# Wronskian/Casoratian Identities and <br> their Application to Quantum Mechanical Systems 

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#### Abstract

Corresponding to a certain Wronskian identity, we present two types of new Casoratian identities. We apply these identities to the Darboux transformations of quantum mechanical systems. The Wronskian identity is applied to the ordinary quantum mechanics, and the two Casoratian identities are applied to the discrete quantum mechanics with pure imaginary and real shifts, respectively.


## 1 Introduction

The Wronski determinant, Wronskian, is a useful tool for analysis. For example, the linear independence of $n$ functions can be checked by calculating their Wronskian. The Wronskian is a determinant of derivatives of functions. Its difference version is the Casorati determinant, Casoratian. Corresponding to the type of difference operations, there are several types of Casoratians. The Wronskian and Casoratians appear in the study of quantum mechanical systems, especially for the deformations by multi-step Darboux transformations.

We have considered three types of quantum mechanical systems: oQM, idQM and rdQM [1]. Based on them we have studied the new type of orthogonal polynomials, exceptional or multi-indexed polynomials [2]-[14]. The Schrödinger equation is a second order differential equation for ordinary quantum mechanics (oQM), and a second order difference equation for discrete quantum mechanics (dQM). Discrete quantum mechanics with pure imaginary shifts (idQM) is dQM for the continuous coordinate, and discrete quantum mechanics with real shifts (rdQM) is dQM for the discrete coordinate. In our study of deformations of these systems by multi-step Darboux transformations [4, 5, , 7, 8, , 9, 13, 14, the following Wronskian
and Casoratian identities $(n \geq 0)$ have played a very important role (See $\S 2$ for definitions of Casoratians $\mathrm{W}_{\gamma}$ and $\mathrm{W}_{\mathrm{C}}$ ):

$$
\begin{align*}
\mathrm{oQM}: & \mathrm{W}\left[\mathrm{~W}\left[f_{1}, f_{2}, \ldots, f_{n}, g\right], \mathrm{W}\left[f_{1}, f_{2}, \ldots, f_{n}, h\right]\right](x) \\
& =\mathrm{W}\left[f_{1}, f_{2}, \ldots, f_{n}\right](x) \mathrm{W}\left[f_{1}, f_{2}, \ldots, f_{n}, g, h\right](x),  \tag{1}\\
\operatorname{idQM}: & \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n}, g\right], \mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n}, h\right]\right](x) \\
& =\mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n}\right](x) \mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n}, g, h\right](x),  \tag{2}\\
\operatorname{rdQM}: & \mathrm{W}_{\mathrm{C}}\left[\mathrm{~W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}, g\right], \mathrm{W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}, h\right]\right](x) \\
& =\mathrm{W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}\right](x+1) \mathrm{W}_{\mathrm{C}}\left[f_{1}, f_{2}, \ldots, f_{n}, g, h\right](x) . \tag{3}
\end{align*}
$$

There is a nice generalization of the Wronskian identity (11) [15]. It is Theorem1(10), and the above identity corresponds to $m=2$ case. It is expected that Casoratian identities (2) and (3) have also similar generalizations. The first purpose of this paper is to find Casoratian identities corresponding to Theorem1. They are presented as Theorem2 and 3,

The second purpose of this paper is the application of Theorem13. We apply them to the deformation of quantum mechanical systems by multi-step Darboux transformations. We consider quantum mechanical systems, whose Schrödinger equation is (27). For any solution $\tilde{\phi}(x)$ of the Schrödinger equation $\mathcal{H} \tilde{\phi}(x)=\tilde{\mathcal{E}} \tilde{\phi}(x)$, which may not belong to the Hilbert space (namely, may not be square integrable), the Hamiltonian can be written as $\mathcal{H}=\hat{\mathcal{A}}^{\dagger} \hat{\mathcal{A}}+\tilde{\mathcal{E}}$, where $\hat{\mathcal{A}}$ is some operator depending on $\tilde{\phi}$ and satisfies $\hat{\mathcal{A}} \tilde{\phi}(x)=0$ (some modification is needed for rdQM). The Darboux transformation with the seed solution $\tilde{\phi}$ maps the Hamiltonian $\mathcal{H}$ to $\mathcal{H}^{\text {new }}=\hat{\mathcal{A}} \hat{\mathcal{A}}^{\dagger}+\tilde{\mathcal{E}}$, and the transformed eigenfunctions $\phi_{n}^{\text {new }}(x)=$ $\hat{\mathcal{A}} \phi_{n}(x)$ satisfy $\mathcal{H}^{\text {new }} \phi_{n}^{\text {new }}(x)=\mathcal{E}_{n} \phi_{n}^{\text {new }}(x)$. The operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^{\dagger}$ may have zero modes (in the Hilbert space). For example, when the eigenfunction $\phi_{n}$ is taken as a seed solution, $\hat{\mathcal{A}}$ has a zero mode, $\hat{\mathcal{A}} \phi_{n}(x)=0$. Therefore, a Darboux transformation deforms a system almost isospectrally. The property of the deformed system depends on the employed seed solution:

$$
\begin{array}{ccc}
\text { seed solution } & & \text { deformed system } \\
\text { virtual state wavefuntion } & \Rightarrow & \text { isospectral } \\
\text { eigenstate wavefunction } & \Rightarrow & \text { state deleted } \\
\text { pseudo virtual state wavefunction } & \Rightarrow & \text { state added }
\end{array}
$$

(Both virtual and pseudo virtual states do not belong to the Hilbert space. For a virtual state, both $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^{\dagger}$ have no zero mode. For a pseudo virtual state, $\hat{\mathcal{A}}$ has no zero mode but $\hat{\mathcal{A}}^{\dagger}$ has a zero mode with new energy eigenvalue. See [16] for more explicit conditions
for (pseudo) virtual state in oQM.) When the eigenfunctions of the original system are described by the orthogonal polynomial $P_{n}$, those of the deformed system are described by the multi-indexed orthogonal polynomials $P_{\mathcal{D}, n}$, where $\mathcal{D}$ is the set of the labels of the seed solutions. The characteristic feature of the multi-indexed orthogonal polynomials is the missing of degrees. When the set of missing degrees $\mathcal{I}=\mathbb{Z}_{\geq 0} \backslash\left\{\operatorname{deg} P_{\mathcal{D}, n} \mid n \in \mathbb{Z}_{\geq 0}\right\}$ is $\mathcal{I}=\{0,1, \ldots, \ell-1\}(\ell:$ a positive integer $)$, we call $P_{\mathcal{D}, n}$ a case-(1) multi-indexed polynomial, and otherwise we call it a case-(2) polynomial. The situation of case-(1) is called stable in [17]. When only the virtual state wavefunctions are used as seed solutions, the case-(1) multiindexed polynomials are obtained, and in the other combinations, the case-(2) multi-indexed polynomials are obtained. We consider the multi-step Darboux transformations using both virtual state wavefunctions labeled by $\mathcal{D}_{\mathrm{v}}$ and eigenstate wavefunctions labeled by $\mathcal{D}_{\mathrm{e}}$ as seed solutions. In this case, no state with new energy eigenvalue is added. We interpret this in two ways:

$$
\begin{align*}
& \text { (i) }: \mathcal{H} \xrightarrow{\text { virtual states and eigenstates of } \mathcal{H}} \mathcal{H}_{\mathcal{D}_{\mathrm{v}} \cup \mathcal{D}_{\mathrm{e}}} \\
& \text { (ii) }: \mathcal{H} \xrightarrow{\text { virtual states of } \mathcal{H}} \mathcal{H}_{\mathcal{D}_{\mathrm{v}}} \xrightarrow{\text { eigenstates of } \mathcal{H}_{\mathcal{D}_{\mathrm{v}}}} \mathcal{H}_{\mathcal{D}_{\mathrm{v}} \cup \mathcal{D}_{\mathrm{e}}} \tag{4}
\end{align*}
$$

The first interpretation (i) is straightforward. The second one (ii) consists of two steps. After deforming the original Hamiltonian $\mathcal{H}$ by the Darboux transformations with only the virtual state wavefunctions as seed solutions, we deform the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ by the Darboux transformations with the eigenstate wavefunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ as seed solutions. Corresponding to these two interpretations, the eigenfunctions of the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \cup \mathcal{D}_{\mathrm{e}}}$ are expressed in two ways, and they should agree. The agreement of these two expressions is shown by using the Wronskian and Casoratian identities Theorem 1 3,

This paper is organized as follows. In section 2 the Wronskian identities are recapitulated and two types of the Casoratian identities are presented. In section 3, Theorem 1 , 2 and 3 are applied to quantum mechanical systems, oQM, idQM and rdQM, respectively. Section 4 is for a summary and comments.

## 2 Wronskian and Casoratian Identities

In this section, after recapitulating the known Wronskian identities, we derive two types of Casoratian identities. To the best of our knowledge, Theorem 2 and 3 are new results.

### 2.1 Wronskian identities

In our study of the deformations of oQM systems [4, 7], the Wronskian identity (1) has been used extensively. This identity (11) has an interesting generalization [15], Theorem1, whose $m=2$ case corresponds to (11). We present the definition of the Wronskian, its basic properties and Theorem 1 , which is proved in [15]. We also present its Corollary.

