Markov Chains Generated by Convolutions of Orthogonality Measures

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Abstract

About two dozens of exactly solvable Markov chains on one-dimensional finite and semi-infinite integer lattices are constructed in terms of convolutions of orthogonality measures of the Krawtchouk, Hahn, Meixner, Charlier, q-Hahn, q-Meixner and little q-Jacobi polynomials. By construction, the stationary probability distributions, the complete sets of eigenvalues and eigenvectors are provided by the polynomials and the orthogonality measures. An interesting property possessed by these stationary probability distributions, called 'convolutional self-similarity,' is demonstrated.

1 Introduction

In this paper we study one mathematical aspect of Markov chains on one-dimensional integer lattices. The goal is constructing plenty of examples of workable Markov chains containing enough adjustable parameters so that rich and functional applications could be made in diverse disciplines. Markov chains are most easy to handle when, on top of the stationary probability distributions, the complete set of eigenvectors and corresponding eigenvalues are known. The eigenvectors discussed in the present paper form an orthogonal basis as they belong to real symmetric matrices (2.16).

We pursue this goal within the framework of orthogonal polynomials of a discrete variable [1]–[5]. To be more specific, we construct the basic transition matrix K by certain convolutions of the 'orthogonality measures = stationary probability distributions.' The information of the stationary distributions and the orthogonal eigenvectors are built in the scheme. In most cases, the eigenvalues can be extracted directly from the convolutions. In all cases, the eigenvalues are calculated exactly, see **Theorem 2.2**. In some respects, in particular, in incorporating many adjustable parameters, the present method is more advantageous than

some sophisticated procedures, for example those based on the finite group actions. The presentation of this paper is simple and plain so that non-experts can understand.

This paper is organised as follows. In section two, after a common problem setting, the basic properties of K(x, y), the transition probability matrix from y to x, are stated in **Lemma** and four **Theorems**. Lemma states K(x, y) is triangular in a certain basis. The most fundamental one is **Theorem 2.1** stating K is related to a real symmetric matrix \mathcal{H} by a similarity transformation. The complete set of eigenvectors of K is identified based on Lemma. It is followed by the eigenvalue formula **Theorem 2.2**, the spectral representation **Theorem 2.3** and solutions of the initial value problem and the ℓ -step transition matrix in **Theorem 2.4**. Section three provides fundamental data. The list of five types of convolutions can be found in $\S3.1$, the polynomial data in $\S3.2$, the convolutional self-similarity of various stationary distributions are explored in §3.3. Many explicit examples of K(x, y)constructed by convolutions are demonstrated in $\S4.1-\$4.5$, corresponding to each type of convolutions. Markov chains constructed by convolutions of type (i) and (iii) have the special property that the eigenvalues are directly obtained from the determinant of K's, which have a factorised form of an upper and lower triangular matrix. The polynomials defined on a one-dimensional integer lattice $\mathcal{X} = \{0, 1, \dots, N\}$, the Krawtchouk, Hahn and q-Hahn are the main players. Those K's lead to a wider class of Markov chains on a semi-infinite lattice by limiting procedures $N \to \infty$, described by (q-)Meixner and Charlier polynomials. Markov chains constructed by repeated convolutions of type (i) and (iii) for the Krawtchouk are presented in §4.6. Several examples of one parameter families of commuting K's are presented in $\S4.7$. Section five deals with two related topics. The dual convolutions obtained by mirroring $\{0, 1, \ldots, N\} \rightarrow \{N, N-1, \ldots, 0\}$ are explored in §5.1. Several semi-infinite Markov chains involving the little q-Jacobi polynomials are derived in $\S5.1.1$ through dual convolutions based on q-Hahn polynomials. The repeated discrete time birth and death processes are presented in \S 5.2. The final section is for comments on the salient properties of the eigenvalues of the K's constructed in this project. The proof of the triangularity **Lemma** is provided as Appendix.

2 Main Theorems

2.1 Problem setting

We discuss stationary Markov chains on a one-dimensional finite integer lattice \mathcal{X} ,

$$\mathcal{X} \stackrel{\text{def}}{=} \{0, 1, \dots, N\},\tag{2.1}$$

with its points denoted by x, y, z, etc. The main ingredient of the theory is the transition probability matrix K(x, y) on \mathcal{X} which specifies the transition probability from y to xsatisfying the basic conditions of probability and its conservation,

$$K(x,y) \ge 0, \quad \sum_{x \in \mathcal{X}} K(x,y) = 1.$$
 (2.2)

For given K(x, y), much useful information can be extracted depending on the specific needs of the applications. The most basic ones would be the solutions of the initial value problem and the determination of the ℓ -step transition probability:

• Initial value problem : one step time evolution $\mathcal{P}(x; \ell+1) = \sum_{y \in \mathcal{X}} K(x, y) \mathcal{P}(y; \ell),$

$$\mathcal{P}(x;0) \ge 0, \ \sum_{x \in \mathcal{X}} \mathcal{P}(x;0) = 1 \Rightarrow \mathcal{P}(x;\ell) = \sum_{y \in \mathcal{X}} K^{\ell}(x,y) \mathcal{P}(y;0),$$
(2.3)

• ℓ -step transition probability from y to $x : \mathcal{P}(x, y; \ell)$,

$$\mathcal{P}(x;0) = \delta_{xy} \Rightarrow \mathcal{P}(x,y;\ell) = K^{\ell}(x,y).$$
(2.4)

They can be obtained based on the *complete set of eigenvalues and the corresponding eigen*vectors of K(x, y). It is well known that K(x, y) has always a maximal eigenvalue 1 and the range of spectrum

$$-1 \leq$$
 The moduli of the eigenvalues of $K(x, y) \leq 1$, (2.5)

as a consequence of the non-negativity, *i.e.* Perron-Frobenius theorem, and probability conservation (2.2).

Among many known strategies of procuring K(x, y) with explicit forms of eigensystems, one very promising plan is to construct K(x, y) within the framework of orthogonal polynomials on \mathcal{X} . Simple examples are Birth and Death (BD) processes, well known processes of nearest neighbour hopping. All hypergeometric orthogonal polynomials of a discrete variable belonging to Askey scheme provide exactly solvable continuous time BD processes [6, 7] and a good part of it solves the discrete time versions [8], typical Markov chains. The normalised orthogonality measures always provide the stationary probability distributions of the corresponding BD and the polynomials the eigenvectors.

2.2 Basic properties of K(x, y)

Here we develop a rather general method of building K(x, y) by 'convolutions' of the normalised orthogonality measures. There are many different types of convolutions available, but the main structure of the logic is common, which we will describe by choosing one typical convolution. In §4 many explicit examples of K(x, y) together with the eigensystems, etc, constructed by various types of convolutions, will be displayed together with multitudes of derivative forms obtained by the limiting procedures of $N \to \infty$.

Let us introduce some notation. The normalised orthogonality measure of orthogonal polynomials $\check{P}_n(x) \stackrel{\text{def}}{=} P_n(\eta(x))$ (deg $P_n(\eta) = n$) on \mathcal{X} is denoted by $\pi(x, N, \lambda)$ ($x \in \mathcal{X}$)

$$\pi(x, N, \boldsymbol{\lambda}) > 0, \quad \sum_{x \in \mathcal{X}} \pi(x, N, \boldsymbol{\lambda}) = 1, \quad \sum_{x \in \mathcal{X}} \pi(x, N, \boldsymbol{\lambda}) \check{P}_m(x, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) = 0 \quad (m, n \in \mathcal{X}, m \neq n),$$
(2.6)

in which $\eta(x)$ is called the sinusoidal coordinate [9]. For the explicit forms, see (2.11). Here λ stands for the set of parameters other than the size of the lattice N. The N dependence of the polynomials is suppressed for simplicity. Throughout this paper we adopt the *universal* normalisation of the polynomials,

$$\check{P}_0(x,\boldsymbol{\lambda}) = 1 \quad (x \in \mathcal{X}), \quad \check{P}_n(0,\boldsymbol{\lambda}) = 1 \quad (n \in \mathcal{X}),$$
(2.7)

and $\eta(0) = 0$. As mentioned above, later π will denote the stationary probability distribution of the Markov chain. The formulation presented in this subsection would in principle apply for any polynomials of a discrete variable [1] in the Askey family [2]–[5]. But for the actual construction of K(x, y) we deal with three kinds of orthogonal polynomials, the Krawtchouk (K), Hahn (H) and q-Hahn (qH), and the initial is attached, e.g. $\pi_{\rm K}(x, N, \lambda)$, $\check{P}_{{\rm K}n}(x, \lambda)$ when formulas need specification of the polynomials. In later subsections, more polynomials defined on a semi-infinite lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$ will be discussed. They are the Charlier (C), Meixner (M), and q-Meixner (qM), which are obtained from (K), (H) and (qH) by limiting processes of some parameters in λ and $N \to \infty$.

The rules of the game is to construct K(x, y) by a convolution of two or more $\pi(x)$ with certain choice of parameter dependence λ_1 and λ_2 , for example,

(i):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, \lambda_2) \pi(z, y, \lambda_1) \quad (x, y \in \mathcal{X}),$$
 (2.8)

in such a way that K(x, y) satisfies a symmetry condition with another π with a parameter dependence λ ,

$$K(x,y)\pi(y,N,\boldsymbol{\lambda}) = K(y,x)\pi(x,N,\boldsymbol{\lambda}) \quad (x,y \in \mathcal{X}).$$
(2.9)

The explicit forms of five types of convolutions K(x, y) are listed in § 3.1. It is easy to see that the above K(x, y) (2.8) and four other types listed in § 3.1 satisfy the basic condition of probability conservation $\sum_{x \in \mathcal{X}} K(x, y) = 1$ independently of the choices of parameters λ_1 and λ_2 (and λ_3). That is, the probability conservation condition is satisfied irrespective of the presence of $\pi(x, N, \lambda)$ for the symmetry condition. Now K is a *positive* matrix and its eigenvalues are greater than -1. Let us note that the probability conservation $\sum_{x \in \mathcal{X}} K(x, y) = 1$ means that $\check{\mathcal{V}}_0(x) \stackrel{\text{def}}{=} 1$ is the left eigenvector of K belonging to the highest eigenvalue 1:

$$\sum_{y \in \mathcal{X}} K(y, x) \check{\mathcal{V}}_0(y) = \check{\mathcal{V}}_0(x).$$

For obtaining the rest of left eigenvectors of K, the following **Lemma** is essential.

Lemma The transition matrices K's generated by five types of convolutions (3.1)–(3.5) satisfy the triangularity condition

$$\sum_{y \in \mathcal{X}} K(y, x) \eta(y)^n = \sum_{m=0}^n a_{nm} \eta(x)^m \quad (n \in \mathcal{X}) \quad (a_{nm} = 0 \text{ for } n < m),$$
(2.10)

$$\eta(x) = \begin{cases} x & : (i) - (v) \\ q^{-x} - 1 & : (i), (iii), (iv) \end{cases}$$
(2.11)

That is the r.h.s. of (2.10) is a polynomial in $\eta(x)$ of degree at most n. The vectors $\{\eta(x)^n\}$ $(n \in \mathcal{X}, \eta(x)^0 \stackrel{\text{def}}{=} 1)$ form a basis of \mathbb{R}^{N+1} , because $\eta(x)$ is an increasing function. Since the proof of **Lemma** is of rather technical nature, it is consigned to Appendix. It should be noted that the triangularity of K is the consequence of the explicit forms of the convolutions (3.1)–(3.5) and it is independent of the specific choice of the parameters λ_1 and λ_2 (and λ_3). This **Lemma** means that the left eigenvalue of K is $\kappa(n) = a_{nn}$ and the corresponding left eigenvector is given by a certain degree n polynomial in $\eta(x)$, $\check{\mathcal{V}}_n(x) \stackrel{\text{def}}{=} \mathcal{V}_n(\eta(x))$ ($\check{\mathcal{V}}_0(x) =$ $\check{\mathcal{V}}_n(0) = 1$),

$$\sum_{y \in \mathcal{X}} K(y, x) \check{\mathcal{V}}_n(y) = \kappa(n) \check{\mathcal{V}}_n(x) \quad (n \in \mathcal{X}).$$
(2.12)

By taking y summation of the symmetry condition (2.9), we find that $\pi(x, N, \lambda)$ is the eigenvector of K(x, y) with the maximal eigenvalue 1,

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, N, \boldsymbol{\lambda}) = \sum_{y \in \mathcal{X}} K(y, x) \pi(x, N, \boldsymbol{\lambda}) = \pi(x, N, \boldsymbol{\lambda}).$$

The set of all eigenvectors of K is given by

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, N, \boldsymbol{\lambda}) \check{\mathcal{V}}_n(y) = \kappa(n) \pi(x, N, \boldsymbol{\lambda}) \check{\mathcal{V}}_n(x) \quad (n \in \mathcal{X}).$$
(2.13)

Let us introduce the square root of the stationary distribution $\pi(x, N, \lambda)$ and a diagonal matrix Φ on \mathcal{X} ,

$$\hat{\phi}_0(x, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\pi(x, N, \boldsymbol{\lambda})}, \quad \Phi(x, x) \stackrel{\text{def}}{=} \hat{\phi}_0(x, \boldsymbol{\lambda}), \quad \Phi(x, y) \stackrel{\text{def}}{=} 0 \quad (x \neq y).$$
 (2.14)

By a similarity transformation in terms of this Φ , we define a matrix \mathcal{H} as follows:

$$\mathcal{H} \stackrel{\text{def}}{=} \Phi^{-1} K \Phi, \quad \mathcal{H}(x, y) = \frac{\hat{\phi}_0(y, \lambda)}{\hat{\phi}_0(x, \lambda)} K(x, y).$$
(2.15)

By dividing both sides of the symmetry condition (2.9) by $\hat{\phi}_0(x, \lambda)\hat{\phi}_0(y, \lambda)$, we obtain

$$\mathcal{H}(x,y) = \mathcal{H}(y,x) \quad (x,y \in \mathcal{X}), \tag{2.16}$$

namely \mathcal{H} is a real symmetric matrix. Hence it is diagonalizable and its eigenvectors can be taken to be orthogonal with each other. On the other hand, (2.12) and (2.9) imply

$$\sum_{y \in \mathcal{X}} \mathcal{H}(x, y) \hat{\phi}_0(y, \boldsymbol{\lambda}) \check{\mathcal{V}}_n(y) = \kappa(n) \hat{\phi}_0(x, \boldsymbol{\lambda}) \check{\mathcal{V}}_n(x) \quad (n \in \mathcal{X}),$$
(2.17)

namely $\hat{\phi}_0(x, \boldsymbol{\lambda})\check{\mathcal{V}}_n(x)$'s are eigenvectors of \mathcal{H} . The polynomial $\check{\mathcal{V}}_n(x)$ of degree n in $\eta(x)$, being orthogonal with others with respect to the measure $\hat{\phi}_0(x, \boldsymbol{\lambda})^2 = \pi(x, N, \boldsymbol{\lambda})$, should be $\check{P}_n(x, \boldsymbol{\lambda})$. Therefore we obtain the following theorem. **Theorem 2.1** The eigenvectors of the two matrices $\mathcal{H} = \Phi^{-1}K\Phi$ and K are described by the orthogonal polynomials $\check{P}_n(x, \lambda) = P_n(\eta(x), \lambda)$ $(n \in \mathcal{X})$, belonging to $\pi(x, N, \lambda)$,

$$\sum_{y \in \mathcal{X}} \mathcal{H}(x, y) \hat{\phi}_0(y, \boldsymbol{\lambda}) \check{P}_n(y, \boldsymbol{\lambda}) = \kappa(n) \hat{\phi}_0(x, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}),$$
(2.18)

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, N, \boldsymbol{\lambda}) \check{P}_n(y, \boldsymbol{\lambda}) = \kappa(n) \pi(x, N, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$
(2.19)

Here we suppress the parameter dependence of the eigenvalues $\{\kappa(n)\}\$ for the simplicity of presentation.

