathcal{D}$ is an index set of the virtual state wavefunctions used in M-step Darboux transformations. The degree of $\Xi_{\mathcal{D}}(\eta)$ is given by $\ell_{\mathcal{D}}$ (A.26) and there is no restriction on $\ell_{\mathcal{D}}$ for L, J, W and AW cases. However, for MP and cH cases, the degree $\ell_{\mathcal{D}}$ must be even in order that the Hamiltonian $\mathcal{H}_{\mathcal{D}}$ is hermitian. The multi-indexed MP and cH orthogonal polynomials are constructed for even $\ell_{\mathcal{D}}$ (and some conditions) [36]. In this paper we define

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the multi-indexed MP and cH polynomials for any index set $\mathcal{D}$, namely $\ell_{\mathcal{D}}$ may be odd and they may not be orthogonal polynomials. We conjecture that the recurrence relations for the multi-indexed MP and cH polynomials hold for non-orthogonal case.

This paper is organized as follows. In section 2 the case-(1) multi-indexed MP and cH polynomials are recapitulated. The definitions of the multi-indexed MP and cH polynomials are extended to non-orthogonal case. In section 3 the recurrence relations of the case-(1) multi-indexed MP and cH orthogonal polynomials are derived. The recurrence relations with variable dependent coefficients are presented in section 3.1, and those with constant coefficients in section 3.2. Some explicit examples are presented in section 3.3. In section 4 the generalized closure relations and the creation/annihilation operators are presented. Section 5 is for a summary and comments. In Appendix A formulation of idQM and deformed systems are recapitulated. Some properties of the multi-indexed MP polynomials are presented in Appendix B, and those of the multi-indexed cH polynomials are presented in Appendix C. In Appendix D more examples for section 3 are presented, which correspond to non-orthogonal case.

2 Multi-indexed Meixner-Pollaczek and Continuous Hahn polynomials

In this section we recapitulate the case-(1) multi-indexed Meixner-Pollaczek and continuous Hahn polynomials [36]. Generalizing the result of [36], we first define the multi-indexed MP and cH polynomials for any $\mathcal{D}$, and then consider the condition that they become orthogonal polynomials.

The case-(1) multi-indexed orthogonal polynomials [15, 17, 19, 22, 36] and those of case-(2) [3, 4, 7, 37, 38, 39] are constructed based on the quantum mechanical formulations [5]. For the Meixner-Pollaczek (MP) and Continuous Hahn (cH) polynomials, we use the discrete quantum mechanics with pure imaginary shifts (idQM) [2, 5]. The formulation of idQM is presented in Appendix A and we follow the notation there. For MP and cH cases, the lower bound x_1, upper bound x_2, the parameter γ, the sinusoidal coordinate $\eta(x)$ and the auxiliary function $\varphi(x)$ are

$$x_1 = -\infty, \quad x_2 = \infty, \quad \gamma = 1, \quad \eta(x) = x, \quad \varphi(x) = 1. \tag{2.1}$$

Namely, the physical range of the coordinate x is the whole real line. It is not necessary to

distinguish $\check{P}_n$ and P_n since $\eta(x) = x$, but we will use both notations to compare with other cases in [17].

2.1 Meixner-Pollaczek and continuous Hahn polynomials

First we take a set of parameters $\boldsymbol{\lambda}$ as follows:

$$\text{MP}: \ \boldsymbol{\lambda} = (a, \phi), \ \ a, \phi \in \mathbb{R}, \qquad \text{cH}: \ \boldsymbol{\lambda} = (a_1, a_2), \ \ a_1, a_2 \in \mathbb{C}. \tag{2.2}$$

The fundamental data are the following [2]: $(n \in \mathbb{Z}_{\geq 0})$

$$V(x; \boldsymbol{\lambda}) = \begin{cases} e^{i(\frac{\pi}{2} - \phi)}(a + ix) & : \text{MP} \\ (a_1 + ix)(a_2 + ix) & : \text{cH} \end{cases}, \tag{2.3}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = \begin{cases} 2n \sin \phi & : \text{MP} \\ n(n + b_1 - 1) & : \text{cH} \end{cases}, \quad b_1 \overset{\text{def}}{=} a_1 + a_2 + a_1^* + a_2^*, \tag{2.4}$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = P_n\big(\eta(x); \boldsymbol{\lambda}\big) = \begin{cases} P_n^{(a)}\big(\eta(x); \phi\big) & : \text{MP} \\ p_n\big(\eta(x); a_1, a_2, a_1^*, a_2^*\big) & : \text{cH} \end{cases} \tag{2.5}$$

$$= \begin{cases} \dfrac{(2a)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix} -n, \ a + ix \\ 2a \end{matrix} \ \middle| \ 1 - e^{-2i\phi}\right) & : \text{MP} \\ i^n \dfrac{(a_1 + a_1^*, a_1 + a_2^*)_n}{n!} {}_3F_2\left(\begin{matrix} -n, \ n + b_1 - 1, \ a_1 + ix \\ a_1 + a_1^*, \ a_1 + a_2^* \end{matrix} \ \middle| \ 1\right) & : \text{cH} \end{cases} \tag{2.6}$$

$$= c_n(\boldsymbol{\lambda})\eta(x)^n + (\text{lower order terms}), \quad c_n(\boldsymbol{\lambda}) = \begin{cases} \frac{1}{n!}(2 \sin \phi)^n & : \text{MP} \\ \frac{1}{n!}(n + b_1 - 1)_n & : \text{cH} \end{cases}, \tag{2.7}$$

$$\boldsymbol{\delta} = \begin{cases} (\frac{1}{2}, 0) & : \text{MP} \\ (\frac{1}{2}, \frac{1}{2}) & : \text{cH} \end{cases}, \quad \kappa = 1, \quad f_n(\boldsymbol{\lambda}) = \begin{cases} 2 \sin \phi & : \text{MP} \\ n + b_1 - 1 & : \text{cH} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) = n. \tag{2.8}$$

(Although the notation b_1 conflicts with $b_{n-1}(\boldsymbol{\lambda})$, we think this does not cause any confusion.) Here $P_n^{(a)}(\eta; \phi)$ and $p_n(\eta; a_1, a_2, a_3, a_4)$ in (2.5) are the Meixner-Pollaczek and continuous Hahn polynomials of degree n in η, respectively [1]. Note that $\check{P}_n^*(x; \boldsymbol{\lambda}) = \check{P}_n(x; \boldsymbol{\lambda})$. The Meixner-Pollaczek polynomial with $\phi = \frac{\pi}{2}$ has a definite parity,

$$\text{MP}: \ \ \check{P}_n(-x; \boldsymbol{\lambda}) = (-1)^n \check{P}_n(x; \boldsymbol{\lambda}) \ \ \text{for } \boldsymbol{\lambda} = (a, \tfrac{\pi}{2}). \tag{2.9}$$

The polynomials $\check{P}_n(x)$ satisfy the forward and backward shift relations (A.10), hence the second order difference equation (A.14).

Next let us restrict a set of parameters $\boldsymbol{\lambda}$ as follows:

$$\text{MP}: \ a > 0, \ \ 0 < \phi < \pi, \qquad \text{cH}: \ \text{Re} \, a_i > 0 \ \ (i = 1, 2). \tag{2.10}$$

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Then the Hamiltonian of idQM system is well-defined and hermitian. Additional fundamental data are the following [2]:

$$\phi_n(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}), \tag{2.11}$$

$$\phi_0(x;\boldsymbol{\lambda}) = \begin{cases} e^{(\phi-\frac{\pi}{2})x}\sqrt{\Gamma(a+ix)\Gamma(a-ix)} & : \text{MP} \\ \sqrt{\Gamma(a_1+ix)\Gamma(a_2+ix)\Gamma(a_1^*-ix)\Gamma(a_2^*-ix)} & : \text{cH} \end{cases}, \tag{2.12}$$

$$h_n(\boldsymbol{\lambda}) = \begin{cases} 2\pi\dfrac{\Gamma(n+2a)}{n!\,(2\sin\phi)^{2a}} & : \text{MP} \\ 2\pi\dfrac{\prod_{j,k=1}^2 \Gamma(n+a_j+a_k^*)}{n!\,(2n+b_1-1)\Gamma(n+b_1-1)} & : \text{cH} \end{cases}. \tag{2.13}$$

Note that $\phi_0^*(x;\boldsymbol{\lambda}) = \phi_0(x;\boldsymbol{\lambda})$. The eigenfunctions $\phi_n(x)$ are orthogonal each other (A.4), which gives the orthogonality relation of $\check{P}_n(x)$ (A.7).

The MP and cH idQM systems have shape invariance property (A.8), which gives the forward and backward shift relations (A.10). The second order difference equation (A.14) is a rewrite of the Schrödinger equation (A.3).

2.2 Multi-indexed Meixner-Pollaczek and continuous Hahn polynomials

The case-(1) multi-indexed Meixner-Pollaczek and continuous Hahn polynomials were constructed by deforming the idQM systems in §2.1 [36]. The index set $\mathcal{D} = \{d_1, \ldots, d_M\}$ (d_j: mutually distinct) labels the virtual state wavefunctions used in the M-step Darboux transformations. For cH system, there are two types of virtual states (type I and II) and $\mathcal{D}$ is $\mathcal{D} = \{d_1, \ldots, d_M\} = \{d_1^{\text{I}}, \ldots, d_{M_\text{I}}^{\text{I}}, d_1^{\text{II}}, \ldots, d_{M_\text{II}}^{\text{II}}\}$ ($M = M_\text{I} + M_\text{II}$, d_j^{I}: mutually distinct, d_j^{II} : mutually distinct).

First we define multi-indexed MP and cH polynomials for any $\boldsymbol{\lambda}$ and $\mathcal{D}$, which may not be orthogonal polynomials. Then we present a sufficient condition for them to become orthogonal polynomials.

2.2.1 definitions (for any $\boldsymbol{\lambda}$ and $\mathcal{D}$)

A set of parameters $\boldsymbol{\lambda}$ is taken as (2.2) and d_j's are non-negative integers. The twist operations $\mathfrak{t}$ and constants $\tilde{\boldsymbol{\delta}}$ are defined by

$$\text{MP}: \quad \mathfrak{t}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-a, \phi), \quad \tilde{\boldsymbol{\delta}} \stackrel{\text{def}}{=} (-\tfrac{1}{2}, 0), \tag{2.14}$$

$$\text{cH}: \quad \text{type I}: \quad \mathfrak{t}^{\text{I}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (1-a_1^*, a_2), \quad \tilde{\boldsymbol{\delta}}^{\text{I}} \stackrel{\text{def}}{=} (-\tfrac{1}{2}, \tfrac{1}{2}),$$

$$\text{type II}: \quad \mathbf{t}^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (a_1, 1 - a_2^*), \quad \tilde{\boldsymbol{\delta}}^{\text{II}} \stackrel{\text{def}}{=} (\tfrac{1}{2}, -\tfrac{1}{2}). \tag{2.15}$$

For cH case, corresponding to the type I and type II, we add superscripts I and II. The virtual state energies $\tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda})$ and virtual state polynomials $\xi_{\text{v}}(\eta; \boldsymbol{\lambda})$ are defined by

$$\text{MP}: \quad \tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -2(2a - \text{v} - 1)\sin\phi,$$

$$\check{\xi}_{\text{v}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\text{v}}\big(\eta(x); \boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} \check{P}_{\text{v}}\big(x; \mathbf{t}(\boldsymbol{\lambda})\big) = P_{\text{v}}\big(\eta(x); \mathbf{t}(\boldsymbol{\lambda})\big), \tag{2.16}$$

$$\text{cH}: \quad \tilde{\mathcal{E}}_{\text{v}}^{\text{I}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(a_1 + a_1^* - \text{v} - 1)(a_2 + a_2^* + \text{v}),$$

$$\tilde{\mathcal{E}}_{\text{v}}^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(a_2 + a_2^* - \text{v} - 1)(a_1 + a_1^* + \text{v}),$$

$$\check{\xi}_{\text{v}}^{\text{I}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\text{v}}^{\text{I}}\big(\eta(x); \boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} \check{P}_{\text{v}}\big(x; \mathbf{t}^{\text{I}}(\boldsymbol{\lambda})\big) = P_{\text{v}}\big(\eta(x); \mathbf{t}^{\text{I}}(\boldsymbol{\lambda})\big), \tag{2.17}$$

$$\check{\xi}_{\text{v}}^{\text{II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\text{v}}^{\text{II}}\big(\eta(x); \boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} \check{P}_{\text{v}}\big(x; \mathbf{t}^{\text{II}}(\boldsymbol{\lambda})\big) = P_{\text{v}}\big(\eta(x); \mathbf{t}^{\text{II}}(\boldsymbol{\lambda})\big).$$

The virtual state polynomials $\xi_{\text{v}}(\eta; \boldsymbol{\lambda})$ are polynomials of degree v in η and satisfy $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})\check{\xi}_{\text{v}}(x; \boldsymbol{\lambda})$ $= \tilde{\mathcal{E}}_{\text{v}}(\boldsymbol{\lambda})\check{\xi}_{\text{v}}(x; \boldsymbol{\lambda})$. Note that $\check{\xi}_{\text{v}}^*(x; \boldsymbol{\lambda}) = \check{\xi}_{\text{v}}(x; \boldsymbol{\lambda})$. The functions $r_j(x_j^{(M)}; \boldsymbol{\lambda}, M)$ $(j = 1, 2, \ldots, M)$ are defined by

$$\text{MP}: \quad r_j(x_j^{(M)}; \boldsymbol{\lambda}, M) \stackrel{\text{def}}{=} (-1)^{j-1}i^{1-M}\big(a - \tfrac{M-1}{2} + ix\big)_{j-1}\big(a - \tfrac{M-1}{2} - ix\big)_{M-j}, \tag{2.18}$$

$$\text{cH}: \quad r_j^{\text{I}}(x_j^{(M)}; \boldsymbol{\lambda}, M) \stackrel{\text{def}}{=} (-1)^{j-1}i^{1-M}\big(a_1 - \tfrac{M-1}{2} + ix\big)_{j-1}\big(a_1^* - \tfrac{M-1}{2} - ix\big)_{M-j},$$

$$r_j^{\text{II}}(x_j^{(M)}; \boldsymbol{\lambda}, M) \stackrel{\text{def}}{=} (-1)^{j-1}i^{1-M}\big(a_2 - \tfrac{M-1}{2} + ix\big)_{j-1}\big(a_2^* - \tfrac{M-1}{2} - ix\big)_{M-j}, \tag{2.19}$$

where $x_j^{(n)} \stackrel{\text{def}}{=} x + i(\tfrac{n+1}{2} - j)\gamma$. The auxiliary function $\varphi_M(x)$ introduced in [4] is $\varphi_M(x) = 1$ in the present case.

