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Modular Invariance and Two-loop Bosonic String Vacuum Amplitude

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Abstract:

We obtain an explicit formula for the two-loop vacuum amplitude of closed bosonic string in terms of a Siegel modular form (cusp form) of weight ten .

Recognition of the possibility of superstring theory as a "theory of everything" has resulted in a great deal of activities in theoretical particle physics. It could be a finite theory free from divergences. So it is a pressing problem to investigate quantum corrections in string amplitudes.

One-loop amplitudes are explicitly calculated using either operator formalism [1] or path-integral formalism [2] , and the finiteness of the one-loop corrections for heterotic string amplitudes has been established [3].

The evaluation of multiloop amplitudes is much more difficult. Firstly, when we express the amplitude as an integral over the moduli space, we do not know the explicit dependence of the integrand on the moduli coordinates. Secondly, boundary of fundamental region is not so clear as in the one-loop case.

Recently, much progress has been made in our understanding of the structure of multiloop amplitudes. Remarkable work of Belavin and Knižhnik [4] has made it clear that the integration measure is a squared modulus of a holomorphic function on $\mathcal{M}_g$, the moduli space of genus g. Using their result, Manin [5] succeeded in obtaining multiloop bosonic string vacuum amplitudes using algebraic geometrical techniques. His results are very important, however, physicists would be happier if they have more "explicit" formula with which they can calculate the amplitude in practice. Is it possible to extract such a concrete object from rather abstract results of algebraic geometry ?

In this paper we obtain the two-loop bosonic string vacuum amplitude using the period matrix τ_{ij} as the moduli coordinate. The period matrix is a "good" coordinate because modular transformations and its automorphic forms are known to be easily expressed in terms of it.

According to ref.[4] the g-loop vacuum amplitude --the sum over random genus g surfaces -- has the following form (D=26)

$$Z_g = \int \prod_{i=1}^{3g-3} dy_i \wedge d\bar{y}_i \, (\det \mathrm{Im}\, \tau)^{-13} |F(y_1, \cdots, y_{3g-3})|^2 \quad (1)$$

where $y_1, y_2, \ldots, y_{3g-3}$ are some complex coordinates on the Teichmüller space $\mathcal{T}_g$. (τ_{ij}) is the period matrix of the Riemann surface whose complex structure is specified by the Teichmüller coordinates. F(y) is a nowhere-zero holomorphic functionon $\mathcal{M}_g$ and has second order poles at the infinity D_g of $\mathcal{M}_g$, where surfaces degenerate.

The period matrix belongs to the Siegel upper half space of degree g, $\mathfrak{S}_g$, the variety of complex symmetric g x g matrices with positive definite imaginary parts. The modular group Γ_g = Sp(2g,Z) acts biholomorphically on $\mathfrak{S}_g$, i.e. every symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acts as [6,7]

$$\tau \rightarrow \tau' = (A\tau + B)(C\tau + D)^{-1} \quad (2)$$

Let $\mathfrak{S}_g^*$ denote the Siegel fundamental domain $\mathfrak{S}_g/\Gamma_g$.

There is a canonical holomorphic map :

$$i \; : \; \mathcal{M}_g \longrightarrow \mathfrak{S}_g^* \quad (3)$$

The Torelli's theorem asserts that this map i is injective.

On the other hand, the complex dimensions of $\mathcal{M}_g$ and $\mathfrak{S}^*_g$ are 3g-3 and g(g+1)/2 respectively, and they coincide for g=2,3. In fact, for g=2 it has been shown that $\overline{\mathcal{M}}_2$ (the stable curve compactification of $\mathcal{M}_g$ [8]) and $\overline{\mathfrak{S}}^*_2$ (a suitable compactification of $\mathfrak{S}^*_2$) are isomorphic ! [9].