The set of orthogonal vectors $\{\hat{\phi}_0(x, \boldsymbol{\lambda})\check{P}_n(x, \boldsymbol{\lambda})\},\$

$$\sum_{x \in \mathcal{X}} \hat{\phi}_0(x, \boldsymbol{\lambda})^2 \check{P}_m(x, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) = \sum_{x \in \mathcal{X}} \pi(x, N, \boldsymbol{\lambda}) P_m(\eta(x), \boldsymbol{\lambda}) P_n(\eta(x), \boldsymbol{\lambda})$$
$$= \sum_{x \in \mathcal{X}} \pi(x, N, \boldsymbol{\lambda}) \check{P}_m(x, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) = \frac{\delta_{mn}}{d_n^2} \quad (m, n \in \mathcal{X}), \quad d_0 = 1,$$
(2.20)

form the complete set of eigenvectors of \mathcal{H} . Now the scale of the polynomials $\{\check{P}_n(x, \boldsymbol{\lambda})\}$ is fixed by the universal normalisation (2.7), the normalisation constant d_n^2 is uniquely determined by the above formula. It should be noted that, because of the context, the present definition of d_n^2 corresponds to d_n^2/d_0^2 in our previous series of papers [6, 7, 9]. The parameter dependence of d_n^2 is also suppressed. Based on the universal normalisation condition of the polynomials (2.7), we arrive at the *universal formula for the eigenvalues* { $\kappa(n)$ } in terms of $K(x, y), \pi(x, N, \boldsymbol{\lambda})$ and { $\check{P}_n(x, \boldsymbol{\lambda})$ }.

Theorem 2.2 By setting x = 0 in (2.19), we obtain the universal expression of the eigenvalues

$$\kappa(n) = \sum_{y \in \mathcal{X}} K(0, y) \frac{\pi(y, N, \boldsymbol{\lambda})}{\pi(0, N, \boldsymbol{\lambda})} \check{P}_n(y, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}), \quad \kappa(0) = 1.$$
(2.21)

Let us introduce the set of orthonormal eigenvectors of \mathcal{H} ,

$$\hat{\phi}_n(x,\boldsymbol{\lambda}) \stackrel{\text{def}}{=} d_n \hat{\phi}_0(x,\boldsymbol{\lambda}) \check{P}_n(x,\boldsymbol{\lambda}), \quad \sum_{x \in \mathcal{X}} \hat{\phi}_m(x,\boldsymbol{\lambda}) \hat{\phi}_n(x,\boldsymbol{\lambda}) = \delta_{mn} \quad (n,m \in \mathcal{X}).$$
(2.22)

Theorem 2.3 The spectral representation of the real symmetric matrix \mathcal{H} provides that of K,

$$\mathcal{H}(x,y) = \sum_{n \in \mathcal{X}} \kappa(n) \hat{\phi}_n(x, \boldsymbol{\lambda}) \hat{\phi}_n(y, \boldsymbol{\lambda}), \qquad (2.23)$$

$$K(x,y) = \hat{\phi}_0(x,\boldsymbol{\lambda}) \sum_{n \in \mathcal{X}} \kappa(n) \hat{\phi}_n(x,\boldsymbol{\lambda}) \hat{\phi}_n(y,\boldsymbol{\lambda}) \cdot \hat{\phi}_0(y,\boldsymbol{\lambda})^{-1}$$
$$= \sum_{n \in \mathcal{X}} \kappa(n) d_n^2 \pi(x,N,\boldsymbol{\lambda}) \check{P}_n(x,\boldsymbol{\lambda}) \check{P}_n(y,\boldsymbol{\lambda}).$$
(2.24)

Theorem 2.4 The solution of the initial value problem of the Markov chain with the transition rate K(x, y) after ℓ steps is given by

$$\mathcal{P}(x;\ell) = \hat{\phi}_0(x,\boldsymbol{\lambda}) \sum_{n \in \mathcal{X}} c_n \kappa(n)^\ell \, \hat{\phi}_n(x,\boldsymbol{\lambda}) = \pi(x,N,\boldsymbol{\lambda}) \sum_{n \in \mathcal{X}} c_n d_n \kappa(n)^\ell \check{P}_n(x,\boldsymbol{\lambda}), \qquad (2.25)$$

in which $\{c_n\}$ are determined by the expansion of the initial distribution $\mathcal{P}(x; 0)$,

$$\mathcal{P}(x;0) = \hat{\phi}_0(x,\boldsymbol{\lambda}) \sum_{n \in \mathcal{X}} c_n \hat{\phi}_n(x,\boldsymbol{\lambda})$$
$$\Rightarrow c_n = \sum_{x \in \mathcal{X}} \hat{\phi}_n(x,\boldsymbol{\lambda}) \hat{\phi}_0(x,\boldsymbol{\lambda})^{-1} \mathcal{P}(x;0) = d_n \sum_{x \in \mathcal{X}} \check{P}_n(x,\boldsymbol{\lambda}) \mathcal{P}(x;0) \quad (n \in \mathcal{X}), \quad c_0 = 1. \quad (2.26)$$

The ℓ -step transition matrix from y to x is

$$\mathcal{P}(x,y;\ell) = K^{\ell}(x,y) = \hat{\phi}_0(x,\boldsymbol{\lambda}) \sum_{n\in\mathcal{X}} \kappa(n)^{\ell} \hat{\phi}_n(x,\boldsymbol{\lambda}) \hat{\phi}_n(y,\boldsymbol{\lambda}) \hat{\phi}_0(y,\boldsymbol{\lambda})^{-1}$$
$$= \pi(x,N,\boldsymbol{\lambda}) \sum_{n\in\mathcal{X}} d_n^2 \kappa(n)^{\ell} \check{P}_n(x,\boldsymbol{\lambda}) \check{P}_n(y,\boldsymbol{\lambda}).$$
(2.27)

Since $\kappa(0) = 1$ and $-1 < \kappa(n) < 1$ $(n \neq 0)$, the stationary distribution is reached asymptotically,

$$\lim_{\ell \to \infty} \mathcal{P}(x;\ell) = \lim_{\ell \to \infty} \mathcal{P}(x,y;\ell) = \pi(x,N,\boldsymbol{\lambda}).$$
(2.28)

It should be stressed that the results and theorems derived in this section are valid for other choices of convolutions than (2.8) so long as the basic condition (2.2) and the symmetry condition (2.9) and Lemma are satisfied.

When a good $N \to \infty$ limit exists, it leads to a Markov chain on a semi-infinite lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$, and the above theorems also hold. That is, the symmetry condition (2.9), the eigenvectors (2.19) and the eigenvalue formula (2.21) are

$$K(x,y)\pi(y,\boldsymbol{\lambda}) = K(y,x)\pi(x,\boldsymbol{\lambda}) \quad (x,y \in \mathcal{X}),$$
(2.29)

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, \boldsymbol{\lambda}) \check{P}_n(y, \boldsymbol{\lambda}) = \kappa(n) \pi(x, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}),$$
(2.30)

$$\kappa(n) = \sum_{y \in \mathcal{X}} K(0, y) \frac{\pi(y, \boldsymbol{\lambda})}{\pi(0, \boldsymbol{\lambda})} \check{P}_n(y, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}), \quad \kappa(0) = 1,$$
(2.31)

where $\mathcal{X} = \mathbb{Z}_{\geq 0}$.

3 Fundamental Data

Here we present fundamental data for constructing and displaying the explicit forms of various realisations of K(x, y). Starting with the list of convolutions in §3.1, the basic data of 'orthogonality measures = stationary distributions,' polynomials and the normalisation constants d_n^2 etc are presented in §3.2.

3.1 List of convolutions

Here we list five forms of 'convolutions' used for the construction of K(x, y). The list is not exhaustive at all. A new and interesting convolution might be added in future.

(i):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, \lambda_2) \pi(z, y, \lambda_1),$$
 (3.1)

(ii):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{\substack{z=\max(0,x+y-N)\\N}}^{\min(x,y)} \pi(x-z,N-y,\boldsymbol{\lambda}_2)\pi(z,y,\boldsymbol{\lambda}_1),$$
 (3.2)

(iii):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=\max(x,y)}^{N} \pi(x,z,\boldsymbol{\lambda}_2)\pi(z-y,N-y,\boldsymbol{\lambda}_1),$$
 (3.3)

(iv):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, \lambda_1) \sum_{z_1=\max(x,y)}^N \pi(x - z_2, z_1 - z_2, \lambda_3) \pi(z_1 - y, N - y, \lambda_2), \quad (3.4)$$

(v):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, \lambda_1) \sum_{z_1=x+y-z_2}^N \pi(x-z_2, z_1-y, \lambda_3) \pi(z_1-y, N-y, \lambda_2).$$
 (3.5)

It is easy to convince oneself that the basic condition (2.2) is satisfied for each convolution.

It is obvious that these are very different from the standard forms of convolutions, e.g.

$$(f * g)(x) = \sum_{z \in \mathcal{X}} f(x - z)g(z),$$

since the formulas (3.1)–(3.5) must contain x and y. The above forms could be considered as deformations of convolutions containing x and y, like

$$(f * g)(x, y) = \sum_{z \in \mathcal{X}} f(x - z)g(z - y).$$

Similar expressions appear in section 4 during the reduction, $N \to \infty$, processes.

The stationary probability measures $\pi(x, N, \lambda)$ presented in the subsequent subsection have a remarkable property of 'self-similarity' under 'ordinary' convolutions. This will be demonstrated in §3.3.

3.2 Polynomials data

Here we provide the basic data of the participating polynomials. Most are known facts collected for the consistency of notation, which is standard. For the explicit definitions of the basic quantities, e.g. $(a)_n$, $(a;q)_n$, $_rF_s$ and $_r\phi_s$, consult [2, 3]. The data of the second family of orthogonal polynomials of the q-Meixner (3.34)–(3.38) are not reported in standard references [1]–[5]. We believe some explicit expressions of the general formulas (A.1)–(A.3), e.g. (3.8), (3.9) etc. are new. The parametrisation of some polynomials [9], (H) and (qH), is different from the conventional ones. There are many equivalent and different looking expressions. We adopt the ones easy to grasp and simple to use. Recall that $d_n > 0$.

The measure $\pi(x, N, \lambda)$ is defined for $N \in \mathbb{Z}_{\geq 0}$ and $x \in \{0, 1, ..., N\}$. For simplicity in presentation, we extend the domain of definition to $x, N \in \mathbb{Z}$ by setting $\pi(x, N, \lambda) = 0$ for otherwise. Similarly, the domain of definition of $\pi(x, \lambda)$ ($x \in \mathbb{Z}_{\geq 0}$) is extended to $x \in \mathbb{Z}$ by setting $\pi(x, \lambda) = 0$ for otherwise.

3.2.1 Krawtchouk (K)

The polynomial depends on one positive parameter $\lambda = p \ (0 ,$

$$\pi(x, N, p) = \binom{N}{x} p^x (1-p)^{N-x}, \quad \binom{N}{x} = \frac{N!}{x! (N-x)!}, \quad d_n^2 = \binom{N}{n} \left(\frac{p}{1-p}\right)^n, \quad (3.6)$$

$$\pi(N - x, N, p) \equiv \pi(x, N, 1 - p),$$

$$s_{1}n(x)\pi(x, N, p) = -\pi(x - 1, N - 1, p), \quad n(x) = x, \quad s_{1} \stackrel{\text{def}}{=} -\frac{1}{1}$$
(3.8)

$$s_1\eta(x)\pi(x,N,p) = -\pi(x-1,N-1,p), \quad \eta(x) = x, \quad s_1 = -\frac{1}{pN},$$
(3.8)

$$\eta(z)\pi(z,x,p) = p\eta(x)\pi(z-1,x-1,p),$$
(3.9)

$$\check{P}_n(x,p) = P_n(x,p) = {}_2F_1\begin{pmatrix} -n, -x \\ -N \end{pmatrix} p^{-1}, \quad P_n(x,p) = P_x(n,p),$$
(3.10)

$$\check{P}_n(N-x,p) = (-1)^n (p^{-1}-1)^n \check{P}_n(x,1-p).$$
(3.11)

 $P_n(x,p)$ is a self-dual $P_n(x,p) = P_x(n,p)$ [9] degree *n* polynomial in *x* and π is the binomial distribution.

3.2.2 Charlier (C)

This polynomial is defined on a semi-infinite integer lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$ with $\boldsymbol{\lambda} = a \ (a > 0)$,

$$\pi(x,a) = \frac{a^x e^{-a}}{x!}, \quad d_n^2 = \frac{a^n}{n!},$$
(3.12)

$$s_1\eta(x)\pi(x,a) = -\pi(x-1,a), \quad \eta(x) = x, \quad s_1 \stackrel{\text{def}}{=} -\frac{1}{a},$$
(3.13)

$$\check{P}_n(x,a) = P_n(x,a) = {}_2F_0\begin{pmatrix} -n, -x \\ - \end{pmatrix} - a^{-1}, \quad P_n(x,a) = P_x(n,a).$$
(3.14)

 $P_n(x,a)$ is a degree *n* polynomial in *x* and π is the Poisson distribution. It is self-dual $P_n(x,a) = P_x(n,a)$, too. By the replacement $p \to pN^{-1}$ and the limit $N \to \infty$, the Krawtchouk (K) goes to Charlier (C),

$$\check{P}_{\mathrm{K}\,n}(x,p) \to \check{P}_{\mathrm{C}\,n}(x,p), \quad \pi_{\mathrm{K}}(x,N,p) \to \pi_{\mathrm{C}}(x,p), \quad d^{2}_{\mathrm{K}\,n} \to d^{2}_{\mathrm{C}\,n}.$$

3.2.3 Hahn (H)

The polynomial depends on two positive parameters $\boldsymbol{\lambda} = (a, b) \ (a, b > 0)$,

$$\pi(x, N, a, b) = \binom{N}{x} \frac{(a)_x (b)_{N-x}}{(a+b)_N}, \quad d_n^2 = \binom{N}{n} \frac{(a)_n (2n+a+b-1)(a+b)_N}{(b)_n (n+a+b-1)_{N+1}}, \tag{3.15}$$

$$\pi(N - x, N, a, b) = \pi(x, N, b, a), \tag{3.16}$$

$$s_1\eta(x)\pi(x,N,a,b) = -\pi(x-1,N-1,a+1,b), \quad \eta(x) = x, \quad s_1 \stackrel{\text{def}}{=} -\frac{a+b}{aN}, \quad (3.17)$$

$$\eta(z)\pi(z,x,a,b) = \frac{a}{a+b}\eta(x)\pi(z-1,x-1,a+1,b),$$
(3.18)

$$\check{P}_n(x,a,b) = P_n(x,a,b) = {}_3F_2 \Big(\begin{array}{c} -n, \ n+a+b-1, \ -x \\ a, \ -N \end{array} \Big| 1 \Big), \tag{3.19}$$

$$\check{P}_n(N-x,a,b) = (-1)^n \frac{(b)_n}{(a)_n} \check{P}_n(x,b,a).$$
(3.20)

 $P_n(x, a, b)$ is a degree *n* polynomial in *x* and π is connected with the hypergeometric distribution or the Polya distribution.