Let us define the denominator polynomial $\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})$ and the multi-indexed polynomial $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$:

$$\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \Xi_{\mathcal{D}}\big(\eta(x); \boldsymbol{\lambda}\big), \quad \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_{\mathcal{D},n}\big(\eta(x); \boldsymbol{\lambda}\big). \tag{2.20}$$

Here $\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})$ are given by determinants as follows. For MP, they are

$$i^{\frac{1}{2}M(M-1)}\det\big(\check{\xi}_{d_k}(x_j^{(M)}; \boldsymbol{\lambda})\big)_{1\le j,k\le M} = \varphi_M(x)\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}), \tag{2.21}$$

$$i^{\frac{1}{2}M(M+1)}\begin{vmatrix} \check{\xi}_{d_1}(x_1^{(M+1)}; \boldsymbol{\lambda}) & \cdots & \check{\xi}_{d_M}(x_1^{(M+1)}; \boldsymbol{\lambda}) & r_1(x_1^{(M+1)}; \boldsymbol{\lambda}, M+1)\check{P}_n(x_1^{(M+1)}; \boldsymbol{\lambda}) \\ \check{\xi}_{d_1}(x_2^{(M+1)}; \boldsymbol{\lambda}) & \cdots & \check{\xi}_{d_M}(x_2^{(M+1)}; \boldsymbol{\lambda}) & r_2(x_2^{(M+1)}; \boldsymbol{\lambda}, M+1)\check{P}_n(x_2^{(M+1)}; \boldsymbol{\lambda}) \\ \vdots & \cdots & \vdots & \vdots \\ \check{\xi}_{d_1}(x_{M+1}^{(M+1)}; \boldsymbol{\lambda}) & \cdots & \check{\xi}_{d_M}(x_{M+1}^{(M+1)}; \boldsymbol{\lambda}) & r_{M+1}(x_{M+1}^{(M+1)}; \boldsymbol{\lambda}, M+1)\check{P}_n(x_{M+1}^{(M+1)}; \boldsymbol{\lambda}) \end{vmatrix}$$

$$= \varphi_{M+1}(x)\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}). \tag{2.22}$$

For cH, they are

$$i^{\frac{1}{2}M(M-1)} \left| \begin{array}{ccccccc} \vec{X}_{d_1^{\mathrm{I}}}^{(M)} & \cdots & \vec{X}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{(M)} & \vec{Y}_{d_1^{\mathrm{II}}}^{(M)} & \cdots & \vec{Y}_{d_{M_{\mathrm{II}}}^{\mathrm{II}}}^{(M)} \end{array} \right| = \varphi_M(x)\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}) \times A, \qquad (2.23)$$

$$i^{\frac{1}{2}M(M+1)} \left| \begin{array}{ccccccccc} \vec{X}_{d_1^{\mathrm{I}}}^{(M+1)} & \cdots & \vec{X}_{d_{M_{\mathrm{I}}}^{\mathrm{I}}}^{(M+1)} & \vec{Y}_{d_1^{\mathrm{II}}}^{(M+1)} & \cdots & \vec{Y}_{d_{M_{\mathrm{II}}}^{\mathrm{II}}}^{(M+1)} & \vec{Z}_n^{(M+1)} \end{array} \right|$$

$$= \varphi_{M+1}(x)\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) \times B, \qquad (2.24)$$

where A and B are

$$A = \prod_{j=1}^{M_{\mathrm{I}}-1} (a_2 - \tfrac{M-1}{2} + ix, a_2^* - \tfrac{M-1}{2} - ix)_j \cdot \prod_{j=1}^{M_{\mathrm{II}}-1} (a_1 - \tfrac{M-1}{2} + ix, a_1^* - \tfrac{M-1}{2} - ix)_j, \quad (2.25)$$

$$B = \prod_{j=1}^{M_{\mathrm{I}}} (a_2 - \tfrac{M}{2} + ix, a_2^* - \tfrac{M}{2} - ix)_j \cdot \prod_{j=1}^{M_{\mathrm{II}}} (a_1 - \tfrac{M}{2} + ix, a_1^* - \tfrac{M}{2} - ix)_j, \qquad (2.26)$$

and $\vec{X}_{\mathrm{v}}^{(M)}$, $\vec{Y}_{\mathrm{v}}^{(M)}$ and $\vec{Z}_{\mathrm{v}}^{(M)}$ are

$$\begin{aligned} \left(\vec{X}_{\mathrm{v}}^{(M)} \right)_j &= r_j^{\mathrm{II}}(x_j^{(M)};\boldsymbol{\lambda}, M)\check{\xi}_{\mathrm{v}}^{\mathrm{I}}(x_j^{(M)};\boldsymbol{\lambda}), \qquad (j = 1, 2, \ldots, M), \\ \left(\vec{Y}_{\mathrm{v}}^{(M)} \right)_j &= r_j^{\mathrm{I}}(x_j^{(M)};\boldsymbol{\lambda}, M)\check{\xi}_{\mathrm{v}}^{\mathrm{II}}(x_j^{(M)};\boldsymbol{\lambda}), \\ \left(\vec{Z}_n^{(M)} \right)_j &= r_j^{\mathrm{II}}(x_j^{(M)};\boldsymbol{\lambda}, M)r_j^{\mathrm{I}}(x_j^{(M)};\boldsymbol{\lambda}, M)\check{P}_n(x_j^{(M)};\boldsymbol{\lambda}). \end{aligned} \qquad (2.27)$$

(For the cases of type I only ($M_{\mathrm{II}} = 0$) or type II only ($M_{\mathrm{I}} = 0$), the expressions (2.23) and (2.24) are rewritten as (2.21) and (2.22), see [36].) The denominator polynomial $\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})$ and the multi-indexed polynomial $P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda})$ are polynomials in η and their degrees are $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}}+n$, respectively (we assume $c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) \neq 0$ and $c_{\mathcal{D},n}^P(\boldsymbol{\lambda}) \neq 0$, see (B.1)–(B.2) and (C.1)–(C.2)). Here $\ell_{\mathcal{D}}$ is given by (A.26). Note that $\check{\Xi}_{\mathcal{D}}^*(x;\boldsymbol{\lambda}) = \check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$ and $\check{P}_{\mathcal{D},n}^*(x;\boldsymbol{\lambda}) = \check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$. Under the permutation of d_j's, $\check{\Xi}_{\mathcal{D}}(x)$ and $\check{P}_{\mathcal{D},n}(x)$ change their signs. The parity property of the Meixner-Pollaczek polynomial (2.9) is inherited by the multi-indexed polynomials of even $\ell_{\mathcal{D}}$,

$$\text{MP}: \quad \check{P}_{\mathcal{D},n}(-x;\boldsymbol{\lambda}) = (-1)^n \check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) \ \text{ for } \boldsymbol{\lambda} = (a, \tfrac{\pi}{2}) \text{ and even } \ell_{\mathcal{D}}. \qquad (2.28)$$

The deformed potential functions $V_{\mathcal{D}}(x;\boldsymbol{\lambda})$ are defined by (A.21) and (A.24). The multi-indexed polynomials $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$ satisfy the forward and backward shift relations (A.34), hence the second order difference equation (A.38).

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2.2.2 orthogonal polynomials

The deformed idQM systems should be well-defined, namely the deformed Hamiltonian $\mathcal{H}_\mathcal{D}(\boldsymbol{\lambda})$ should be hermitian. We restrict $\boldsymbol{\lambda}$ as (2.10) and impose the following conditions,

$$\text{MP}: \quad \max_j\{d_j\} < 2a - 1, \tag{2.29}$$

$$\text{cH}: \quad \max_j\{d_j^{\text{I}}\} < a_1 + a_1^* - 1, \quad \max_j\{d_j^{\text{II}}\} < a_2 + a_2^* - 1, \tag{2.30}$$

under which the virtual state energies become negative,

$$\text{MP}: \quad \tilde{\mathcal{E}}_v(\boldsymbol{\lambda}) < 0 \iff 2a > v + 1,$$

$$\text{cH}: \quad \tilde{\mathcal{E}}_v^{\text{I}}(\boldsymbol{\lambda}) < 0 \iff a_1 + a_1^* > v + 1, \quad \tilde{\mathcal{E}}_v^{\text{II}}(\boldsymbol{\lambda}) < 0 \iff a_2 + a_2^* > v + 1.$$

The deformed Hamiltonian $\mathcal{H}_\mathcal{D}(\boldsymbol{\lambda})$ is hermitian, if the condition (A.27) is satisfied [36]. Since the rectangular domain D_γ contains the whole real axis, the degree of $\Xi_\mathcal{D}(\eta; \boldsymbol{\lambda})$, $\ell_\mathcal{D}$ (A.26), should be even. Although we have no analytical proof that there exists a range of parameters $\boldsymbol{\lambda}$ satisfying the condition (A.27), we can verify that there exists such a range of $\boldsymbol{\lambda}$ by numerical calculation (for small M and d_j). We have observed various sufficient conditions for the parameter range satisfying (A.27), see [36]. In the rest of this section we assume that the condition (A.27) is satisfied.

The eigenfunctions of the deformed Hamiltonian $\mathcal{H}_\mathcal{D}(\boldsymbol{\lambda})$ have the form (A.22). Note that $\psi_\mathcal{D}^*(x; \boldsymbol{\lambda}) = \psi_\mathcal{D}(x; \boldsymbol{\lambda})$. The eigenfunctions $\phi_{\mathcal{D}\,n}(x)$ are orthogonal each other, which gives the orthogonality relation of $\check{P}_{\mathcal{D},n}(x)$, (A.28)–(A.29). The multi-indexed orthogonal polynomial $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ has n zeros in the physical region $\eta \in \mathbb{R}$ ($\iff \eta(x_1) < \eta < \eta(x_2)$), which interlace the $n + 1$ zeros of $P_{\mathcal{D},n+1}(\eta; \boldsymbol{\lambda})$ in the physical region, and $\ell_\mathcal{D}$ zeros in the unphysical region $\eta \in \mathbb{C} \backslash \mathbb{R}$. These properties and (A.28) can be verified by numerical calculation.

Since the deformed Hamiltonian $\mathcal{H}_\mathcal{D}(\boldsymbol{\lambda})$ (A.19) is expressed in terms of the potential function $V_\mathcal{D}(x; \boldsymbol{\lambda})$ (A.21), $\mathcal{H}_\mathcal{D}(\boldsymbol{\lambda})$ is determined by the denominator polynomial $\check{\Xi}_\mathcal{D}(x; \boldsymbol{\lambda})$, whose normalization is irrelevant. Under the permutation of d_j's, the deformed Hamiltonian $\mathcal{H}_\mathcal{D}$ is invariant.

The deformed MP and cH idQM systems have also shape invariance property (A.30), which gives the forward and backward shift relations (A.34). The second order difference equation (A.38) is a rewrite of the Schrödinger equation (A.18). The properties (B.4) and (C.4)–(C.5) are also the consequences of the shape invariance. By (B.3) and (C.3), the similar property holds for the denominator polynomial $\check{\Xi}_\mathcal{D}(x; \boldsymbol{\lambda})$. The M-step deformed

system labeled by $\mathcal{D}$ with 0 is equivalent to the $(M-1)$-step deformed system labeled by $\mathcal{D}'$ with shifted parameters $\boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}}$. For the multi-indexed W and AW polynomials, there are equivalence among the index sets $\mathcal{D}$ [40]. The multi-indexed MP and cH polynomials have also equivalence in the same form as W and AW cases, which is derived from the properties (B.4) and (C.4)–(C.5).

3   Recurrence Relations

In this section we present the recurrence relations of the case-(1) multi-indexed Meixner-Pollaczek and continuous Hahn polynomials. There are two types of recurrence relations: variable dependent coefficients and constant coefficients.

The three term recurrence relations of the Meixner-Pollaczek and continuous Hahn polynomials are [1]

$$\eta P_n(\eta; \boldsymbol{\lambda}) = A_n(\boldsymbol{\lambda}) P_{n+1}(\eta; \boldsymbol{\lambda}) + B_n(\boldsymbol{\lambda}) P_n(\eta; \boldsymbol{\lambda}) + C_n(\boldsymbol{\lambda}) P_{n-1}(\eta; \boldsymbol{\lambda}), \tag{3.1}$$

where A_n, B_n and C_n are

$$\text{MP}: \ A_n(\boldsymbol{\lambda}) = \frac{n+1}{2\sin\phi}, \quad B_n(\boldsymbol{\lambda}) = -(n+a)\cot\phi, \quad C_n(\boldsymbol{\lambda}) = \frac{n+2a-1}{2\sin\phi}, \tag{3.2}$$

$$\text{cH}: \ A_n(\boldsymbol{\lambda}) = \frac{(n+1)(n+b_1-1)}{(2n+b_1-1)(2n+b_1)},$$

$$B_n(\boldsymbol{\lambda}) = i\Big(a_1 - \frac{(n+b_1-1)(n+a_1+a_1^*)(n+a_1+a_2^*)}{(2n+b_1-1)(2n+b_1)}$$

$$+ \frac{n(n+a_2+a_1^*-1)(n+a_2+a_2^*-1)}{(2n+b_1-2)(2n+b_1-1)}\Big), \tag{3.3}$$

$$C_n(\boldsymbol{\lambda}) = \frac{(n+a_1+a_1^*-1)(n+a_1+a_2^*-1)(n+a_2+a_1^*-1)(n+a_2+a_2^*-1)}{(2n+b_1-2)(2n+b_1-1)}.$$

For simplicity of presentation, we set $P_n(\eta) = 0$ for $n \in \mathbb{Z}_{<0}$ and define A_n, B_n and C_n for $n \in \mathbb{Z}_{<0}$ by (3.2)–(3.3). Then, (3.1) hold for $n \in \mathbb{Z}$. Similarly we set $P_{\mathcal{D},n}(\eta) = 0$ for $n \in \mathbb{Z}_{<0}$.

3.1   Recurrence relations with variable dependent coefficients

We present $3 + 2M$ term recurrence relations with variable dependent coefficients. For the case-(1) multi-indexed Wilson and Askey-Wilson polynomials, such recurrence relations are

shown in [28]. Since the derivation can be applied in the present case without difficulty, we present only the results here.

Let us define $\check{R}^{[s]}_{n,k}(x)$ $(n, k \in \mathbb{Z}, \ s \in \mathbb{Z}_{\geq -1})$ as follows:

$$\check{R}^{[s]}_{n,k}(x) = 0 \ \ (|k| > s+1 \text{ or } n+k < 0), \quad \check{R}^{[-1]}_{n,0}(x) = 1 \ \ (n \geq 0),$$

$$\check{R}^{[s]}_{n,k}(x) = A_n \check{R}^{[s-1]}_{n+1,k-1}(x + i\tfrac{\gamma}{2}) + \big(B_n - \eta(x - i\tfrac{s}{2}\gamma)\big)\check{R}^{[s-1]}_{n,k}(x + i\tfrac{\gamma}{2}) \tag{3.4}$$

$$+ \, C_n \check{R}^{[s-1]}_{n-1,k+1}(x + i\tfrac{\gamma}{2}) \ \ (s \geq 0).$$

Here A_n, B_n and C_n are given by (3.2)–(3.3). Note that A_n $(n < -1)$, B_n $(n < 0)$ and C_n $(n < 0)$ do not appear, because $A_{-1} = 0$ (and we regard $A_{-1} \times (\cdots) = 0$). For example, non-trivial $\check{R}^{[s]}_{n,k}(x)$ $(n + k \geq 0)$ for $s = 0, 1$ are

$$s = 0 : \quad \check{R}^{[0]}_{n,1}(x) = A_n, \quad \check{R}^{[0]}_{n,0}(x) = B_n - \eta(x), \quad \check{R}^{[0]}_{n,-1}(x) = C_n,$$

$$s = 1 : \quad \check{R}^{[1]}_{n,2}(x) = A_n A_{n+1}, \quad \check{R}^{[1]}_{n,1}(x) = A_n\big(B_n + B_{n+1} - \eta(x - i\tfrac{\gamma}{2}) - \eta(x + i\tfrac{\gamma}{2})\big),$$

$$\check{R}^{[1]}_{n,0}(x) = A_n C_{n+1} + A_{n-1} C_n + \big(B_n - \eta(x - i\tfrac{\gamma}{2})\big)\big(B_n - \eta(x + i\tfrac{\gamma}{2})\big),$$

$$\check{R}^{[1]}_{n,-2}(x) = C_n C_{n-1}, \quad \check{R}^{[1]}_{n,-1}(x) = C_n\big(B_n + B_{n-1} - \eta(x - i\tfrac{\gamma}{2}) - \eta(x + i\tfrac{\gamma}{2})\big).$$

It is easy to see that $\check{R}^{[s]}_{n,k}(x)$ is a polynomial in $x = \eta(x)$. We define $R^{[s]}_{n,k}(\eta)$ as follows:

$$\check{R}^{[s]}_{n,k}(x) = R^{[s]}_{n,k}\big(\eta(x)\big) \ \ (|k| \leq s+1) : \text{a polynomial of degree } s+1-|k| \text{ in } \eta(x). \tag{3.5}$$

Note that $\check{R}^{[s]\,*}_{n,k}(x) = \check{R}^{[s]}_{n,k}(x)$. Then we have the following result.

Theorem 1 *The multi-indexed Meixner-Pollaczek and continuous Hahn polynomials satisfy the $3 + 2M$ term recurrence relations with variable dependent coefficients:*

$$\sum_{k=-M-1}^{M+1} R^{[M]}_{n,k}(\eta) P_{\mathcal{D},n+k}(\eta) = 0, \tag{3.6}$$

which holds for $n \in \mathbb{Z}$.

Remark 1 The derivation in [28] uses only the algebraic property of the M-step Darboux transformations. Hence Theorem 1 holds for $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ with any $\boldsymbol{\lambda}$ and $\mathcal{D}$ (namely $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ may not be orthogonal polynomials).

Remark 2 The multi-indexed polynomials $P_{\mathcal{D},n}(\eta)$ $(n \geq M + 1)$ are determined by the $3 + 2M$ term recurrence relations (3.6) with $M + 1$ "initial data" [28],

$$P_{\mathcal{D},0}(\eta), \ P_{\mathcal{D},1}(\eta), \ \ldots, \ P_{\mathcal{D},M}(\eta). \tag{3.7}$$

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After calculating the initial data (3.7) by (2.22) and (2.24), we can obtain $P_{\mathcal{D},n}(\eta)$ through the $3 + 2M$ term recurrence relations (3.6). The calculation cost of this method is much less than the original determinant expression (2.22) and (2.24) for large M.

3.2   Recurrence relations with constant coefficients

We present $1 + 2L$ term recurrence relations with constant coefficients. For the case-(1) multi-indexed Wilson and Askey-Wilson polynomials, such recurrence relations are presented in [31] and shown in Appendix B of [35]. Since the derivation can be applied in the present case without difficulty, we present only the results here.

We want to find the following recurrence relations,

$$X(\eta)P_{\mathcal{D},n}(\eta) = \sum_{k=-n}^{L} r_{n,k}^{X,\mathcal{D}} P_{\mathcal{D},n+k}(\eta),$$

where $r_{n,k}^{X,\mathcal{D}}$'s are constants and $X(\eta)$ is some polynomial of degree L in η. The overall normalization and the constant term of $X(\eta)$ are not important, because the change of the former induces that of the overall normalization of $r_{n,k}^{X,\mathcal{D}}$ and the shift of the latter induces that of $r_{n,0}^{X,\mathcal{D}}$.