Definition 1 The Wronski determinant of a set of $n$ functions $\left\{f_{k}(x)\right\}_{k=1}^{n}$, W, is defined by

$$
\begin{equation*}
\mathrm{W}\left[f_{1}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} \operatorname{det}\left(\frac{d^{j-1} f_{k}(x)}{d x^{j-1}}\right)_{1 \leq j, k \leq n}, \tag{5}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}[\cdot](x)=1$ ).
Lemma 1.1 For functions $f(x)$ and $g(x)$,

$$
\begin{equation*}
\frac{d}{d x} \frac{f(x)}{g(x)}=\frac{\mathrm{W}[g, f](x)}{g(x)^{2}} \tag{6}
\end{equation*}
$$

Lemma 1.2 For functions $f_{1}(x), \ldots, f_{n}(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}\left[1, f_{1}, \ldots, f_{n}\right](x)=\mathrm{W}\left[f_{1}^{\prime}, \ldots, f_{n}^{\prime}\right](x) \tag{7}
\end{equation*}
$$

where $f_{k}^{\prime}(x) \stackrel{\text { def }}{=} \frac{d}{d x} f_{k}(x)$.
Proposition 1.1 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}\left[g f_{1}, \ldots, g f_{n}\right](x)=(g(x))^{n} \mathrm{~W}\left[f_{1}, \ldots, f_{n}\right](x) \tag{8}
\end{equation*}
$$

Proposition 1.2 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}\left[g, f_{1}, \ldots, f_{n}\right](x)=(g(x))^{1-n} \mathrm{~W}\left[\mathrm{~W}\left[g, f_{1}\right], \ldots, \mathrm{W}\left[g, f_{n}\right]\right](x) \tag{9}
\end{equation*}
$$

Theorem 1 [15] For functions $f_{1}(x), \ldots, f_{n}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(n \geq 0, m \geq 1)$,

$$
\begin{align*}
& \left(\mathrm{W}\left[f_{1}, \ldots, f_{n}\right](x)\right)^{m-1} \mathrm{~W}\left[f_{1}, \ldots, f_{n}, u_{1}, \ldots, u_{m}\right](x) \\
= & \mathrm{W}\left[\mathrm{~W}\left[f_{1}, \ldots, f_{n}, u_{1}\right], \ldots, \mathrm{W}\left[f_{1}, \ldots, f_{n}, u_{m}\right]\right](x) \tag{10}
\end{align*}
$$

This theorem is proved by induction on $n$. By applying Proposition 1.1 to Theorem (for later use, $n$ is changed to $l$ ), we obtain the following.

Corollary 1 For functions $f_{1}(x), \ldots, f_{l}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(l \geq 0, m \geq 1)$,

$$
\begin{equation*}
\frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x)}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right](x)}=\mathrm{W}\left[\frac{\mathrm{~W}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \ldots, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}\right](x) \tag{11}
\end{equation*}
$$

### 2.2 Casoratian identities for idQM

Next let us consider the Casoratian appearing in idQM. In our study of the deformations of idQM systems [4, 9, 14], the Casoratian identity (2) has been used extensively. Parallel to the Wronskian in $\S 2.1$, we present the definition of the Casoratian, its basic properties, Theorem and Corollary. Here we present their proofs. We use the convention $\prod_{j=n}^{n-1} a_{j}=1$.

Definition 2 The Casorati determinant of a set of $n$ functions $\left\{f_{k}(x)\right\}_{k=1}^{n}, \mathrm{~W}_{\gamma}$, is defined by

$$
\begin{equation*}
\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} i^{\frac{1}{2} n(n-1)} \operatorname{det}\left(f_{k}\left(x_{j}^{(n)}\right)\right)_{1 \leq j, k \leq n}, \quad x_{j}^{(n)} \stackrel{\text { def }}{=} x+i\left(\frac{n+1}{2}-j\right) \gamma \tag{12}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}_{\gamma}[\cdot](x)=1$ ). Here $\gamma$ is a nonzero real constant and $i$ is the imaginary unit.

Lemma 2.1 For functions $f(x)$ and $g(x)$,

$$
\begin{equation*}
\frac{f\left(x-i \frac{\gamma}{2}\right)}{g\left(x-i \frac{\gamma}{2}\right)}-\frac{f\left(x+i \frac{\gamma}{2}\right)}{g\left(x+i \frac{\gamma}{2}\right)}=\frac{\mathrm{W}_{\gamma}[g, f](x)}{i g\left(x-i \frac{\gamma}{2}\right) g\left(x+i \frac{\gamma}{2}\right)} \tag{13}
\end{equation*}
$$

Proof: Direct calculation shows this lemma.

Lemma 2.2 For functions $f_{1}(x), \ldots, f_{n}(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\gamma}\left[1, f_{1}, \ldots, f_{n}\right](x)=i^{n} \mathrm{~W}_{\gamma}\left[D f_{1}, \ldots, D f_{n}\right](x) \tag{14}
\end{equation*}
$$

where $D f_{k}(x) \stackrel{\text { def }}{=} f_{k}\left(x-i \frac{\gamma}{2}\right)-f_{k}\left(x+i \frac{\gamma}{2}\right)$.
Proof: By definition, the LHS is written as a determinant. In the determinant, subtract the $j$-th row from the $(j+1)$-th row $(j=n, \ldots, 2,1$ in turn $)$, and expand the determinant along the 1-st column. Since $x_{j+1}^{(n+1)}=x_{j}^{(n)}-i \frac{\gamma}{2}$ and $x_{j}^{(n+1)}=x_{j}^{(n)}+i \frac{\gamma}{2}$, we obtain the RHS.
Remark: The LHS of (13) is expressed as $D \frac{f}{g}(x)$.
Proposition 2.1 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\gamma}\left[g f_{1}, \ldots, g f_{n}\right](x)=\prod_{j=1}^{n} g\left(x_{j}^{(n)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x) \tag{15}
\end{equation*}
$$

Proof: By definition, the LHS is written as a determinant. In the determinant, for each $j$-th row, move the factor $g\left(x_{j}^{(n)}\right)$ out of the determinant.

Proposition 2.2 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\gamma}\left[g, f_{1}, \ldots, f_{n}\right](x)=g\left(x_{1}^{(n+1)}\right) \prod_{j=1}^{n} \frac{1}{g\left(x_{j}^{(n+1)}\right)} \cdot \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[g, f_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[g, f_{n}\right]\right](x) \tag{16}
\end{equation*}
$$

Remark: The overall factor in the RHS is written as $\prod_{j=2}^{n} g\left(x_{j}^{(n+1)}\right)^{-1}$ for $n \geq 1$.
Proof:

$$
\begin{aligned}
\text { LHS } & \stackrel{(\mathrm{i})}{=} \prod_{j=1}^{n+1} g\left(x_{j}^{(n+1)}\right) \cdot \mathrm{W}_{\gamma}\left[1, \frac{f_{1}}{g}, \cdots, \frac{f_{n}}{g}\right](x) \stackrel{(\mathrm{ii})}{=} \prod_{j=1}^{n+1} g\left(x_{j}^{(n+1)}\right) \cdot i^{n} \mathrm{~W}_{\gamma}\left[D \frac{f_{1}}{g}, \cdots, D \frac{f_{n}}{g}\right](x) \\
& \stackrel{(\mathrm{iii)})}{=} \prod_{j=1}^{n+1} g\left(x_{j}^{(n+1)}\right) \cdot i^{n} \prod_{j=1}^{n} \frac{1}{i g\left(x_{j}^{(n)}-i \frac{\gamma}{2}\right) g\left(x_{j}^{(n)}+i \frac{\gamma}{2}\right)} \cdot \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[g, f_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[g, f_{n}\right]\right](x) \\
& \stackrel{(\mathrm{iv})}{=} \mathrm{RHS},
\end{aligned}
$$

where we have used (i): Proposition2.1, (ii): Lemma2.2, (iii): Lemma 2.1 (with the remark below Lemma 2.2) and Proposition2.1, (iv): $x_{j}^{(n)}-i \frac{\gamma}{2}=x_{j+1}^{(n+1)}$ and $x_{j}^{(n)}+i \frac{\gamma}{2}=x_{j}^{(n+1)}$.

The following theorem is a new result.
Theorem 2 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(n \geq 0, m \geq 1)$,

$$
\begin{align*}
& \prod_{j=1}^{m-1} \mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right]\left(x_{j}^{(m-1)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}, u_{1}, \ldots, u_{m}\right](x) \\
= & \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}, u_{m}\right]\right](x) . \tag{17}
\end{align*}
$$

Proof: Let us prove this theorem by induction on $n$. It is trivial for $n=0$. For $n>0$, since it is trivial for $f_{1}(x)=0$, we assume $f_{1}(x) \neq 0$. For $n=1$, we have

$$
\begin{aligned}
& \prod_{j=1}^{m-1} \mathrm{~W}_{\gamma}\left[f_{1}\right]\left(x_{j}^{(m-1)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, u_{1}, \ldots, u_{m}\right](x) \\
& \stackrel{(\mathrm{i})}{m-1} \prod_{j=1}^{m} f_{1}\left(x_{j}^{(m-1)}\right) \cdot \prod_{j=2}^{m} \frac{1}{f_{1}\left(x_{j}^{(m+1)}\right)} \cdot \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, u_{m}\right]\right](x) \\
& \stackrel{(\mathrm{iii})}{=} \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, u_{m}\right]\right](x),
\end{aligned}
$$

where we have used (i): Proposition[2.2, (ii): $x_{j}^{(m-1)}=x_{j+1}^{(m+1)}$. Hence $n=1$ case holds. Assume that (17) holds till $n(n \geq 1)$. Then we have

$$
\mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n+1}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n+1}, u_{m}\right]\right](x)
$$