3.2.4 Meixner (M)

This polynomial is defined on a semi-infinite integer lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$ with $\lambda = (a, b)$ (a > 0, 0 < b < 1),

$$\pi(x,a,b) = \frac{(a)_x b^x (1-b)^a}{x!}, \quad d_n^2 = \frac{(a)_n b^n}{n!}, \tag{3.21}$$

$$s_1\eta(x)\pi(x,a,b) = -\pi(x-1,a+1,b), \quad \eta(x) = x, \quad s_1 \stackrel{\text{def}}{=} -\frac{b^{-1}-1}{a}, \quad (3.22)$$

$$\check{P}_n(x,a,b) = P_n(x,a,b) = {}_2F_1\left(\begin{array}{c} -n, -x \\ a \end{array} \middle| 1 - b^{-1}\right), \quad P_n(x,a,b) = P_x(n,a,b).$$
(3.23)

 $P_n(x, a, b)$ is a self-dual degree n polynomial in x and π is connected with the negative binomial distribution. By the replacement $b \to N(1-b)b^{-1}$ and the limit $N \to \infty$, the Hahn (H) goes to Meixner (M),

$$\check{P}_{\mathrm{H}n}(x,a,b) \to \check{P}_{\mathrm{M}n}(x,a,b), \quad \pi_{\mathrm{H}}(x,N,a,b) \to \pi_{\mathrm{M}}(x,a,b), \quad d^{2}_{\mathrm{H}n} \to d^{2}_{\mathrm{M}n}$$

By the replacement $b \to b/(a+b)$ and the limit $a \to \infty$, the Meixner (M) goes to Charlier (C)

$$\check{P}_{\mathrm{M}\,n}(x,a,b) \to \check{P}_{\mathrm{C}\,n}(x,b), \quad \pi_{\mathrm{M}}(x,a,b) \to \pi_{\mathrm{C}}(x,b), \quad d^{2}_{\mathrm{M}\,n} \to d^{2}_{\mathrm{C}\,n}.$$

3.2.5 q-Hahn (qH)

The three polynomials, the q-Hahn (qH), q-Meixner (qM) and q-Charlier (qC), to be discussed hereafter, depend on q, 0 < q < 1 on top of the other parameters. The q dependence of π and \check{P}_n is suppressed. The limiting processes of these q-polynomials to non q-polynomials will not be discussed here. It should be stressed that these three polynomials $\check{P}_n(x)$ are degree n polynomials in $q^{-x} - 1$, not in x. The q-Hahn is defined on a finite integer lattice with two positive parameters $\lambda = (a, b)$ (0 < a < 1, b < 1),

$$\pi(x, N, a, b) = \begin{bmatrix} N \\ x \end{bmatrix} \frac{(a; q)_x (b; q)_{N-x} a^{N-x}}{(ab; q)_N}, \quad \begin{bmatrix} N \\ x \end{bmatrix} \stackrel{\text{def}}{=} \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}}, \quad (3.24)$$

$$d_n^2 = \begin{bmatrix} N\\n \end{bmatrix} \frac{(a, abq^{-1}; q)_n}{(abq^N, b; q)_n a^n} \frac{1 - abq^{2n-1}}{1 - abq^{-1}},$$
(3.25)

$$s_1\eta(x)\pi(x,N,a,b) = -\pi(x-1,N-1,aq,b), \quad s_1 \stackrel{\text{def}}{=} -\frac{1-ab}{(1-a)(q^{-N}-1)}, \quad (3.26)$$

$$\eta(z)\pi(z,x,a,b) = \frac{1-a}{1-ab}\eta(x)\pi(z-1,x-1,aq,b), \quad \eta(x) = q^{-x} - 1, \tag{3.27}$$

$$\check{P}_n(x,a,b) = P_n(\eta(x),a,b) = {}_3\phi_2 \Big(\begin{array}{c} q^{-n}, \ abq^{n-1}, \ q^{-x} \\ a, \ q^{-N} \end{array} \Big| \ q \,; q \Big), \tag{3.28}$$

$$\pi(N-x, N, a, b) = \frac{(ab)^x}{b^N} \pi(x, N, b, a), \quad \check{P}_n(N-x, N, a, b) \not\propto \check{P}_n(x, N, b, a).$$
(3.29)

3.2.6 q-Meixner (qM)

This is a polynomial in $\eta(x) = q^{-x} - 1$ defined on a semi-infinite integer lattice $\mathcal{X} = \mathbb{Z}_{\geq 0}$ with $\boldsymbol{\lambda} = (b, c) \ (0 < b < q^{-1}, c > 0),$

$$\pi(x,b,c) = \frac{(bq;q)_x}{(q,-bcq;q)_x} c^x q^{\binom{x}{2}} \frac{(-bcq;q)_\infty}{(-c;q)_\infty}, \quad d_n^2 = \frac{q^n (bq;q)_n}{(q,-c^{-1}q;q)_n}, \tag{3.30}$$

$$s_1\eta(x)\pi(x,b,c) = -\pi(x-1,bq,c), \quad \eta(x) = q^{-x} - 1, \quad s_1 \stackrel{\text{def}}{=} -\frac{q}{c(1-bq)}, \quad (3.31)$$

$$\check{P}_n(x,b,c) = P_n(\eta(x),b,c) = {}_2\phi_1 \left(\begin{array}{c} q^{-n}, q^{-x} \\ bq \end{array} \middle| q; -c^{-1}q^{n+1} \right),$$
(3.32)

$$\phi_0(x, b, c) = \sqrt{\pi(x, b, c)}, \quad \phi_n(x, b, c) = d_n \phi_0(x, b, c) P_n(x, b, c),$$

$$\sum_{x \in \mathcal{X}} \hat{\phi}_n(x, b, c) \hat{\phi}_m(x, b, c) = \delta_{nm} \quad (n, m \in \mathcal{X}).$$
(3.33)

The completeness relation is not satisfied

$$\sum_{n \in \mathcal{X}} \hat{\phi}_n(x, b, c) \hat{\phi}_n(y, b, c) \neq \delta_{xy} \quad (x, y \in \mathcal{X}),$$

by these polynomials [10] as seen clearly by (3.15) of [7]. Another set of orthogonal polynomials obtained from the original set by the parameter change (involution)

$$(b,c) \to (-bc,c^{-1}),$$

is needed for the completeness,

$$\pi^{(-)}(x,b,c) = \frac{(-bcq;q)_x}{(q,bq;q)_x} c^{-x} q^{\binom{x}{2}} \frac{(bq;q)_\infty}{(-c^{-1};q)_\infty}, \quad d_n^{(-)\,2} = \frac{q^n (-bcq;q)_n}{(q,-cq;q)_n} \ (d_n^{(-)} > 0), \quad (3.34)$$

$$s_1 \eta(x) \pi^{(-)}(x, b, c) = -\pi^{(-)}(x - 1, bq, c), \quad s_1 \stackrel{\text{def}}{=} -\frac{cq}{1 + bcq}, \tag{3.35}$$

$$\check{P}_{n}^{(-)}(x,b,c) = P_{n}^{(-)}(\eta(x),b,c) = {}_{2}\phi_{1} \begin{pmatrix} q^{-n}, q^{-x} \\ -bcq \end{pmatrix} q; -cq^{n+1} \end{pmatrix},$$
(3.36)

$$\hat{\phi}_{0}^{(-)}(x,b,c) \stackrel{\text{def}}{=} (-1)^{x} \sqrt{\pi^{(-)}(x,b,c)}, \quad \hat{\phi}_{n}^{(-)}(x,b,c) \stackrel{\text{def}}{=} d_{n}^{(-)} \hat{\phi}_{0}^{(-)}(x,b,c) \check{P}_{n}^{(-)}(x,b,c), \quad (3.37)$$

$$\sum_{x \in \mathcal{X}} \hat{\phi}_n(x, b, c) \hat{\phi}_m^{(-)}(x, b, c) = 0, \quad \sum_{x \in \mathcal{X}} \hat{\phi}_n^{(-)}(x, b, c) \hat{\phi}_m^{(-)}(x, b, c) = \delta_{nm} \quad (n, m \in \mathcal{X}).$$
(3.38)

qM (3.30)–(3.38) is obtained from qH by the replacement $a \to bq$, $b \to -b^{-1}c^{-1}q^{-N}$ and the limit $N \to \infty$. The q-Charlier with $\lambda = c$ (c > 0) is obtained from q-Meixner by setting b = 0.

3.3 Convolutional self-similarity of stationary distributions

It is well known that the Gaussian distribution

$$\pi_{\rm G}(x,\sigma) \stackrel{\rm def}{=} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad (\sigma > 0),$$

keeps its form under the standard convolution

$$\int_{-\infty}^{\infty} \pi_{\rm G}(x-z,\sigma)\pi_{\rm G}(z,\tau)dz = \pi_{\rm G}(x,\sqrt{\sigma^2+\tau^2}).$$

Here we show that the π 's listed in the previous subsection also keep their forms under several forms of convolutions. This means repeating these convolutions as a part of constructing K(x, y) would be redundant, as using two π 's is the same as one π . The following formulas are verified easily by straightforward calculation.

Charlier (C)

$$\sum_{z=0}^{x} \pi(x-z, a_2)\pi(z, a_1) = \pi(x, a_1 + a_2), \qquad (3.39)$$

$$\sum_{z=y}^{x} \pi(x-z, a_2) \pi(z-y, a_1) = \pi(x-y, a_1+a_2).$$
(3.40)

These are classical results obtained by the binomial theorem.

Meixner (M)

$$\sum_{z=0}^{x} \pi(x-z, a_2, b) \pi(z, a_1, b) = \pi(x, a_1 + a_2, b),$$
(3.41)

$$\sum_{z=y}^{x} \pi(x-z, a_2, b) \pi(z-y, a_1, b) = \pi(x-y, a_1+a_2, b).$$
(3.42)

These are obtained by the sum formula of $\pi_{\rm H}$

$$\sum_{x \in \mathcal{X}} \pi_{\mathrm{H}}(x, N, a, b) = 1 \Longleftrightarrow \sum_{k=0}^{n} \binom{n}{k} (a)_{k} (b)_{n-k} = (a+b)_{n} \quad (n \in \mathbb{Z}_{\geq 0}), \tag{3.43}$$

which is derived from the formula [4](1.5.4),

$${}_{2}F_{1}\binom{-n, \ b}{c} \mid 1 = \frac{(c-b)_{n}}{(c)_{n}} \ (n \in \mathbb{Z}_{\geq 0}).$$
(3.44)

These four formulas (C) and (M) are symmetric in λ_1 and λ_2 .

Krawtchouk (K)

$$\sum_{z=0}^{\infty} \pi(x-z, N-z, a_2) \pi(z, N, a_1) = \pi(x, N, 1 - (1 - a_1)(1 - a_2)), \qquad (3.45)$$

$$\sum_{z=x}^{y} \pi(x, z, a_2) \pi(z, y, a_1) = \pi(x, y, a_1 a_2), \qquad (3.46)$$

$$\sum_{z=y}^{x} \pi(x-z, N-z, a_2) \pi(z-y, N-y, a_1) = \pi \left(x-y, N-y, 1-(1-a_1)(1-a_2)\right). \quad (3.47)$$

In all formulas the binomial theorem is used. The result (3.45) was reported in [11]p115, (2.3) together with the *n*-fold repetition in (2.6). All three formulas work when $\lambda_1 = a_1$ and $\lambda_2 = a_2$ are interchanged.

Hahn (H)

$$\sum_{z=0}^{x} \pi(x-z, N-z, a_1, b_1) \pi(z, N, a_2, a_1+b_1) = \pi(x, N, a_1+a_2, b_1),$$
(3.48)

$$\sum_{z=x}^{y} \pi(x, z, a_1, b_1) \pi(z, y, a_1 + b_1, b_2) = \pi(x, y, a_1, b_1 + b_2),$$
(3.49)

$$\sum_{z=y}^{x} \pi(x-z, N-z, a_1, b_1) \pi(z-y, N-y, a_2, a_1+b_1) = \pi(x-y, N-y, a_1+a_2, b_1).$$
(3.50)

These formulas are obtained by the sum formula of $\pi_{\rm H}$ (3.43). It is very interesting to note that, like (C), (M) and (K), (3.49) and (3.50) work when λ_1 and λ_2 are interchanged,

$$\sum_{z=x}^{y} \pi(x, z, a_1 + b_1, b_2) \pi(z, y, a_1, b_1) = \pi(x, y, a_1, b_1 + b_2),$$
(3.51)

$$\sum_{z=y}^{x} \pi(x-z, N-z, a_2, a_1+b_1) \pi(z-y, N-y, a_1, b_1) = \pi(x-y, N-y, a_1+a_2, b_1).$$
(3.52)

These are obtained by the sum formula $(n, m \in \mathbb{Z}_{\geq 0})$,

$$\sum_{k=0}^{n} \binom{n}{k} (b_1)_{n-k} (b_2)_k \frac{(a)_{m+k}}{(a+b_1+b_2)_{m+k}} = \frac{(a)_m (b_1+b_2)_n}{(a+b_1+b_2)_{m+n}} \frac{(a+b_1)_{m+n}}{(a+b_1)_m},$$
(3.53)

which is derived from the Pfaff-Saalschütz formula [2](2.2.8),

$${}_{3}F_{2}\left(\begin{array}{c}-n, a, b\\c, 1+a+b-c-n\end{array}\middle|1\right) = \frac{(c-a, c-b)_{n}}{(c, c-a-b)_{n}} \quad (n \in \mathbb{Z}_{\geq 0}).$$
(3.54)

q-Hahn (qH)

$$\sum_{z=0}^{x} \pi(x-z, N-z, a_1, b_1) \pi(z, N, a_2, a_1 b_1) = \pi(x, N, a_1 a_2, b_1),$$
(3.55)

$$\sum_{z=x}^{y} \pi(x, z, a_1, b_1) \pi(z, y, a_1 b_1, b_2) = \pi(x, y, a_1, b_1 b_2),$$
(3.56)

$$\sum_{z=y}^{x} \pi(x-z, N-z, a_1, b_1) \pi(z-y, N-y, a_2, a_1b_1) = \pi(x-y, N-y, a_1a_2, b_1).$$
(3.57)

These are obtained by the sum formula for π_{qH}

$$\sum_{x \in \mathcal{X}} \pi_{q\mathrm{H}}(x, N, a, b) = 1 \Longleftrightarrow \sum_{k=0}^{n} {n \brack k} (a; q)_{k} (b; q)_{n-k} a^{n-k} = (ab; q)_{n} \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.58)$$

which is derived from the formula [4](1.11.4),

$${}_{2}\phi_{1}\left(\begin{array}{c}q^{-n}, \ b\\c\end{array} \middle| \ q \ ; \frac{cq^{n}}{b}\right) = \frac{(b^{-1}c \ ; q)_{n}}{(c \ ; q)_{n}} \quad (n \in \mathbb{Z}_{\geq 0}).$$
(3.59)

Similarly to the Hahn cases, (3.56) and (3.57) work when λ_1 and λ_2 are interchanged,

$$\sum_{z=x}^{y} \pi(x, z, a_1b_1, b_2)\pi(z, y, a_1, b_1) = \pi(x, y, a_1, b_1b_2),$$
(3.60)

$$\sum_{z=y}^{x} \pi(x-z, N-z, a_2, a_1b_1)\pi(z-y, N-y, a_1, b_1) = \pi(x-y, N-y, a_1a_2, b_1).$$
(3.61)

These are obtained by the sum formula $(n, m \in \mathbb{Z}_{\geq 0})$

$$\sum_{k=0}^{n} {n \brack k} (b_1;q)_{n-k} (b_2;q)_k \frac{b_1^k (a;q)_{m+k}}{(ab_1b_2;q)_{m+k}} = \frac{(a;q)_m (b_1b_2;q)_n}{(ab_1b_2;q)_{m+n}} \frac{(ab_1;q)_{m+n}}{(ab_1;q)_m},$$
(3.62)

which is derived from the q-Pfaff-Saalschütz formula [4](1.11.9),

$${}_{3}\phi_{2}\left(\begin{array}{c}q^{-n}, a, b\\c, abc^{-1}q^{1-n} \end{array} \middle| q;q\right) = \frac{(a^{-1}c, b^{-1}c; q)_{n}}{(c, a^{-1}b^{-1}c; q)_{n}} \quad (n \in \mathbb{Z}_{\geq 0}).$$

$$(3.63)$$

4 Many Examples of K(x, y)

In this section we present various examples of K(x, y) constructed by the five types of convolutions listed in §3.1 applied to the polynomials Krawtchouk (K), Hahn (H) and q-Hahn (qH). For each polynomial, the limiting forms obtained by $N \to \infty$ are displayed, Charlier (C), Meixner (M) and q-Meixner (qM). We do strongly hope that these examples would enrich many related disciplines.

The symmetry condition (2.9) or (2.29) is easily verified without evaluating the sum(s) in K(x, y) for all examples in this section except for those in §4.6.