The sinusoidal coordinate $\eta(x)$ ($\eta(x) = x$ in the present case) satisfies [41, 31]

$$\frac{\eta(x - i\frac{\gamma}{2})^{n+1} - \eta(x + i\frac{\gamma}{2})^{n+1}}{\eta(x - i\frac{\gamma}{2}) - \eta(x + i\frac{\gamma}{2})} = \sum_{k=0}^{n} g_n'^{(k)} \eta(x)^{n-k} \quad (n \in \mathbb{Z}_{\geq 0}), \tag{3.8}$$

where $g_n'^{(k)}$ is given by [42]

$$g_n'^{(k)} = \theta(k : \text{even}) \, (-1)^{\frac{k}{2}} 2^{-k} \binom{n+1}{k+1}. \tag{3.9}$$

Here $\theta(P)$ is a step function for a proposition P ; $\theta(P) = 1$ for P : true, $\theta(P) = 0$ for P : false. For a polynomial $p(\eta)$ in η, let us define a polynomial in η, $I[p](\eta)$, as follows:

$$p(\eta) = \sum_{k=0}^{n} a_k \eta^k \mapsto I[p](\eta) = \sum_{k=0}^{n+1} b_k \eta^k, \tag{3.10}$$

where b_k's are defined by

$$b_{k+1} = \frac{1}{g_k'^{(0)}} \left(a_k - \sum_{j=k+1}^{n} g_j'^{(j-k)} b_{j+1} \right) \quad (k = n, n-1, \ldots, 1, 0), \quad b_0 = 0. \tag{3.11}$$

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The constant term of $I[p](\eta)$ is chosen to be zero. It is easy to show that this polynomial $I[p](\eta) = P(\eta)$ satisfies

$$\frac{\check{P}(x - i\frac{\gamma}{2}) - \check{P}(x + i\frac{\gamma}{2})}{\eta(x - i\frac{\gamma}{2}) - \eta(x + i\frac{\gamma}{2})} = \check{p}(x), \tag{3.12}$$

where $\check{P}(x) = P(\eta(x))$ and $\check{p}(x) = p(\eta(x))$.

For the denominator polynomial $\Xi_{\mathcal{D}}(\eta)$ and a polynomial in η, $Y(\eta)(\neq 0)$, we set $X(\eta) = X^{\mathcal{D},Y}(\eta)$ as

$$X(\eta) = I[\Xi_{\mathcal{D}}Y](\eta), \quad \deg X(\eta) = L = \ell_{\mathcal{D}} + \deg Y(\eta) + 1, \tag{3.13}$$

where $\Xi_{\mathcal{D}}Y$ means a polynomial $(\Xi_{\mathcal{D}}Y)(\eta) = \Xi_{\mathcal{D}}(\eta)Y(\eta)$. Note that $L \geq M + 1$ because of $\ell_{\mathcal{D}} \geq M$. The minimal degree one, which corresponds to $Y(\eta) = 1$, is

$$X_{\min}(\eta) = I[\Xi_{\mathcal{D}}](\eta), \quad \deg X_{\min}(\eta) = \ell_{\mathcal{D}} + 1. \tag{3.14}$$

Then we have the following theorem.

Theorem 2 *For any polynomial $Y(\eta)(\neq 0)$, we take $X(\eta) = X^{\mathcal{D},Y}(\eta)$ as (3.13). Then the multi-indexed Meixner-Pollaczek and continuous Hahn orthogonal polynomials $P_{\mathcal{D},n}(\eta)$ satisfy $1 + 2L$ term recurrence relations with constant coefficients:*

$$X(\eta)P_{\mathcal{D},n}(\eta) = \sum_{k=-L}^{L} r_{n,k}^{X,\mathcal{D}} P_{\mathcal{D},n+k}(\eta), \tag{3.15}$$

which hold for $n \in \mathbb{Z}_{\geq 0}$. Here $r_{n,k}^{X,\mathcal{D}}$'s are constants.

Remark 1 By defining $r_{n,k}^{X,\mathcal{D}} = 0$ for $n < 0$, (3.15) holds for $n \in \mathbb{Z}$.

Remark 2 Any polynomial $X(\eta)$ giving the recurrence relations with constant coefficients must have the form (3.13) [31].

Remark 3 Many parts of the derivation in [31] and Appendix B of [35] are done algebraically, but some parts use the orthogonality. So we can not conclude that Theorem 2 holds for $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ with any $\boldsymbol{\lambda}$ and $\mathcal{D}$. However, explicit calculation for small M, d_j, n and $\deg Y$ suggests the following conjecture.

Conjecture 1 *Theorem 2 holds for $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ with any $\boldsymbol{\lambda}$ and $\mathcal{D}$ (namely $P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda})$ may not be orthogonal polynomials).*

Remark 4 Direct verification of this theorem is rather straightforward for lower M and smaller d_j, n and $\deg Y$, by a computer algebra system, e.g. Mathematica. The coefficients

$r_{n,k}^{X,\mathcal{D}}$ are explicitly obtained for small d_j and n. However, to obtain the closed expression of $r_{n,k}^{X,\mathcal{D}}$ for general n is not an easy task even for small d_j, and it is a different kind of problem. We present some examples in §3.3 and Appendix D.

Remark 5 Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. However not all of them are independent. The relations among them are unclear. For '$M = 0$ case' (namely, ordinary orthogonal polynomials), it is trivial that recurrence relations obtained from arbitrary $Y(\eta)$ ($\deg Y \geq 1$) are derived by the three term recurrence relations.

Let us consider some properties of the coefficient $r_{n,k}^{X,\mathcal{D}}$. By using the orthogonality relation (A.28) and the recurrence relations (3.15), we obtain the relations among them,

$$r_{n+k,-k}^{X,\mathcal{D}} = \frac{h_{\mathcal{D},n+k}}{h_{\mathcal{D},n}} r_{n,k}^{X,\mathcal{D}} \quad (1 \leq k \leq L). \tag{3.16}$$

Explicit forms of $\frac{h_{\mathcal{D},n+k}}{h_{\mathcal{D},n}}$ with $k \geq 0$ are

$$\text{MP} : \quad \frac{h_{\mathcal{D},n+k}}{h_{\mathcal{D},n}} = \frac{(n+2a)_k}{(n+1)_k} \prod_{j=1}^{M} \frac{n+2a-d_j-1+k}{n+2a-d_j-1}, \tag{3.17}$$

$$\text{cH} : \quad \frac{h_{\mathcal{D},n+k}}{h_{\mathcal{D},n}} = \frac{(n+a_1+a_1^*, n+a_1+a_2^*, n+a_2+a_1^*, n+a_2+a_2^*)_k}{(n+1, n+b_1-1)_k} \frac{2n+b_1-1}{2n+b_1-1+2k}$$

$$\times \prod_{j=1}^{M_{\mathrm{I}}} \frac{(n+a_1+a_1^*-1-d_j^{\mathrm{I}}+k)(n+a_2+a_2^*+d_j^{\mathrm{I}}+k)}{(n+a_1+a_1^*-1-d_j^{\mathrm{I}})(n+a_2+a_2^*+d_j^{\mathrm{I}})}$$

$$\times \prod_{j=1}^{M_{\mathrm{II}}} \frac{(n+a_2+a_2^*-1-d_j^{\mathrm{II}}+k)(n+a_1+a_1^*+d_j^{\mathrm{II}}+k)}{(n+a_2+a_2^*-1-d_j^{\mathrm{II}})(n+a_1+a_1^*+d_j^{\mathrm{II}})}. \tag{3.18}$$

The values of $P_{\mathcal{D},n}(\eta)$ at some specific values η_0 are known explicitly as (B.6) and (C.7). By substituting η_0 for η in (3.15), we have $X(\eta_0)P_{\mathcal{D},n}(\eta_0) = \sum_{k=-L}^{L} r_{n,k}^{X,\mathcal{D}} P_{\mathcal{D},n+k}(\eta_0)$, which gives

$$r_{n,0}^{X,\mathcal{D}} = X(\eta_0) - \sum_{\substack{k=-L \\ k \neq 0}}^{L} \frac{P_{\mathcal{D},n+k}(\eta_0)}{P_{\mathcal{D},n}(\eta_0)} r_{n,k}^{X,\mathcal{D}}. \tag{3.19}$$

Therefore it is sufficient to find $r_{n,k}^{X,\mathcal{D}}$ ($1 \leq k \leq L$). The top coefficient $r_{n,L}^{X,\mathcal{D}}$ is easily obtained by comparing the highest degree terms,

$$r_{n,L}^{X,\mathcal{D}} = \frac{c^X c_{\mathcal{D},n}^P}{c_{\mathcal{D},n+L}^P}, \tag{3.20}$$

where c^X is the coefficient of the highest term of $X(\eta) = c^X \eta^L + $ (lower order terms) and $c_{\mathcal{D},n}^P$ is given by (B.2) and (C.2). For later use, we provide a conjecture about $r_{n,0}^{X,\mathcal{D}}$.

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Conjecture 2 *As a function of n, the coefficients $r_{n,0}^{X,\mathcal{D}}$ has the following form,*

$$r_{n,0}^{X,\mathcal{D}} = -\frac{I(z)}{\prod_{j=1}^{L}\alpha_j(z)\alpha_{2L+1-j}(z)}\Big|_{z=\mathcal{E}_n}, \qquad I(z) : a\ polynomial\ in\ z, \qquad (3.21)$$

where $\alpha_j(z)\alpha_{2L+1-j}(z)$ will be given in (4.10). The degree of $I(z)$ is $\deg I = L$ for MP, $\deg I \leq 2L$ for cH.

Note that this polynomial $I(z)$ is nothing to do with the map $I[\cdot]$ in (3.10).

3.3 Examples

For illustration, we present some examples of the coefficients $r_{n,k}^{X,\mathcal{D}}$ of the recurrence relations (3.15) for multi-indexed orthogonal polynomials. See Appendix D for non-orthogonal case. Except for Ex.1 in §3.3.1, we present only $r_{n,k}^{X,\mathcal{D}}$ $(1 \leq k \leq L)$, because $r_{n,k}^{X,\mathcal{D}}$ $(-L \leq k \leq 0)$ are obtained by (3.16)–(3.19).

3.3.1 multi-indexed Meixner-Pollaczek polynomials

<u>Ex.1</u> $\mathcal{D} = \{2\}$, $Y(\eta) = 1$ $(\Rightarrow \ell_{\mathcal{D}} = 2,\ X(\eta) = X_{\min}(\eta),\ L = 3)$: 7-term recurrence relations

$$X(\eta) = \frac{\eta}{12}\big(8\sin^2\phi\cdot\eta^2 - 6(2a-3)\sin 2\phi\cdot\eta + 12a^2 - 24a + 13 + (12a^2 - 36a + 23)\cos 2\phi\big),$$

$$r_{n,3}^{X,\mathcal{D}} = \frac{(n+1)_3}{12\sin\phi}\frac{2a+n-3}{2a+n}, \qquad r_{n,-3}^{X,\mathcal{D}} = \frac{(2a+n-3)_3}{12\sin\phi},$$

$$r_{n,2}^{X,\mathcal{D}} = -\frac{1}{2}(n+1)_2(2a+n-3)\cot\phi, \qquad r_{n,-2}^{X,\mathcal{D}} = -\frac{1}{2}(2a+n-3)_3\cot\phi,$$

$$r_{n,1}^{X,\mathcal{D}} = \frac{(n+1)(2a+n-3)}{4\sin\phi}\big(4a+3n+2(2a+n-1)\cos 2\phi\big),$$

$$r_{n,-1}^{X,\mathcal{D}} = \frac{(2a+n-3)(2a+n-1)}{4\sin\phi}\big(4a+3n-3+2(2a+n-2)\cos 2\phi\big),$$

$$r_{n,0}^{X,\mathcal{D}} = -\frac{1}{6}\big((2a-3)(2a-1)(7a-1) + 2(36a^2 - 60a + 19)n + 6(8a-7)n^2 + 10n^3\big)\cot\phi$$
$$+ \frac{1}{24}(2a+2n-1)\big(4a(7a+5n-16) + 33 - 22n + 4n^2\big)\sin 2\phi.$$

The polynomial $I(z)$ (3.21) is

$$I(z) = -48\sin\phi\sin 2\phi\Big((4+\cos 2\phi)z^3 + 6\big(6a - 5 + 2(a-1)\cos 2\phi\big)\sin\phi\cdot z^2$$
$$+ 4\big(4(6a^2 - 9a + 2) + (12a^2 - 24a + 11)\cos 2\phi\big)\sin^2\phi\cdot z$$
$$+ (2a-3)(2a-1)\big(14a + 7 + (14a - 11)\cos 2\phi\big)\sin^3\phi\Big).$$

<u>Ex.2</u> $\mathcal{D} = \{1, 2\}$, $Y(\eta) = 1$ ($\Rightarrow \ell_{\mathcal{D}} = 2$, $X(\eta) = X_{\min}(\eta)$, $L = 3$): 7-term recurrence relations

$$X(\eta) = \frac{2\sin\phi}{3}\eta\big(2\sin^2\phi \cdot \eta^2 - 3(a-1)\sin 2\phi \cdot \eta + 3a^2 - 9a + 7 + (3a^2 - 6a + 2)\cos 2\phi\big),$$

$$r_{n,3}^{X,\mathcal{D}} = \frac{1}{6}(n+1)_3\frac{(2a+n-3)_2}{(2a+n)_2}, \quad r_{n,2}^{X,\mathcal{D}} = -(n+1)_2\frac{(2a+n-3)_2}{2a+n}\cos\phi,$$

$$r_{n,1}^{X,\mathcal{D}} = \frac{1}{2}(n+1)(2a+n-3)\big(4a+3n-4+2(2a+n-2)\cos 2\phi\big).$$

The polynomial $I(z)$ (3.21) is

$$\begin{aligned}
I(z) = -96\sin^2\phi\sin 2\phi\Big(&(4+\cos 2\phi)z^3 + 12(a-1)(3+\cos 2\phi)\sin\phi \cdot z^2 \\
&+ 4\big(4(6a^2-12a+5)+(12a^2-24a+11)\cos 2\phi\big)\sin^2\phi \cdot z \\
&+ 8(a-1)\big(a(7a-11)+(7a^2-14a+6)\cos 2\phi\big)\sin^3\phi\Big).
\end{aligned}$$

<u>Ex.3</u> $\mathcal{D} = \{2\}$, $Y(\eta) = \eta$ ($\Rightarrow \ell_{\mathcal{D}} = 2$, $L = 4$): 9-term recurrence relations

$$\begin{aligned}
X(\eta) = \frac{\eta}{24}\Big(&12\sin^2\phi \cdot \eta^3 - 8(2a-3)\sin 2\phi \cdot \eta^2 \\
&+ 3\big(4a^2 - 8a + 5 + (4a^2 - 12a + 7)\cos 2\phi\big)\eta - 2(2a-3)\sin 2\phi\Big),
\end{aligned}$$

$$r_{n,4}^{X,\mathcal{D}} = \frac{(n+1)_4}{32\sin^2\phi}\frac{2a+n-3}{2a+n+1}, \quad r_{n,3}^{X,\mathcal{D}} = -\frac{(n+1)_3\cos\phi}{12\sin^2\phi}\frac{2a+n-3}{2a+n}(5a+3n),$$

$$r_{n,2}^{X,\mathcal{D}} = \frac{(n+1)_2}{8\sin^2\phi}(2a+n-3)\big(2(2a+2n+1)+(4a+3n)\cos 2\phi\big),$$

$$\begin{aligned}
r_{n,1}^{X,\mathcal{D}} = -\frac{(n+1)\cos\phi}{4\sin^2\phi}&(2a+n-3)\big(4a(a+1)+(11a+1)n+5n^2 \\
&+ 2(a+n)(2a+n-1)\cos 2\phi\big).
\end{aligned}$$

The polynomial $I(z)$ (3.21) is

$$\begin{aligned}
I(z) = -192\sin^2\phi\Big(&3(18+16\cos 2\phi+\cos 4\phi)z^4 \\
&+ 4\big(36(4a-3)+4(34a-27)\cos 2\phi+(10a-9)\cos 4\phi\big)\sin\phi \cdot z^3 \\
&+ 12\big(2(86a^2-119a+30)+8(22a^2-33a+10)\cos 2\phi+(16a^2-28a+11)\cos 4\phi\big)\sin^2\phi \cdot z^2 \\
&+ 16\big(3(56a^3-98a^2+39a-9)+4(48a^3-96a^2+50a-9)\cos 2\phi \\
&\quad + (24a^3-60a^2+44a-9)\cos 4\phi\big)\sin^3\phi \cdot z \\
&+ (2a-3)\big(3(2a+1)(68a^2-4a+1)+4(2a-1)(2a+1)(34a-3)\cos 2\phi \\
&\quad + (2a-1)(68a^2-80a+15)\cos 4\phi\big)\sin^4\phi\Big).
\end{aligned}$$