$$
\begin{aligned}
& \stackrel{(\mathrm{i})}{=} \mathrm{W}_{\gamma}\left[g \mathrm{~W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right], \mathrm{W}_{\gamma}\left[f_{1}, u_{1}\right]\right], \ldots,\right. \\
& \\
& \left.\quad g \mathrm{~W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right], \mathrm{W}_{\gamma}\left[f_{1}, u_{m}\right]\right]\right](x) \quad\left(g(x) \stackrel{\text { def }}{=} \prod_{j=2}^{n+1} \frac{1}{f_{1}\left(x_{j}^{(n+2)}\right)}\right) \\
& \stackrel{\text { (ii) }}{=} \prod_{l=1}^{m} g\left(x_{l}^{(m)}\right) \cdot \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right], \mathrm{W}_{\gamma}\left[f_{1}, u_{1}\right]\right], \ldots,\right. \\
& \\
& \left.\qquad \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right], \mathrm{W}_{\gamma}\left[f_{1}, u_{m}\right]\right]\right](x) \\
& \stackrel{\text { (iii) }}{=} \prod_{l=1}^{m} g\left(x_{l}^{(m)}\right) \cdot \prod_{j=1}^{m-1} \mathrm{~W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right]\right]\left(x_{j}^{(m-1)}\right) \\
& \quad \times \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, f_{2}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, f_{n+1}\right], \mathrm{W}_{\gamma}\left[f_{1}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, u_{m}\right]\right](x) \\
& \left.\stackrel{\text { (iv) }}{=} \prod_{l=1}^{m} g\left(x_{l}^{(m)}\right) \cdot \prod_{j=1}^{m-1}\left(\prod_{l=2}^{n} f_{1}\left(x_{j}^{(m-1)}+i\left(\frac{n+2}{2}-l\right) \gamma\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n+1}\right]\right]\left(x_{j}^{(m-1)}\right)\right) \\
& \quad \times \prod_{l=2}^{n+m} f_{1}\left(x_{l}^{(n+m+1)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n+1}, u_{1}, \ldots, u_{m}\right](x) \\
& \stackrel{\text { (v) }}{=} \prod_{j=1}^{m-1} \mathrm{~W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n+1}\right]\left(x_{j}^{(m-1)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, f_{2}, \ldots, f_{n+1}, u_{1}, \ldots, u_{m}\right](x),
\end{aligned}
$$

where we have used (i): Proposition2.2, (ii): Proposition2.1, (iii): induction assumption, (iv): Proposition[2.2, (v): calculation of $f_{1}$ factors. Therefore $n+1$ case also holds. This concludes the induction proof of (17).
The Casoratian identity (2) corresponds to $m=2$ case of Theorem2,
We present a corollary of Theorem2 (for later use, $n$ is changed to $l$ ).

Corollary 2 For functions $f_{1}(x), \ldots, f_{l}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(l \geq 0, m \geq 1)$,

$$
\begin{align*}
& \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m}{2} \gamma\right)}} \\
& =\mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x) \tag{18}
\end{align*}
$$

where $w(x) \stackrel{\text { def }}{=} \sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{\gamma}{2}\right)}$.
Proof:

$$
\operatorname{RHS} \stackrel{(\mathrm{i})}{=} \prod_{j=1}^{m} \frac{1}{w\left(x_{j}^{(l)}\right)} \cdot \mathrm{W}_{\gamma}\left[\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right], \ldots, \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]\right](x)
$$

$$
\begin{aligned}
& \stackrel{(\text { iii })}{=} \prod_{j=1}^{m} \frac{1}{w\left(x_{j}^{(l)}\right)} \cdot \prod_{j=1}^{m-1} \mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x_{j}^{(m-1)}\right) \cdot \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x) \\
& \stackrel{(\text { iii) }}{=} \text { LHS, }
\end{aligned}
$$

where we have used (i): Proposition[2.1, (ii): Theorem2 (iii): direct calculation.
Remark: We regard the square root function $\sqrt{ }$ in Corollary 2 as a complex function.
The Casoratian $\mathrm{W}_{\gamma}$ reduces to the Wronskian W in the $\gamma \rightarrow 0$ limit.

## Proposition 2.3

$$
\begin{equation*}
\lim _{\gamma \rightarrow 0} \gamma^{-\frac{1}{2} n(n-1)} \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x)=\mathrm{W}\left[f_{1}, \ldots, f_{n}\right](x) . \tag{19}
\end{equation*}
$$

Proof: For a determinant of an $n \times n$ matrix, let us define the operation $O_{m}(1 \leq m \leq n-1)$ as follows: Subtract the $j$-th row from the $(j+1)$-th row $(j=n-1, n-2, \ldots, m$ in turn). By definition, $\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x)$ is written as a determinant. Apply the operations $O_{m}$ ( $m=1,2, \ldots, n-1$ in turn) to the determinant. Then the ( $j, k$ )-element of the determinant becomes

$$
\begin{aligned}
& \sum_{r=0}^{j-1}(-1)^{r}\binom{j-1}{r} f_{k}\left(x_{j-r}^{(n)}\right)=\sum_{r=0}^{j-1}(-1)^{r}\binom{j-1}{r} f_{k}\left(x+i \frac{n-j}{2} \gamma+i\left(r-\frac{j-1}{2}\right) \gamma\right) \\
& \stackrel{(\mathrm{i})}{=} \sum_{r=0}^{j-1}(-1)^{r}\binom{j-1}{r}\left(\sum_{s=0}^{j-1} \frac{1}{s!} \frac{d^{s}}{d x^{s}} f_{k}\left(x+i \frac{n-j}{2} \gamma\right)\left(i\left(r-\frac{j-1}{2}\right) \gamma\right)^{s}+O\left(\gamma^{j}\right)\right) \\
&= \sum_{s=0}^{j-1} \frac{(i \gamma)^{s}}{s!} \frac{d^{s}}{d x^{s}} f_{k}\left(x+i \frac{n-j}{2} \gamma\right) \sum_{r=0}^{j-1}(-1)^{r}\binom{j-1}{r}\left(r-\frac{j-1}{2}\right)^{s}+O\left(\gamma^{j}\right) \\
& \stackrel{(\text { ii })}{=}(-i \gamma)^{j-1} \frac{d^{j-1}}{d x^{j-1}} f_{k}\left(x+i \frac{n-j}{2} \gamma\right)+O\left(\gamma^{j}\right) \\
&=(-i \gamma)^{j-1} \frac{d^{j-1} f_{k}(x)}{d x^{j-1}} \times(1+O(\gamma)),
\end{aligned}
$$

where we have used (i): Taylor expansion, (ii): the following sum formula

$$
\sum_{r=0}^{j-1}(-1)^{r}\binom{j-1}{r}\left(r-\frac{j-1}{2}\right)^{s}=(-1)^{j-1}(j-1)!\delta_{s, j-1} \quad(0 \leq s \leq j-1) .
$$

Thus we have

$$
\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{n}\right](x)=i^{\frac{1}{2} n(n-1)} \operatorname{det}\left((-i \gamma)^{j-1} \frac{d^{j-1} f_{k}(x)}{d x^{j-1}} \times(1+O(\gamma))\right)_{1 \leq j, k \leq n}
$$

$$
\begin{aligned}
& =i^{\frac{1}{2} n(n-1)} \prod_{j=1}^{n}(-i \gamma)^{j-1} \cdot \operatorname{det}\left(\frac{d^{j-1} f_{k}(x)}{d x^{j-1}} \times(1+O(\gamma))\right)_{1 \leq j, k \leq n} \\
& =\gamma^{\frac{1}{2} n(n-1)} \mathrm{W}\left[f_{1}, \ldots, f_{n}\right](x) \times(1+O(\gamma)) .
\end{aligned}
$$

By multiplying $\gamma^{-\frac{1}{2} n(n-1)}$ and taking the $\gamma \rightarrow 0$ limit, we obtain (19).
By multiplying appropriate powers of $\gamma$ and taking the $\gamma \rightarrow 0$ limit, the properties of the Casoratian $\mathrm{W}_{\gamma}$ presented in this subsection reduce to those of the Wronskian W in $\S$ [2.1.

### 2.3 Casoratian identities for rdQM

Next let us consider the Casoratian appearing in rdQM. In our study of the deformations of rdQM systems [5, 8, 13], the Casoratian identity (3) has been used extensively. Parallel to the Wronskian in $\S 2.1$, we present the definition of the Casoratian, its basic properties, Theorem and Corollary. Since their proofs are similar to those of $\S 2.2$, we omit them.

Definition 3 The Casorati determinant of a set of $n$ functions $\left\{f_{k}(x)\right\}_{k=1}^{n}, \mathrm{~W}_{\mathrm{C}}$, is defined by

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}\right](x) \stackrel{\text { def }}{=} \operatorname{det}\left(f_{k}(x+j-1)\right)_{1 \leq j, k \leq n} \tag{20}
\end{equation*}
$$

(for $n=0$, we set $\mathrm{W}_{\mathrm{C}}[\cdot](x)=1$ ).

Lemma 3.1 For functions $f(x)$ and $g(x)$,

$$
\begin{equation*}
\frac{f(x+1)}{g(x+1)}-\frac{f(x)}{g(x)}=\frac{\mathrm{W}_{\mathrm{C}}[g, f](x)}{g(x) g(x+1)} \tag{21}
\end{equation*}
$$

Lemma 3.2 For functions $f_{1}(x), \ldots, f_{n}(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[1, f_{1}, \ldots, f_{n}\right](x)=\mathrm{W}_{\mathrm{C}}\left[D f_{1}, \ldots, D f_{n}\right](x) \tag{22}
\end{equation*}
$$

where $D f_{k}(x) \stackrel{\text { def }}{=} f_{k}(x+1)-f_{k}(x)$.