So, hereafter, we investigate the two-loop vacuum amplitude directly on $\mathfrak{S}^*_g$, where the good set of the coordinates τ_{ij} $(i \leq j)$ is available. Then we have, instead of (1)

$$Z_2 = \int d\tau_{11} \wedge d\bar{\tau}_{11} \wedge d\tau_{12} \wedge d\bar{\tau}_{12} \wedge d\tau_{22} \wedge d\bar{\tau}_{22} \, (\det \mathrm{Im}\, \tau)^{-13} |F(\tau)|^2 \quad (4)$$

In order to explore the form of F(τ), we need some basic facts from symplectic geometry. First, note that under the action (2), det(Im τ) transforms as

$$\det(\mathrm{Im}\tau) \longrightarrow \det(\mathrm{Im}\,\tau') = \frac{\det \mathrm{Im}\,\tau}{|\det(C\tau + D)|^2} \quad (5)$$

The invariant line element

$$ds^2 = \mathrm{Tr}((\mathrm{Im}\,\tau)^{-1} d\tau \, (\mathrm{Im}\,\tau)^{-1} d\bar{\tau}) \quad (6)$$

on $\mathfrak{S}_g$ determines the invariant volume element [7]

$$dV = \frac{\prod_{i \leq j} d\tau_{ij} \wedge d\bar{\tau}_{ij}}{(\det \mathrm{Im}\,\tau)^{g+1}} \quad (7)$$

For g=2, the combination

$$dV = \frac{d\tau_{11} \wedge d\bar{\tau}_{11} \wedge d\tau_{12} \wedge d\bar{\tau}_{12} \wedge d\tau_{22} \wedge d\bar{\tau}_{22}}{(\det \mathrm{Im}\,\tau)^3} \quad (8)$$

is the modular invariant volume element.

Clearly, in order to determine the function f(τ) = 1/F(τ) we need another condition on it. In ref.[10], it is proved that there is no global obstruction for modular invariance in multi-loop amplitudes. So it is natural to require that $f(\tau)$ is a Siegel modular form that is explained below. From (4), (5) and (8) its weight is ten.

Siegel modular forms are natural generalization of elliptic modular forms to higher genus case. A modular form of degree g $\psi(\tau)$ is defined by the following two conditions:

(i) For every element M of Sp(2g,Z), $\psi(\tau)$ satisfies a functional equation of the form

$$\psi(M\cdot\tau) = \det(C\tau + D)^k \psi(\tau) \tag{9}$$

for some even integer k ;

(ii) it is holomorphic in $\mathfrak{S}_g$.

(plus a condition at infinity for g=1)

The integer k in (i) is called the weight of the modular form. (Remark: The term "degree" is more frequently used in the literature on modular forms, but it is identical with the term "genus" in this context.) The best known example of Siegel modular form is the Eisenstein series [7]. In general, if τ is a point of $\mathfrak{S}_g$, the Eisenstein series of degree g and of weight k is defined as follows:

$$\Psi_k(\tau) = \sum_{C,D} \det(C\tau + D)^{-k} \tag{10}$$

The summation extends over all classes of coprime pairs, i.e. over all inequivalent bottom rows of elements of Sp(2g,Z) with respect to left multiplications by unimodular integer matrices of

degree g.

The set of all Siegel modular forms of degree g and weight k forms a finite dimensional complex vector space, denoted by $\mathfrak{M}_k^g$. For g=1, it is well known [11] that the graded ring of the modular forms are generated by two Eisenstein series of degree one, E_4 and E_6 (algebraically independent over C),

$$\sum_{k=0}^{\infty} \mathfrak{M}_k^1 = \mathbb{C}[E_4, E_6] \tag{11}$$

An analogous result for genus two has been obtained by Igusa [12]: The Eisenstein series of degree two, $\Psi_4, \Psi_6, \Psi_{10}$ and Ψ_{12} are algebraically independent over C and

$$\sum_{k=0}^{\infty} \mathfrak{M}_k^2 = \mathbb{C}[\Psi_4, \Psi_6, \Psi_{10}, \Psi_{12}] \tag{12}$$

For g=1, the vacuum amplitude is expressed in terms of the modular form of weight 12, $\Delta(\tau) = (2\pi)^{12}\eta^{24}(\tau)$ (η is the Dedekind eta function). There are two linearly independent modular forms of weight 12, but the particular combination $\Delta(\tau) = (2\pi)^{12}\, 2^{-6}\, 3^{-3}\, (E_4^3 - E_6^2)$ appears in the formula. This is the only cusp form of weight 12, which is characterized by the condition $\Delta(i\infty) = 0$. The point $i\infty$ corresponds to a degenerate torus where we see the divergence due to the tachyon pole.