Ex.4 $\mathcal{D} = \{4\}$, $Y(\eta) = 1$ ($\Rightarrow \ell_\mathcal{D} = 4$, $X(\eta) = X_{\min}(\eta)$, $L = 5$): 11-term recurrence relations

$$
\begin{aligned}
X(\eta) = \frac{\eta}{960}\Big(& 128\sin^4\phi \cdot \eta^4 - 320(2a-5)\cos\phi\sin^3\phi \cdot \eta^3 \\
& + 160\big(4a^2 - 16a + 17 + (4a^2 - 20a + 23)\cos 2\phi\big)\sin^2\phi \cdot \eta^2 \\
& - 80(2a-5)\big(4a^2 - 8a + 5 + (4a^2 - 20a + 19)\cos 2\phi\big)\cos\phi\sin\phi \cdot \eta \\
& + 240a^4 - 1440a^3 + 3160a^2 - 3000a + 1067 \\
& + 4(80a^4 - 560a^3 + 1360a^2 - 1380a + 511)\cos 2\phi \\
& + (80a^4 - 800a^3 + 2760a^2 - 3800a + 1689)\cos 4\phi\Big),
\end{aligned}
$$

$$
r_{n,5}^{X,\mathcal{D}} = \frac{(n+1)_5}{240\sin\phi}\frac{2a+n-5}{2a+n}, \quad r_{n,4}^{X,\mathcal{D}} = -\frac{1}{24}(n+1)_4\,(2a+n-5)\cot\phi,
$$

$$
r_{n,3}^{X,\mathcal{D}} = \frac{(n+1)_3}{48\sin\phi}(2a+n-5)\big(8a+5n+4(2a+n-1)\cos 2\phi\big),
$$

$$
r_{n,2}^{X,\mathcal{D}} = -\frac{1}{6}(n+1)_2\,(2a+n-5)(2a+n-1)\cot\phi\big(2a+2n+1+(2a+n-2)\cos 2\phi\big),
$$

$$
\begin{aligned}
r_{n,1}^{X,\mathcal{D}} = \frac{n+1}{24\sin\phi}(2a+n-5)(2a+n-1)\big(& 12a^2 + 6a(4n-1) + 10n(n-1) \\
& + 2(2a+n-2)(4a+5n)\cos 2\phi + (2a+n-3)_2\cos 4\phi\big).
\end{aligned}
$$

The polynomial $I(z)$ (3.21) is

$$
\begin{aligned}
I(z) = -3840\sin^3\phi\sin 2\phi\Big(& (38 + 24\cos 2\phi + \cos 4\phi)z^5 \\
& + 10\big(8(7a-8) + 4(10a-13)\cos 2\phi + (2a-3)\cos 4\phi\big)\sin\phi \cdot z^4 \\
& + 20\big(2(78a^2 - 173a + 78) + 4(32a^2 - 82a + 47)\cos 2\phi \\
& \quad + (8a^2 - 24a + 17)\cos 4\phi\big)\sin^2\phi \cdot z^3 \\
& + 40\big(4(2a-1)(25a^2 - 67a + 35) + 4(48a^3 - 180a^2 + 200a - 65)\cos 2\phi \\
& \quad + (2a-3)(8a^2 - 24a + 15)\cos 4\phi\big)\sin^3\phi \cdot z^2 \\
& + 32\big(280a^4 - 1100a^3 + 1310a^2 - 645a + 131 \\
& \quad + 2(160a^4 - 760a^3 + 1180a^2 - 690a + 119)\cos 2\phi \\
& \quad + (40a^4 - 240a^3 + 510a^2 - 450a + 137)\cos 4\phi\big)\sin^4\phi \cdot z \\
& + (2a-5)(2a-1)\big(744a^3 - 828a^2 + 406a - 297 \\
& \quad + 4(2a-3)(2a+1)(62a-57)\cos 2\phi \\
& \quad + (2a-3)(124a^2 - 352a + 193)\cos 4\phi\big)\sin^5\phi\Big).
\end{aligned}
$$

We have also obtained 9-term recurrence relations for $\mathcal{D} = \{1,2\}$ with non-minimal $X(\eta)$

$(Y(\eta) = \eta)$, and 11-term recurrence relations for $\mathcal{D} = \{1,4\}, \{2,3\}, \{1,2,4\}, \{1,2,3,4\}$ with $X(\eta) = X_{\min}(\eta)$ and $\mathcal{D} = \{2\}, \{1,2\}$ with non-minimal $X(\eta)$ $(Y(\eta) = \eta^2)$. Since the explicit forms of $r_{n,k}^{X,\mathcal{D}}$ are somewhat lengthy, we do not write down them here.

3.3.2 multi-indexed continuous Hahn polynomials

We set $\sigma_1 = a_1 + a_1^*$, $\sigma_2 = a_1 a_1^*$, $\sigma_1' = a_2 + a_2^*$ and $\sigma_2' = a_2 a_2^*$.

<u>Ex.1</u> $\mathcal{D} = \{2^{\mathrm{I}}\}$ $(M_{\mathrm{I}} = 1, M_{\mathrm{II}} = 0)$, $Y(\eta) = 1$ $(\Rightarrow \ell_{\mathcal{D}} = 2, X(\eta) = X_{\min}(\eta), L = 3)$: 7-term recurrence relations

$$
\begin{aligned}
X(\eta) = \frac{\eta}{24}\Big(& 4(\sigma_1 - \sigma_1' - 4)_2\,\eta^2 + 6i(\sigma_1 - \sigma_1' - 3)\big(a_1 - a_1^* + 2(a_1 a_2^* - a_1^* a_2) + 3(a_2 - a_2^*)\big)\eta \\
& + 12 + \big(36a_2(a_2 + 1) - 2\sigma_1' - 24\sigma_2' - 7\big)\sigma_1 + \big(1 - 12a_2(a_2 + 1)\big)\sigma_1^2 \\
& + 24a_2(a_2 + 1)(a_1^2 - 3a_1 + \sigma_2) - \sigma_1'(23\sigma_1' + 17) - 12a_1(a_1 - 3)\sigma_1'(\sigma_1' + 1) \\
& + 24(a_1^2 - 3a_1 + 3 + \sigma_2)\sigma_2'\Big),
\end{aligned}
$$

$$
r_{n,3}^{X,\mathcal{D}} = (\sigma_1 - \sigma_1' - 4)_2\,(n+1)_3 \frac{(n + \sigma_1 - 3)(n + b_1 - 1)_3}{6(n + \sigma_1)(2n + b_1 - 1)_6},
$$

$$
\begin{aligned}
r_{n,2}^{X,\mathcal{D}} = {}& i(a_1 - a_1^* - a_2 + a_2^*)(\sigma_1 - \sigma_1' - 3)(b_1 - 2)(n+1)_2 \\
& \times \frac{(n + \sigma_1 - 3)(n + \sigma_1' + 2)(n + b_1 - 1)_2}{2(2n + b_1 - 2)_5\,(2n + b_1 + 4)},
\end{aligned}
$$

$$
r_{n,1}^{X,\mathcal{D}} = \frac{(n + \sigma_1 - 3)(n + \sigma_1' + 2)(n + b_1 - 1)}{2(2n + b_1 - 3)_4\,(2n + b_1 + 2)_2} \times A, \tag{3.22}
$$

where A is a polynomial of degree 5 in n and we present it in Appendix D.3, (D.1), because it is a bit long. The polynomial $I(z)$ of degree 4 (3.21) is omitted because it has a lengthy expression.

The example with $\mathcal{D} = \{2^{\mathrm{II}}\}$ $(M_{\mathrm{I}} = 0, M_{\mathrm{II}} = 1)$ and $Y(\eta) = 1$ can be obtained by exchanging a_1 and a_2.

We have also obtained 11-term recurrence relations for $\mathcal{D} = \{4^{\mathrm{I}}\}, \{4^{\mathrm{II}}\}$ with $X(\eta) = X_{\min}(\eta)$. Since the explicit forms of $r_{n,k}^{X,\mathcal{D}}$ are somewhat lengthy, we do not write down them here.

4 Generalized Closure Relations and Creation/Annihilation Operators

In this section we discuss the generalized closure relations and the creation/annihilation operators of the multi-indexed Meixner-Pollaczek and continuous Hahn idQM systems described

by $\mathcal{H}_\mathcal{D}$ (A.19).

First let us recapitulate the essence of the (generalized) closure relation [34]. The closure relation of order K is an algebraic relation between a Hamiltonian $\mathcal{H}$ and some operator X ($= X(\eta(x)) = \check{X}(x)$) [34]:

$$(\operatorname{ad}\mathcal{H})^K X = \sum_{i=0}^{K-1} (\operatorname{ad}\mathcal{H})^i X \cdot R_i(\mathcal{H}) + R_{-1}(\mathcal{H}), \tag{4.1}$$

where $(\operatorname{ad}\mathcal{H})X = [\mathcal{H}, X]$, $(\operatorname{ad}\mathcal{H})^0 X = X$ and $R_i(z) = R_i^X(z)$ is a polynomial in z. The original closure relation [43, 6] corresponds to $K = 2$. Since the closure relation of order K implies that of order $K' > K$, we are interested in the smallest integer K satisfying (4.1). We assume that the matrix $A = (a_{ij})_{1 \le i,j \le K}$ ($a_{i+1,i} = 1$ ($1 \le i \le K - 1$), $a_{i+1,K} = R_i(z)$ ($0 \le i \le K - 1$), $a_{ij} = 0$ (others)) has K distinct real non-vanishing eigenvalues $\alpha_i = \alpha_i(z)$ for $z \ge 0$, which are indexed in decreasing order $\alpha_1(z) > \alpha_2(z) > \cdots > \alpha_K(z)$. Then we obtain the exact Heisenberg solution of X,

$$X_{\mathrm{H}}(t) \stackrel{\text{def}}{=} e^{i\mathcal{H}t} X e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\operatorname{ad}\mathcal{H})^n X = \sum_{j=1}^{K} a^{(j)} e^{i\alpha_j(\mathcal{H})t} - R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1}. \tag{4.2}$$

Here $a^{(j)} = a^{(j)}(\mathcal{H}, X)$ ($1 \le j \le K$) are creation or annihilation operators,

$$a^{(j)} = \left(\sum_{i=1}^{K} (\operatorname{ad}\mathcal{H})^{i-1} X \cdot p_{ij}(\mathcal{H}) + R_{-1}(\mathcal{H})\alpha_j(\mathcal{H})^{-1} \right) \prod_{\substack{k=1 \\ k \ne j}}^{K} (\alpha_j(\mathcal{H}) - \alpha_k(\mathcal{H}))^{-1}, \tag{4.3}$$

where $p_{ij}(z)$ ($1 \le i, j \le K$) are

$$p_{ij}(z) = \alpha_j(z)^{K-i} - \sum_{k=1}^{K-i} R_{K-k}(z)\, \alpha_j(z)^{K-i-k}. \tag{4.4}$$

Let us consider the idQM systems described by the multi-indexed Meixner-Pollaczek and continuous Hahn polynomials. The Hamiltonian is $\mathcal{H}_\mathcal{D}$ (A.19) and a candidate of the operator X is a polynomial $X(\eta(x)) = \check{X}(x)$ discussed in §3.2. The closure relation (4.1) is now

$$(\operatorname{ad}\mathcal{H}_\mathcal{D})^K X = \sum_{i=0}^{K-1} (\operatorname{ad}\mathcal{H}_\mathcal{D})^i X \cdot R_i(\mathcal{H}_\mathcal{D}) + R_{-1}(\mathcal{H}_\mathcal{D}). \tag{4.5}$$

From the form of $\mathcal{H}_\mathcal{D}$, the polynomials $R_i(z) = R_i^X(z)$ have at most the following degrees,

$$R_i(z) = \sum_{j=0}^{K-i} r_i^{(j)} z^j \ \ (0 \le i \le K - 1), \quad R_{-1}(z) = \sum_{j=0}^{K} r_{-1}^{(j)} z^j, \tag{4.6}$$

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where $r_i^{(j)} = r_i^{X(j)}$ are coefficients.

Let us define $\alpha_j(z)$ $(1 \leq j \leq 2L)$ as follows:

$$\text{MP}: \quad \alpha_j(z) = \begin{cases} 2(L+1-j)\sin\phi & (1 \leq j \leq L) \\ -2(j-L)\sin\phi & (L+1 \leq j \leq 2L) \end{cases}, \tag{4.7}$$

$$\text{cH}: \quad \alpha_j(z) = \begin{cases} (L+1-j)^2 + (L+1-j)\sqrt{4z+(b_1-1)^2} & (1 \leq j \leq L) \\ (j-L)^2 - (j-L)\sqrt{4z+(b_1-1)^2} & (L+1 \leq j \leq 2L) \end{cases}. \tag{4.8}$$

For MP, $\alpha_j(z)$'s are constant functions. The pair of $\alpha_j(z)$ and $\alpha_{2L+1-j}(z)$ $(1 \leq j \leq L)$ satisfies

$$\alpha_j(z) + \alpha_{2L+1-j}(z) = \begin{cases} 0 & : \text{MP} \\ 2(L+1-j)^2 & : \text{cH} \end{cases}, \tag{4.9}$$

$$\alpha_j(z)\alpha_{2L+1-j}(z) = \begin{cases} -4(L+1-j)^2 \sin^2\phi & : \text{MP} \\ (L+1-j)^2\big((L+1-j)^2 - 4z - (b_1-1)^2\big) & : \text{cH} \end{cases}. \tag{4.10}$$

These $\alpha_j(z)$ satisfy

$$\alpha_1(z) > \alpha_2(z) > \cdots > \alpha_L(z) > 0 > \alpha_{L+1}(z) > \alpha_{L+2}(z) > \cdots > \alpha_{2L}(z) \quad (z \geq 0), \tag{4.11}$$

for $0 < \phi < \pi$ (MP) and $b_1 > 2L$ (cH). We remark that $\alpha_j(\mathcal{E}_n)$ for cH is square root free, $\sqrt{4\mathcal{E}_n + (b_1-1)^2} = 2n + b_1 - 1$. It is easy to show the following:

$$\alpha_j(\mathcal{E}_n) = \begin{cases} \mathcal{E}_{n+L+1-j} - \mathcal{E}_n > 0 & (1 \leq j \leq L) \\ \mathcal{E}_{n-(j-L)} - \mathcal{E}_n < 0 & (L+1 \leq j \leq 2L) \end{cases}. \tag{4.12}$$

Like the Wilson and Askey-Wilson cases [34], we conjecture the following.

Conjecture 3 *Take $X(\eta)$ as Theorem 2 and take $R_i(z)$ $(-1 \leq i \leq 2L-1)$ as follows:*

$$R_i(z) = (-1)^{i+1} \sum_{1 \leq j_1 < j_2 < \cdots < j_{2L-i} \leq 2L} \alpha_{j_1}(z)\alpha_{j_2}(z)\cdots\alpha_{j_{2L-i}}(z) \quad (0 \leq i \leq 2L-1), \tag{4.13}$$

$$R_{-1}(z) = -I(z), \tag{4.14}$$

where $I(z)$ is given by (3.21). Then the closure relation of order $K = 2L$ (4.5) holds.

We remark that $R_i(z)$ in (4.13) are indeed polynomials in z, because RHS of (4.13) are symmetric under the exchange of α_j and α_{2L+1-j} and their sum and product are polynomials in z, (4.9)–(4.10). For MP, $R_i(z)$ $(0 \leq i \leq 2L-1)$ are constant functions; $R_{2j}(z) = (-1)^{L-j-1}(2\sin\phi)^{2L-2j} \times$ (positive integer) and $R_{2j+1}(z) = 0$ $(0 \leq j \leq L-1)$. Since $R_i(z)$

$(0 \leq i \leq 2L-1)$ are expressed in terms of $\alpha_j(z)$, they do not depend on $\mathcal{D}$ and X (except for $\deg X = L$). Only $R_{-1}(z)$ depends on $\mathcal{D}$ and X. For '$L = 1$ case', namely the original system $(\mathcal{D} = \emptyset, \, \ell_\mathcal{D} = 0, \, \Xi_\mathcal{D}(\eta) = 1, \, X(\eta) = X_{\min}(\eta) = \eta)$, this generalized closure relation reduces to the original closure relation [43]. Direct verification of this conjecture is straightforward for lower M and smaller d_j and $\deg Y$, by a computer algebra system.

If Conjecture 3 is true, we have the exact Heisenberg operator solution $X_\mathrm{H}(t)$ (4.2) and the creation/annihilation operators $a^{(j)} = a^{\mathcal{D},X(j)}$ (4.3). Action of (4.2) on $\phi_{\mathcal{D}\,n}(x)$ (A.22) is

$$e^{i\mathcal{H}_\mathcal{D}t}Xe^{-i\mathcal{H}_\mathcal{D}t}\phi_{\mathcal{D}\,n}(x) = \sum_{j=1}^{2L} e^{i\alpha_j(\mathcal{E}_n)t}a^{(j)}\phi_{\mathcal{D}\,n}(x) - R_{-1}(\mathcal{E}_n)R_0(\mathcal{E}_n)^{-1}\phi_{\mathcal{D}\,n}(x).$$

On the other hand the LHS turns out to be

$$e^{i\mathcal{H}_\mathcal{D}t}Xe^{-i\mathcal{H}_\mathcal{D}t}\phi_{\mathcal{D}\,n}(x) = e^{i\mathcal{H}_\mathcal{D}t}Xe^{-i\mathcal{E}_nt}\phi_{\mathcal{D}\,n}(x) = e^{-i\mathcal{E}_nt}e^{i\mathcal{H}_\mathcal{D}t}\sum_{k=-L}^{L} r_{n,k}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+k}(x)$$

$$= \sum_{k=-L}^{L} e^{i(\mathcal{E}_{n+k}-\mathcal{E}_n)t}\, r_{n,k}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+k}(x),$$

where we have used (3.15). Comparing these t-dependence (we assume $b_1 > 2L$ for cH), we obtain (4.12) and

$$a^{(j)}\phi_{\mathcal{D}\,n}(x) = \begin{cases} r_{n,L+1-j}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n+L+1-j}(x) & (1 \leq j \leq L) \\ r_{n,-(j-L)}^{X,\mathcal{D}}\phi_{\mathcal{D}\,n-(j-L)}(x) & (L+1 \leq j \leq 2L) \end{cases}, \tag{4.15}$$

$$- R_{-1}(\mathcal{E}_n)R_0(\mathcal{E}_n)^{-1} = r_{n,0}^{X,\mathcal{D}}, \tag{4.16}$$

where $r_{n,k}^{X,\mathcal{D}} = 0$ for $n + k < 0$. Note that (4.16) is consistent with Conjecture 2. Therefore $a^{(j)}$ $(1 \leq j \leq L)$ and $a^{(j)}$ $(L + 1 \leq j \leq 2L)$ are creation and annihilation operators, respectively. Among them, $a^{(L)}$ and $a^{(L+1)}$ are fundamental, $a^{(L)}\phi_{\mathcal{D},n}(x) \propto \phi_{\mathcal{D}\,n+1}(x)$ and $a^{(L+1)}\phi_{\mathcal{D},n}(x) \propto \phi_{\mathcal{D}\,n-1}(x)$. Furthermore, $X = X_{\min}$ case is the most basic.