Remark: The LHS of (21) is expressed as $D \frac{f}{g}(x)$.
Proposition 3.1 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[g f_{1}, \ldots, g f_{n}\right](x)=\prod_{j=1}^{n} g(x+j-1) \cdot \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}\right](x) \tag{23}
\end{equation*}
$$

Proposition 3.2 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $g(x)(n \geq 0)$,

$$
\begin{equation*}
\mathrm{W}_{\mathrm{C}}\left[g, f_{1}, \ldots, f_{n}\right](x)=g(x) \prod_{j=1}^{n} \frac{1}{g(x+j-1)} \cdot \mathrm{W}_{\mathrm{C}}\left[\mathrm{~W}_{\mathrm{C}}\left[g, f_{1}\right], \ldots, \mathrm{W}_{\mathrm{C}}\left[g, f_{n}\right]\right](x) \tag{24}
\end{equation*}
$$

Remark: The overall factor in the RHS is written as $\prod_{j=2}^{n} g(x+j-1)^{-1}$ for $n \geq 1$.
The following theorem is a new result.
Theorem 3 For functions $f_{1}(x), \ldots, f_{n}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(n \geq 0, m \geq 1)$,

$$
\begin{align*}
& \prod_{j=1}^{m-1} \mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}\right](x+j) \cdot \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}, u_{1}, \ldots, u_{m}\right](x) \\
= & \mathrm{W}_{\mathrm{C}}\left[\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}, u_{1}\right], \ldots, \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{n}, u_{m}\right]\right](x) . \tag{25}
\end{align*}
$$

The Casoratian identity (3) corresponds to $m=2$ case of Theorem3,
We present a corollary of Theorem3 (for later use, $n$ is changed to $l$ ).
Corollary 3 For functions $f_{1}(x), \ldots, f_{l}(x)$ and $u_{1}(x), \ldots, u_{m}(x)(l \geq 0, m \geq 1)$,

$$
\begin{align*}
& \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m)}} \\
& =\mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x), \tag{26}
\end{align*}
$$

where $w(x) \stackrel{\text { def }}{=} \sqrt{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+1)}$.
Remark: We regard the square root function $\sqrt{ }$ in Corollary 3 as a real function. We have assumed $\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x)>0$.

## 3 Application to Quantum Mechanical Systems

In this section we consider the application of Theorem 1 3 to the deformation of quantum mechanical systems by multi-step Darboux transformations. As quantum mechanical systems, we consider oQM, idQM and rdQM, to which Theorem 1, 2 and 3 are applied respectively. For simplicity of presentation, we assume that rdQM systems are semi-infinite systems.

We assume that the original system with the Hamiltonian $\mathcal{H}$, which is hermitian and positive semi-definite, has the eigenfunctions (eigenstate wavefunctions) $\phi_{n}(x)$,

$$
\begin{equation*}
\mathcal{H} \phi_{n}(x)=\mathcal{E}_{n} \phi_{n}(x), \quad 0=\mathcal{E}_{0}<\mathcal{E}_{1}<\cdots \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{27}
\end{equation*}
$$

and the virtual state wavefunctions $\tilde{\phi}_{\mathrm{v}}(x)$ [7, 8, 9, 13, 14],

$$
\begin{equation*}
\mathcal{H} \tilde{\phi}_{\mathrm{v}}(x)=\tilde{\mathcal{E}}_{\mathrm{v}} \tilde{\phi}_{\mathrm{v}}(x), \quad \tilde{\mathcal{E}}_{\mathrm{v}}<0 \tag{28}
\end{equation*}
$$

The virtual state wavefunction has a definite sign for the physical value of $x$. As seed solutions of the multi-step Darboux transformations, we take both the virtual state wavefunctions $\tilde{\phi}_{\mathrm{v}}(x)\left(\mathrm{v} \in \mathcal{D}_{\mathrm{v}}\right)$ and the eigenfunctions $\phi_{n}(x)\left(n \in \mathcal{D}_{\mathrm{e}}\right)$. Here $\mathcal{D}_{\mathrm{v}}$ and $\mathcal{D}_{\mathrm{e}}$ are sets of labels of the virtual states and the eigenstates respectively, and we set them as

$$
\begin{equation*}
\mathcal{D}_{\mathrm{v}} \stackrel{\text { def }}{=}\left\{\mathrm{v}_{1}, \ldots, \mathrm{v}_{M_{\mathrm{v}}}\right\} \quad\left(\mathrm{v}_{j} \in \mathbb{Z}_{\geq 0}\right), \quad \mathcal{D}_{\mathrm{e}} \stackrel{\text { def }}{=}\left\{e_{1}, \ldots, e_{M_{\mathrm{e}}}\right\} \quad\left(e_{j} \in \mathbb{Z}_{\geq 0}\right) \tag{29}
\end{equation*}
$$

where $\mathrm{v}_{j}$ 's are mutually distinct and $e_{j}$ 's are mutually distinct. If there are two types of the virtual states, the label includes the type. By combining these, we set

$$
\begin{equation*}
\mathcal{D} \stackrel{\text { def }}{=} \mathcal{D}_{\mathrm{v}} \cup \mathcal{D}_{\mathrm{e}} \stackrel{\text { def }}{=}\left\{d_{1}, \ldots, d_{M}\right\}, \quad M \stackrel{\text { def }}{=} M_{\mathrm{v}}+M_{\mathrm{e}} \tag{30}
\end{equation*}
$$

(Exactly speaking, the index sets $\mathcal{D}_{\mathrm{v}}, \mathcal{D}_{\mathrm{e}}$ and $\mathcal{D}$ are ordered sets, but we do not care much about the order, because the deformed Hamiltonians $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}, \mathcal{H}_{\mathcal{D}_{\mathrm{e}}}$ and $\mathcal{H}_{\mathcal{D}}$ do not depend on the order.) We set seed solutions as $\psi_{j}(x)(j=1, \ldots, M)$, namely, $\psi_{j}(x)=\tilde{\phi}_{\mathrm{v}_{k}}(x)$ for $d_{j}=\mathrm{v}_{k}$ and $\psi_{j}(x)=\phi_{e_{k}}(x)$ for $d_{j}=e_{k}$. By the multi-step Darboux transformations with the seed solutions $\psi_{j}(x)(j \in \mathcal{D})$, the Hamiltonian $\mathcal{H}$ is deformed to $\mathcal{H}_{\mathcal{D}}$. The Schrödinger equation of the deformed system is

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}} \phi_{\mathcal{D} n}(x)=\mathcal{E}_{n} \phi_{\mathcal{D} n}(x) \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) \tag{31}
\end{equation*}
$$

If the Krein-Adler condition [18, 19],

$$
\begin{equation*}
\prod_{j=1}^{M_{\mathrm{e}}}\left(m-e_{j}\right) \geq 0 \quad\left(\forall m \in \mathbb{Z}_{\geq 0}\right) \tag{32}
\end{equation*}
$$

is satisfied (it is trivial for $\mathcal{D}_{\mathrm{e}}=\emptyset$ ), the norm of $\phi_{\mathcal{D} n}(x)$ becomes positive definite, $\left(\phi_{\mathcal{D} n}, \phi_{\mathcal{D} n}\right)>$ $0\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right)$ [18, 19, 4, 5]. This condition (32) means $\mathcal{D}_{\mathrm{e}}=\left\{0,1, \ldots, n_{0}\right\} \cup \bigcup_{l=1}^{L}\left\{j_{l}, j_{l}+1\right\}$ $\left(n_{0}+1, L, j_{l} \in \mathbb{Z}_{\geq 0}, n_{0}+1<j_{1}, j_{l}+2 \leq j_{l+1}, n_{0}+1+2 L=M_{\mathrm{e}},\left\{0,1, \ldots, n_{0}\right\}=\emptyset\right.$ for $n_{0}=-1$ and $\bigcup_{l=1}^{L} A_{l}=\emptyset$ for $L=0$ ), or equivalently, $\mathcal{D}_{\mathrm{e}}=\left\{0,1, \ldots, n_{0}\right\} \cup \bigcup_{l=1}^{L^{\prime}}\left\{k_{l}, k_{l}+1, \ldots, k_{l}+\right.$ $\left.2 r_{l}-1\right\}\left(n_{0}+1, L^{\prime}, k_{l}, r_{l}-1 \in \mathbb{Z}_{\geq 0}, n_{0}+1<k_{1}, k_{l}+2 r_{l}<k_{l+1}, n_{0}+1+\sum_{l=1}^{L^{\prime}} 2 r_{l}=M_{\mathrm{e}}\right)$. For oQM, it is shown that the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ with (32) is well-defined and hermitian (for an appropriate range of the parameters) [18, 19]. For dQM, it is conjectured that the
deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ with (32) is well-defined and hermitian (for an appropriate range of the parameters) [4, 5]. This is strongly supported by the positive definiteness of the norm. It is also supported by numerical calculation for each system. The eigenfunctions $\phi_{\mathcal{D} n}(x)$ are expressed in terms of the Wronskian/Casoratian.