For g=2, we saw that the vacuum amplitude should be expressed by the modular form of weight 10. Now $\dim \mathfrak{M}_{10}^2 = 2$, and $\mathfrak{M}_{10}^2$ is spanned by two modular forms, $\Psi_4\Psi_6$ and Ψ_{10}. So we must look for $f(\tau)$ among the linear combinations of the these two modular forms.

According to ref [4,13], at the corner of the moduli space corresponding to the degeneration of Riemann surface illustrated in fig.1 (pinching a cycle homologous to zero) or in fig.2 (pinching a nonzero homology cycle), we have

$$F(\tau(t)) \sim t^{-2} \quad (t \sim 0) \tag{13}$$

where t is the coordinate in $\overline{\mathcal{M}}_g$ transversal to the subvariety D_g of the degenerate surface [4,14]. Intuitively, t can be taken as $|t| = \exp(-T)$ where T is the length of the cylinder that connects two tori (fig.1.(c)) [4]. $F(\tau)$ is uniquely determined up to a multiple constant by this behavior at the corner of the moduli space when $F(\tau)$ is defined as a function of the moduli space.

The period matrix $\tau(t)$ as a function of t has an expansion [14] for fig.1:

$$\tau(t) = \begin{array}{c} \\ g_1\{ \\ g_2\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{g_1} & \overbrace{\quad}^{g_2} \end{array} \\ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \end{array} + \begin{pmatrix} \text{constant} \\ \text{matrix} \end{pmatrix} \cdot t + O(t^2) \quad (t \to 0) \tag{14}$$

and for fig.2:

$$\tau(t) = \begin{pmatrix} \tau_{ij} + t\sigma_{ij} & a_i + t\sigma_{ig} \\ a_i + t\sigma_{gi} & \frac{1}{2\pi}\log\frac{1}{t} + C_1 + C_2 t \end{pmatrix} + O(t^2) \tag{15}$$

where C_1, C_2 are some constants, and σ_{ij} are constants determined by Abelian integrals on the Riemann surface(s) in the $t \to 0$ limit.

We can use (13) and (15) to get the following constraint on f :

$$\lim_{\mathrm{Im}\lambda \to \infty} f\begin{pmatrix} \tau^{(g-1)} & a \\ {}^t a & \lambda \end{pmatrix} = 0 \tag{16}$$

This condition on $f(\tau)$ is nothing but the definition of a cusp form [7,12]. For g=2, there exists only one cusp form of weight 10,

$$\chi_{10} = -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (\Psi_4\Psi_6 - \Psi_{10}) \tag{17}$$

which is the lowest weight cusp form of genus two. Therefore $f(\tau)$ is χ_{10} up to some numerical constant. So our conclusion is:

$$F(\tau) = \text{const} \times (\chi_{10}(\tau))^{-1}$$
$$Z_2 = \text{const} \times \int_{\mathfrak{S}_2^*} dV \,(\det \mathrm{Im}\,\tau)^{-10} |\chi_{10}(\tau)|^{-2} \tag{18}$$

According to ref.[15], $|\chi_{10}|$ is bounded in $\mathfrak{S}^*_2$, so $F(\tau)$ is nonzero in accordance with [4]. For $F(\tau)$ to be holomorphic on $\mathcal{M}_2$, χ_{10} cannot have zeros on $\mathcal{M}_2$. This is not so obvious in (17). But Igusa [16] has also obtained a beautiful formula which expresses χ_{10} as a product of theta constants $\vartheta[{}^{\vec a}_{\vec b}](0|\tau)$.

The theta function with characteristics $(\vec a, \vec b)$ for $\vec a, \vec b \in \mathbb{R}^g$ is defined by the sum

$$\vartheta\begin{bmatrix} \vec a \\ \vec b \end{bmatrix}(\vec z|\tau) = \sum_{\vec n \in \mathbb{Z}^g} e^{2\pi i\left[\frac{1}{2}(\vec n + \vec a)\tau(\vec n + \vec a) + (\vec n + \vec a)(\vec z + \vec b)\right]} \tag{19}$$

where $\vec z \in \mathbb{C}^g$, $\tau \in \mathfrak{S}_g$

We call characteristics $(\vec a, \vec b) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^{2g}$ even or odd depending on whether $4\vec a \cdot \vec b$ is even or odd. Then we have [16]

$$\chi_{10}(\tau) = -2^{-14} \prod_{(\vec a, \vec b) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^4} \vartheta\begin{bmatrix} \vec a \\ \vec b \end{bmatrix}(0|\tau) \tag{20}$$

where $(\vec{a},\vec{b})$ runs over ten even characteristics. It is known that for genus two, the even theta constants do not vanish except when τ is equivalent to a diagonal matrix by Γ_2 [17]. Therefore χ_{10} has no zeros on $\mathcal{M}_2$.