By the similarity transformation (see (A.35)), the closure relation (4.5) becomes

$$(\operatorname{ad}\widetilde{\mathcal{H}}_\mathcal{D})^K X = \sum_{i=0}^{K-1}(\operatorname{ad}\widetilde{\mathcal{H}}_\mathcal{D})^i X \cdot R_i(\widetilde{\mathcal{H}}_\mathcal{D}) + R_{-1}(\widetilde{\mathcal{H}}_\mathcal{D}), \tag{4.17}$$

and the creation/annihilation operators for eigenpolynomials can be obtained,

$$\tilde{a}^{(j)} \stackrel{\text{def}}{=} \psi_\mathcal{D}(x)^{-1} \circ a^{(j)}(\mathcal{H}_\mathcal{D}, X) \circ \psi_\mathcal{D}(x) = a^{(j)}(\widetilde{\mathcal{H}}_\mathcal{D}, X), \tag{4.18}$$

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$$\tilde{a}^{(j)}\check{P}_{\mathcal{D},n}(x) = \begin{cases} r^{X,\mathcal{D}}_{n,L+1-j}\check{P}_{\mathcal{D},n+L+1-j}(x) & (1 \le j \le L) \\ r^{X,\mathcal{D}}_{n,-(j-L)}\check{P}_{\mathcal{D},n-(j-L)}(x) & (L+1 \le j \le 2L) \end{cases}. \tag{4.19}$$

Remark Since the closure relations (4.5) (or (4.17)) is an algebraic relation between $\mathcal{H}_{\mathcal{D}}$ (or $\widetilde{\mathcal{H}}_{\mathcal{D}}$) and X, it is expected to hold even when $\mathcal{H}_{\mathcal{D}}$ (or $\widetilde{\mathcal{H}}_{\mathcal{D}}$) is singular. Especially we conjecture that $\tilde{a}^{(j)}$ (4.18) are the creation/annihilation operators for eigenpolynomials $\check{P}_{\mathcal{D},n}(x)$ with any $\mathcal{D}$, (4.19), which can be verified by direct calculation for small d_j, n and deg Y.

For an illustration, we present an example. Let us consider Ex.1 in § 3.3.1. The denominator polynomial $\Xi_{\mathcal{D}}(\eta)$ is

$$\Xi_{\mathcal{D}}(\eta) = (1 - \cos 2\phi)\eta^2 - (2a - 3)\sin 2\phi \cdot \eta + (a-1)\big(a - 1 + (a-2)\cos 2\phi\big),$$

and $R_i(z)$ are

$$R_5(z) = R_3(z) = R_1(z) = 0,$$
$$R_4(z) = 56\sin^2\phi, \quad R_2(z) = -784\sin^4\phi, \quad R_0(z) = 2304\sin^6\phi,$$
$$R_{-1}(z) = 48\sin\phi\,\sin 2\phi\Big((4 + \cos 2\phi)z^3 + 6\big(6a - 5 + 2(a-1)\cos 2\phi\big)\sin\phi \cdot z^2$$
$$+ 4\big(4(6a^2 - 9a + 2) + (12a^2 - 24a + 11)\cos 2\phi\big)\sin^2\phi \cdot z$$
$$+ (2a-3)(2a-1)\big(14a + 7 + (14a - 11)\cos 2\phi\big)\sin^3\phi\Big).$$

The creation/annihilation operators for eigenpolynomials $\tilde{a}^{(j)}$ are

$$\tilde{a}^{(1)} = \frac{1}{7680}\Big(384X + \frac{64}{\sin\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})X - \frac{120}{\sin^2\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^2X - \frac{20}{\sin^3\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^3X$$
$$+ \frac{6}{\sin^4\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^4X + \frac{1}{\sin^5\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^5X + \frac{1}{6\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big),$$
$$\tilde{a}^{(2)} = \frac{1}{1920}\Big(-576X - \frac{144}{\sin\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})X + \frac{160}{\sin^2\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^2X + \frac{40}{\sin^3\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^3X$$
$$- \frac{4}{\sin^4\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^4X - \frac{1}{\sin^5\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^5X - \frac{1}{4\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big),$$
$$\tilde{a}^{(3)} = \frac{1}{1536}\Big(1152X + \frac{576}{\sin\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})X - \frac{104}{\sin^2\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^2X - \frac{52}{\sin^3\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^3X$$
$$+ \frac{2}{\sin^4\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^4X + \frac{1}{\sin^5\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^5X + \frac{1}{2\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big),$$
$$\tilde{a}^{(4)} = \frac{1}{1536}\Big(1152X - \frac{576}{\sin\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})X - \frac{104}{\sin^2\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^2X + \frac{52}{\sin^3\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^3X$$
$$+ \frac{2}{\sin^4\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^4X - \frac{1}{\sin^5\phi}(\mathrm{ad}\,\widetilde{\mathcal{H}})^5X + \frac{1}{2\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big),$$

$$\tilde{a}^{(5)} = \frac{1}{1920}\Big(-576X + \frac{144}{\sin\phi}(\text{ad }\widetilde{\mathcal{H}})X + \frac{160}{\sin^2\phi}(\text{ad }\widetilde{\mathcal{H}})^2X - \frac{40}{\sin^3\phi}(\text{ad }\widetilde{\mathcal{H}})^3X$$
$$- \frac{4}{\sin^4\phi}(\text{ad }\widetilde{\mathcal{H}})^4X + \frac{1}{\sin^5\phi}(\text{ad }\widetilde{\mathcal{H}})^5X - \frac{1}{4\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big),$$
$$\tilde{a}^{(6)} = \frac{1}{7680}\Big(384X - \frac{64}{\sin\phi}(\text{ad }\widetilde{\mathcal{H}})X - \frac{120}{\sin^2\phi}(\text{ad }\widetilde{\mathcal{H}})^2X + \frac{20}{\sin^3\phi}(\text{ad }\widetilde{\mathcal{H}})^3X$$
$$+ \frac{6}{\sin^4\phi}(\text{ad }\widetilde{\mathcal{H}})^4X - \frac{1}{\sin^5\phi}(\text{ad }\widetilde{\mathcal{H}})^5X + \frac{1}{6\sin^6\phi}R_{-1}(\widetilde{\mathcal{H}})\Big).$$

Here $(\text{ad }\widetilde{\mathcal{H}})^iX$ are

$$(\text{ad }\widetilde{\mathcal{H}})X = V'_{\mathcal{D}}(x)\big(X(x-i\gamma)-X(x)\big)e^{\gamma p} + V'^*_{\mathcal{D}}(x)\big(X(x+i\gamma)-X(x)\big)e^{-\gamma p},$$
$$(\text{ad }\widetilde{\mathcal{H}})^2X = V'_{\mathcal{D}}(x)V'_{\mathcal{D}}(x-i\gamma)\big(X(x)-2X(x-i\gamma)+X(x-2i\gamma)\big)e^{2\gamma p}$$
$$+ V'_{\mathcal{D}}(x)\big(V_{\mathcal{D}}(x)+V^*_{\mathcal{D}}(x)-V_{\mathcal{D}}(x-i\gamma)-V^*_{\mathcal{D}}(x-i\gamma)\big)\big(X(x)-X(x-i\gamma)\big)e^{\gamma p}$$
$$+ 2V'_{\mathcal{D}}(x)V'^*_{\mathcal{D}}(x-i\gamma)\big(X(x)-X(x-i\gamma)\big) + 2V'^*_{\mathcal{D}}(x)V'_{\mathcal{D}}(x+i\gamma)\big(X(x)-X(x+i\gamma)\big)$$
$$+ V'^*_{\mathcal{D}}(x)\big(V_{\mathcal{D}}(x)+V^*_{\mathcal{D}}(x)-V_{\mathcal{D}}(x+i\gamma)-V^*_{\mathcal{D}}(x+i\gamma)\big)\big(X(x)-X(x+i\gamma)\big)e^{-\gamma p}$$
$$+ V'^*_{\mathcal{D}}(x)V'^*_{\mathcal{D}}(x+i\gamma)\big(X(x)-2X(x+i\gamma)+X(x+2i\gamma)\big)e^{-2\gamma p},$$

and so on (we omit them because they are somewhat lengthy). By direct calculation, the relations (4.19) can be verified for small n.

5   Summary and Comments

Following the preceding papers on the case-(1) multi-indexed orthogonal polynomials (Laguerre and Jacobi cases in oQM [28, 31, 33], Wilson and Askey-Wilson cases in idQM [28, 31, 35] and Racah and q-Racah cases in rdQM [35]), we have discussed the recurrence relations for the case-(1) multi-indexed Meixner-Pollaczek and continuous Hahn orthogonal polynomials in idQM, whose physical range of the coordinate is the whole real line. The $3+2M$ term recurrence relations with variable dependent coefficients (3.6) (Theorem 1) provide an efficient method to calculate the multi-indexed MP and cH polynomials. The $1+2L$ term ($L \geq M+1$) recurrence relations with constant coefficients (3.15) (Theorem 2), and their examples are presented. Since $Y(\eta)$ is arbitrary, we obtain infinitely many recurrence relations. Not all of them are independent, but the relations among them are unclear. To clarify their relations is an important problem. Corresponding to the recurrence relations with constant coefficients, the idQM systems described by the multi-indexed MP and cH

orthogonal polynomials satisfy the generalized closure relations (4.5) (Conjecture 3), from which the creation and annihilation operators are obtained. There are many creation and annihilation operators and it is an interesting problem to study their relations.

The Hamiltonian of the deformed system is determined by the denominator polynomial $\Xi_{\mathcal{D}}(\eta)$, whose degree is $\ell_{\mathcal{D}}$ (A.26). There is no restriction on $\ell_{\mathcal{D}}$ for L, J, W and AW cases, whereas the degree $\ell_{\mathcal{D}}$ must be even for MP and cH cases, in order that the deformed Hamiltonian is hermitian. This is because the physical range of the coordinate of the deformed MP and cH systems is the whole real line, see (A.27). The range of the coordinate of the harmonic oscillator, whose eigenstates are described by the Hermite polynomial, is also the whole real line, but the case-(1) multi-indexed Hermite orthogonal polynomials do not exist. In §2.2.1 we have defined the multi-indexed MP and cH polynomials for any index set $\mathcal{D}$, namely $\ell_{\mathcal{D}}$ may be odd and they may not be orthogonal polynomials. The recurrence relations with variable dependent coefficients (3.6) for the multi-indexed MP and cH polynomials hold even for non-orthogonal case. We conjecture that the recurrence relations with constant coefficients (3.15), the generalized closure relations (4.17) and the creation/annihilation operators (4.19) also hold even for non-orthogonal case.

Acknowledgments

A   Discrete Quantum Mechanics With Pure Imaginary Shifts and Deformed Systems

In this Appendix we recapitulate the discrete quantum mechanics with pure imaginary shifts (idQM) and deformed systems [2, 5, 17, 36].

The dynamical variables of idQM are the real coordinate x ($x_1 < x < x_2$) and the conjugate momentum $p = -i\partial_x$, which are governed by the following factorized positive semi-definite Hamiltonian:

$$\mathcal{H} \overset{\text{def}}{=} \sqrt{V(x)}\, e^{\gamma p} \sqrt{V^*(x)} + \sqrt{V^*(x)}\, e^{-\gamma p} \sqrt{V(x)} - V(x) - V^*(x) = \mathcal{A}^\dagger \mathcal{A}, \qquad (\text{A.1})$$

$$\mathcal{A} \overset{\text{def}}{=} i\big(e^{\frac{\gamma}{2}p}\sqrt{V^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V(x)}\big), \quad \mathcal{A}^\dagger \overset{\text{def}}{=} -i\big(\sqrt{V(x)}\, e^{\frac{\gamma}{2}p} - \sqrt{V^*(x)}\, e^{-\frac{\gamma}{2}p}\big). \qquad (\text{A.2})$$

Here the potential function $V(x)$ is an analytic function of x and γ is a real constant. The $*$-operation on an analytic function $f(x) = \sum_n a_n x^n$ ($a_n \in \mathbb{C}$) is defined by $f^*(x) = \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n. Since the momentum operator appears in exponentiated forms, the Schrödinger equation

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n \phi_n(x) \quad (n = 0, 1, 2, \ldots), \tag{A.3}$$

is an analytic difference equation with pure imaginary shifts instead of a differential equation. We consider those systems which have a square-integrable groundstate together with an infinite number of discrete energy levels: $0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots$. The orthogonality relation reads

$$(\phi_n, \phi_m) \stackrel{\text{def}}{=} \int_{x_1}^{x_2} dx\, \phi_n^*(x)\phi_m(x) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \ldots), \quad 0 < h_n < \infty. \tag{A.4}$$

The eigenfunctions $\phi_n(x)$ can be chosen 'real', $\phi_n^*(x) = \phi_n(x)$, and the groundstate wavefunction $\phi_0(x)$ is determined as the zero mode of the operator $\mathcal{A}$, $\mathcal{A}\phi_0(x) = 0$. The norm of a function $f(x)$ is $\|f\| \stackrel{\text{def}}{=} (f, f)^{\frac{1}{2}}$.

The Hamiltonian $\mathcal{H}$ should be hermitian. From its form $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$, it is formally hermitian, $\mathcal{H}^\dagger = (\mathcal{A}^\dagger \mathcal{A})^\dagger = (\mathcal{A})^\dagger (\mathcal{A}^\dagger)^\dagger = \mathcal{A}^\dagger \mathcal{A} = \mathcal{H}$. However, the true hermiticity is defined in terms of the inner product, $(f_1, \mathcal{H}f_2) = (\mathcal{H}f_1, f_2)$ [2, 41, 17]. To show the hermiticity of $\mathcal{H}$, singularities of some functions in the rectangular domain D_γ are important. Here D_γ is defined by [17]

$$D_\gamma \stackrel{\text{def}}{=} \left\{ x \in \mathbb{C} \mid x_1 \leq \operatorname{Re} x \leq x_2, |\operatorname{Im} x| \leq \tfrac{1}{2}|\gamma| \right\}. \tag{A.5}$$

The Meixner-Pollaczek (MP), continuous Hahn (cH), Wilson (W), Askey-Wilson (AW) polynomials etc. are members of the Askey-scheme of the (basic) hypergeometric orthogonal polynomials and satisfy the second order analytic difference equation with pure imaginary shifts [1]. These orthogonal polynomials can be studied in the framework of idQM, in which they appear as part of the eigenfunction as follows:

$$\phi_n(x) = \phi_0(x)\check{P}_n(x), \quad \check{P}_n(x) \stackrel{\text{def}}{=} P_n\big(\eta(x)\big) \quad (n = 0, 1, 2, \ldots). \tag{A.6}$$

Here $\eta(x)$ is a sinusoidal coordinate [43, 41] and $P_n(\eta)$ is a orthogonal polynomial of degree n in η. The orthogonality relation (A.4) gives that of $\check{P}_n(x)$,

$$\int_{x_1}^{x_2} dx\, \phi_0(x)^2 \check{P}_n(x)\check{P}_m(x) = h_n \delta_{nm} \quad (n, m = 0, 1, 2, \ldots). \tag{A.7}$$

We call this idQM system by the name of the orthogonal polynomial: MP system, cH system, W system, AW system etc. These idQM systems have the property of shape invariance, which is a sufficient condition for exact solvability. Concrete idQM systems have a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$. Various quantities depend on them and their dependence is expressed like, $f = f(\boldsymbol{\lambda})$, $f(x) = f(x; \boldsymbol{\lambda})$. (We sometimes omit writing $\boldsymbol{\lambda}$-dependence, when it does not cause confusion.)