Let us reinterpret this deformation as (4). First, by the multi-step Darboux transformations with the seed solutions $\tilde{\phi}_{\mathrm{v}}(x)\left(\mathrm{v} \in \mathcal{D}_{\mathrm{v}}\right)$, the Hamiltonian $\mathcal{H}$ is deformed to $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$. The Schrödinger equation of this deformed system is

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}_{\mathrm{v}}} \phi_{\mathcal{D}_{\mathrm{v}} n}(x)=\mathcal{E}_{n} \phi_{\mathcal{D}_{\mathrm{v}} n}(x) \quad\left(n \in \mathbb{Z}_{\geq 0}\right) \tag{33}
\end{equation*}
$$

The eigenfunctions $\phi_{\mathcal{D}_{\mathrm{v}} n}(x)$ are expressed in terms of the Wronskian/Casoratian, and the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ is well-defined and hermitian (for an appropriate range of the parameters). Second, by the multi-step Darboux transformations with the seed solutions $\phi_{\mathcal{D}_{\mathrm{v}} n}(x)\left(n \in \mathcal{D}_{\mathrm{e}}\right)$, the Hamiltonian $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ is deformed to $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}$. The Schrödinger equation of this deformed system is

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}_{\mathfrak{v}} \mathcal{D}_{\mathrm{e}}} \phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}(x)=\mathcal{E}_{n} \phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x) \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) . \tag{34}
\end{equation*}
$$

The eigenfunctions $\phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x)$ are expressed in terms of the Wronskian/Casoratian. If the Krein-Adler condition (32) is satisfied, the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}_{v} \mathcal{D}_{\mathrm{e}}}$ is well-defined and hermitian (for an appropriate range of the parameters). Since two deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}$ should be the same, two eigenfunctions $\phi_{\mathcal{D} n}(x)$ and $\phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x)$ must be the same (proportional). We will show this equality $\phi_{\mathcal{D} n}(x)=\phi_{\mathcal{D}_{\mathfrak{v}} \mathcal{D}_{\mathrm{e}} n}(x)$ by using the Wronskian/Casoratian identities Theorem13 (Corollary 13).

### 3.1 Application to oQM

First let us consider oQM. The virtual states are studied for the exactly solvable systems whose eigenfunctions are described by the Laguerre and Jacobi polynomials [7]. The Hamiltonian $\mathcal{H}$ of oQM has the following form:

$$
\begin{equation*}
\mathcal{H}=p^{2}+U(x) \tag{35}
\end{equation*}
$$

where $x$ is the coordinate and $p$ is the momentum, $p=-i \frac{d}{d x}$. The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ (31) is given by

$$
\begin{equation*}
\mathcal{H}_{\mathcal{D}}=p^{2}+U_{\mathcal{D}}(x), \quad U_{\mathcal{D}}(x)=U(x)-2 \partial_{x}^{2} \log \left|\mathrm{~W}\left[\psi_{1}, \ldots, \psi_{M}\right](x)\right| \tag{36}
\end{equation*}
$$

and its eigenfunctions $\phi_{\mathcal{D} n}(x)$ are given by (for example, see $\S 2$ of [4] and Appendix A of [20])

$$
\begin{equation*}
\phi_{\mathcal{D} n}(x)=\frac{\mathrm{W}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{n}\right](x)}{\mathrm{W}\left[\psi_{1}, \ldots, \psi_{M}\right](x)} \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) \tag{37}
\end{equation*}
$$

On the other hand, the eigenfunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ (33) are given by [7]

$$
\begin{equation*}
\phi_{\mathcal{D}_{\mathrm{v}} n}(x)=\frac{\mathrm{W}\left[\tilde{\phi}_{\mathrm{v}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{n}\right](x)}{\mathrm{W}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x)} \quad\left(n \in \mathbb{Z}_{\geq 0}\right) . \tag{38}
\end{equation*}
$$

So the eigenfunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}$ (34) are expressed as

$$
\begin{equation*}
\phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x)=\frac{\mathrm{W}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}, \phi_{\mathcal{D}_{\mathrm{v}} n}\right](x)}{\mathrm{W}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right](x)} \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) . \tag{39}
\end{equation*}
$$

We will show that two expressions (37) and (39) are actually identical by using the Wronskian identity, Corollary 1 .

Corollary 1 with the replacements $m \rightarrow m+1$ and $u_{m+1}=v$ becomes

$$
\begin{aligned}
& \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right](x)} \\
= & \mathrm{W}\left[\frac{\mathrm{~W}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \ldots, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, v\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}\right](x) .
\end{aligned}
$$

Dividing this equation by (11), we obtain

$$
\begin{equation*}
\frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x)}=\frac{\mathrm{W}\left[\frac{\mathrm{~W}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \ldots, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, v\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}\right](x)}{\mathrm{W}\left[\frac{\mathrm{~W}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}, \ldots, \frac{\mathrm{W}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{\mathrm{W}\left[f_{1}, \ldots, f_{l}\right]}\right](x)} \tag{40}
\end{equation*}
$$

This shows the equality $\phi_{\mathcal{D} n}(x)=\phi_{\mathcal{D}_{\mathfrak{v}} \mathcal{D}_{\mathrm{e}} n}(x)$ by the following replacements:

$$
\begin{equation*}
l=M_{\mathrm{v}}, \quad m=M_{\mathrm{e}}, \quad f_{j}=\tilde{\phi}_{\mathrm{v}_{j}}, \quad u_{j}=\phi_{e_{j}}, \quad v=\phi_{n} \tag{41}
\end{equation*}
$$

### 3.2 Application to idQM

Next let us consider idQM. The virtual states are studied for the exactly solvable systems whose eigenfunctions are described by the Wilson and Askey-Wilson 9], Meixner-Pollaczek and continuous Hahn [14] polynomials. The Hamiltonian $\mathcal{H}$ of idQM has the following form:

$$
\begin{equation*}
\mathcal{H}=\sqrt{V(x)} e^{\gamma p} \sqrt{V^{*}(x)}+\sqrt{V^{*}(x)} e^{-\gamma p} \sqrt{V(x)}-V(x)-V^{*}(x), \tag{42}
\end{equation*}
$$

where $x$ is the coordinate and $p$ is the momentum, $p=-i \frac{d}{d x}$, and $\gamma$ is a nonzero real constant. The potential function $V(x)$ is an analytic function of $x$ and the $*$-operation on an
analytic function $f(x)=\sum_{n} a_{n} x^{n}\left(a_{n} \in \mathbb{C}\right)$ is defined by $f^{*}(x)=\sum_{n} a_{n}^{*} x^{n}$, in which $a_{n}^{*}$ is the complex conjugation of $a_{n}$. The function $\sqrt{ }$ is the square root function as a complex function. The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ (31) is given by [4, (9, 14

$$
\begin{align*}
\mathcal{H}_{\mathcal{D}}= & \sqrt{V_{\mathcal{D}}(x)} e^{\gamma p} \sqrt{V_{\mathcal{D}}^{*}(x)}+\sqrt{V_{\mathcal{D}}^{*}(x)} e^{-\gamma p} \sqrt{V_{\mathcal{D}}(x)}-V_{\mathcal{D}}(x)-V_{\mathcal{D}}^{*}(x)+\mathcal{E}_{\mu}  \tag{43}\\
V_{\mathcal{D}}(x)= & \sqrt{V\left(x-i \frac{M}{2} \gamma\right) V^{*}\left(x-i \frac{M+2}{2} \gamma\right)} \quad\left(\mu \stackrel{\text { def }}{=} \min \left\{n \mid n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right\}\right) \\
& \times \frac{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}\right]\left(x+i \frac{\gamma}{2}\right)}{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}\right]\left(x-i \frac{\gamma}{2}\right)} \frac{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x-i \gamma)}{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x)} \tag{44}
\end{align*}
$$

and its eigenfunctions $\phi_{\mathcal{D} n}(x)$ are given by

$$
\begin{align*}
\phi_{\mathcal{D} n}(x)= & \left(\prod_{j=0}^{M-1} V\left(x+i\left(\frac{M}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{M}{2}-j\right) \gamma\right)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\psi_{1}, \ldots, \psi_{M}\right]\left(x+i \frac{\gamma}{2}\right)}}\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) . \tag{45}
\end{align*}
$$

On the other hand, the eigenfunctions and the potential function of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ (33) are given by [9, 14]

$$
\begin{align*}
\phi_{\mathcal{D}_{\mathrm{v}} n}(x)= & \left(\prod_{j=0}^{M_{\mathrm{v}}-1} V\left(x+i\left(\frac{M_{\mathrm{v}}}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{M_{\mathrm{v}}}{2}-j\right) \gamma\right)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right]\left(x+i \frac{\gamma}{2}\right)}}\left(n \in \mathbb{Z}_{\geq 0}\right),  \tag{46}\\
V_{\mathcal{D}_{\mathrm{v}}}(x)= & \sqrt{V\left(x-i \frac{M_{\mathrm{v}}}{2} \gamma\right) V^{*}\left(x-i \frac{M_{\mathrm{v}}+2}{2} \gamma\right)} \\
& \times \frac{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{v}}}\right]\left(x+i \frac{\gamma}{2}\right)}{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right]\left(x-i \frac{\gamma}{2}\right)} \frac{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x-i \gamma)}{\mathrm{W}_{\gamma}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x)} \tag{47}
\end{align*}
$$

So the eigenfunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}$ (34) are expressed as

$$
\begin{align*}
\phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x)= & \left(\prod_{j=0}^{M_{\mathrm{e}}-1} V_{\mathcal{D}_{\mathrm{v}}}\left(x+i\left(\frac{M_{\mathrm{e}}}{2}-j\right) \gamma\right) V_{\mathcal{D}_{\mathrm{v}}}^{*}\left(x-i\left(\frac{M_{\mathrm{e}}}{2}-j\right) \gamma\right)\right)^{\frac{1}{4}} \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) \\
& \times \frac{\mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}, \phi_{\mathcal{D}_{\mathrm{v}} n}\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right]\left(x+i \frac{\gamma}{2}\right)}} . \tag{48}
\end{align*}
$$

We will show that two expressions (45) and (48) are actually identical by using the Casoratian identity, Corollary2.