Let us investigate the behavior of the integrand at the degeneration of Riemann surface. For the deformation depicted in fig.1 , we substitute τ by (14) and take the limit $t \to 0$. We get

$$\chi_{10}(\tau(t)) \propto \Delta(\tau_{11})\Delta(\tau_{22})t^2 + O(t^3) \tag{21}$$

Therefore the integrand factorizes into the product of two one-loop amplitudes together with the second order pole coming from thin tube in between.

Similarly for the deformation depicted in fig.2, we substitute τ by (15) and take the limit $t \to 0$. We get

$$\begin{aligned}\chi_{10}(\tau(t)) = {}& 2^{-6} e^{2\pi i c_1} t \left(\vartheta_2(0|\tau)\vartheta_3(0|\tau)\vartheta_4(0|\tau)\right)^4 \\ & \times \left(\vartheta_1\left(\tfrac{a}{2}|\tau\right)\vartheta_2\left(\tfrac{a}{2}|\tau\right)\vartheta_3\left(\tfrac{a}{2}|\tau\right)\vartheta_4\left(\tfrac{a}{2}|\tau\right)\right)^2 \\ & + O(t^{\frac{3}{2}})\end{aligned} \tag{22}$$

After some calculation we find

$$\begin{aligned} & dV(\det \mathrm{Im}\,\tau)^{-10}|\chi_{10}|^{-2} \\ & \sim d^2\tau\, d^2a\, d^2t\, [|t|^{-4}(-\log|t|)^{-13}] \exp(4\pi \,\mathrm{Im}\, C_1) \\ & \times \left| qv \prod_{n=1}^{\infty}(1-q^n)^{20}(1-q^n v)^2(1-q^{n-1}v^{-1})^2 \right|^{-2}\end{aligned} \tag{23}$$

where $q=\exp(2\pi i\tau)$, $v=\exp(2\pi i a)$.

In (23), the behavior at $t \sim 0$ is

$$|t|^{-4}(-\log|t|)^{-13} \tag{24}$$

which is exactly the same behavior as eq. (22) in [4]. For the coefficient of the leading term, compare it with that of 1-loop amplitude with two tachyon insertions. (see for example [1])

$$\begin{aligned} A_2 \sim & \int d^2\tau\, d^2a\, (\mathrm{Im}\,\tau)^{-13} \exp[4\pi(\mathrm{Im}\, a)^2/\mathrm{Im}\,\tau] \\ & \times \left| qv \prod_{n=1}^{\infty}(1-q^n)^{20}(1-q^n v)^2(1-q^{n-1}v^{-1}) \right|^{-2}\end{aligned} \tag{25}$$

From (23),(25), they coincide except for the zero mode factor.

For zero mode factor, the correspondence between $\mathrm{Im}\, C_1$ and $(\mathrm{Im}\,a)^2/\mathrm{Im}\,\tau$ is easily verified by examining their behavior under 1-loop modular transformations. We consider subgroup $SL(2,Z) \subset Sp(4,Z)$ generated by following two generators S,T.

$$S = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

Transformations of τ_{ij} ($t \sim 0$) are

$$S: \begin{pmatrix} \tau & a \\ a & \frac{i}{2\pi}\log\frac{1}{t} + C_1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{\tau} & \frac{a}{\tau} \\ \frac{a}{\tau} & \frac{i}{2\pi}\log\frac{1}{t} + C_1 - \frac{a^2}{\tau} \end{pmatrix} \tag{27a}$$

$$T: \begin{pmatrix} \tau & a \\ a & \frac{i}{2\pi}\log\frac{1}{t} + C_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \tau+1 & a \\ a & \frac{i}{2\pi}\log\frac{1}{t} + C_1 \end{pmatrix} \tag{27b}$$

(we neglect higher order terms in t.)