The shape invariant condition is the following [2, 41, 5]:

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^{\dagger} = \kappa\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta})^{\dagger}\mathcal{A}(\boldsymbol{\lambda}+\boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}), \tag{A.8}$$

where κ is a real positive constant and $\boldsymbol{\delta}$ is the shift of the parameters. This condition combined with the Crum's theorem allows the wavefunction $\phi_n(x)$ and energy eigenvalue $\mathcal{E}_n$ of the excited states to be expressed in terms of the ground state wavefunction $\phi_0(x)$ and the first excited state energy eigenvalue $\mathcal{E}_1$ with shifted parameters. As a consequence of the shape invariance, we have

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\phi_{n-1}(x; \boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{A}(\boldsymbol{\lambda})^{\dagger}\phi_{n-1}(x; \boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}), \tag{A.9}$$

where $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are some constants satisfying $f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})$. These relations can be rewritten as the forward and backward shift relations:

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda}+\boldsymbol{\delta}), \quad \mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}). \tag{A.10}$$

Here the forward and backward shift operators $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$ are defined by

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) = i\varphi(x)^{-1}(e^{\frac{\gamma}{2}p} - e^{-\frac{\gamma}{2}p}), \tag{A.11}$$

$$\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^{\dagger} \circ \phi_0(x; \boldsymbol{\lambda}+\boldsymbol{\delta}) = -i\big(V(x; \boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - V^*(x; \boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\big)\varphi(x), \tag{A.12}$$

where $\varphi(x)$ is an auxiliary function $(\varphi(x) \propto \eta(x - i\frac{\gamma}{2}) - \eta(x + i\frac{\gamma}{2}))$. The difference operator $\widetilde{\mathcal{H}}$ acting on the polynomial eigenfunctions is square root free:

$$\widetilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = \mathcal{B}\mathcal{F}$$

$$= V(x)(e^{\gamma p} - 1) + V^*(x)(e^{-\gamma p} - 1), \tag{A.13}$$

$$\widetilde{\mathcal{H}}\check{P}_n(x) = \mathcal{E}_n\check{P}_n(x) \quad (n = 0, 1, 2, \ldots). \tag{A.14}$$

By the Darboux transformation, we can deform idQM systems keeping their exact solvability. The multi-step Darboux transformations with virtual state wavefunctions as seed

solutions give iso-spectral deformations and the case-(1) multi-indexed orthogonal polynomials are obtained [17, 36]. The virtual state wavefunctions are obtained by using the twist operation. The twist operation $\mathfrak{t}$ is a map for parameters $\boldsymbol{\lambda}$ and gives a linear relation between two Hamiltonians:

$$\mathcal{H}(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\lambda})\mathcal{H}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \alpha'(\boldsymbol{\lambda}), \tag{A.15}$$

where α and α' are constants. The constant $\tilde{\boldsymbol{\delta}}$ is introduced as $\mathfrak{t}(\boldsymbol{\lambda} + \beta\boldsymbol{\delta}) = \mathfrak{t}(\boldsymbol{\lambda}) + \beta\tilde{\boldsymbol{\delta}}$ ($\forall \beta \in \mathbb{R}$). There are two types of twist operations (type I and II) for cH, W and AW systems, and one type of twist operation for MP system. The virtual state wavefunctions $\tilde{\phi}_{\mathrm{v}}(x)$ are obtained from the eigenfunction $\phi_n(x)$ (A.6) as follows:

$$\tilde{\phi}_{\mathrm{v}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_{\mathrm{v}}\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big) = \phi_0\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big)\check{P}_{\mathrm{v}}\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big),$$
$$\tilde{\xi}_{\mathrm{v}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \xi_{\mathrm{v}}\big(\eta(x); \boldsymbol{\lambda}\big) \stackrel{\text{def}}{=} \check{P}_{\mathrm{v}}\big(x; \mathfrak{t}(\boldsymbol{\lambda})\big) = P_{\mathrm{v}}\big(\eta(x); \mathfrak{t}(\boldsymbol{\lambda})\big), \tag{A.16}$$

which satisfies the Schrödinger equation $\mathcal{H}\tilde{\phi}_{\mathrm{v}}(x) = \tilde{\mathcal{E}}_{\mathrm{v}}\tilde{\phi}_{\mathrm{v}}(x)$ with the virtual state energy $\tilde{\mathcal{E}}_{\mathrm{v}}$,

$$\tilde{\mathcal{E}}_{\mathrm{v}}(\boldsymbol{\lambda}) = \alpha(\boldsymbol{\lambda})\mathcal{E}_{\mathrm{v}}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) + \alpha'(\boldsymbol{\lambda}). \tag{A.17}$$

The Hamiltonian is deformed as $\mathcal{H} \to \mathcal{H}_{d_1} \to \mathcal{H}_{d_1 d_2} \to \cdots \to \mathcal{H}_{d_1 \ldots d_s} \to \cdots \to \mathcal{H}_{d_1 \ldots d_M} = \mathcal{H}_{\mathcal{D}}$ by M-step Darboux transformations with virtual state wavefunctions as seed solutions. Here the index set $\mathcal{D} = \{d_1, \ldots, d_M\}$ (d_j: mutually distinct) labels the virtual state wavefunctions used in the transformations. Exactly speaking, $\mathcal{D}$ is an ordered set. For cH, W and AW systems, there are two types of virtual states (type I and II) and $\mathcal{D}$ is $\mathcal{D} = \{d_1, \ldots, d_M\} = \{d_1^{\mathrm{I}}, \ldots, d_{M_{\mathrm{I}}}^{\mathrm{I}}, d_1^{\mathrm{II}}, \ldots, d_{M_{\mathrm{II}}}^{\mathrm{II}}\}$ ($M = M_{\mathrm{I}} + M_{\mathrm{II}}$, d_j^{I} : mutually distinct, d_j^{II} : mutually distinct). Various quantities of the deformed systems are denoted as $\mathcal{H}_{\mathcal{D}}$, $\phi_{\mathcal{D}n}$, $\mathcal{A}_{\mathcal{D}}$, etc.

The Schrödinger equation of the deformed system is

$$\mathcal{H}_{\mathcal{D}}\phi_{\mathcal{D}n}(x) = \mathcal{E}_n\phi_{\mathcal{D}n}(x) \quad (n = 0, 1, 2, \ldots). \tag{A.18}$$

The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ and eigenfunctions $\phi_{\mathcal{D}n}(x)$ are given by

$$\mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \sqrt{V_{\mathcal{D}}(x)}\, e^{\gamma p}\sqrt{V_{\mathcal{D}}^*(x)} + \sqrt{V_{\mathcal{D}}^*(x)}\, e^{-\gamma p}\sqrt{V_{\mathcal{D}}(x)} - V_{\mathcal{D}}(x) - V_{\mathcal{D}}^*(x) = \mathcal{A}_{\mathcal{D}}^{\dagger}\mathcal{A}_{\mathcal{D}}, \tag{A.19}$$

$$\mathcal{A}_{\mathcal{D}} \stackrel{\text{def}}{=} i\big(e^{\frac{\gamma}{2}p}\sqrt{V_{\mathcal{D}}^*(x)} - e^{-\frac{\gamma}{2}p}\sqrt{V_{\mathcal{D}}(x)}\big), \quad \mathcal{A}_{\mathcal{D}}^{\dagger} \stackrel{\text{def}}{=} -i\big(\sqrt{V_{\mathcal{D}}(x)}\, e^{\frac{\gamma}{2}p} - \sqrt{V_{\mathcal{D}}^*(x)}\, e^{-\frac{\gamma}{2}p}\big), \tag{A.20}$$

$$V_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x; \boldsymbol{\lambda}')\frac{\check{\Xi}_{\mathcal{D}}(x + i\frac{\gamma}{2}; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x - i\frac{\gamma}{2}; \boldsymbol{\lambda})}\frac{\check{\Xi}_{\mathcal{D}}(x - i\gamma; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}, \tag{A.21}$$

$$\phi_{\mathcal{D}\,n}(x) \stackrel{\text{def}}{=} A\psi_{\mathcal{D}}(x)\check{P}_{\mathcal{D},n}(x), \quad \psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x;\boldsymbol{\lambda}')}{\sqrt{\check{\Xi}_{\mathcal{D}}(x - i\frac{\gamma}{2};\boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x + i\frac{\gamma}{2};\boldsymbol{\lambda})}}, \tag{A.22}$$

where A and $\boldsymbol{\lambda}'$ are

$$A = \begin{cases} \kappa^{-\frac{1}{4}M(M+1)}\alpha(\boldsymbol{\lambda}')^{\frac{1}{2}M} & : \text{MP} \\ \kappa^{-\frac{1}{4}M_{\text{I}}(M_{\text{I}}+1)-\frac{1}{4}M_{\text{II}}(M_{\text{II}}+1)+\frac{5}{2}M_{\text{I}}M_{\text{II}}}\alpha^{\text{I}}(\boldsymbol{\lambda}')^{\frac{1}{2}M_{\text{I}}}\alpha^{\text{II}}(\boldsymbol{\lambda}')^{\frac{1}{2}M_{\text{II}}} & : \text{cH,W,AW} \end{cases}, \tag{A.23}$$

$$\boldsymbol{\lambda}' = \begin{cases} \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}} & : \text{MP} \\ \boldsymbol{\lambda}^{[M_{\text{I}},M_{\text{II}}]} \stackrel{\text{def}}{=} \boldsymbol{\lambda} + M_{\text{I}}\tilde{\boldsymbol{\delta}}^{\text{I}} + M_{\text{II}}\tilde{\boldsymbol{\delta}}^{\text{II}} & : \text{cH,W,AW} \end{cases}. \tag{A.24}$$

Note that $A = 1$ for MP, cH and W. Here $\check{\Xi}_{\mathcal{D}}(x)$ and $\check{P}_{\mathcal{D},n}(x)$ are polynomials in $\eta(x)$,

$$\check{\Xi}_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \Xi_{\mathcal{D}}\big(\eta(x)\big), \quad \check{P}_{\mathcal{D},n}(x) \stackrel{\text{def}}{=} P_{\mathcal{D},n}\big(\eta(x)\big), \tag{A.25}$$

and their explicit forms are given in [17, 36]. The denominator polynomial $\Xi_{\mathcal{D}}(\eta)$ and the multi-indexed polynomial $P_{\mathcal{D},n}(\eta)$ are polynomials in η and their degrees are $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}} + n$, respectively. Here $\ell_{\mathcal{D}}$ is

$$\ell_{\mathcal{D}} \stackrel{\text{def}}{=} \sum_{j=1}^{M} d_j - \frac{1}{2}M(M-1) + \begin{cases} 0 & : \text{MP} \\ 2M_{\text{I}}M_{\text{II}} & : \text{cH,W,AW} \end{cases}. \tag{A.26}$$

The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ is hermitian, if the following condition is satisfied [17, 36]:

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ has no zero in D_{γ} (A.5). (A.27)

The eigenfunctions $\phi_{\mathcal{D}\,n}(x)$ are orthogonal each other, which gives the orthogonality relation of $\check{P}_{\mathcal{D},n}(x)$:

$$\int_{x_1}^{x_2} dx\, \psi_{\mathcal{D}}(x)^2 \check{P}_{\mathcal{D},n}(x)\check{P}_{\mathcal{D},m}(x) = h_{\mathcal{D},n}\delta_{nm} \quad (n, m = 0, 1, 2, \ldots), \tag{A.28}$$

$$h_{\mathcal{D},n} = A^{-2}h_n \times \begin{cases} \prod_{j=1}^{M}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) & : \text{MP} \\ \prod_{j=1}^{M_{\text{I}}}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j^{\text{I}}}^{\text{I}}) \cdot \prod_{j=1}^{M_{\text{II}}}(\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j^{\text{II}}}^{\text{II}}) & : \text{cH,W,AW} \end{cases}, \tag{A.29}$$

where A is given by (A.23). The multi-indexed orthogonal polynomial $P_{\mathcal{D},n}(\eta)$ has n zeros in the physical region ($\eta(x_1) < \eta < \eta(x_2)$ for MP, cH, W, $\eta(x_2) < \eta < \eta(x_1)$ for AW), which interlace the $n + 1$ zeros of $P_{\mathcal{D},n+1}(\eta)$ in the physical region, and $\ell_{\mathcal{D}}$ zeros in the unphysical region ($\eta \in \mathbb{C}\backslash\{\text{physical region of } \eta\}$).

The shape invariance of the original system is inherited by the deformed systems. By the argument of [17], the Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})$ is shape invariant:

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger} = \kappa\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta})^{\dagger}\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \tag{A.30}$$

As a consequence of the shape invariance, the actions of $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}$ on the eigen-functions $\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda})$ are

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}) = \kappa^{\frac{M}{2}}f_n(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),$$
$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger}\phi_{\mathcal{D}\,n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = \kappa^{-\frac{M}{2}}b_{n-1}(\boldsymbol{\lambda})\phi_{\mathcal{D}\,n}(x;\boldsymbol{\lambda}). \tag{A.31}$$

The forward and backward shift operators are defined by

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}\,(x;\boldsymbol{\lambda}+\boldsymbol{\delta})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}\,(x;\boldsymbol{\lambda})$$
$$= \frac{i}{\varphi(x)\breve{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})}\Big(\breve{\Xi}_{\mathcal{D}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda}+\boldsymbol{\delta})e^{\frac{\gamma}{2}p} - \breve{\Xi}_{\mathcal{D}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda}+\boldsymbol{\delta})e^{-\frac{\gamma}{2}p}\Big), \tag{A.32}$$

$$\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}\,(x;\boldsymbol{\lambda})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^{\dagger} \circ \psi_{\mathcal{D}}\,(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) \tag{A.33}$$
$$= \frac{-i}{\breve{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\Big(V(x;\boldsymbol{\lambda}')\breve{\Xi}_{\mathcal{D}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} - V^*(x;\boldsymbol{\lambda}')\breve{\Xi}_{\mathcal{D}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\Big)\varphi(x),$$

($\boldsymbol{\lambda}'$ is given by (A.24)) and their actions on $\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})$ are

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}),$$
$$\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda}). \tag{A.34}$$

The similarity transformed Hamiltonian is square root free:

$$\widetilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})$$
$$= V(x;\boldsymbol{\lambda}')\frac{\breve{\Xi}_{\mathcal{D}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}{\breve{\Xi}_{\mathcal{D}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\left(e^{\gamma p} - \frac{\breve{\Xi}_{\mathcal{D}}(x-i\gamma;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\breve{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\right)$$
$$+ V^*(x;\boldsymbol{\lambda}')\frac{\breve{\Xi}_{\mathcal{D}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})}{\breve{\Xi}_{\mathcal{D}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}\left(e^{-\gamma p} - \frac{\breve{\Xi}_{\mathcal{D}}(x+i\gamma;\boldsymbol{\lambda}+\boldsymbol{\delta})}{\breve{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}\right). \tag{A.35}$$

By defining $V_{\mathcal{D}}'(x)$ as

$$V_{\mathcal{D}}'(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} V(x;\boldsymbol{\lambda}')\frac{\breve{\Xi}_{\mathcal{D}}(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})}{\breve{\Xi}_{\mathcal{D}}(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})}, \tag{A.36}$$

it is written as

$$\widetilde{\mathcal{H}}_{\mathcal{D}} = V_{\mathcal{D}}'(x)e^{\gamma p} + V_{\mathcal{D}}'^*(x)e^{-\gamma p} - V_{\mathcal{D}}(x) - V_{\mathcal{D}}^*(x). \tag{A.37}$$

The multi-indexed orthogonal polynomials $\check{P}_{\mathcal{D},n}(x)$ are its eigenpolynomials:

$$\widetilde{\mathcal{H}}_{\mathcal{D}}\check{P}_{\mathcal{D},n}(x) = \mathcal{E}_n\check{P}_{\mathcal{D},n}(x) \quad (n = 0, 1, 2, \ldots). \tag{A.38}$$

B Some Properties of the Multi-indexed Meixner-Pollaczek Polynomials

We present some properties of the multi-indexed Meixner-Pollaczek polynomials [36].

- coefficients of the highest degree terms :

$$\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) = c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})\eta^{\ell_{\mathcal{D}}} + \text{(lower order terms)},$$

$$c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) = \prod_{j=1}^{M} c_{d_j}\big(\mathfrak{t}(\boldsymbol{\lambda})\big) \cdot \prod_{1 \le j < k \le M} (d_k - d_j), \tag{B.1}$$

$$P_{\mathcal{D}}(\eta;\boldsymbol{\lambda}) = c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})\eta^{\ell_{\mathcal{D}}+n} + \text{(lower order terms)},$$

$$c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda}) = c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda}) \prod_{j=1}^{M} (-2a - n + d_j + 1). \tag{B.2}$$

- $\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda})$ vs $\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda})$:

$$\check{P}_{\mathcal{D},0}(x;\boldsymbol{\lambda}) = A\,\check{\Xi}_{\mathcal{D}}(x;\boldsymbol{\lambda}+\boldsymbol{\delta}), \quad A = \prod_{j=1}^{M} (-2a + d_j + 1). \tag{B.3}$$

- $d_j = 0$ case :

$$\check{P}_{\mathcal{D},n}(x;\boldsymbol{\lambda})\Big|_{d_M=0} = A\,\check{P}_{\mathcal{D}',n}(x;\boldsymbol{\lambda}+\tilde{\boldsymbol{\delta}}), \quad \mathcal{D}' = \{d_1 - 1, \ldots, d_{M-1} - 1\},$$

$$A = (-1)^M (2a + n - 1)(2\sin\phi)^{M-1}. \tag{B.4}$$

- values at special points :

Let x_0 and η_0 be

$$x_0 \stackrel{\text{def}}{=} -i\gamma(a - \tfrac{1}{2}M), \quad \eta_0 \stackrel{\text{def}}{=} \eta(x_0). \tag{B.5}$$

Note that, as coordinates x and η, these values x_0 and η_0 are unphysical (they are imaginary). The multi-indexed polynomials $P_{\mathcal{D},n}(\eta)$ take 'simple' values at these 'unphysical' values η_0:

$$P_{\mathcal{D},n}(\eta_0;\boldsymbol{\lambda}) = c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})e^{i\phi(\ell_{\mathcal{D}}-n)}\frac{(2a)_n}{(2\sin\phi)^{\ell_{\mathcal{D}}+n}} \prod_{j=1}^{M} \frac{(1-2a)_{d_j}}{(1-2a)_{j-1}} \cdot \prod_{j=1}^{M} \frac{d_j + 1 - 2a}{d_j + 1 - n - 2a}, \tag{B.6}$$

where we have assumed $0 \le d_1 < \cdots < d_M$.