4.1 Type (i) convolution

This convolution

(i):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=0}^{\min(x,y)} \pi(x-z,N-z,\boldsymbol{\lambda}_2)\pi(z,y,\boldsymbol{\lambda}_1),$$

has been applied to (K) and (H) in many papers [11]–[13] in connection with "cumulative Bernoulli trials." It has a factorised form, $K(x,y) = \sum_{z=0}^{N} A(x,z)B(z,y)$, the non-vanishing

elements of A(x, z) and B(z, y) being $\pi(x - z, N - z, \lambda_2)$ $(x \ge z)$ and $\pi(z, y, \lambda_1)$ $(z \le y)$, respectively. Namely, A is a lower triangular matrix and B is upper triangular. The determinant of K is easily obtained as

$$\prod_{n \in \mathcal{X}} \kappa(n) = \det K = \prod_{x \in \mathcal{X}} A(x, x) \cdot \prod_{x \in \mathcal{X}} B(x, x) = \prod_{n \in \mathcal{X}} \pi(0, n, \lambda_2) \pi(n, n, \lambda_1),$$

from which eigenvalues $\kappa(n)$ are obtained, if it is known that the eigenvalues are independent of $N \iff \kappa(n) = \pi(0, n, \lambda_2)\pi(n, n, \lambda_1)$. In fact, as we will see shortly, the eigenvalues are N independent for all the examples in this section. Therefore for type (i) and (iii) convolutions, the determinant formulas give eigenvalues. Moreover, for x = 0 only one term z = 0 contributes to K(0, y). This greatly simplifies the eigenvalue formula (2.21)

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(0, N, \boldsymbol{\lambda}_2) \pi(0, y, \boldsymbol{\lambda}_1) \frac{\pi(y, N, \boldsymbol{\lambda})}{\pi(0, N, \boldsymbol{\lambda})} \check{P}_n(y, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$
(4.1)

4.1.1 Krawtchouk (K)

By taking $\lambda_1 = a$ and $\lambda_2 = b$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b) \pi(z, y, a) \quad \left(\Rightarrow \det K = \prod_{n \in \mathcal{X}} a^n (1-b)^n\right).$$
(4.2)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = p \stackrel{\text{def}}{=} \frac{b}{1 - a + ab},\tag{4.3}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19),

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, N, p) \check{P}_n(y, p) = \kappa(n) \pi(x, N, p) \check{P}_n(x, p).$$

By writing down the eigenvalue formula (4.1), we have

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, b) \check{P}_n(y, p) = a^n (1 - b)^n, \qquad (4.4)$$

by using a generating function of Krawtchouk $P_n(x)$ [4](9.11.11),

$$\sum_{n=0}^{N} \binom{N}{n} P_n(x,p) t^n = \left(1 - \frac{1-p}{p}t\right)^x (1+t)^{N-x},$$
(4.5)

together with the self-duality of (K) $P_n(x,p) = P_x(n,p)$. In a summation formula like (4.4) y summation is easily performed as the only y dependence of $\check{P}_n(y, \lambda)$ in this paper is due to $(-y)_k$ (or $(q^{-y};q)_k$ for qH, qM and qC) (k = 0, 1, ..., n), which gives

$$\sum_{y \in \mathcal{X}} \pi(y, N, b) (-y)_k = (-N)_k b^k.$$
(4.6)

This cancels the N dependence in the hypergeometric summation of $\check{P}_n(y,p)$. This is the general mechanism leading to the N-independence of the eigenvalues for (K), (H) and (qH). By direct calculation using (4.6), we obtain

$$\kappa(n) = \sum_{k=0}^{n} \frac{(-n)_k p^{-k}}{(-N)_k k!} (-N)_k b^k = \sum_{k=0}^{n} (-n)_k \frac{(bp^{-1})^k}{k!} = {}_1F_0 {\binom{-n}{-}} bp^{-1}.$$
(4.7)

As we will show in the following, all the eigenvalues $\kappa(n)$ of K's constructed in this section are expressed by a terminating (q-)hypergeometric series, except for those corresponding to the extra eigenvectors of (qM).

Krawtchouk \rightarrow **Charlier** This is achieved by $b \rightarrow bN^{-1}$, $N \rightarrow \infty$,

$$\check{P}_{n}(x,p) \to \check{P}_{Cn}(x,p'), \quad p' \stackrel{\text{def}}{=} \frac{b}{1-a},
\pi(x,N,p) \to \pi_{C}(x,p'), \quad \kappa(n) \to \kappa_{C}(n) = a^{n},
K(x,y) \to K_{C}(x,y,a,b) = \sum_{z=0}^{\min(x,y)} \pi_{C}(x-z,b)\pi(z,y,a),$$
(4.8)

and the relations (2.9), (2.19) and (2.21) of (K) reduce to those of (C). The resulting $K_{\rm C}$ is a standard convolution of the π 's of (C) and (K). This was discussed in [11]§4. Based on $K_{\rm C}$ (4.8), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = p'$ and **Theorem 2.1** gives (2.30). The direct calculation of the eigenvalue formula (2.31) gives

$$\kappa_{\rm C}(n) = \sum_{y=0}^{\infty} \pi_{\rm C}(y,b) \check{P}_{{\rm C}\,n}(y,p') = a^n = {}_1F_0 {\binom{-n}{-}} bp'^{-1}.$$
(4.9)

It can also be obtained by using a generating function of (C) [4](9.14.11) together with the self-duality of (C).

4.1.2 Hahn (H)

By taking $\lambda_1 = (a, b)$ and $\lambda_2 = (b, c)$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b, c) \pi(z, y, a, b) \quad \left(\Rightarrow \det K = \prod_{n \in \mathcal{X}} \frac{(a)_n(c)_n}{(a+b)_n(b+c)_n}\right).$$
(4.10)

For the following λ ,

$$\boldsymbol{\lambda} = (a+b,c),\tag{4.11}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19),

$$\sum_{y \in \mathcal{X}} K(x, y)\pi(y, N, a+b, c)\check{P}_n(y, a+b, c) = \kappa(n)\pi(x, N, a+b, c)\check{P}_n(x, a+b, c).$$

By evaluating the eigenvalue formula (4.1), we obtain a balanced ${}_{3}F_{2}$,

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, b, c) \check{P}_n(y, a + b, c)$$

= ${}_3F_2 \begin{pmatrix} -n, n + a + b + c - 1, b \\ a + b, b + c \end{pmatrix} | 1 = \frac{(a)_n(c)_n}{(a + b)_n(b + c)_n}.$ (4.12)

The last equality is due to the Pfaff-Saalschütz formula (3.54). This provides another sum formula for the dual Hahn polynomial $\check{Q}_n(x, a + b, c)$ [9],

$$\sum_{n=0}^{N} \binom{N}{n} (b)_n (c)_{N-n} \check{Q}_n (x, a+b, c) = \frac{(b+c)_N (a)_x (c)_x}{(a+b)_x (b+c)_x},$$
(4.13)

$$\check{Q}_n(x, a+b, c) \stackrel{\text{def}}{=} {}_3F_2 \Big(\begin{array}{c} -n, \ x+a+b+c-1, \ -x \\ a+b, \ -N \end{array} \Big| 1 \Big).$$
(4.14)

By rewriting $(a)_x = \Gamma(a+x)/\Gamma(a)$, this sum formula is valid for $\forall x \in \mathbb{C}$.

Hahn \rightarrow **Meixner** This is achieved by fixing *a* and *b* with $c \rightarrow N(1-c)c^{-1}$ ($\Rightarrow 0 < c < 1$), $N \rightarrow \infty$,

$$\dot{P}_{n}(x, a + b, c) \to \dot{P}_{Mn}(x, a + b, c),
\pi(x, N, a + b, c) \to \pi_{M}(x, a + b, c), \quad \kappa(n) \to \kappa_{M}(n) = \frac{(a)_{n}}{(a + b)_{n}},
K(x, y) \to K_{M}(x, y, a, b, c) = \sum_{z=0}^{\min(x, y)} \pi_{M}(x - z, b, c)\pi(z, y, a, b),$$
(4.15)

and the relations (2.9), (2.19) and (2.21) of (H) reduce to those of (M). This $K_{\rm M}$ is a standard convolution of π 's of (M) and (H). Based on $K_{\rm M}$ (4.15), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (a + b, c)$ and **Theorem 2.1** gives (2.30). The direct calculation of the eigenvalue formula (2.31) gives

$$\kappa_{\rm M}(n) = \sum_{y=0}^{\infty} \pi_{\rm M}(y, b, c) \check{P}_{{\rm M}\,n}(y, a+b, c) = \frac{(a)_n}{(a+b)_n} = {}_2F_1\Big(\frac{-n, b}{a+b} \Big| 1\Big). \tag{4.16}$$

It can be obtained by using a generating function [4](9.10.13) together with the self-duality of (M). Of course, $_2F_1$ form is also obtained from the balanced $_3F_2$ form (4.12) by the above limit.

Hahn \rightarrow **Meixner** \rightarrow **Charlier** This is achieved by $a \rightarrow aN, b \rightarrow bN, c \rightarrow \frac{c}{c+N}, N \rightarrow \infty$,

$$P_{\mathrm{M}n}(x, (a+b)N, \frac{c}{c+N}) \to P_{\mathrm{C}n}(x, (a+b)c),$$

$$\pi_{\mathrm{M}}(x, (a+b)N, \frac{c}{c+N}) \to \pi_{\mathrm{C}}(x, (a+b)c),$$

$$\kappa_{\mathrm{M}}(n) = \frac{(aN)_{n}}{\left((a+b)N\right)_{n}} \to \kappa_{\mathrm{C}}(n) = \left(\frac{a}{a+b}\right)^{n},$$

$$\pi_{\mathrm{M}}(x, bN, \frac{c}{c+N}) \to \pi_{\mathrm{C}}(x, bc), \quad \pi(z, y, aN, bN) \to \pi_{\mathrm{K}}(z, y, \frac{a}{a+b}),$$

$$K_{\mathrm{M}}(x, y, a, b, c) \to K_{\mathrm{C}}(x, y, a, b, c) = \sum_{z=0}^{\min(x,y)} \pi_{\mathrm{C}}(x-z, bc)\pi_{\mathrm{K}}(z, y, \frac{a}{a+b}),$$
(4.17)

and the relations (2.29), (2.30) and (2.31) of (M) reduce to those of (C). This $K_{\rm C}$ agrees with (4.8) with the replacement $(a, b) \rightarrow (\frac{a}{a+b}, bc)$.

4.1.3 q-Hahn (qH)

We believe the explicit examples of Markov chains described by the q-Hahn polynomial and its reduction, q-Meixner are new.

By taking $\lambda_1 = (a, b)$ and $\lambda_2 = (b, c)$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=0}^{\min(x,y)} \pi(x-z, N-z, b, c) \pi(z, y, a, b) \quad \left(\Rightarrow \det K = \prod_{n \in \mathcal{X}} \frac{b^n(a;q)_n(c;q)_n}{(ab;q)_n(bc;q)_n}\right).$$
(4.18)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = (ab, c), \tag{4.19}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19),

$$\sum_{y \in \mathcal{X}} K(x, y)\pi(y, N, ab, c)\check{P}_n(y, ab, c) = \kappa(n)\pi(x, N, ab, c)\check{P}_n(x, ab, c).$$

By evaluating the eigenvalue formula (4.1), we obtain a balanced $_{3}\phi_{2}$

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, b, c) \check{P}_n(y, ab, c) = {}_3\phi_2 \Big(\begin{array}{c} q^{-n}, \ abcq^{n-1}, \ b \\ ab, \ bc \end{array} \Big| \ q \, ; q \Big) = \frac{b^n(a \, ; q)_n(c \, ; q)_n}{(ab \, ; q)_n(bc \, ; q)_n}.$$
(4.20)

The last equality is due to the q-Pfaff-Saalschütz formula (3.63).

q-Hahn $\rightarrow q$ -Meixner This limit is achieved by fixing a, b with $c \rightarrow -c^{-1}q^{1-N}, N \rightarrow \infty$, (c > 0),

$$\check{P}_{n}(x, ab, c) \to \check{P}_{qMn}(x, abq^{-1}, (ab)^{-1}c),
\pi(x, N, ab, c) \to \pi_{qM}(x, abq^{-1}, (ab)^{-1}c), \quad \kappa(n) \to \kappa_{qM}(n) = \frac{(a;q)_{n}}{(ab;q)_{n}},
\pi(x - z, N - z, b, c)$$
(4.21)

$$\rightarrow \pi'_{qM}(x, z, bq^{-1}, b^{-1}c) = \frac{(-b^{-1}c; q)_z(b; q)_{x-z}(b^{-1}c)^{x-z} q^{\binom{x}{2} - \binom{z}{2}}}{(-c; q)_x(q; q)_{x-z}} \frac{(-c; q)_\infty}{(-b^{-1}c; q)_\infty}, \quad (4.22)$$

$$K(x,y) \to K_{qM}(x,y,a,b,c) = \sum_{z=0}^{\min(x,y)} \pi'_{qM}(x,z,bq^{-1},b^{-1}c)\pi(z,y,a,b).$$
(4.23)

The above K_{qM} (4.23) is not a standard convolution of π 's of (qM) and (qH), as the form of π'_{qM} (4.22) is markedly different from that of (4.21). The relations (2.9), (2.19) and (2.21) of (qH) reduce to those of (qM). Based on K_{qM} (4.23), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (abq^{-1}, (ab)^{-1}c)$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) takes a neat form and the sum is directly evaluated

$$\kappa_{qM}(n) = \sum_{y=0}^{\infty} \pi_{qM}(y, bq^{-1}, b^{-1}c) \check{P}_{qMn}(y, abq^{-1}, (ab)^{-1}c)$$
$$= {}_{2}\phi_{1} {\binom{q^{-n}, b}{ab}} | q; aq^{n} = \frac{(a; q)_{n}}{(ab; q)_{n}}.$$
(4.24)

In the last equality (3.59) is used. The $_2\phi_1$ form is also obtained from the balanced $_3\phi_2$ form (4.20) by the above limit.

The other set of eigenvectors is

$$(-1)^{x} \sqrt{\pi_{qM}(x, abq^{-1}, (ab)^{-1}c)\pi_{qM}(x, -cq^{-1}, abc^{-1})} \check{P}_{qMn}(x, -cq^{-1}, abc^{-1})$$

= $(-1)^{x} \frac{q^{\binom{x}{2}}}{(q;q)_{x}} \check{P}_{qMn}(x, -cq^{-1}, abc^{-1}) \times \sqrt{\frac{(ab, -c;q)_{\infty}}{(-abc^{-1}, -(ab)^{-1}c;q)_{\infty}}},$ (4.25)

and the corresponding eigenvalue formula (2.31) reads

$$\kappa_{qM}^{(-)}(n) = \sum_{y=0}^{\infty} K_{qM}(0,y)(-1)^{y} \sqrt{\frac{\pi_{qM}(y,\boldsymbol{\lambda})\pi_{qM}^{(-)}(y,\boldsymbol{\lambda})}{\pi_{qM}(0,\boldsymbol{\lambda})\pi_{qM}^{(-)}(0,\boldsymbol{\lambda})}} \check{P}_{qMn}^{(-)}(y,\boldsymbol{\lambda}).$$
(4.26)

After a few lines of direct calculation, we obtain

$$\kappa_{qM}^{(-)}(n) = \frac{(a, -c; q)_{\infty}}{(ab, -b^{-1}c; q)_{\infty}} {}_{2}\phi_{1} \left(\begin{array}{c} q^{-n}, b \\ -c \end{array} \middle| q; -b^{-1}cq^{n} \right) = \frac{(a, -c; q)_{\infty}}{(ab, -b^{-1}c; q)_{\infty}} \frac{(-b^{-1}c; q)_{n}}{(-c; q)_{n}}, \quad (4.27)$$

with

$$0 < \kappa_{q\mathbf{M}}^{(-)}(n) < 1, \quad 0 < \frac{\kappa_{q\mathbf{M}}^{(-)}(n)}{\kappa_{q\mathbf{M}}(n)} = \frac{(aq^n, -cq^n; q)_{\infty}}{(abq^n, -b^{-1}cq^n; q)_{\infty}} < 1.$$

The reduction to q-Charlier is not feasible, as it requires $ab \to 0$ in $\check{P}_n(y, ab, c)$, leading $(ab)^{-1}c$ to diverge.

4.2 Type (ii) convolution

Since this convolution

(ii):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z,N-y,\boldsymbol{\lambda}_2)\pi(z,y,\boldsymbol{\lambda}_1)$$

is not of a factorised form, the determinant of K is not obtained easily. For x = 0 only one term z = 0 contributes and the general eigenvalue formula (2.21) takes a simple form

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(0, N - y, \boldsymbol{\lambda}_2) \pi(0, y, \boldsymbol{\lambda}_1) \frac{\pi(y, N, \boldsymbol{\lambda})}{\pi(0, N, \boldsymbol{\lambda})} \check{P}_n(y, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$
(4.28)

This convolution was employed for Markov chains related with the Hahn polynomial in [14], but the use of the convolution was not stated explicitly.