Note that transformations of τ and a are exactly that of 1-loop modular transformations and transformations of C_1 is

$$S: \; C_1 \to C_1 - \frac{a^2}{\tau} \qquad T: \; C_1 \to C_1 \tag{28}$$

From (28), it can be easily verified that modular transformations of Im C_1 are same as that of $(\mathrm{Im}\ a)^2/\mathrm{Im}\tau$. As a result, although the forms of zero mode factor seem different, there is no problem in modular invariance.

In either case, for fig.1 or fig.2, we get natural results from physical point of view, i.e. two loop amplitudes reduce to desired one loop amplitudes.

We have argued that the integrand of the two-loop vacuum amplitude for closed bosonic strings can be expressed in terms of a cusp form of weight ten, using the elements of the period matrix as coordinates on the moduli space.

Our arguement does not directly generalize to higher genus cases, because we relied on some special properties of g=2. However, our results may bear some hints on the explicit calculations of the general multiloop amplitudes.

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Figure captions

Fig.1.(a),(b) Pinching a homologically trivial cycle.

(c) Deformation which is conformally equivalent to (b).

Fig.2.(a),(b) Pinching a homologically nontrivial cycle.

(c) Deformation which is conformally equivalent to (b).

a

b

c

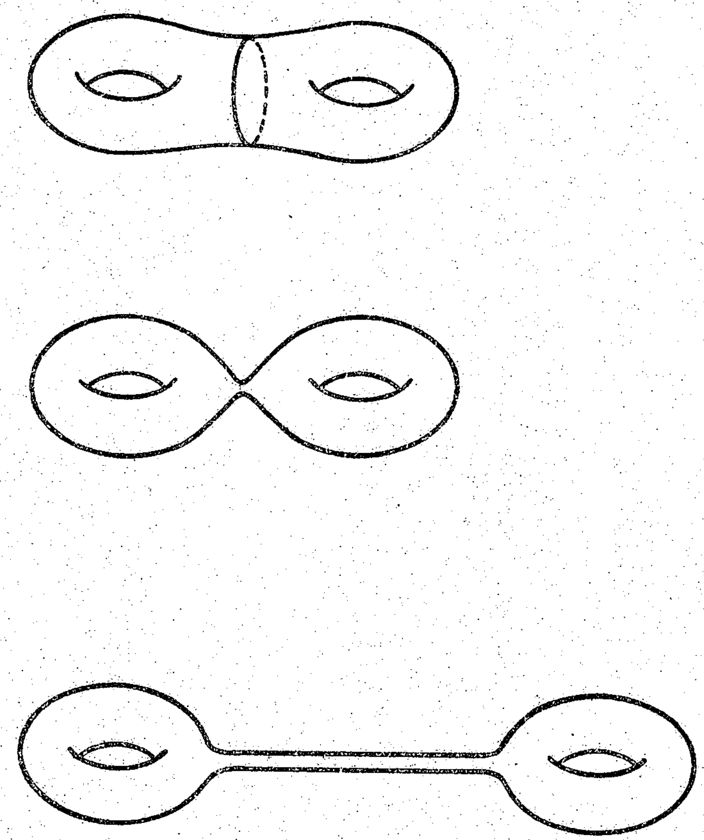

fig. 1

a

b

c

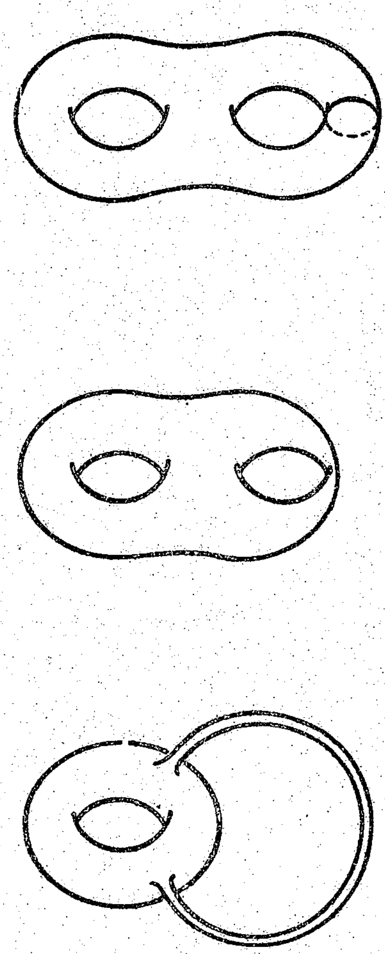

fig. 2