Corollary 2 is

$$
\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x) \quad\left(w(x) \stackrel{\text { def }}{=} \sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{\gamma}{2}\right)}\right)
$$

$$
\begin{aligned}
= & \sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m}{2} \gamma\right)} \\
& \times \mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x)
\end{aligned}
$$

and, by the replacements $m \rightarrow m+1$ and $u_{m+1}=v$, it becomes

$$
\begin{aligned}
& \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x) \\
& = \\
& \sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m+1}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m+1}{2} \gamma\right)} \\
& \\
& \quad \times \mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x) .
\end{aligned}
$$

From these two equations, we obtain

$$
\begin{align*}
& \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right]\left(x+i \frac{\gamma}{2}\right)}} \\
= & \left(\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m+1}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m+1}{2} \gamma\right)}{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m-1}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m-1}{2} \gamma\right)}\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x)}{\left.\sqrt{\mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}\right.}, \ldots, \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right]\left(x+i \frac{\gamma}{2}\right)} \tag{49}
\end{align*} .
$$

In the following, we consider the replacements (identification) (41). The eigenfunctions (46) of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ are expressed as

$$
\begin{align*}
& \phi_{\mathcal{D}_{\mathrm{v}} n}(x)=G(x) \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, v\right](x)}{w(x)}, \quad \phi_{\mathcal{D}_{\mathrm{v}} e_{j}}(x)=G(x) \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{j}\right](x)}{w(x)}  \tag{50}\\
& G(x) \stackrel{\text { def }}{=}\left(\prod_{j=0}^{l-1} V\left(x+i\left(\frac{l}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l}{2}-j\right) \gamma\right)\right)^{\frac{1}{4}} \tag{51}
\end{align*}
$$

By Proposition2.1, we have

$$
\begin{align*}
& \frac{\mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}, \phi_{\mathcal{D}_{\mathrm{v}} n}\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right]\left(x+i \frac{\gamma}{2}\right)}} \\
&=\left.\frac{\mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}\right.}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x) \\
& \sqrt{\mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[\frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right]\left(x+i \frac{\gamma}{2}\right)} \\
& \times \frac{\prod_{j=1}^{m+1} G\left(x_{j}^{(m+1)}\right)}{\sqrt{\prod_{j=1}^{m} G\left(x_{j}^{(m)}-i \frac{\gamma}{2}\right) G\left(x_{j}^{(m)}+i \frac{\gamma}{2}\right)}} \tag{52}
\end{align*}
$$

and a short calculation shows

$$
\begin{align*}
\frac{\prod_{j=1}^{m+1} G\left(x_{j}^{(m+1)}\right)}{\sqrt{\prod_{j=1}^{m} G\left(x_{j}^{(m)}-i \frac{\gamma}{2}\right) G\left(x_{j}^{(m)}+i \frac{\gamma}{2}\right)}}= & \left(\prod_{j=0}^{l-1} V\left(x+i\left(\frac{l+m}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l+m}{2}-j\right) \gamma\right)\right.  \tag{53}\\
& \left.\times \prod_{j=m}^{l+m-1} V\left(x+i\left(\frac{l+m}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l+m}{2}-j\right) \gamma\right)\right)^{\frac{1}{8}}
\end{align*}
$$

For the potential function $V_{\mathcal{D}_{\mathrm{v}}}(x)$ (47), a short calculation shows

$$
\begin{align*}
& \prod_{j=0}^{m-1} V_{\mathcal{D}_{\mathrm{v}}}\left(x+i\left(\frac{m}{2}-j\right) \gamma\right) V_{\mathcal{D}_{\mathrm{v}}}^{*}\left(x-i\left(\frac{m}{2}-j\right) \gamma\right) \\
&=\left(\prod_{j=0}^{m-1} V\left(x+i\left(\frac{l+m}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l+m}{2}-j\right) \gamma\right)\right. \\
&\left.\quad \times \prod_{j=l}^{l+m-1} V\left(x+i\left(\frac{l+m}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l+m}{2}-j\right) \gamma\right)\right)^{\frac{1}{2}} \\
& \times \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m+1}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m+1}{2} \gamma\right)}{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x-i \frac{m-1}{2} \gamma\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}\right]\left(x+i \frac{m-1}{2} \gamma\right)} . \tag{54}
\end{align*}
$$

From (154), (52) -(53) and (49), we obtain

$$
\begin{align*}
(48)= & \left(\prod_{j=0}^{l+m-1} V\left(x+i\left(\frac{l+m}{2}-j\right) \gamma\right) V^{*}\left(x-i\left(\frac{l+m}{2}-j\right) \gamma\right)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\sqrt{\mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right]\left(x-i \frac{\gamma}{2}\right) \mathrm{W}_{\gamma}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right]\left(x+i \frac{\gamma}{2}\right)}}  \tag{55}\\
= & (45),
\end{align*}
$$

namely the equality $\phi_{\mathcal{D} n}(x)=\phi_{\mathcal{D}_{\mathfrak{v}} \mathcal{D}_{\mathrm{e}} n}(x)$.

### 3.3 Application to rdQM

Next let us consider rdQM. The virtual states are studied for the exactly solvable systems whose eigenfunctions are described by the Racah and $q$-Racah [8, Meixner and little $q$-Jacobi (Laguerre) [13] polynomials. The Hamiltonian of $\operatorname{rdQM}, \mathcal{H}=\left(\mathcal{H}_{x, y}\right)$, is a tri-diagonal real symmetric (Jacobi) matrix and its rows and columns are indexed by integers $x$ and $y$, which take values in $\left\{0,1, \ldots, x_{\max }\right\}$ (finite) or $\mathbb{Z}_{\geq 0}$ (semi-infinite) or $\mathbb{Z}$ (full infinite),

$$
\begin{equation*}
\mathcal{H}_{x, y}=-\sqrt{B(x) D(x+1)} \delta_{x+1, y}-\sqrt{B(x-1) D(x)} \delta_{x-1, y}+(B(x)+D(x)) \delta_{x, y} . \tag{56}
\end{equation*}
$$

There exist finite and semi-infinite rdQM systems with virtual states [8, 13], but we do not know full infinite rdQM systems with virtual states. In the following, for simplicity of presentation, we consider semi-infinite systems only (For finite systems, some modification is needed). The potential functions $B(x)$ and $D(x)$ are real and positive but vanish at the boundary: $B(x)>0\left(n \in \mathbb{Z}_{\geq 0}\right), D(x)>0\left(n \in \mathbb{Z}_{\geq 1}\right)$ and $D(0)=0$. The function $\sqrt{ }$ is the square root function as a real function. We take the normalization of $\phi_{n}(x)$ (27) and $\tilde{\phi}_{\mathrm{v}}(x)$ (28) of the original system as $\phi_{n}(0)=\tilde{\phi}_{\mathrm{v}}(0)=1$. For simplicity in notation, we write the matrix $\mathcal{H}$ as follows:

$$
\begin{equation*}
\mathcal{H}=-\sqrt{B(x)} e^{\partial} \sqrt{D(x)}-\sqrt{D(x)} e^{-\partial} \sqrt{B(x)}+B(x)+D(x) \tag{57}
\end{equation*}
$$

where matrices $e^{ \pm \partial}$ are $\left(e^{ \pm \partial}\right)_{x, y} \xlongequal{\text { def }} \delta_{x \pm 1, y}$ and the unit matrix $\mathbf{1}=\left(\delta_{x, y}\right)$ is suppressed. The notation $f(x) A g(x)$, where $f(x)$ and $g(x)$ are functions of $x$ and $A$ is a matrix $A=\left(A_{x, y}\right)$, stands for a matrix whose $(x, y)$-element is $f(x) A_{x, y} g(y)$. Note that the matrices $e^{\partial}$ and $e^{-\partial}$ are not inverse to each other: $e^{\partial} e^{-\partial}=\mathbf{1}$ but $e^{-\partial} e^{\partial} \neq \mathbf{1}$. This Hamiltonian can be expressed in a factorized form:

$$
\begin{equation*}
\mathcal{H}=\mathcal{A}^{\dagger} \mathcal{A}, \quad \mathcal{A} \stackrel{\text { def }}{=} \sqrt{B(x)}-e^{\partial} \sqrt{D(x)}, \quad \mathcal{A}^{\dagger}=\sqrt{B(x)}-\sqrt{D(x)} e^{-\partial} \tag{58}
\end{equation*}
$$

The deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ (31) is given by [5, 8, 13 ]

$$
\begin{align*}
\mathcal{H}_{\mathcal{D}} & =-\sqrt{B_{\mathcal{D}}(x)} e^{\partial} \sqrt{D_{\mathcal{D}}(x)}-\sqrt{D_{\mathcal{D}}(x)} e^{-\partial} \sqrt{B_{\mathcal{D}}(x)}+B_{\mathcal{D}}(x)+D_{\mathcal{D}}(x)+\mathcal{E}_{\mu}  \tag{59}\\
& =\mathcal{A}_{\mathcal{D}}^{\dagger} \mathcal{A}_{\mathcal{D}}+\mathcal{E}_{\mu}, \quad\left(\mu \stackrel{\text { def }}{=} \min \left\{n \mid n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right\}\right)
\end{align*}
$$

where the potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ are

$$
\begin{align*}
& B_{\mathcal{D}}(x)=\sqrt{B(x+M) D(x+M+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x)} \\
& D_{\mathcal{D}}(x)=\sqrt{B(x-1) D(x)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x-1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{\mu}\right](x)} \tag{60}
\end{align*}
$$

Its eigenfunctions $\phi_{\mathcal{D} n}(x)$ are given by

$$
\begin{align*}
\phi_{\mathcal{D} n}(x)= & (-1)^{M} \epsilon_{\mathcal{D}}\left(\prod_{j=1}^{M} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x+1)}} \tag{61}
\end{align*}
$$

where the sign factor $\epsilon_{\mathcal{D}}$ is defined by

$$
\begin{equation*}
\epsilon_{\mathcal{D}}=\epsilon_{d_{1} \ldots d_{M}} \stackrel{\text { def }}{=} \prod_{1 \leq i<j \leq M} \operatorname{sgn}\left(\mathcal{E}_{\psi_{i}}-\mathcal{E}_{\psi_{j}}\right) \tag{62}
\end{equation*}
$$

(for $M=0,1$, we set $\epsilon_{\mathcal{D}}=1$. $\mathcal{D}$ is regarded as an ordered set.). Here $\mathcal{E}_{\psi_{j}}$ is $\mathcal{E}_{\psi_{j}}=\tilde{\mathcal{E}}_{\mathrm{v}_{k}}$ for $d_{j}=\mathrm{v}_{k}$ and $\mathcal{E}_{\psi_{j}}=\mathcal{E}_{e_{k}}$ for $d_{j}=e_{k}$. This sign factor $\epsilon_{\mathcal{D}}$ was written as $(-1)^{M} \mathcal{S}_{d_{1} \ldots d_{M}}$ in [13], but we missed it in [5, 8]. The sign factor $\epsilon_{\mathcal{D}}$ is important for Darboux transformations, but not as an eigenfunction.