4.2.1 Krawtchouk (K)

By taking $\lambda_1 = a$ and $\lambda_2 = b$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z,N-y,b)\pi(z,y,a).$$
(4.29)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = p \stackrel{\text{def}}{=} \frac{b}{1 - a + b},\tag{4.30}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By directly evaluating the eigenvalue formula (4.28), we obtain

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, b) \check{P}_n(y, p) = (a - b)^n = {}_1F_0 {\binom{-n}{-}} bp^{-1}.$$
(4.31)

The use of the generating function (4.5) and the self-duality (K) leads to the above result, too. It is interesting to note that odd eigenvalues are all negative if 0 < a < b < 1.

Krawtchouk \rightarrow **Charlier** This is exactly the same situation of (K) \rightarrow (C) (4.8) in the type (i) convolution.

4.2.2 Hahn (H)

By taking $\lambda_1 = (a, b)$ and $\lambda_2 = (b, c)$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=\max(0,x+y-N)}^{\min(x,y)} \pi(x-z,N-y,b,c)\pi(z,y,a,b).$$
(4.32)

The special cases of this convolution with $a = b = c = \frac{1}{2}$ corresponding to the discrete Chebyshev polynomials and $a = b = c = \frac{1}{2}\theta$ are discussed in detail in [14]. For the following λ ,

$$\boldsymbol{\lambda} = (a+b,b+c),\tag{4.33}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By writing down the eigenvalue formula (4.28), we have

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, b, c) \check{P}_n(y, a+b, b+c) = {}_3F_2 \left(\begin{array}{c} -n, \ n+a+2b+c-1, \ b \\ a+b, \ b+c \end{array} \right)$$
(4.34)

$$=\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \frac{(b)_{k}(n+a+2b+c-1)_{k}}{(a+b)_{k}(b+c)_{k}}.$$
(4.35)

The above explicit eigenvalue formula (4.35) reproduces the known results (5.9) in [14], which corresponds to the special case $a = b = c = \frac{1}{2}\theta$.

Hahn \rightarrow **Meixner** By fixing a, b with $c \rightarrow N(1-c)c^{-1}$ ($\Rightarrow 0 < c < 1$) and taking the limit $N \rightarrow \infty$, one obtains the same Meixner limit $K_{\rm M}(x, y)$ as in (4.15)

$$K(x,y) \to K_{\mathrm{M}}(x,y,a,b,c) = \sum_{z=0}^{\min(x,y)} \pi_{\mathrm{M}}(x-z,b,c)\pi(z,y,a,b).$$

It is interesting to note that the q-Hahn version of this convolution does not work, due to the presence of the extra factor a^{N-x} in $\pi(x, N, a, b)$ in (3.24).

4.3 Type (iii) convolution

Since this convolution

(iii):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z=\max(x,y)}^{N} \pi(x,z,\boldsymbol{\lambda}_2)\pi(z-y,N-y,\boldsymbol{\lambda}_1),$$

is of a factorised form, the first factor $\pi(x, z, \lambda_2)$ being upper triangular and the second factor $\pi(z - y, N - y, \lambda_1)$ lower triangular, the eigenvalues can be guessed from the determinant,

$$\prod_{n \in \mathcal{X}} \kappa(n) = \det K = \prod_{n \in \mathcal{X}} \pi(n, n, \lambda_2) \pi(0, n, \lambda_1).$$

4.3.1 Krawtchouk (K)

By taking $\lambda_1 = a$ and $\lambda_2 = b$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=\max(x,y)}^{N} \pi(x,z,b)\pi(z-y,N-y,a) \quad \left(\Rightarrow \det K = \prod_{n\in\mathcal{X}} (1-a)^n b^n\right). \tag{4.36}$$

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = p \stackrel{\text{def}}{=} \frac{ab}{1 - b + ab},\tag{4.37}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By evaluating the eigenvalue formula (2.21), we have

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, a) \sum_{y=0}^{z} \pi(y, z, b) \check{P}_{n}(y, p) = (1-a)^{n} b^{n} = {}_{1}F_{0} {\binom{-n}{-}} abp^{-1}.$$
(4.38)

The use of the generating function (4.5) and the self-duality (K) give the same result.

Krawtchouk \rightarrow **Charlier** This is achieved by $a \rightarrow aN^{-1}$, $N \rightarrow \infty$,

$$\check{P}_{n}(x,p) \rightarrow P_{Cn}(x,p'), \quad p' \stackrel{\text{def}}{=} \frac{ab}{1-b},$$

$$\pi(x,N,p) \rightarrow \pi_{C}(x,p'), \quad \kappa(n) \rightarrow \kappa_{C}(n) = b^{n},$$

$$K(x,y) \rightarrow K_{C}(x,y,a,b) = \sum_{z=\max(x,y)}^{\infty} \pi(x,z,b)\pi_{C}(z-y,a),$$
(4.39)

and the relations (2.9), (2.19) and (2.21) of (K) reduce to those of (C). Based on $K_{\rm C}$ (4.39), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = p'$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) reads

$$\kappa_{\rm C}(n) = \sum_{z=0}^{\infty} \pi_{\rm C}(z,a) \sum_{y=0}^{z} \pi(y,z,b) \check{P}_{{\rm C}\,n}(y,p') = b^n = {}_1F_0 \binom{-n}{-} abp'^{-1}, \qquad (4.40)$$

by using the generating function [4](9.14.11) and the self-duality (C).

4.3.2 Hahn (H)

By taking $\lambda_1 = (a, b)$ and $\lambda_2 = (c, a)$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=\max(x,y)}^{N} \pi(x,z,c,a)\pi(z-y,N-y,a,b) \quad \left(\Rightarrow \det K = \prod_{n\in\mathcal{X}} \frac{(b)_n(c)_n}{(a+b)_n(a+c)_n}\right).$$
(4.41)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = (c, a + b), \tag{4.42}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By evaluating the eigenvalue formula (2.21), we obtain a balanced ${}_{3}F_{2}$

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, a, b) \sum_{y=0}^{z} \pi(y, z, c, a) \check{P}_{n}(y, c, a + b)$$

= ${}_{3}F_{2} \begin{pmatrix} -n, n+a+b+c-1, a\\ a+b, a+c \end{pmatrix} | 1 \end{pmatrix} = \frac{(b)_{n}(c)_{n}}{(a+b)_{n}(a+c)_{n}}.$ (4.43)

In the last equality Pfaff-Saalschütz formula (3.54) is used.

Hahn \rightarrow **Meixner** This is achieved by fixing *a* and *c* with $b \rightarrow N(1-b)b^{-1} (\Rightarrow 0 < b < 1)$, $N \rightarrow \infty$,

$$\dot{P}_{n}(x, c, a + b) \to \dot{P}_{Mn}(x, c, b),
\pi(x, N, c, a + b) \to \pi_{M}(x, c, b), \quad \kappa(n) \to \kappa_{M}(n) = \frac{(c)_{n}}{(a + c)_{n}},
K(x, y) \to K_{M}(x, y, a, b, c) = \sum_{z=\max(x, y)}^{\infty} \pi(x, z, c, a) \pi_{M}(z - y, a, b),$$
(4.44)

and the relations (2.9), (2.19) and (2.21) of (H) reduce to those of (M). This is a standard convolution of π 's of (H) and (M), but the order is opposite from that of (4.15). Based on $K_{\rm M}$ (4.44), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (c, b)$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) reads

$$\kappa_{\rm M}(n) = \sum_{z=0}^{\infty} \pi_{\rm M}(z,a,b) \sum_{y=0}^{z} \pi(y,z,c,a) \check{P}_{{\rm M}\,n}(y,c,b) = {}_2F_1 \begin{pmatrix} -n, \ a \\ a+c \end{pmatrix} \left| 1 \right) = \frac{(c)_n}{(a+c)_n}.$$
 (4.45)

Of course the $_2F_1$ form is also obtained from the balanced $_3F_2$ form (4.43) by the above limit.

Hahn \rightarrow **Meixner** \rightarrow **Charlier** This is achieved by $a \rightarrow aN, c \rightarrow cN, b \rightarrow \frac{b}{b+N}, N \rightarrow \infty$,

$$P_{\mathrm{M}n}(x,cN,\frac{b}{b+N}) \to P_{\mathrm{C}n}(x,bc),$$

$$\pi_{\mathrm{M}}(x,cN,\frac{b}{b+N}) \to \pi_{\mathrm{C}}(x,bc), \quad \kappa_{\mathrm{M}}(n) = \frac{(cN)_{n}}{\left((a+c)N\right)_{n}} \to \kappa_{\mathrm{C}}(n) = \left(\frac{c}{a+c}\right)^{n},$$

$$\pi(x,z,cN,aN) \to \pi_{\mathrm{K}}(x,z,\frac{c}{a+c}), \quad \pi_{\mathrm{M}}(x,aN,\frac{b}{b+N}) \to \pi_{\mathrm{C}}(x,ab),$$

$$K_{\mathrm{M}}(x,y,a,b,c) \to K_{\mathrm{C}}(x,y,a,b,c) = \sum_{z=\max(x,y)}^{\infty} \pi_{\mathrm{K}}(x,z,p)\pi_{\mathrm{C}}(z-y,ab), \quad (4.46)$$

and the relations (2.29), (2.30) and (2.31) of (M) reduce to those of (C). This $K_{\rm C}$ agrees with (4.39) with the replacement $(a, b) \rightarrow (ab, p)$.

4.3.3 q-Hahn (qH)

By taking $\lambda_1 = (a, b)$ and $\lambda_2 = (c, a)$, the matrix K(x, y) is

$$K(x,y) = \sum_{z=\max(x,y)}^{N} \pi(x,z,c,a)\pi(z-y,N-y,a,b) \quad \left(\Rightarrow \det K = \prod_{n\in\mathcal{X}} \frac{a^n(b\,;\,q)_n(c\,;\,q)_n}{(ab\,;\,q)_n(ac\,;\,q)_n}\right).$$
(4.47)

For the following λ ,

$$\boldsymbol{\lambda} = (c, ab), \tag{4.48}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By evaluating the eigenvalue formula (2.21), we obtain a balanced $_3\phi_2$

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, a, b) \sum_{y=0}^{z} \pi(y, z, c, a) \check{P}_{n}(y, c, ab)$$

= ${}_{3}\phi_{2} \begin{pmatrix} q^{-n}, abcq^{n-1}, a \\ ac, ab \end{pmatrix} | q;q \end{pmatrix} = \frac{a^{n}(b;q)_{n}(c;q)_{n}}{(ab;q)_{n}(ac;q)_{n}}.$ (4.49)

The last equality is due to the q-Pfaff-Saalschütz formula (3.63).

q-Hahn $\rightarrow q$ -Meixner This limit is achieved by fixing a, c with $b \rightarrow -b^{-1}q^{1-N}, N \rightarrow \infty$, (b > 0),

$$\check{P}_{n}(x,c,ab) \to \check{P}_{qMn}(x,cq^{-1},(ac)^{-1}b),
\pi(x,N,c,ab) \to \pi_{qM}(x,cq^{-1},(ac)^{-1}b), \quad \kappa(n) \to \kappa_{qM}(n) = \frac{(c;q)_{n}}{(ac;q)_{n}},
\pi(z-y,N-y,a,b)$$

$$\rightarrow \pi'_{qM}(z, y, aq^{-1}, a^{-1}b) = \frac{(-a^{-1}b; q)_y(a; q)_{z-y}(a^{-1}b)^{z-y}q^{\binom{z}{2}-\binom{y}{2}}}{(-b; q)_{z-y}} \frac{(-b; q)_{\infty}}{(-a^{-1}b; q)_{\infty}},$$

$$K(x, y) \rightarrow K_{qM}(x, y, a, b, c) = \sum_{k=1}^{\infty} \pi(x, z, c, a)\pi'_{qM}(z, y, aq^{-1}, a^{-1}b).$$

$$(4.50)$$

This is not a standard convolution as π'_{qM} is not an orthogonality measure of (qM). The relations (2.9), (2.19) and (2.21) of (qH) reduce to those of (qM). Based on K_{qM} (4.50), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (cq^{-1}, (ac)^{-1}b)$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) is written down as

 $z = \max(x, y)$

$$\kappa_{qM}(n) = \sum_{z=0}^{\infty} \pi_{qM}(z, aq^{-1}, a^{-1}b) \sum_{y=0}^{z} \pi(y, z, c, a) \check{P}_{qMn}(y, cq^{-1}, (ac)^{-1}b)$$
$$= {}_{2}\phi_{1} \left(\begin{array}{c} q^{-n}, a \\ ac \end{array} \middle| q; cq^{n} \right) = \frac{(c; q)_{n}}{(ac; q)_{n}}.$$
(4.51)

In the last equality (3.59) is used. The $_2\phi_1$ form is also obtained from the balanced $_3\phi_2$ form (4.49) by the above limit.

The other set of eigenvectors of K_{qM} is

$$(-1)^{x} \sqrt{\pi_{qM}(x, cq^{-1}, (ac)^{-1}b)\pi_{qM}(x, -a^{-1}bq^{-1}, acb^{-1})} \check{P}_{qMn}(x, -a^{-1}bq^{-1}, acb^{-1})$$

$$= (-1)^{x} \frac{q^{\binom{x}{2}}}{(q;q)_{x}} \check{P}_{qMn}(x, -a^{-1}bq^{-1}, acb^{-1}) \times \sqrt{\frac{(-a^{-1}b, c; q)_{\infty}}{(-acb^{-1}, -(ac)^{-1}b; q)_{\infty}}},$$
(4.52)

and the corresponding eigenvalue formula (2.31) takes the same form as (4.26). After a few lines of direct calculation, we obtain

$$\kappa_{qM}^{(-)}(n) = \frac{(c, -b; q)_{\infty}}{(ac, -a^{-1}b; q)_{\infty}} {}_{2}\phi_{1} \begin{pmatrix} q^{-n}, a \\ -b \end{pmatrix} | q; -a^{-1}bq^{n} \end{pmatrix} = \frac{(c, -b; q)_{\infty}}{(ac, -a^{-1}b; q)_{\infty}} \frac{(-a^{-1}b; q)_{n}}{(-b; q)_{n}},$$
(4.53)

with

$$0 < \kappa_{qM}^{(-)}(n) < 1, \quad 0 < \frac{\kappa_{qM}^{(-)}(n)}{\kappa_{qM}(n)} = \frac{(cq^n, -bq^n; q)_{\infty}}{(acq^n, -a^{-1}bq^n; q)_{\infty}} < 1$$

The q-Charlier limit does not exist, as it requires $c \to 0$. This causes $\kappa_{qM}(n) \to 1$ and $\pi(x, z, c, a) \to 0$ in K_{qM} .

4.4 Type (iv) convolution

This type of convolutions

(iv):
$$K(x,y) \stackrel{\text{def}}{=} \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, \lambda_1) \sum_{z_1=\max(x,y)}^N \pi(x-z_2, z_1-z_2, \lambda_3) \pi(z_1-y, N-y, \lambda_2),$$

was reported in [11] for (K). The eigenvalues were derived by a different method from that given below. At x = 0 only the $z_2 = 0$ term contributes and the eigenvalue formula becomes a manageable double sum formula

$$\kappa(n) = \sum_{z=0}^{N} \sum_{y=0}^{z} \pi(0, y, \boldsymbol{\lambda}_{1}) \pi(0, z, \boldsymbol{\lambda}_{3}) \pi(z - y, N - y, \boldsymbol{\lambda}_{2}) \frac{\pi(y, N, \boldsymbol{\lambda})}{\pi(0, N, \boldsymbol{\lambda})} \check{P}_{n}(y, \boldsymbol{\lambda}).$$
(4.54)

4.4.1 Krawtchouk (K)

By taking $\lambda_1 = a$, $\lambda_2 = b$ and $\lambda_3 = c$, the matrix K(x, y) is

$$K(x,y) = \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, a) \sum_{z_1=\max(x,y)}^{N} \pi(x - z_2, z_1 - z_2, c) \pi(z_1 - y, N - y, b).$$
(4.55)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = p \stackrel{\text{def}}{=} \frac{bc}{bc + (1 - a)(1 - c)},\tag{4.56}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By writing down the eigenvalue formula (4.54), we have

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, b) \sum_{y=0}^{z} \pi(y, z, c) \check{P}_{n}(y, p) = {}_{1}F_{0} \begin{pmatrix} -n \\ - \end{pmatrix} bcp^{-1}$$

= $(a + c - ac - bc)^{n} = (1 - bc - (1 - a)(1 - c))^{n},$ (4.57)

by the generating function (4.5) and the self-duality (K).