C Some Properties of the Multi-indexed Continuous Hahn Polynomials

We present some properties of the multi-indexed continuous Hahn polynomials [36].

- coefficients of the highest degree terms :

$$\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) = c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})\eta^{\ell_{\mathcal{D}}} + \text{(lower order terms)},$$

$$c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) = \prod_{j=1}^{M_{\mathrm{I}}} c_{d_j^{\mathrm{I}}}\big(\mathsf{t}^{\mathrm{I}}(\boldsymbol{\lambda})\big) \cdot \prod_{j=1}^{M_{\mathrm{II}}} c_{d_j^{\mathrm{II}}}\big(\mathsf{t}^{\mathrm{II}}(\boldsymbol{\lambda})\big) \cdot \prod_{1 \leq j < k \leq M_{\mathrm{I}}} (d_k^{\mathrm{I}} - d_j^{\mathrm{I}}) \cdot \prod_{1 \leq j < k \leq M_{\mathrm{II}}} (d_k^{\mathrm{II}} - d_j^{\mathrm{II}})$$

$$\times \prod_{j=1}^{M_{\mathrm{I}}} \prod_{k=1}^{M_{\mathrm{II}}} (-a_2 - a_2^* - d_j^{\mathrm{I}} + a_1 + a_1^* + d_k^{\mathrm{II}}), \tag{C.1}$$

$$P_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) = c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda})\eta^{\ell_{\mathcal{D}}+n} + \text{(lower order terms)},$$

$$c_{\mathcal{D},n}^{P}(\boldsymbol{\lambda}) = c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda}) \prod_{j=1}^{M_{\mathrm{I}}}(-a_1 - a_1^* - n + d_j^{\mathrm{I}} + 1) \cdot \prod_{j=1}^{M_{\mathrm{II}}}(-a_2 - a_2^* - n + d_j^{\mathrm{II}} + 1). \tag{C.2}$$

- $\check{P}_{\mathcal{D},0}(x; \boldsymbol{\lambda})$ vs $\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})$:

$$\check{P}_{\mathcal{D},0}(x; \boldsymbol{\lambda}) = A\,\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}),$$

$$A = \prod_{j=1}^{M_{\mathrm{I}}}(-a_1 - a_1^* + d_j^{\mathrm{I}} + 1) \cdot \prod_{j=1}^{M_{\mathrm{II}}}(-a_2 - a_2^* + d_j^{\mathrm{II}} + 1). \tag{C.3}$$

- $d_j = 0$ case :

$$\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})\,\Big|_{d_{M_{\mathrm{I}}}^{\mathrm{I}}=0} = A\,\check{P}_{\mathcal{D}',n}(x; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}}^{\mathrm{I}}),$$

$$\mathcal{D}' = \{d_1^{\mathrm{I}} - 1, \ldots, d_{M_{\mathrm{I}}-1}^{\mathrm{I}} - 1, d_1^{\mathrm{II}} + 1, \ldots, d_{M_{\mathrm{II}}}^{\mathrm{II}} + 1\},$$

$$A = (-1)^{M_{\mathrm{I}}}(a_1 + a_1^* + n - 1) \prod_{j=1}^{M_{\mathrm{I}}-1}(-a_1 - a_1^* + a_2 + a_2^* + d_j^{\mathrm{I}} + 1) \cdot \prod_{j=1}^{M_{\mathrm{II}}}(d_j^{\mathrm{II}} + 1), \tag{C.4}$$

$$\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})\,\Big|_{d_{M_{\mathrm{II}}}^{\mathrm{II}}=0} = B\,\check{P}_{\mathcal{D}',n}(x; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}}^{\mathrm{II}}),$$

$$\mathcal{D}' = \{d_1^{\mathrm{I}} + 1, \ldots, d_{M_{\mathrm{I}}}^{\mathrm{I}} + 1, d_1^{\mathrm{II}} - 1, \ldots, d_{M_{\mathrm{II}}-1}^{\mathrm{II}} - 1\},$$

$$B = (-1)^{M}(a_2 + a_2^* + n - 1) \prod_{j=1}^{M_{\mathrm{II}}-1}(-a_2 - a_2^* + a_1 + a_1^* + d_j^{\mathrm{II}} + 1) \cdot \prod_{j=1}^{M_{\mathrm{I}}}(d_j^{\mathrm{I}} + 1). \tag{C.5}$$

- values at special points :

Let x_0 and η_0 be

$$x_0 \overset{\text{def}}{=} -i\gamma\big(a_2^* + \tfrac{1}{2}(M_{\mathrm{I}} - M_{\mathrm{II}})\big), \quad \eta_0 \overset{\text{def}}{=} \eta(x_0). \tag{C.6}$$

Note that, as coordinates x and η, these values x_0 and η_0 are unphysical (they are imaginary). The multi-indexed polynomials $P_{\mathcal{D},n}(\eta)$ take 'simple' values at these 'unphysical' values η_0:

$$P_{\mathcal{D},n}(\eta_0; \boldsymbol{\lambda})$$

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$$= (-i)^{\ell_{\mathcal{D}}+n} c_{\mathcal{D},n}^P(\boldsymbol{\lambda})(-1)^{\sum_{j=1}^{M_{\mathrm{II}}} d_j^{\mathrm{II}} - \frac{1}{2}M_{\mathrm{II}}(M_{\mathrm{II}}-1)}$$

$$\times \prod_{j=1}^{M_{\mathrm{I}}} \frac{(a_2^* - a_1^* + 1, a_2 + a_2^*)_{d_j^{\mathrm{I}}}}{(a_2 + a_2^* - a_1 - a_1^* + d_j^{\mathrm{I}} + 1)_{d_j^{\mathrm{I}}}} \cdot \frac{\displaystyle\prod_{1 \le j < k \le M_{\mathrm{I}}} (a_2 + a_2^* - a_1 - a_1^* + d_j^{\mathrm{I}} + d_k^{\mathrm{I}} + 1)}{\displaystyle\prod_{j=1}^{M_{\mathrm{I}}} (a_2^* - a_1^* + 1, a_2 + a_2^*)_{j-1}}$$

$$\times \prod_{j=1}^{M_{\mathrm{II}}} \frac{(a_1^* - a_2^* + 1, 1 - a_2 - a_2^*)_{d_j^{\mathrm{II}}}}{(a_1 + a_1^* - a_2 - a_2^* + d_j^{\mathrm{II}} + 1)_{d_j^{\mathrm{II}}}} \cdot \frac{\displaystyle\prod_{1 \le j < k \le M_{\mathrm{II}}} (a_1 + a_1^* - a_2 - a_2^* + d_j^{\mathrm{II}} + d_k^{\mathrm{II}} + 1)}{\displaystyle\prod_{j=1}^{M_{\mathrm{II}}} (a_1^* - a_2^* + 1, 1 - a_2 - a_2^*)_{j-1}}$$

$$\times \prod_{j=1}^{M_{\mathrm{I}}} \prod_{k=1}^{M_{\mathrm{II}}} \frac{(a_2^* - a_1^* + j - k)(a_2 + a_2^* + j - k)}{a_2 + a_2^* - a_1 - a_1^* + d_j^{\mathrm{I}} - d_k^{\mathrm{II}}}$$

$$\times \frac{(a_1 + a_2^*, a_2 + a_2^*)_n}{(a_1 + a_1^* + a_2 + a_2^* + n - 1)_n} \prod_{j=1}^{M_{\mathrm{I}}} \frac{a_2 + a_2^* + d_j^{\mathrm{I}} + n}{a_2 + a_2^* + j - 1} \cdot \prod_{j=1}^{M_{\mathrm{II}}} \frac{d_j^{\mathrm{II}} + 1 - a_2 - a_2^*}{d_j^{\mathrm{II}} + 1 - n - a_2 - a_2^*}, \quad \text{(C.7)}$$

where we have assumed $0 \le d_1^{\mathrm{I}} < \cdots < d_{M_{\mathrm{I}}}^{\mathrm{I}}$ and $0 \le d_1^{\mathrm{II}} < \cdots < d_{M_{\mathrm{II}}}^{\mathrm{II}}$. We remark that a similar formula can be obtained by interchanging $a_1 \leftrightarrow a_2$ and I $\leftrightarrow$ II.

D   More Examples for §3

We present more examples for §3. Unlike in §3.3, the examples presented here do not satisfy the condition (A.27). Namely, the multi-indexed polynomials $P_{\mathcal{D},n}(\eta)$ are not orthogonal polynomials. Except for Ex.1 in §D.1, we present only $r_{n,k}^{X,\mathcal{D}}$ ($1 \le k \le L$), because $r_{n,k}^{X,\mathcal{D}}$ ($-L \le k \le 0$) are obtained by (3.16)–(3.19).

An explicit form of A in (3.22) is also presented.

D.1   Examples for §3.3.1

<u>Ex.1</u> $\mathcal{D} = \{1\}$, $Y(\eta) = 1$ ($\Rightarrow \ell_{\mathcal{D}} = 1$, $X(\eta) = X_{\min}(\eta)$, $L = 2$): 5-term recurrence relations

$$X(\eta) = \eta\big(\sin\phi \cdot \eta + 2(1 - a)\cos\phi\big),$$

$$r_{n,2}^{X,\mathcal{D}} = \frac{(n+1)_2}{4\sin\phi} \frac{2a + n - 2}{2a + n}, \quad r_{n,-2}^{X,\mathcal{D}} = \frac{(2a + n - 2)_2}{4\sin\phi},$$

$$r_{n,1}^{X,\mathcal{D}} = -(n+1)(2a + n - 2)\cot\phi, \quad r_{n,-1}^{X,\mathcal{D}} = -(2a + n - 2)_2 \cot\phi,$$

$$r_{n,0}^{X,\mathcal{D}} = \frac{a(6a + 10n - 7) + 3n^2 - 7n + 1}{2\sin\phi} - \frac{1}{4}(2a + 2n - 1)(6a + 2n - 5)\sin\phi.$$

The polynomial $I(z)$ (3.21) is

$$I(z) = -8\sin\phi\Big((2 + \cos 2\phi)z^2 + 2\big(6a - 4 + (4a - 3)\cos 2\phi\big)\sin\phi \cdot z$$

$$+ \left(12a(a-1) - 1 + (2a-1)(6a-5)\cos 2\phi\right)\sin^2\phi\right).$$

<u>Ex.2</u> $\mathcal{D} = \{1\}$, $Y(\eta) = \eta$ ($\Rightarrow \ell_{\mathcal{D}} = 1$, $L = 3$): 7-term recurrence relations

$$X(\eta) = \frac{\eta}{6}\left(4\sin\phi \cdot \eta^2 + 6(1-a)\cos\phi \cdot \eta + \sin\phi\right),$$

$$r_{n,3}^{X,\mathcal{D}} = \frac{(n+1)_3}{12\sin^2\phi}\frac{2a+n-2}{2a+n+1}, \quad r_{n,2}^{X,\mathcal{D}} = -\frac{(n+1)_2(3a+2n)\cos\phi}{4\sin^2\phi}\frac{2a+n-2}{2a+n},$$

$$r_{n,1}^{X,\mathcal{D}} = \frac{(n+1)(2a+n-2)}{4\sin^2\phi}\left(2a+3n+1+2(a+n)\cos 2\phi\right).$$

The polynomial $I(z)$ (3.21) is

$$I(z) = -48\sin 2\phi\Big((4+\cos 2\phi)z^3 + 3\big(10a - 6 + (3a-2)\cos 2\phi\big)\sin\phi \cdot z^2$$
$$+ 2\big(30a(a-1) + 4 + (12a^2 - 15a + 4)\cos 2\phi\big)\sin^2\phi \cdot z$$
$$+ (a-1)\big(20a(a+1) - 3 + (2a-1)(10a-3)\cos 2\phi\big)\sin^3\phi\Big).$$

<u>Ex.3</u> $\mathcal{D} = \{1\}$, $Y(\eta) = \eta^2$ ($\Rightarrow \ell_{\mathcal{D}} = 1$, $L = 4$): 9-term recurrence relations

$$X(\eta) = \frac{\eta}{12}\left(6\sin\phi \cdot \eta^3 + 8(1-a)\cos\phi \cdot \eta^2 + 3\sin\phi \cdot \eta + 2(1-a)\cos\phi\right),$$

$$r_{n,4}^{X,\mathcal{D}} = \frac{(n+1)_4}{32\sin^3\phi}\frac{2a+n-2}{2a+n+2}, \quad r_{n,3}^{X,\mathcal{D}} = -\frac{(n+1)_3(4a+3n+2)\cos\phi}{12\sin^3\phi}\frac{2a+n-2}{2a+n+1},$$

$$r_{n,2}^{X,\mathcal{D}} = \frac{(n+1)_2}{8\sin^3\phi}\frac{2a+n-2}{2a+n}\left(5a(a+1) + 2(5a+1)n + 4n^2 + (5a^2 + 2a + 8an + 3n^2 + n)\cos 2\phi\right),$$

$$r_{n,1}^{X,\mathcal{D}} = -\frac{(n+1)(2a+n-2)\cos\phi}{4\sin^3\phi}\left(2a^2 + 4a + 8an + 5n^2 + 2n + 1 + 2(a+n)^2\cos 2\phi\right).$$

The polynomial $I(z)$ (3.21) is

$$I(z) = -192\sin\phi\Big(3(18 + 16\cos 2\phi + \cos 4\phi)z^4$$
$$+ 4\big(18(7a-4) + 4(29a-17)\cos 2\phi + (8a-5)\cos 4\phi\big)\sin\phi \cdot z^3$$
$$+ 12\big(126a(a-1) + 30 + 4(31a^2 - 33a + 8)\cos 2\phi + (10a^2 - 12a + 3)\cos 4\phi\big)\sin^2\phi \cdot z^2$$
$$+ 16\big(3(32a^3 - 33a^2 + 13a - 6) + 2(54a^3 - 69a^2 + 29a - 8)\cos 2\phi$$
$$+ (a-1)(12a^2 - 9a + 1)\cos 4\phi\big)\sin^3\phi \cdot z$$
$$+ \big(3(2a+1)(56a^3 + 4a^2 - 58a - 3) + 4(112a^4 - 16a^3 - 112a^2 + 16a + 3)\cos 2\phi$$
$$+ (2a-1)(56a^3 - 100a^2 + 38a + 3)\cos 4\phi\big)\sin^4\phi\Big).$$

<u>Ex.4</u> $\mathcal{D} = \{3\}$, $Y(\eta) = 1$ ($\Rightarrow \ell_{\mathcal{D}} = 3$, $X(\eta) = X_{\min}(\eta)$, $L = 4$): 9-term recurrence relations

$$X(\eta) = \frac{\eta}{12}\left(4\sin^3\phi \cdot \eta^3 - 16(a-2)\cos\phi\sin^2\phi \cdot \eta^2\right.$$

$$+ \big(12a(a-3) + 29 + (12a(a-4) + 43)\cos 2\phi\big)\sin\phi \cdot \eta$$

$$- 2(a-2)\big((2a-1)^2 + (4a(a-4)+11)\cos 2\phi\big)\cos\phi\Big),$$

$$r_{n,4}^{X,\mathcal{D}} = \frac{(n+1)_4}{48\sin\phi}\frac{2a+n-4}{2a+n}, \quad r_{n,3}^{X,\mathcal{D}} = -\frac{1}{6}(n+1)_3\,(2a+n-4)\cot\phi,$$

$$r_{n,2}^{X,\mathcal{D}} = \frac{(n+1)_2}{12\sin\phi}(2a+n-4)\big(6a+4n+3(2a+n-1)\cos 2\phi\big),$$

$$r_{n,1}^{X,\mathcal{D}} = -\frac{1}{6}(n+1)(2a+n-1)(2a+n-4)\cot\phi\big(4a+5n+2+2(2a+n-2)\cos 2\phi\big).$$

The polynomial $I(z)$ (3.21) is

$$I(z) = -384\sin^3\phi\Big((18 + 16\cos 2\phi + \cos 4\phi)z^4$$

$$+ 4\big(54(a-1) + 4(13a-14)\cos 2\phi + (4a-5)\cos 4\phi\big)\sin\phi \cdot z^3$$

$$+ 4\big(6(2a-1)(19a-27) + 8(30a^2 - 63a + 28)\cos 2\phi + (24a^2 - 60a + 35)\cos 4\phi\big)\sin^2\phi \cdot z^2$$

$$+ 16\big(3(2a-1)(16a^2 - 35a + 12) + 8(14a^3 - 42a^2 + 35a - 8)\cos 2\phi$$

$$+ (4a-5)(4a^2 - 10a + 5)\cos 4\phi\big)\sin^3\phi \cdot z$$

$$+ 3(2a-1)\big((2a-1)(60a^2 - 116a - 9) + 4(40a^3 - 116a^2 + 70a + 5)\cos 2\phi$$

$$+ (2a-3)(20a^2 - 56a + 31)\cos 4\phi\big)\sin^4\phi\Big).$$

We have also obtained 9-term recurrence relations for $\mathcal{D} = \{1,3\}, \{1,2,3\}$ with $X(\eta) = X_{\min}(\eta)$. Since the explicit forms of $r_{n,k}^{X,\mathcal{D}}$ are somewhat lengthy, we do not write down them here.