Before we go any further, let us mention the square root function and the sign of $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)$. If the Krein-Adler condition (32) is satisfied and the range of parameters is chosen appropriately, we have the following two facts (conjectures for $\mathcal{D}_{\mathrm{e}} \neq \emptyset$ case, which are supported by numerical calculation). (i) : The potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ are real and positive (except for $D_{\mathcal{D}}(0)=0$ ). (ii) : The function $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)$ has a definite $\operatorname{sign} \epsilon_{d_{1} \ldots d_{M}}$, namely sgn $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)=\epsilon_{d_{1} \ldots d_{M}}\left(x \in \mathbb{Z}_{\geq 0}\right)$. The fact (i) means that $\mathcal{H}_{\mathcal{D}}$ (59) is well-defined and hermitian, and (ii) implies that $\phi_{\mathcal{D} n}(x)$ (61) is real, because $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x+1)$ in the square root is positive. However, in the intermediate steps of the multi-step Darboux transformations with $\mathcal{D}_{\mathrm{e}} \neq \emptyset$, the KreinAdler condition (32) may not be satisfied. This means that the function $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M^{\prime}}\right](x)$ $\left(M^{\prime}<M\right)$ may not have a definite sign. If so, the argument of the square root in (61) (with $M \rightarrow M^{\prime}$ ) becomes negative, and the potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ (with $M \rightarrow M^{\prime}$ ) also become negative. Since we regard $\sqrt{ }$ as a real function, its argument should be real and non-negative, and its value is also real and non-negative. We remark that the final result (31), which is obtained by the $M$-step Darboux transformations with $\mathcal{D}$ satisfying the KreinAdler condition (32), is correct, because the calculation of the Darboux transformation is purely algebraic. Since the argument of $\sqrt{ }$ may be negative in the intermediate steps, we have to specify how to treat $\sqrt{f(x)}$ for the function $f(x)$ that does not have a definite sign. We missed pointing out this remark in [5].

We adopt the following rule for $\sqrt{f(x)}$. If it is not necessary, the value of $\sqrt{f(x)}$ is not evaluated and is left as it is. By using the property $\sqrt{a} \sqrt{b}=\sqrt{a b}$, the calculation is continued as follows: $\sqrt{f(x)} / \sqrt{f(x)}=\sqrt{f(x) / f(x)}=\sqrt{1}=1$ and $\sqrt{f(x)} \sqrt{f(x)}=$ $\sqrt{f(x)^{2}}=\operatorname{sgn} f(0) \cdot f(x)$. We remark that this rule gives correct results for the function with a definite sign. Let us illustrate this rule by the calculation on the sign factor $\epsilon_{\mathcal{D}}$. We assume that the virtual state energy $\tilde{\mathcal{E}}_{\mathrm{v}}(28)$ is a monotonically increasing or decreasing
function of v , which is possible by choosing the range of parameters appropriately. We assume sgn $\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{M}\right](x)=\epsilon_{\mathcal{D}}$ for $x=0,1$ (for an appropriate range of the parameters) even if the Krein-Adler condition (32) is not satisfied. This assumption can be verified by numerical calculation. In the intermediate steps of the Darboux transformations, the deformed Hamiltonian $\mathcal{H}_{d_{1} \ldots d_{s}}$, which may be singular, is [5, 8, 13]

$$
\begin{align*}
& \mathcal{H}_{d_{1} \ldots d_{s}}=\hat{\mathcal{A}}_{d_{1} \ldots d_{s}} \hat{\mathcal{A}}_{d_{1} \ldots d_{s}}^{\dagger}+\mathcal{E}_{\psi_{s}}  \tag{63}\\
& \hat{\mathcal{A}}_{d_{1} \ldots d_{s}}=\sqrt{\hat{B}_{d_{1} \ldots d_{s}}(x)}-e^{\partial} \sqrt{\hat{D}_{d_{1} \ldots d_{s}}(x)}, \quad \hat{\mathcal{A}}_{d_{1} \ldots d_{s}}^{\dagger}=\sqrt{\hat{B}_{d_{1} \ldots d_{s}}(x)}-\sqrt{\hat{D}_{d_{1} \ldots d_{s}}(x)} e^{-\partial}, \tag{64}
\end{align*}
$$

where the potential functions $\hat{B}_{d_{1} \ldots d_{s}}(x)$ and $\hat{D}_{d_{1} \ldots d_{s}}(x)$ are

$$
\begin{align*}
& \hat{B}_{d_{1} \ldots d_{s}}(x)=\sqrt{B(x+s-1) D(x+s)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s-1}\right](x)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s-1}\right](x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x)} \\
& \hat{D}_{d_{1} \ldots d_{s}}(x)=\sqrt{B(x-1) D(x)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s-1}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s-1}\right](x)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x-1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x)} \tag{65}
\end{align*}
$$

Its "eigenfunctions" $\phi_{d_{1} \ldots d_{s} n}(x)$ are

$$
\begin{align*}
& \phi_{d_{1} \ldots d_{s} n}(x) \stackrel{\text { def }}{=} \hat{\mathcal{A}}_{d_{1} \ldots d_{s}} \phi_{d_{1} \ldots d_{s-1} n}  \tag{66}\\
= & (-1)^{s} \epsilon_{d_{1} \ldots d_{s}}\left(\prod_{j=1}^{s} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \frac{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)}}
\end{align*}
$$

By calculation with careful treatment of the square root, the next step "eigenfunction" $\phi_{d_{1} \ldots d_{s+1} n}(x)$ becomes

$$
\begin{aligned}
& \phi_{d_{1} \ldots d_{s+1} n}(x)=\hat{\mathcal{A}}_{d_{1} \ldots d_{s+1}} \phi_{d_{1} \ldots d_{s} n}(x) \\
&= \sqrt{\sqrt{B(x+s) D(x+s+1)}} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x)} \\
& \times(-1)^{s} \epsilon_{d_{1} \ldots d_{s}}\left(\prod_{j=1}^{s} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \frac{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)}} \\
&-\sqrt{\sqrt{B(x) D(x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+2)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x)}{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)}} \\
& \times(-1)^{s} \epsilon_{d_{1} \ldots d_{s}}\left(\prod_{j=1}^{s} B(x+j) D(x+j+1)\right)^{\frac{1}{4}} \frac{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}, \phi_{n}\right](x+1)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+2)}} \\
& \stackrel{(\mathrm{i})}{=}(-1)^{s+1} \epsilon_{d_{1} \ldots d_{s}}\left(\prod_{j=1}^{s+1} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \frac{1}{\sqrt{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)}}
\end{aligned}
$$

$$
\begin{align*}
& \times \frac{1}{\sqrt{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}\right](x+1)^{2}}}\left(\sqrt{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x)^{2}} \mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}, \phi_{n}\right](x+1)\right. \\
& \\
& \left.-\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)^{2}} \mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s}, \phi_{n}\right](x)\right) \\
& \stackrel{\text { (ii) })}{=}(-1)^{s+1} \epsilon_{d_{1} \ldots d_{s}}\left(\prod_{j=1}^{s+1} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \frac{1}{\sqrt{\mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)}} \\
& \times \frac{\epsilon_{d_{1} \ldots d_{s+1}}^{\epsilon_{d_{1} \ldots d_{s}}} \mathrm{~W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}, \phi_{n}\right](x)}{=} \\
& (-1)^{s+1} \epsilon_{d_{1} \ldots d_{s+1}}\left(\prod_{j=1}^{s+1} B(x+j-1) D(x+j)\right)^{\frac{1}{4}}  \tag{67}\\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x) \mathrm{W}_{\mathrm{C}}\left[\psi_{1}, \ldots, \psi_{s+1}\right](x+1)}},
\end{align*}
$$

where we have used (i): $\sqrt{a} \sqrt{b}=\sqrt{a b}$, (ii): the rule $\sqrt{f(x)^{2}}=\operatorname{sgn} f(0) \cdot f(x)$ and the Casoratian identity (3). This calculation establishes the sign factor $\epsilon_{\mathcal{D}}$ in (61).