Krawtchouk \rightarrow **Charlier** This is achieved by fixing a and c with $b \rightarrow bN^{-1}$, $N \rightarrow \infty$,

$$\check{P}_{n}(x,p) \to \check{P}_{Cn}(x,p'), \quad p' \stackrel{\text{def}}{=} \frac{bc}{(1-a)(1-c)}, \\
\pi(x,N,p) \to \pi_{C}(x,p'), \quad \kappa(n) \to \kappa_{C}(n) = (a+c-ac)^{n} = \left(1-(1-a)(1-c)\right)^{n}, \\
K(x,y) \to K_{C}(x,y,a,b,c) = \sum_{z_{2}=0}^{\min(x,y)} \pi(z_{2},y,a) \sum_{z_{1}=\max(x,y)}^{\infty} \pi(x-z_{2},z_{1}-z_{2},c)\pi_{C}(z_{1}-y,b), \quad (4.58)$$

and the relations (2.9), (2.19) and (2.21) of (K) reduce to those of (C). Based on $K_{\rm C}$ (4.58), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = p'$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) reads

$$\kappa_{\rm C}(n) = \sum_{z=0}^{\infty} \pi_{\rm C}(z,b) \sum_{y=0}^{z} \pi(y,z,c) \check{P}_{{\rm C}\,n}(y,p') = (a+c-ac)^n = {}_1F_0 \begin{pmatrix} -n \\ - \end{pmatrix} bcp'^{-1}, \quad (4.59)$$

by using the generating function [4](9.14.11) and the self-duality (C).

4.4.2 Hahn (H)

By taking $\lambda_1 = (a_1, b_1)$, $\lambda_2 = (a_2, b_2)$ and $\lambda_3 = (b_1, a_2)$, the matrix K(x, y) with four parameters is

$$K(x,y) = \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, a_1, b_1) \sum_{z_1=\max(x,y)}^{N} \pi(x - z_2, z_1 - z_2, b_1, a_2) \pi(z_1 - y, N - y, a_2, b_2).$$
(4.60)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = (a_1 + b_1, a_2 + b_2), \tag{4.61}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By evaluating the eigenvalue formula (4.54), we obtain a balanced $_4F_3$

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, a_2, b_2) \sum_{y=0}^{z} \pi(y, z, b_1, a_2) \check{P}_n(y, a_1 + b_1, a_2 + b_2)$$

= ${}_4F_3 \Big(\frac{-n, n + a_1 + b_1 + a_2 + b_2 - 1, b_1, a_2}{a_1 + b_1, b_1 + a_2, a_2 + b_2} \Big| 1 \Big).$ (4.62)

Hahn \rightarrow **Meixner** This is achieved by fixing a_1 , b_1 and a_2 with $b_2 \rightarrow N(1-b_2)b_2^{-1}$ ($\Rightarrow 0 < b_2 < 1$), $N \rightarrow \infty$,

$$\check{P}_{n}(x, a_{1} + b_{1}, a_{2} + b_{2}) \to \check{P}_{Mn}(x, a_{1} + b_{1}, b_{2}),$$

$$\pi(x, N, a_{1} + b_{1}, a_{2} + b_{2}) \to \pi_{M}(x, a_{1} + b_{1}, b_{2}),$$

$$K(x, y) \to K_{M}(x, y, a_{1}, b_{1}, a_{2}, b_{2})$$

$$= \sum_{z_{2}=0}^{\min(x, y)} \pi(z_{2}, y, a_{1}, b_{1}) \sum_{z_{1}=\max(x, y)}^{\infty} \pi(x - z_{2}, z_{1} - z_{2}, b_{1}, a_{2}) \pi_{M}(z_{1} - y, a_{2}, b_{2}),$$
(4.63)

and the relations (2.9), (2.19) and (2.21) of (H) reduce to those of (M). Based on $K_{\rm M}$ (4.63), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (a_1 + b_1, b_2)$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) is written down as

$$\kappa_{\mathrm{M}}(n) = \sum_{z=0}^{\infty} \pi_{\mathrm{M}}(z, a_{2}, b_{2}) \sum_{y=0}^{z} \pi(y, z, b_{1}, a_{2}) \check{P}_{\mathrm{M}\,n}(y, a_{1} + b_{1}, b_{2})$$

= ${}_{3}F_{2} {\binom{-n, b_{1}, a_{2}}{a_{1} + b_{1}, b_{1} + a_{2}}} | 1 \Big).$ (4.64)

Hahn \rightarrow **Meixner** \rightarrow **Charlier** This is achieved by $a_1 \rightarrow a_1 N$, $b_1 \rightarrow b_1 N$, $a_2 \rightarrow a_2 N$, $b_2 \rightarrow \frac{b_2}{b_2 + N}$, $N \rightarrow \infty$,

$$\check{P}_{Mn}(x, (a_{1}+b_{1})N, \frac{b_{2}}{b_{2}+N}) \to P_{Cn}(x, (a_{1}+b_{1})b_{2}),
\pi_{M}(x, a_{2}N, \frac{b_{2}}{b_{2}+N}) \to \pi_{C}(x, a_{2}b_{2}),
\pi(z, y, a_{1}N, b_{1}N) \to \pi_{K}(z, y, \frac{a_{1}}{a_{1}+b_{1}}), \quad \pi(x, z, b_{1}N, a_{2}N) \to \pi_{K}(x, z, \frac{b_{1}}{b_{1}+a_{2}}),
K_{M}(x, y) \to K_{C}(x, y)
= \sum_{z_{2}=0}^{\min(x,y)} \pi_{K}(z_{2}, y, \frac{a_{1}}{a_{1}+b_{1}}) \sum_{z_{1}=\max(x,y)}^{\infty} \pi_{K}(x - z_{2}, z_{1} - z_{2}, \frac{b_{1}}{b_{1}+a_{2}}) \pi_{C}(z_{1} - y, a_{2}b_{2}),$$
(4.65)

and the relations (2.29), (2.30) and (2.31) of (M) reduce to those of (C). This $K_{\rm C}$ agrees with (4.58) with the replacement $(a, b, c) \rightarrow (\frac{a_1}{a_1+b_1}, a_2b_2, \frac{b_1}{b_1+a_2}).$

4.4.3 q-Hahn (qH)

This convolution for q-Hahn has almost the same structure as that for Hahn. By taking $\lambda_1 = (a_1, b_1), \lambda_2 = (a_2, b_2)$ and $\lambda_3 = (b_1, a_2)$, the matrix K(x, y) with four parameters is

$$K(x,y) = \sum_{z_2=0}^{\min(x,y)} \pi(z_2, y, a_1, b_1) \sum_{z_1=\max(x,y)}^N \pi(x - z_2, z_1 - z_2, b_1, a_2) \pi(z_1 - y, N - y, a_2, b_2).$$
(4.66)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = (a_1 b_1, a_2 b_2), \tag{4.67}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). The eigenvalue formula (4.54) takes a neat form and after a few lines of direct calculation, we obtain a balanced $_4\phi_3$,

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, a_2, b_2) \sum_{y=0}^{z} \pi(y, z, b_1, a_2) \check{P}_n(y, a_1b_1, a_2b_2)$$

= $_4\phi_3 \Big(\begin{array}{c} q^{-n}, a_1b_1a_2b_2q^{n-1}, b_1, a_2 \\ a_1b_1, b_1a_2, a_2b_2 \end{array} \mid q;q \Big).$ (4.68)

q-Hahn $\rightarrow q$ -Meixner This limit is achieved by fixing a_1 , b_1 , a_2 with $b_2 \rightarrow -b_2^{-1}q^{1-N}$, $N \rightarrow \infty$,

$$\check{P}_n(x, a_1b_1, a_2b_2) \to \check{P}_{qMn}(x, a_1b_1q^{-1}, (a_1b_1a_2)^{-1}b_2),$$

$$\pi(x, N, a_{1}b_{1}, a_{2}b_{2}) \to \pi_{qM}(x, a_{1}b_{1}q^{-1}, (a_{1}b_{1}a_{2})^{-1}b_{2}),$$

$$\pi(z_{1} - y, N - y, a_{2}, b_{2})$$

$$\to \pi'_{qM}(z_{1}, y, a_{2}q^{-1}, a_{2}^{-1}b_{2}) = \frac{(-a_{2}^{-1}b_{2}; q)_{y}(a_{2}; q)_{z_{1}-y}(a_{2}^{-1}b_{2})^{z_{1}-y}q^{\binom{z_{1}}{2}} - \binom{y}{2}}{(-b_{2}; q)_{\infty}},$$

$$K(x, y) \to K_{qM}(x, y) = \sum_{z_{2}=0}^{\min(x, y)} \pi(z_{2}, y, a_{1}, b_{1}) \sum_{z_{1}=\max(x, y)}^{\infty} \pi(x - z_{2}, z_{1} - z_{2}, b_{1}, a_{2})$$

$$\times \pi'_{qM}(z_{1}, y, a_{2}q^{-1}, a_{2}^{-1}b_{2}), \qquad (4.69)$$

and the relations (2.9), (2.19) and (2.21) of (qH) reduce to those of (qM). The above K_{qM} (4.69) is not a convolution of the orthogonality measures as π'_{qM} is not. Based on K_{qM} (4.69), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = (a_1b_1q^{-1}, (a_1b_1a_2)^{-1}b_2)$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) takes a neat form and several lines of direct calculation gives

$$\kappa_{qM}(n) = \sum_{z=0}^{\infty} \pi_{qM}(z, a_2 q^{-1}, a_2^{-1} b_2) \sum_{y=0}^{z} \pi(y, z, b_1, a_2) \check{P}_{qMn}(y, a_1 b_1 q^{-1}, (a_1 b_1 a_2)^{-1} b_2)$$

= ${}_{3}\phi_2 \Big(\frac{q^{-n}, b_1, a_2}{a_1 b_1, b_1 a_2} \mid q; a_1 b_1 q^n \Big).$ (4.70)

The $_{3}\phi_{2}$ form (4.70) is also obtained from the $_{4}\phi_{3}$ form (4.68) by the above mentioned limit.

The other set of eigenvectors of K_{qM} is

$$(-1)^{x} \sqrt{\pi_{qM}(x, a_{1}b_{1}q^{-1}, (a_{1}b_{1}a_{2})^{-1}b_{2})\pi_{qM}(x, -a_{2}^{-1}b_{2}q^{-1}, a_{1}b_{1}a_{2}b_{2}^{-1})} \times \check{P}_{qMn}(x, -a_{2}^{-1}b_{2}q^{-1}, a_{1}b_{1}a_{2}b_{2}^{-1})} = (-1)^{x} \frac{q^{\binom{x}{2}}}{(q;q)_{x}} \check{P}_{qMn}(x, -a_{2}^{-1}b_{2}q^{-1}, a_{1}b_{1}a_{2}b_{2}^{-1}) \times \sqrt{\frac{(a_{1}b_{1}, -a_{2}^{-1}b_{2};q)_{\infty}}{(-a_{1}b_{1}a_{2}b_{2}^{-1}, -(a_{1}b_{1}a_{2})^{-1}b_{2};q)_{\infty}}}.$$
 (4.71)

After a few pages of long calculation, we obtain the corresponding eigenvalue

$$\kappa_{qM}^{(-)}(n) = \frac{(b_1, -b_2; q)_{\infty}}{(b_1 a_2, -a_2^{-1} b_2; q)_{\infty}} \sum_{z=0}^{\infty} \pi_{qM}(z, a_2 q^{-1}, -b_1) \sum_{y=0}^{z} \pi(y, z, -a_2^{-1} b_2, a_2) \pi(0, y, a_1, b_1) \\ \times \check{P}_{qMn}(y, -a_2^{-1} b_2 q^{-1}, a_1 b_1 a_2 b_2^{-1}) \\ = \frac{(b_1, -b_2; q)_{\infty}}{(b_1 a_2, -a_2^{-1} b_2; q)_{\infty}} \sum_{k=0}^{n} \frac{(q^{-n}, b_1, a_2; q)_k}{(-b_2, a_1 b_1; q)_k} \frac{(-a_2^{-1} b_2 q^n)^k}{(q; q)_k} \\ \times {}_{3}\phi_2 \binom{a_1, a_2 q^k, -a_2^{-1} b_2 q^k}{-b_2 q^k, a_1 b_1 q^k} \mid q; b_1 \end{pmatrix}.$$
(4.72)

The q-Charlier limit does not exist, as it requires $a_1b_1 \to 0$ in $\pi_{qM}(x, a_1b_1, a_2^{-1}b_2)$. This causes $\pi(z_2, y, a_1, b_1)\pi(x - z_2, z_1 - z_2, b_1, a_2) \to 0$ in K_{qM} .

4.5 Type (v) convolution

We find only one K(x, y) by this convolution for Krawtchouk (K). By taking $\lambda_1 = a$, $\lambda_2 = b$ and $\lambda_3 = c$, the matrix K(x, y) is

(v):
$$K(x,y) = \sum_{z_2=0}^{\min(x,y)} \pi(z_2,y,a) \sum_{z_1=x+y-z_2}^{N} \pi(x-z_2,z_1-y,c)\pi(z_1-y,N-y,b).$$
 (4.73)

For the following $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = p \stackrel{\text{def}}{=} \frac{bc}{1 - a + bc},\tag{4.74}$$

the symmetry condition (2.9) is satisfied and **Theorem 2.1** gives (2.19). By writing down the eigenvalue formula (2.21), in which only the $z_2 = 0$ term contributes, we have

$$\kappa(n) = \sum_{z=0}^{N} \pi(z, N, b) \sum_{y=0}^{z} \pi(y, z, c) \check{P}_{n}(y, p) = (a - bc)^{n} = {}_{1}F_{0} {\binom{-n}{-}} bcp^{-1}, \qquad (4.75)$$

by using the generating function (4.5) and the self-duality (K). It is interesting to note that these expressions are all symmetric in b and c. Odd eigenvalues are negative if a < bc.

Krawtchouk \rightarrow **Charlier** This is achieved by fixing a and c with $b \rightarrow bN^{-1}$, $N \rightarrow \infty$,

$$\check{P}_{n}(x,p) \to \check{P}_{Cn}(x,p'), \quad p' \stackrel{\text{def}}{=} \frac{bc}{1-a},
\pi(x,N,p) \to \pi_{C}(x,p'), \quad \kappa(n) \to \kappa_{C}(n) = a^{n},
K(x,y) \to K_{C}(x,y,a,b,c) = \sum_{z_{2}=0}^{\min(x,y)} \pi(z_{2},y,a) \sum_{z_{1}=x+y-z_{2}}^{\infty} \pi(x-z_{2},z_{1}-y,c)\pi_{C}(z_{1}-y,b), \quad (4.76)$$

and the relations (2.9), (2.19) and (2.21) of (K) reduce to those of (C). Based on $K_{\rm C}$ (4.76), let us rederive these results. The symmetry condition (2.29) is satisfied for $\lambda = p'$ and **Theorem 2.1** gives (2.30). The eigenvalue formula (2.31) reads

$$\kappa_{\rm C}(n) = \sum_{z=0}^{\infty} \pi_{\rm C}(z,b) \sum_{y=0}^{z} \pi(y,z,c) \check{P}_{{\rm C}\,n}(y,p') = a^n = {}_1F_0 \binom{-n}{-} |bcp'^{-1}|, \qquad (4.77)$$

by using the generating function [4](9.14.11) and the self-duality (C).

4.6 Multiple convolutions of type (i) and (iii)

For the Krawtchouk (K) case, type (i) and (iii) convolutions can be repeated indefinitely.