D.2 Examples for §3.3.2

We set $\sigma_1 = a_1 + a_1^*$, $\sigma_2 = a_1 a_1^*$, $\sigma_1' = a_2 + a_2^*$ and $\sigma_2' = a_2 a_2^*$.

<u>Ex.1</u> $\mathcal{D} = \{1^{\mathrm{I}}\}$ ($M_{\mathrm{I}} = 1$, $M_{\mathrm{II}} = 0$), $Y(\eta) = 1$ ($\Rightarrow \ell_{\mathcal{D}} = 1$, $X(\eta) = X_{\min}(\eta)$, $L = 2$): 5-term recurrence relations

$$X(\eta) = \frac{\eta}{2}\big((2 - \sigma_1 + \sigma_1')\eta - 2i(a_2 - a_2^* + a_1 a_2^* - a_1^* a_2)\big),$$

$$r_{n,2}^{X,\mathcal{D}} = \frac{(2 - \sigma_1 + \sigma_1')(n+1)_2\,(n + \sigma_1 - 2)(n + b_1 - 1)_2}{2(n + \sigma_1)(2n + b_1 - 1)_4},$$

$$r_{n,1}^{X,\mathcal{D}} = \frac{-i(a_1 - a_1^* - a_2 + a_2^*)(b_1 - 2)(n+1)(n + \sigma_1 - 2)(n + \sigma_1' + 1)(n + b_1 - 1)}{(2n + b_1 - 2)_3(2n + b_1 + 2)}.$$

The polynomial $I(z)$ (3.21) is

$$I(z)$$

$$= -4(2 - \sigma_1 + \sigma_1')z^3$$

$$+ 2\big(16 - \sigma_1^3 - 12\sigma_2 + 12a_2\sigma_2 + 6a_1^2(2a_2 - \sigma_1') + 14\sigma_1' + 4\sigma_2\sigma_1' - 3\sigma_1'^2 + \sigma_1'^3$$

$$+ \sigma_1^2(3 - 6a_2 + 3\sigma_1') + \sigma_1(-10 + 12a_2 + 6a_2^2 + 6\sigma_2 - 16\sigma_1' - 3\sigma_1'^2 - 4\sigma_2') + 20\sigma_2'$$

$$- 6\sigma_1'\sigma_2' - 6a_1(4a_2 + 2a_2^2 - 2\sigma_1' - \sigma_1'^2 + 2\sigma_2'))z^2$$

$$- 2\Big(6 - 24\sigma_2 + 48a_2\sigma_2 + 16a_2^2\sigma_2 + 4a_2^3\sigma_2 + 4a_2^2a_2^*\sigma_2 + 27\sigma_1' - 20\sigma_2\sigma_1' + 4\sigma_1'^2 + 10\sigma_2\sigma_1'^2$$

$$- 11\sigma_1'^3 - 10\sigma_2\sigma_1'^3 + 3\sigma_1'^4 + \sigma_1^4(1 + 2\sigma_1') + 16\sigma_2\sigma_2' + 24\sigma_1'\sigma_2' + 4\sigma_2\sigma_1'\sigma_2' - 14\sigma_1'^2\sigma_2'$$

$$- 2a_1^3(6a_2 + 2a_2^2 - 3\sigma_1' - \sigma_1'^2 + 2\sigma_2') + \sigma_1^3(-1 + 6a_2 + 2a_2^2 - 8\sigma_1' + 10\sigma_2')$$

$$- \sigma_1^2\big(8 + 24a_2 + 2a_2^3 + 2a_2^2(4 + a_2^*) - 19\sigma_1' - 12\sigma_1'^2 + 2\sigma_2(3 + 5\sigma_1') + 46\sigma_2' - 12\sigma_1'\sigma_2'\big)$$

$$+ 2a_1\big(-24a_2 + 2a_2^3 + 2a_2^2(-4 + a_2^*) + 4\sigma_1'^2 - \sigma_1'^3 - 8\sigma_2' + 2\sigma_1'(6 + \sigma_2')\big)$$

$$+ 2a_1^2\big(-6(-4 + a_1^*)a_2 + 2a_2^3 + a_2^2(8 - 2a_1^* + 2a_2^*) + (-4 + a_1^*)\sigma_1'^2 - \sigma_1'^3$$

$$- 2(-4 + a_1^*)\sigma_2' + \sigma_1'(-12 + 3a_1^* + 2\sigma_2')\big)$$

$$- \sigma_1\big(-9 + 2a_2^3 + 12a_2(-2 + \sigma_2) + 2a_2^2(-4 + a_2^* + 2\sigma_2) + 40\sigma_1' + 23\sigma_1'^2 - 12\sigma_1'^3$$

$$+ 2\sigma_1'^4 - 52\sigma_2' + 42\sigma_1'\sigma_2' - 10\sigma_1'^2\sigma_2' + 2\sigma_2(-12 - 15\sigma_1' + 6\sigma_1'^2 + 2\sigma_2'))\Big)z$$

$$- \frac{1}{2}(b_1 - 3)_2\Big(-32a_2^2\sigma_2 - 8a_2^3\sigma_2 - 8a_2^2a_2^*\sigma_2 - 2\sigma_1' + 16\sigma_2\sigma_1' - 7\sigma_1'^2 + 20\sigma_2\sigma_1'^2 - 5\sigma_1'^3$$

$$+ \sigma_1^3\big(-4a_2^2 + (1 + 2\sigma_1')^2\big) - 4a_1^2\big(2a_2^3 + a_2^2(8 - 2a_1^* + 2a_2^*) + (-4 + a_1^* - \sigma_1')(\sigma_1'^2 - 2\sigma_2')\big)$$

$$+ 8a_1\big(2a_2^3 + 2a_2^2(2 + a_2^*) - (2 + \sigma_1')(\sigma_1'^2 - 2\sigma_2')\big) - 32\sigma_2\sigma_2' + 8\sigma_1'\sigma_2' - 8\sigma_2\sigma_1'\sigma_2'$$

$$+ a_1^3(8a_2^2 - 4\sigma_1'^2 + 8\sigma_2') + \sigma_1^2\big(-1 + 4a_2^3 + 4a_2^2(4 + a_2^*) + \sigma_1' - 8\sigma_1'^2 - 4\sigma_1'^3 + 12\sigma_2' + 16\sigma_1'\sigma_2'\big)$$

$$- \sigma_1\big(2 + 8a_2^3 + 8a_2^2(2 + a_2^* - \sigma_2) + (-11 + 16\sigma_2)\sigma_1'^2 - 12\sigma_1'^3 + 24\sigma_2' - 8\sigma_2\sigma_2'$$

$$+ 4\sigma_1'(2 + 5\sigma_2 + 9\sigma_2'))\Big).$$

The example with $\mathcal{D} = \{1^{\text{II}}\}$ ($M_{\text{I}} = 0$, $M_{\text{II}} = 1$) and $Y(\eta) = 1$ can be obtained by exchanging a_1 and a_2.

We have also obtained 9-term recurrence relations for $\mathcal{D} = \{3^{\text{I}}\}, \{3^{\text{II}}\}, \{1^{\text{I}}, 1^{\text{II}}\}$ with $X(\eta) = X_{\min}(\eta)$. Since the explicit forms of $r_{n,k}^{X,\mathcal{D}}$ are somewhat lengthy, we do not write down them here.

D.3   Explicit form of A in (3.22)

A in (3.22)

$$= (\sigma_1 - \sigma_1' - 4)_2\, n^5 + (\sigma_1 - \sigma_1' - 4)_2\,(1 + 2b_1)n^4$$

$$+ \Big(-36 + 12\sigma_2 - 38a_2\sigma_2 + 12a_2^2\sigma_2 + 5a_1^3(2a_2 - \sigma_1') - 5\sigma_1' + 6\sigma_2\sigma_1' + 29\sigma_1'^2 - \sigma_2\sigma_1'^2$$
$$+ 16\sigma_1'^3 + \sigma_1^3(-2 - 5a_2 + \sigma_1') + \big(12 + 12\sigma_2 + \sigma_1'(-13 + 5\sigma_1')\big)\sigma_2' - \sigma_1^2\big(5 + a_2(-19 + 6a_2)$$
$$- 5\sigma_2 + \sigma_1'(19 + 2\sigma_1') + \sigma_2'\big) + a_1^2\big(2a_2(-19 + 5a_1^* + 6a_2) + 19\sigma_1' - 5a_1^*\sigma_1' - 6\sigma_1'^2 + 12\sigma_2'\big)$$
$$+ \sigma_1\big(45 - 5a_2^3 + a_2^2(13 - 5a_2^*) + 2a_2(-6 + 5\sigma_2) + \sigma_2(-19 + \sigma_1') - 6\sigma_2' + \sigma_1'(52$$
$$+ \sigma_1'(5 + \sigma_1') + \sigma_2')\big) + a_1\big(2a_2(12 + a_2(-13 + 5\sigma_1')) - 26\sigma_2' + \sigma_1'(-12 + 13\sigma_1' - 5\sigma_1'^2$$
$$+ 10\sigma_2')\big)\Big)n^3$$

$$+ \Big(-36 + 5a_2^4(2a_1 - \sigma_1) + 10a_2^3a_2^*(2a_1 - \sigma_1) - 15\sigma_1 + 30\sigma_1^2 - 7\sigma_1^3 + 3\sigma_1^4 - \sigma_1^5 + 12\sigma_2$$
$$- 7\sigma_1\sigma_2 - 14\sigma_1^2\sigma_2 + 5\sigma_1^3\sigma_2 + a_2^3\big(2a_1(-8 + 11a_1) + (8 - 11\sigma_1)\sigma_1 + 22\sigma_2\big)$$
$$+ a_2^2a_2^*\big(2a_1(-8 + 11a_1) + (8 - 11\sigma_1)\sigma_1 + 22\sigma_2\big) + a_2\big(2a_1(12 + a_1(-7 - 14a_1 + 5a_1^2$$
$$+ 2(-7 + 5a_1)a_1^*)) - (-3 + \sigma_1)\sigma_1(1 + \sigma_1)(-4 + 5\sigma_1) + 2(-7 + \sigma_1(-14 + 5\sigma_1))\sigma_2$$
$$+ 10\sigma_2^2\big) + a_2^2\big(2a_1(-1 + a_1(-26 + 11\sigma_1)) - 52\sigma_2 + \sigma_1(1 + (26 - 11\sigma_1)\sigma_1 + 22\sigma_2)\big)$$
$$+ \big(-65 + a_1(-12 + a_1(7 + a_1(14 - 5a_1 - 10a_1^*) + 14a_1^*)) + 44\sigma_1 + 8\sigma_1^2 - 6\sigma_1^3$$
$$+ 6(-1 + \sigma_1)^2\sigma_2 - 5\sigma_2^2\big)\sigma_1' + \big(a_1(1 + a_1(26 - 11\sigma_1)) + \sigma_1(38 + (-12 + \sigma_1)\sigma_1)$$
$$+ 2(-7 + 9\sigma_2)\big)\sigma_1'^2 + \big(17 + (8 - 11a_1)a_1 + \sigma_1(8 + \sigma_1) - 6\sigma_2\big)\sigma_1'^3 + (7 - 5a_1)\sigma_1'^4$$
$$- \sigma_1'^5 - 2\big(-6 + a_1 + a_1^2(26 - 11\sigma_1) + 3(-1 + \sigma_1)^2\sigma_1 + 26\sigma_2 - 11\sigma_1\sigma_2\big)\sigma_2'$$
$$+ \big(-1 + 2a_1(-8 + 11a_1) - 18\sigma_1 + 22\sigma_2\big)\sigma_1'\sigma_2' + 2(-4 + 5a_1 + 3\sigma_1)\sigma_1'^2\sigma_2' + 5\sigma_1'^3\sigma_2'$$
$$+ 5(2a_1 - \sigma_1)\sigma_2'^2\Big)n^2$$

$$+ \Big(24 - 50\sigma_1 + 20\sigma_1^2 - 8\sigma_1^3 + 8\sigma_1^4 - 2\sigma_1^5 - 12\sigma_2 + 46\sigma_1\sigma_2 - 41\sigma_1^2\sigma_2 + 9\sigma_1^3\sigma_2$$
$$+ a_2^4\big(6a_1 + 8a_1^2 - \sigma_1(3 + 4\sigma_1) + 8\sigma_2\big) + 2a_2^3a_2^*\big(6a_1 + 8a_1^2 - \sigma_1(3 + 4\sigma_1) + 8\sigma_2\big)$$
$$+ a_2\big(2a_1(-12 + a_1(46 - 41a_1^* + a_1(-41 + 9a_1 + 18a_1^*))) - (-3 + \sigma_1)\sigma_1(4 + \sigma_1(-14$$
$$+ 9\sigma_1)) + 92\sigma_2 + 2\sigma_1(-41 + 9\sigma_1)\sigma_2 + 18\sigma_2^2\big) + a_2^2\big(2a_1(10 + a_1(-22 - 3a_1^* + a_1(-3$$
$$+ 4a_1 + 8a_1^*))) + 3\sigma_1^3 - 4\sigma_1^4 + 4\sigma_2(-11 + 2\sigma_2) - 2\sigma_1(5 + 3\sigma_2) + \sigma_1^2(22 + 8\sigma_2)\big)$$
$$+ a_2^2\sigma_1'\big(2a_1(-5 + a_1(-9 + 8\sigma_1)) - 18\sigma_2 + \sigma_1(5 + (9 - 8\sigma_1)\sigma_1 + 16\sigma_2)\big)$$
$$- \big(14 + a_1(-12 + a_1(46 - 41a_1^* + a_1(-41 + 9a_1 + 18a_1^*))) + \sigma_1(40 + \sigma_1(-46$$
$$+ \sigma_1(18 + (-5 + \sigma_1)\sigma_1))) + 36\sigma_2 + \sigma_1(-19 - 4(-3 + \sigma_1)\sigma_1)\sigma_2 + 9\sigma_2^2\big)\sigma_1' + \big(-36$$
$$+ a_1(-10 + a_1(22 + a_1(3 - 4a_1 - 8a_1^*) + 3a_1^*)) - 3\sigma_1^3 + \sigma_1(14 - 6\sigma_2) + (17 - 4\sigma_2)\sigma_2$$
$$+ \sigma_1^2(6 + 4\sigma_2)\big)\sigma_1'^2 + \big(-4 + a_1(5 + a_1(9 - 8\sigma_1)) + \sigma_1(14 + \sigma_1(-5 + 2\sigma_1) - 4\sigma_2)$$
$$+ 12\sigma_2\big)\sigma_1'^3 + \big(6 - a_1(3 + 4a_1) + 5\sigma_1 - 4\sigma_2\big)\sigma_1'^4 - \sigma_1\sigma_1'^5 + \big(-12 + 2a_1(10 + a_1(-22$$

$$- 3a_1^* + a_1(-3 + 4a_1 + 8a_1^*))) + 12\sigma_1^3 - 4\sigma_1^4 - 6\sigma_1(-6 + \sigma_2) - 44\sigma_2 + 8\sigma_2^2$$

$$+ \sigma_1^2(-19 + 8\sigma_2))\sigma_2' + \big(10 + 2a_1(-5 + a_1(-9 + 8\sigma_1)) - 18\sigma_2 + \sigma_1(-17 + 6\sigma_1 - 4\sigma_1^2$$

$$+ 16\sigma_2))\sigma_1'\sigma_2' + \big(-5 + 6a_1 + 8a_1^2 + 4(-3 + \sigma_1)\sigma_1 + 8\sigma_2\big)\sigma_1'^2\sigma_2' + (3 + 4\sigma_1)\sigma_1'^3\sigma_2'$$

$$+ \big(6a_1 + 8a_1^2 - \sigma_1(3 + 4\sigma_1) + 8\sigma_2\big)\sigma_2'^2\Big)n$$

$$+ (1 + \sigma_1')(b_1 - 3)_2\Big(4 + \sigma_1 - 2\sigma_2 + 2a_2(\sigma_1 - 2\sigma_1^2 + 4\sigma_2) + a_1^2(8a_2 - 4\sigma_1') + 3\sigma_1'$$

$$+ 2a_1(-2a_2 + \sigma_1') - (\sigma_1 - \sigma_1')(\sigma_1^2 - 4\sigma_2 + \sigma_1' - \sigma_1\sigma_1') - 2\sigma_2' + 4\sigma_1\sigma_2'\Big). \tag{D.1}$$

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