Let's return to the main topic of this subsection. The eigenfunctions $\phi_{\mathcal{D} n}(x)$ (31) are given by (61). On the other hand, the eigenfunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ (33) are given by [8, 13]

$$
\begin{align*}
\phi_{\mathcal{D}_{\mathrm{v}} n}(x)= & (-1)^{M_{\mathrm{v}}} \epsilon_{\mathcal{D}_{\mathrm{v}}}\left(\prod_{j=1}^{M_{\mathrm{v}}} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x) \mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x+1)}}\left(n \in \mathbb{Z}_{\geq 0}\right), \tag{68}
\end{align*}
$$

and the potential functions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ are

$$
\begin{align*}
B_{\mathcal{D}_{\mathrm{v}}}(x)= & \sqrt{B\left(x+M_{\mathrm{v}}\right) D\left(x+M_{\mathrm{v}}+1\right)} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{v}}}\right](x)}{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x+1)} \frac{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x)}, \\
D_{\mathcal{D}_{\mathrm{v}}}(x)= & \sqrt{B(x-1) D(x)} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x+1)}{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}\right](x)} \frac{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x-1)}{\mathrm{W}_{\mathrm{C}}\left[\tilde{\phi}_{\mathrm{v}_{1}}, \ldots, \tilde{\phi}_{\mathrm{v}_{M_{\mathrm{v}}}}, \phi_{0}\right](x)} . \tag{69}
\end{align*}
$$

So the eigenfunctions of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}}}$ (34) are expressed as

$$
\begin{align*}
\phi_{\mathcal{D}_{\mathfrak{v}} \mathcal{D}_{\mathrm{e}} n}(x)= & (-1)^{M_{\mathrm{e}}} \epsilon_{\mathcal{D}_{\mathrm{e}}}\left(\prod_{j=1}^{M_{\mathrm{e}}} B_{\mathcal{D}_{\mathrm{v}}}(x+j-1) D_{\mathcal{D}_{\mathrm{v}}}(x+j)\right)^{\frac{1}{4}} \quad\left(n \in \mathbb{Z}_{\geq 0} \backslash \mathcal{D}_{\mathrm{e}}\right) \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}, \phi_{\mathcal{D}_{\mathrm{v}} n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right](x) \mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right](x+1)}} . \tag{70}
\end{align*}
$$

We will show that two expressions (61) and (70) are actually identical by using the Casoratian identity, Corollary3.

As noted in the Remark below Corollary[3, Corollary3 has been shown for $\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x)$ $>0$. If $\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x)$ is a definite sign function with sign $\epsilon$, Corollary3 becomes

$$
\begin{aligned}
& \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x) \quad\left(w(x) \stackrel{\text { def }}{=} \sqrt{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+1)}\right) \\
& =\epsilon^{m-1} \sqrt{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m)} \\
& \quad \times \mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x),
\end{aligned}
$$

and, by the replacements $m \rightarrow m+1$ and $u_{m+1}=v$, it becomes

$$
\begin{aligned}
& \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x) \\
= & \epsilon^{m} \sqrt{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m+1)} \\
& \times \mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x) .
\end{aligned}
$$

From these two equations, we obtain

$$
\begin{align*}
& \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x+1)}} \\
= & \epsilon^{m}\left(\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m+1)}{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+1) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m)}\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x) \mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x+1)}} . \tag{71}
\end{align*}
$$

In the following, we consider the replacements (identification) (41). The sign factor $\epsilon$ in (71) becomes $\epsilon=\epsilon_{\mathcal{D}_{\mathrm{v}}}$. The eigenfunctions (68) of $\mathcal{H}_{\mathcal{D}_{\mathrm{v}}}$ are expressed as

$$
\begin{align*}
& \phi_{\mathcal{D}_{\mathrm{v}} n}(x)=G(x) \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, v\right](x)}{w(x)}, \quad \phi_{\mathcal{D}_{\mathrm{v}} e_{j}}(x)=G(x) \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{j}\right](x)}{w(x)},  \tag{72}\\
& G(x) \stackrel{\text { def }}{=}(-1)^{l} \epsilon_{D_{\mathrm{v}}}\left(\prod_{j=1}^{l} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \tag{73}
\end{align*}
$$

By Proposition 3.1, we have

$$
\frac{\mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}, \phi_{\mathcal{D}_{\mathrm{v}} n}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right](x) \mathrm{W}_{\mathrm{C}}\left[\phi_{\mathcal{D}_{\mathrm{v}} e_{1}}, \ldots, \phi_{\mathcal{D}_{\mathrm{v}} e_{M_{\mathrm{e}}}}\right](x+1)}}
$$

$$
\begin{align*}
= & \frac{\mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, v\right]}{w}\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x) \mathrm{W}_{\mathrm{C}}\left[\frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}\right]}{w}, \ldots, \frac{\mathrm{~W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{m}\right]}{w}\right](x+1)}} \\
& \times \frac{\prod_{j=0}^{m} G(x+j)}{\sqrt{\prod_{j=0}^{m-1} G(x+j) G(x+1+j)}} \tag{74}
\end{align*}
$$

and a short calculation shows

$$
\begin{align*}
& \frac{\prod_{j=0}^{m} G(x+j)}{\sqrt{\prod_{j=0}^{m-1} G(x+j) G(x+1+j)}} \\
= & \left((-1)^{l} \epsilon_{\mathcal{D}_{\mathrm{v}}}\right)^{m+1}\left(\prod_{j=1}^{l} B(x+j-1) D(x+j) \cdot \prod_{j=m+1}^{l+m} B(x+j-1) D(x+j)\right)^{\frac{1}{8}} . \tag{75}
\end{align*}
$$

For the potential functions $B_{\mathcal{D}_{\mathrm{v}}}(x)$ and $D_{\mathcal{D}_{\mathrm{v}}}(x)$ (69), a short calculation shows

$$
\begin{align*}
& \prod_{j=1}^{m} B_{\mathcal{D}_{\mathrm{v}}}(x+j-1) D_{\mathcal{D}_{\mathrm{v}}}(x+j) \\
= & \left(\prod_{j=1}^{m} B(x+j-1) D(x+j) \cdot \prod_{j=l+1}^{l+m} B(x+j-1) D(x+j)\right)^{\frac{1}{2}} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m+1)}{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+1) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}\right](x+m)} \tag{76}
\end{align*}
$$

From (76), (74)-(75) and (71), we obtain

$$
\begin{align*}
(70)= & (-1)^{l+m}(-1)^{l m} \epsilon_{\mathcal{D}_{\mathrm{v}}} \epsilon_{\mathcal{D}_{\mathrm{e}}}\left(\prod_{j=1}^{l+m} B(x+j-1) D(x+j)\right)^{\frac{1}{4}} \\
& \times \frac{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}, v\right](x)}{\sqrt{\mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x) \mathrm{W}_{\mathrm{C}}\left[f_{1}, \ldots, f_{l}, u_{1}, \ldots, u_{m}\right](x+1)}}  \tag{77}\\
& \stackrel{(\mathrm{i})}{=} \text { (61), }
\end{align*}
$$

namely the equality $\phi_{\mathcal{D} n}(x)=\phi_{\mathcal{D}_{\mathrm{v}} \mathcal{D}_{\mathrm{e}} n}(x)$. In (i) we have used $\epsilon_{\mathcal{D}}=(-1)^{l m} \epsilon_{\mathcal{D}_{\mathrm{v}}} \epsilon_{\mathcal{D}_{\mathrm{e}}}$ because an ordered set $\mathcal{D}$ is now $\left\{\mathrm{v}_{1}, \ldots, \mathrm{v}_{M_{\mathrm{v}}}, e_{1}, \ldots, e_{M_{\mathrm{e}}}\right\}$.

## 4 Summary and Comments

The Wronskian and Casoratian identities (11), (2) and (3) have played an important role in the study of deformations of the quantum mechanical systems (oQM, idQM and rdQM,
respectively) by the multi-step Darboux transformations. A generalization of the Wronskian identity (1) is known as Theorem1. Corresponding to this generalization, we have presented similar generalizations of the Casoratian identities (21) and (3) as Theorem2 and 3, respectively.

We have also discussed the application of these Theorem 1 目 to quantum mechanical systems. Multi-step Darboux transformations with both the virtual state wavefunctions and the eigenstate wavefunctions as seed solutions are considered. By interpreting this deformation in two ways, as (4), we obtain two different expressions of the eigenfunctions. The equality of these two expressions is shown by using Theorem 1 [3.

The multi-indexed orthogonal polynomials $P_{\mathcal{D}, n}$, whose characteristic feature is the missing degrees, are obtained from the eigenfunctions $\phi_{\mathcal{D} n}(x)$ by removing the "ground state" part [4, 5, 7, 8, 29, 13, 14]. The multi-indexed polynomials $P_{\mathcal{D}_{\mathrm{v}}, n}$ obtained from (38), (46) and (68) are case-(1) polynomials, namely the set of missing degrees $\mathbb{Z}_{\geq 0} \backslash\left\{\operatorname{deg} P_{\mathcal{D}_{v}, n} \mid n \in \mathbb{Z}_{\geq 0}\right\}$ is $\{0,1, \ldots, \ell-1\}$. For $\mathcal{D}_{\mathrm{e}} \neq \emptyset$, the multi-indexed polynomials $P_{\mathcal{D}, n}$ obtained from (37), (45) and (61) are case-(2) polynomials, namely the set of missing degrees is not $\{0,1, \ldots, \ell-1\}$. Since the expressions (37), (45) and (61) are equal to (39), (48) and (70), respectively, we obtain another expression of $P_{\mathcal{D}, n}$ from (39), (48) and (70). Namely, the case-(2) polynomials $P_{\mathcal{D}, n}$ are expressed in terms of the case-(1) polynomials $P_{\mathcal{D}_{\mathrm{v}}, n}$. For their explicit forms, we leave them as an exercise for interested readers.

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