Let us define $\pi^{(\pm)}(x, y, N, p)$ $(x, y \in \mathbb{Z})$ as follows:

$$\pi^{(+)}(x, y, N, p) \stackrel{\text{def}}{=} \begin{cases} \pi(x - y, N - y, p) &: 0 \le y \le x \le N \\ 0 &: \text{otherwise} \end{cases},$$

$$\pi^{(-)}(x, y, N, p) \stackrel{\text{def}}{=} \begin{cases} \pi(x, y, p) &: 0 \le x \le y \le N \\ 0 &: \text{otherwise} \end{cases}.$$

They are related by

$$\pi^{(-)}(x, y, N, p) = \pi^{(+)}(N - x, N - y, N, 1 - p).$$
(4.78)

For an integer $m \ge 2$, let us define $K(x, y) = K^{(\epsilon_1, \dots, \epsilon_m)}(x, y, N, p_1, \dots, p_m)$ $(x, y \in \mathcal{X}, \epsilon_j = \pm)$ by

$$K(x,y) \stackrel{\text{def}}{=} \sum_{z_1,\dots,z_{m-1}=0}^N \prod_{j=1}^m \pi^{(\epsilon_j)}(z_{j-1},z_j,N,p_j) \quad (z_0=x, \ z_m=y).$$
(4.79)

For example,

$$K^{(+,-,+,-)}(x,y,N,p_1,p_2,p_3,p_4) = \sum_{z_2=0}^{N} \sum_{z_1=0}^{\min(x,z_2)} \sum_{z_3=0}^{\min(z_2,y)} \pi(x-z_1,N-z_1,p_1)\pi(z_1,z_2,p_2)\pi(z_2-z_3,N-z_3,p_3)\pi(z_3,y,p_4).$$

For m = 2 case, $K^{(+,-)}(x, y, N, p_1, p_2)$ and $K^{(-,+)}(x, y, N, p_1, p_2)$ correspond to (4.2) and (4.36) with $(a, b) = (p_2, p_1)$, respectively. From (4.78), we have

$$K^{(\epsilon_1,\dots,\epsilon_m)}(x,y,N,p_1,\dots,p_m) = K^{(-\epsilon_1,\dots,-\epsilon_m)}(N-x,N-y,N,1-p_1,\dots,1-p_m), \quad (4.80)$$

and (4.2) and (4.36) are connected by this relation. Since two successive $\pi^{(+)}\pi^{(+)}$ and $\pi^{(-)}\pi^{(-)}$ can be reduced to one $\pi^{(+)}$ and $\pi^{(-)}$ by (3.47) and (3.46), respectively, it is sufficient to consider $(\epsilon_1, \ldots, \epsilon_m) = (+, -, +, -, \ldots)$ or $(-, +, -, +, \ldots)$. That is, the multiple convolutions of type (i) and (iii), respectively. Since $\pi^{(+)}$ is lower triangular and $\pi^{(-)}$ is upper triangular,

$$\det \pi^{(+)}(*,*,N,p) = \prod_{x \in \mathcal{X}} \pi^{(+)}(x,x,N,p) = \prod_{n \in \mathcal{X}} (1-p)^n,$$
$$\det \pi^{(-)}(*,*,N,p) = \prod_{x \in \mathcal{X}} \pi^{(-)}(x,x,N,p) = \prod_{n \in \mathcal{X}} p^n,$$

the eigenvalues are easily guessed as in the original type (i) and (iii) cases,

$$\det K^{(\epsilon_1,\dots,\epsilon_m)} = \prod_{n\in\mathcal{X}} \left(\prod_{j=1}^m p_j^{(\epsilon_j)}\right)^n, \quad p^{(+)} \stackrel{\text{def}}{=} 1 - p, \quad p^{(-)} \stackrel{\text{def}}{=} p.$$
(4.81)

In order to determine the parameter $\lambda = p$ which satisfies the symmetry condition (2.9), we solve the equation

$$K(0,N)\pi(N,N,p) = K(N,0)\pi(0,N,p) \quad \left(\Rightarrow p = \frac{1}{1 + \left(\frac{K(0,N)}{K(N,0)}\right)^{\frac{1}{N}}}\right),$$

which is obtained by setting x = 0 and y = N in (2.9). Among $(N + 1) \times (N + 1)$ elements of K(x, y), K(0, N) and K(N, 0) are the easiest to evaluate, by successive applications of the binomial theorem, for example,

$$K^{(+,-)}(0, N, N, p_1, p_2) = (1 - p_1)^N (1 - p_2)^N, \quad K^{(+,-)}(N, 0, N, p_1, p_2) = p_1^N,$$

$$K^{(+,-,+)}(0, N, N, p_1, p_2, p_3) = (1 - p_1)^N (1 - p_2)^N,$$

$$K^{(+,-,+)}(N, 0, N, p_1, p_2, p_3) = (p_1 + (1 - p_1)p_2p_3)^N,$$

$$K^{(+,-,+,-)}(0, N, N, p_1, p_2, p_3, p_4) = ((1 - p_1)(1 - p_2p_3 - p_2p_4 + p_2p_3p_4))^N,$$

$$K^{(+,-,+,-)}(N, 0, N, p_1, p_2, p_3, p_4) = (p_1 + (1 - p_1)p_2p_3)^N.$$

We obtain the following $\boldsymbol{\lambda} = p$ and

$$\kappa(n) = \left(\prod_{j=1}^{m} p_{j}^{(\epsilon_{j})}\right)^{n} = {}_{1}F_{0}\binom{-n}{-} | 1 - \kappa(1) \rangle,$$
(4.82)

$$(\epsilon_1, \dots, \epsilon_m) = (+, -, +, -, \dots): \quad p = \frac{1}{1 - \kappa(1)} \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \prod_{j=1}^{2k} p_j^{(\epsilon_j)} \cdot p_{2k+1}, \tag{4.83}$$

$$(\epsilon_1, \dots, \epsilon_m) = (-, +, -, +, \dots): \quad p = \frac{1}{1 - \kappa(1)} \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \prod_{j=1}^{2k-1} p_j^{(\epsilon_j)} \cdot p_{2k}.$$
(4.84)

These two are related by (4.80). The symmetry condition (2.9) is verified by explicit calculation for small m and N. We do not have an analytical proof of the symmetry condition (2.9) for general m and N.

4.7 One-parameter families of commuting K's

We have derived various K(x, y)'s satisfying

$$K(x,y) > 0, \quad \sum_{x \in \mathcal{X}} K(x,y) = 1,$$
 (2.2)

$$\sum_{y \in \mathcal{X}} K(x, y) \pi(y, N, \boldsymbol{\lambda}) \check{P}_n(y, \boldsymbol{\lambda}) = \kappa(n) \pi(x, N, \boldsymbol{\lambda}) \check{P}_n(x, \boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$
(2.19)

It is trivial that the *m*-th power of $K \ (m \ge 1), \ K^m$, also satisfies

$$K^{m}(x,y) > 0, \quad \sum_{x \in \mathcal{X}} K^{m}(x,y) = 1,$$
$$\sum_{y \in \mathcal{X}} K^{m}(x,y)\pi(y,N,\boldsymbol{\lambda})\check{P}_{n}(y,\boldsymbol{\lambda}) = \kappa(n)^{m}\pi(x,N,\boldsymbol{\lambda})\check{P}_{n}(x,\boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$

Namely K^m also gives an exactly solvable Markov chain.

It is interesting to note that by changing the parameters $\{\lambda_j\}$ to $\{\lambda'_j(t)\}$, some K's derived in §4.1–§4.5 can be deformed to create a one parameter (t) family of commuting K's. That is, they share the same eigenvectors but the eigenvalues are different. For examples, the following $\lambda'_j(t)$'s give the same λ in the symmetry condition (2.9),

$$\begin{array}{ll} (4.3): \ \boldsymbol{\lambda}_1'(t) = at, \ \boldsymbol{\lambda}_2'(t) = \frac{(1-at)b}{1-a(1-b(1-t))} & (0 < t \le 1) \Rightarrow \boldsymbol{\lambda} = \frac{b}{1-a+ab}, \\ (4.8): \ \boldsymbol{\lambda}_1'(t) = at, \ \boldsymbol{\lambda}_2'(t) = \frac{(1-at)b}{1-a} & (0 < t \le 1) \Rightarrow \boldsymbol{\lambda} = \frac{b}{1-a}, \\ (4.11): \ \boldsymbol{\lambda}_1'(t) = (a+t,b-t), \ \boldsymbol{\lambda}_2'(t) = (b-t,c) & (-a < t < b) \Rightarrow \boldsymbol{\lambda} = (a+b,c), \\ (4.19): \ \boldsymbol{\lambda}_1'(t) = (at,bt^{-1}), \ \boldsymbol{\lambda}_2'(t) = (bt^{-1},c) & (b < t < a^{-1}) \Rightarrow \boldsymbol{\lambda} = (ab,c), \\ (4.23): \ \boldsymbol{\lambda}_1'(t) = (at,bt^{-1}), \ \boldsymbol{\lambda}_2'(t) = (bt^{-1}q^{-1},b^{-1}tc) & (b < t < a^{-1}) \Rightarrow \boldsymbol{\lambda} = (abq^{-1},(ab)^{-1}c). \\ \text{The two matrices } K(x,y,\{\boldsymbol{\lambda}_j\}) \text{ and } K(x,y,\{\boldsymbol{\lambda}_j'(t)\}) \text{ have the common eigenvectors } \pi(x,\boldsymbol{\lambda}) \\ \tilde{P}_n(x,\boldsymbol{\lambda}) \text{ and they commute as is clear from the spectral representation Theorem 2.3. But \\ \text{the eigenvalues are different, } \kappa(n,\{\boldsymbol{\lambda}_j\}) \neq \kappa(n,\{\boldsymbol{\lambda}_j'(t)\}). \quad \text{Let } \boldsymbol{\lambda}_j'(t^{[i]}) & (i = 1,2,\ldots) \text{ be} \\ \text{the parameters giving the same } \boldsymbol{\lambda}, \text{ and set } K^{[i]}(x,y) = K(x,y,\{\boldsymbol{\lambda}_j'(t^{[i]})\}) \text{ and } \kappa^{[i]}(n) = \\ \kappa(n,\{\boldsymbol{\lambda}_j'(t^{[i]})\}). \text{ For } m \text{ such } K^{[i]}\text{ s, let us consider their matrix product (the order is irrelevant),} \end{array}$$

$$K^{(m)} \stackrel{\text{def}}{=} K^{[m]} \cdots K^{[2]} K^{[1]}, \quad \kappa^{(m)}(n) \stackrel{\text{def}}{=} \kappa^{[m]}(n) \cdots \kappa^{[2]}(n) \kappa^{[1]}(n).$$
(4.85)

Then we have

$$K^{(m)}(x,y) > 0, \quad \sum_{x \in \mathcal{X}} K^{(m)}(x,y) = 1,$$

$$\sum_{y \in \mathcal{X}} K^{(m)}(x,y)\pi(y,N,\boldsymbol{\lambda})\check{P}_n(y,\boldsymbol{\lambda}) = \kappa^{(m)}(n)\pi(x,N,\boldsymbol{\lambda})\check{P}_n(x,\boldsymbol{\lambda}) \quad (n \in \mathcal{X}).$$
(4.86)

Namely $K^{(m)}$ also gives an exactly solvable Markov chain.

5 Other Topics

Here we discuss two related topics.

5.1 Dual Markov chains

For a Markov chain K(x, y) on a finite one dimensional integer lattice \mathcal{X} , its 'dual' Markov chain $K^{d}(x, y)$ is defined by the similarity transformation in terms of the anti-diagonal matrix $J, J(x, y) \stackrel{\text{def}}{=} \delta_{x, N-y}$,

$$K^{\mathrm{d}}(x,y) \stackrel{\mathrm{def}}{=} (JKJ)(x,y) = K(N-x,N-y), \tag{5.1}$$

$$\sum_{y \in \mathcal{X}} K(x, y) v_n(y) = \kappa(n) v_n(x) \Rightarrow \sum_{y \in \mathcal{X}} K^{\mathsf{d}}(x, y) v_n^{\mathsf{d}}(y) = \kappa(n) v_n^{\mathsf{d}}(x), \ v_n^{\mathsf{d}}(x) \stackrel{\text{def}}{=} v_n(N-x).$$
(5.2)

The concept of duality was reported in [12].

For K(x, y) constructed by convolutions listed in §3.1, the dual Markov chains take the following forms,

(di):
$$K^{\mathrm{d}}(x,y) \stackrel{\mathrm{def}}{=} \sum_{\substack{z=\max(x,y)\\ \min(z+y,N)}}^{N} \pi(z-x,z,\boldsymbol{\lambda}_2)\pi(N-z,N-y,\boldsymbol{\lambda}_1),$$
 (5.3)

(dii):
$$K^{\mathrm{d}}(x,y) \stackrel{\mathrm{def}}{=} \sum_{\substack{z=\max(x,y)\\ \min(x,y)}}^{\min(x+y,N)} \pi(z-x,y,\boldsymbol{\lambda}_2)\pi(N-z,N-y,\boldsymbol{\lambda}_1),$$
 (5.4)

(diii):
$$K^{d}(x,y) \stackrel{\text{def}}{=} \sum_{z=0}^{\min(x,y)} \pi(N-x, N-z, \lambda_2) \pi(y-z, y, \lambda_1),$$
 (5.5)

(div):
$$K^{d}(x,y) \stackrel{\text{def}}{=} \sum_{z_{2}=\max(x,y)}^{N} \pi(N-z_{2},N-y,\boldsymbol{\lambda}_{1}) \sum_{z_{1}=0}^{\min(x,y)} \pi(z_{2}-x,z_{2}-z_{1},\boldsymbol{\lambda}_{3}) \pi(y-z_{1},y,\boldsymbol{\lambda}_{2}),$$

(5.6)

(dv):
$$K^{d}(x,y) \stackrel{\text{def}}{=} \sum_{z_{2}=\max(x,y)}^{N} \pi(N-z_{2},N-y,\boldsymbol{\lambda}_{1}) \sum_{z_{1}=0}^{x+y-z_{2}} \pi(z_{2}-x,y-z_{1},\boldsymbol{\lambda}_{3})\pi(y-z_{1},y,\boldsymbol{\lambda}_{2}),$$

(5.7)

$$K^{\mathrm{d}}(x,y)\pi(N-y,N,\boldsymbol{\lambda}) = K^{\mathrm{d}}(y,x)\pi(N-x,N,\boldsymbol{\lambda}).$$
(5.8)

For (K) and (H), the dual eigenvectors take the standard forms with flipped λ , see (3.7), (3.11) for (K) and (3.16), (3.20) for (H),

$$\pi_{\rm K}(N-x,N,p) = \pi_{\rm K}(x,N,1-p), \quad \check{P}_{{\rm K}\,n}(N-x,p) \propto \check{P}_{{\rm K}\,n}(x,1-p),$$

$$\pi_{\rm H}(N-x,N,a,b) = \pi_{\rm H}(x,N,b,a), \quad \check{P}_{{\rm H}\,n}(N-x,a,b) \propto \check{P}_{{\rm H}\,n}(x,b,a),$$

and the limiting procedure for $N \to \infty$ with fixed x goes in a similar way as before. However, for (qH) the situation is different, see (3.29),

$$\pi_{q\mathrm{H}}(N-x,N,a,b) \not\propto \pi_{q\mathrm{H}}(x,N,b,a), \quad \check{P}_{q\mathrm{H}\,n}(N-x,N,a,b) \not\propto \check{P}_{q\mathrm{H}\,n}(x,N,b,a), \tag{5.9}$$

and the reductions to (qM) cannot be carried out.

5.1.1 reduction to little q-Jacobi

It is known [9] (§V.C.1) that the q-Hahn polynomial with the replacement $x \to N - x$ is another polynomial in $\eta(x) = 1 - q^x$ called the alternative q-Hahn (aqH) polynomial. The basic data of (aqH) with $\lambda = (a, b)$ (0 < a < 1, b < 1) are

$$\pi(x, N, a, b) = \begin{bmatrix} N \\ x \end{bmatrix} \frac{(a; q)_{N-x} (b; q)_x a^x}{(ab; q)_N},$$
(5.10)

$$d_n^2 = \begin{bmatrix} N\\ n \end{bmatrix} \frac{(b, abq^{-1}; q)_n a^n q^{n(n-1)}}{(a, abq^N; q)_n} \frac{1 - abq^{2n-1}}{1 - abq^{-1}},$$
(5.11)

$$s_1\eta(x)\pi(x,N,a,b) = -\pi(x-1,N-1,a,bq), \quad s_1 \stackrel{\text{def}}{=} -\frac{1-ab}{a(1-b)(1-q^N)}, \tag{5.12}$$

$$\eta(z)\pi(z,x,a,b) = \frac{a(1-b)}{1-ab}\eta(x)\pi(z-1,x-1,a,bq), \quad \eta(x) = 1-q^x, \tag{5.13}$$

$$\check{P}_n(x,a,b) = P_n(\eta(x),a,b) = {}_3\phi_2 \left(\begin{array}{c} q^{-n}, \ abq^{n-1}, \ q^{-x} \\ b, \ q^{-N} \end{array} \middle| \ q ; a^{-1}q^{x+1-N} \right).$$
(5.14)

The above non-proportionality relation (5.9) is rewritten as the equalities between the q-Hahn (qH) and the alternative q-Hahn (aqH) polynomials

$$\pi_{qH}(N - x, N, a, b) = \pi_{aqH}(x, N, a, b),$$
(5.15)

$$\check{P}_{q\mathrm{H}\,n}(N-x,a,b) = (-a)^n q^{\binom{n}{2}} \frac{(b\,;\,q)_n}{(a\,;\,q)_n} \check{P}_{\mathrm{a}q\mathrm{H}\,n}(x,a,b).$$
(5.16)

The transition matrices K of (aqH) are expressed by the duals of those of (qH), see (5.3)–(5.7),

$$K_{\rm aqH}^{\rm (i)}(x, y, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = K_{\rm qH}^{\rm (diii)}(x, y, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2), \qquad (5.17)$$

$$K_{aqH}^{(iii)}(x, y, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = K_{qH}^{(di)}(x, y, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2), \qquad (5.18)$$

$$K_{aqH}^{(iv)}(x, y, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) = K_{qH}^{(div)}(x, y, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_3),$$
(5.19)

independently of the choice of parameters λ_1 and λ_2 (and λ_3). Therefore the results of (qH) in §4.1.3, §4.3.3 and §4.4.3 are translated to those of (aqH). This is merely rewriting, but their $N \to \infty$ limits give new results. By the limit $N \to \infty$ with fixed a and b, (aqH) goes to the little q-Jacobi (lqJ) polynomial,

$$\lim_{N \to \infty} \pi_{aqH}(x, N, a, b) = \pi_{lqJ}(x, a, b), \quad \lim_{N \to \infty} \check{P}_{aqH\,n}(x, N, a, b) = \check{P}_{lqJ\,n}(x, a, b).$$
(5.20)

The basic data of (lqJ) with $\lambda = (a, b)$ (0 < a < 1, b < 1) is

$$\pi(x,a,b) = \frac{(b;q)_x a^x}{(q;q)_x} \frac{(a;q)_\infty}{(ab;q)_\infty}, \quad d_n^2 = \frac{(b,abq^{-1};q)_n a^n q^{n(n-1)}}{(q,a;q)_n} \frac{1-abq^{2n-1}}{1-abq^{-1}}, \tag{5.21}$$

$$s_1\eta(x)\pi(x,a,b) = -\pi(x-1,a,bq), \quad \eta(x) = 1 - q^x, \quad s_1 \stackrel{\text{def}}{=} -\frac{1 - abq^{-1}}{a(1-b)}, \tag{5.22}$$

$$\check{P}_n(x,a,b) = P_n(\eta(x),a,b) = {}_3\phi_1 \left(\begin{array}{c} q^{-n}, \ abq^{n-1}, \ q^{-x} \\ b \\ \end{array} \middle| \ q ; a^{-1}q^{x+1} \right)$$
(5.23)

$$= (-a)^{-n} q^{-\binom{n}{2}} \frac{(a;q)_n}{(b;q)_n} {}_2\phi_1 \binom{q^{-n}, abq^{n-1}}{a} \mid q;q^{x+1}$$
(5.24)

The conventional little q-Jacobi polynomial is $p_n(q^x; a, b|q) = {}_2\phi_1 \begin{pmatrix} q^{-n}, abq^{n+1} \\ aq \end{pmatrix} |q; q^{x+1}$ and our parametrisation is slightly different from the standard one $(a, b)^{\text{standard}} = (aq^{-1}, bq^{-1})$. Similarly to those examples in §3.3, this π keeps its form under the following convolutions:

$$\sum_{z=0}^{x} \pi(x-z, a_1, b_1) \pi(z, a_1b_1, b_2) = \pi(x, a_1, b_1b_2),$$
(5.25)

$$\sum_{z=y}^{x} \pi(x-z, a_1, b_1) \pi(z-y, a_1b_1, b_2) = \pi(x-y, a_1, b_1b_2).$$
(5.26)

These are obtained by the sum formula for π_{qH} (3.58).

alternative q-Hahn \rightarrow little q-Jacobi From the type (i), (iii), (iv) transition matrices K_{aqH} , we obtain K_{lqJ} whose eigenvectors are described by the little q-Jacobi polynomial,

(i):
$$K_{lqJ}(x,y) = \sum_{z=0}^{\min(x,y)} \pi_{lqJ}(x-z,c,a)\pi_{aqH}(z,y,a,b),$$
 (5.27)

eigenvector : $\pi_{lqJ}(x, c, ab)\check{P}_{lqJn}(x, c, ab), \quad \kappa(n) : (4.49),$ (5.28)

(iii):
$$K_{lqJ}(x,y) = \sum_{z=\max(x,y)}^{\infty} \pi_{aqH}(x,z,b,c)\pi_{lqJ}(z-y,a,b),$$
 (5.29)

eigenvector :
$$\pi_{lqJ}(x, ab, c)\check{P}_{lqJn}(x, ab, c), \quad \kappa(n) : (4.20),$$

$$(5.30)$$

$$\min(x, y) \sim \infty$$

(iv):
$$K_{lqJ}(x,y) = \sum_{z_2=0}^{\min(x,y)} \pi_{aqH}(z_2, y, a_2, b_2) \sum_{z_1=\max(x,y)}^{\infty} \pi_{aqH}(x - z_2, z_1 - z_2, b_1, a_2)$$

$$\times \pi_{lqJ}(z_1 - y, a_1, b_1),$$
 (5.31)

eigenvector :
$$\pi_{lqJ}(x, a_1b_1, a_2b_2)\check{P}_{lqJn}(x, a_1b_1, a_2b_2), \quad \kappa(n) : (4.68).$$
 (5.32)

5.2 Repeated discrete time Birth and Death processes

Exactly solvable discrete time Birth and Death (BD) processes K_{BD} are constructed [8] based on exactly solvable continuous time BD on a one dimensional integer lattice \mathcal{X} ,

$$K_{\rm BD} = I + t_{\rm S} L_{\rm BD}, \text{ i.e. } K_{\rm BD}(x, y) = \delta_{xy} + t_{\rm S} L_{\rm BD}(x, y),$$

$$L_{\rm BD}(x + 1, x) = B(x), \quad L_{\rm BD}(x - 1, x) = D(x), \quad L_{\rm BD}(x, x) = -B(x) - D(x),$$

$$L_{\rm BD}(x, y) = 0 \quad (|x - y| \ge 2), \quad \sum_{x \in \mathcal{X}} L_{\rm BD}(x, y) = 0,$$
(5.34)

in which B(x) and D(x) are the birth and death rates at point x and they are chosen to be the coefficient functions of the difference equations governing the orthogonal polynomials $\{\check{P}_n(x)\}$ belonging to Askey scheme [6, 2, 4, 9],

$$B(x)\big(\check{P}_n(x) - \check{P}_n(x+1)\big) + D(x)\big(\check{P}_n(x) - \check{P}_n(x-1)\big) = \mathcal{E}(n)\check{P}_n(x) \quad (n \in \mathcal{X}),$$
(5.35)

and the time scale parameter $t_{\rm S}$ must satisfy the upper bound condition

$$t_{\rm S} \cdot \max\bigl(B(x) + D(x)\bigr) < 1. \tag{5.36}$$

The eigenvectors of K_{BD} are $\{\pi(x)\check{P}_n(x)\}\ (n \in \mathcal{X})$ and $\pi(x)$ is the normalised orthogonality measure of the polynomial = the stationary probability distribution and

$$\sum_{y \in \mathcal{X}} K_{\rm BD}(x, y) \pi(y) \check{P}_n(y) = \kappa(n) \pi(x) \check{P}_n(x), \quad \kappa(n) \stackrel{\rm def}{=} 1 - t_{\rm S} \mathcal{E}(n) \quad (n \in \mathcal{X}).$$
(5.37)

This applies to a good part of the orthogonal polynomials of a discrete variable in Askey scheme, including the Hahn, q-Hahn and Racah and q-Racah [8].

From the definition (5.34), it is shown that the *m*-th power of $L_{\rm BD}$ ($m \ge 1$), $L_{\rm BD}^m$, has the following form of the matrix elements,

$$L_{\text{BD}}^{m}(x+k,x) = (-1)^{m-k} a_{k}^{(m)}(x) \quad (-m \le k \le m), \quad L_{\text{BD}}^{m}(x,y) = 0 \quad (|x-y| > m),$$

where $a_k^{(m)}(x) > 0$. Let us consider the following matrix X,

$$X = \sum_{j=0}^{m-1} c_j L_{\text{BD}}^{m-j}, \quad c_0 = 1 \quad \left(\Rightarrow \sum_{x \in \mathcal{X}} X(x, y) = 0, \ X(x, y) = 0 \ (|x - y| > m) \right), \tag{5.38}$$

where c_j are constants. Its non zero matrix elements are

$$X(x \pm (m-k), x) = \sum_{j=0}^{k} c_j (-1)^{k-j} a_{\pm (m-k)}^{(m-j)}(x) \quad (0 \le k \le m-1),$$

$$X(x,x) = \sum_{j=0}^{m-1} c_j (-1)^{m-j} a_0^{(m-j)}(x).$$

Starting from $X(x \pm m, x) = a_{\pm m}^{(m)}(x) > 0$, we can tune c_k (k = 1, ..., m - 2 in turn) such that $X(x \pm (m - k), x) > 0$, and tune c_{m-1} such that $X(x \pm 1, x) > 0$ and X(x, x) < 0. For such chosen weights $\{c_j\}$, we define a matrix $K_{\text{BD}}^{(m)}$,

$$K_{\rm BD}^{(m)} \stackrel{\text{def}}{=} I_d + t_{\rm S} X, \quad t_{\rm S} \cdot \max\left(-X(x,x)\right) < 1, \tag{5.39}$$

which satisfies

$$K_{\rm BD}^{(m)}(x,y) \ge 0, \quad \sum_{x \in \mathcal{X}} K_{\rm BD}^{(m)}(x,y) = 1, \quad K_{\rm BD}^{(m)}(x,y) = 0 \quad (|x-y| > m).$$

This gives an exactly solvable Markov chain and the matrices $K_{BD}^{(m)}$'s have common eigenvectors

$$\sum_{y \in \mathcal{X}} K_{\rm BD}^{(m)}(x, y) \pi(y) \check{P}_n(y) = \kappa^{(m)}(n) \pi(x) \check{P}_n(x),$$

$$\kappa^{(m)}(n) \stackrel{\text{def}}{=} 1 + t_{\rm S} \sum_{j=0}^{m-1} (-1)^{m-j} c_j \mathcal{E}(n)^{m-j} \quad (n \in \mathcal{X}).$$
(5.40)

Before closing this section a few remarks on possible applications are in order. As is well known the birth and death processes are recognised to be related to diffusion processes [15]. In other words, the equations for BD processes are space discretisation of 1-d Fokker-Planck equations. Discrete time BD processes are further discretisation in time. As illustrated by the famous Ehrenfest urn model, Markov chains are also related to diffusion processes as well as to the familiar random walks. Various examples in this and preceding sections are multi-parameter generalisations of known Markov chains and BD processes. It is expected that they would find diverse applications in physics, chemistry, etc. in particular, diffusions and random walks.

6 Summary and Comments

Based on the fact that many examples of exactly solvable BD processes/chains [6, 8], typical cases of Markov processes/chains, have been constructed in terms of orthogonal polynomials of a discrete variable [1]–[5], we establish a wide range of generalisations of various Markov

chains/processes [11]–[14] using solvability as a guide. Adopting the convolutions of the orthogonality measures is a new input of the present research. It is an interesting challenge to formulate probabilistic procedures/interpretations like "cumulative Bernoulli trials" for these new examples.

Except for those corresponding to the extra eigenvectors for the q-Meixner (qM) in §4.1.3, §4.3.3, §4.4.3, the eigenvalues of the K's derived in this paper share many remarkable properties;

1. $\kappa(n)$ has a neat sum formula of one or two π 's and \check{P}_n , for example

$$\kappa(n) = \sum_{y \in \mathcal{X}} \pi(y, N, \lambda') \check{P}_n(y, \lambda)$$

- κ(n) is independent of N, the size of the lattice. This reminds us of the similar situation of the exactly solvable BD processes [6, 7, 8], which is due to the fact that the eigenvalues of the difference equations governing these polynomials are N independent [9].
- 3. $\kappa(n)$ has an expression of a terminating (q)-hypergeometric series, for example,

$$\kappa(n) = {}_{3}\phi_{2} \Big(\frac{q^{-n}, b_{1}, a_{2}}{a_{1}b_{1}, b_{1}a_{2}} \mid q; a_{1}b_{1}q^{n} \Big).$$

We do not know how the bounds $-1 < \kappa(n) \le 1$ are ingrained in the hypergeometric expressions, in particular, in $_{3}F_{2}$, $_{4}F_{3}$, $_{3}\phi_{2}$ and $_{4}\phi_{3}$.

Deciphering these curious facts, we believe, would lead to a deeper understanding of the subject.

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Appendix

Here we provide the proof of triangularity **Lemma**

$$\sum_{y \in \mathcal{X}} K(y, x) \eta(y)^n = \sum_{m=0}^n a_{nm} \eta(x)^m \quad (n \in \mathcal{X}) \quad (a_{nm} = 0 \text{ for } n < m),$$
(2.10)

$$\eta(x) = \begin{cases} x & : (i) - (v) \\ q^{-x} - 1 & : (i), (iii), (iv) \end{cases}$$
(2.11)

The proof depends on a universal property of the normalised orthogonality measure $\pi(x, N, \lambda)$ (or $\pi(x, \lambda)$) of all the polynomials of a discrete variable in Askey scheme [9], except for those having the Jackson integral measures:

finite:
$$s_1\eta(x,\boldsymbol{\lambda})\pi(x,N,\boldsymbol{\lambda}) = -\pi(x-1,N-1,\boldsymbol{\lambda}'),$$
 (A.1)

semi-infinite:
$$s_1\eta(x,\boldsymbol{\lambda})\pi(x,\boldsymbol{\lambda}) = -\pi(x-1,\boldsymbol{\lambda}'),$$
 (A.2)

in which s_1 is the coefficient of $\eta(x, \lambda)$ in $P_1(\eta(x, \lambda), \lambda)$,

$$P_1(\eta(x, \boldsymbol{\lambda}), \boldsymbol{\lambda}) = 1 + s_1 \eta(x, \boldsymbol{\lambda}).$$

Note that the Racah, dual Hahn etc. [2, 3] are polynomials in $\eta(x, \lambda)$, depending on parameters λ . It is easy to verify these formulas one by one. In §3.2 the explicit expressions of s_1 and λ' are given in (3.8), (3.13), (3.17), (3.22), (3.26), (3.31), (3.35) for (K), (C), (H), (M), (qH) and (qM), and (5.22) for (lqJ). Among them, the formulas (A.1) for (K), (H) and (qH) can be rewritten as

$$\eta(z)\pi(z,x,\boldsymbol{\lambda}) = \beta \,\eta(x)\pi(z-1,x-1,\boldsymbol{\lambda}'),\tag{A.3}$$

in which β is a constant independent of N. See (3.9), (3.18) and (3.27) for the explicit forms for (K), (H) and (qH). To the best of our knowledge, the formulas (A.1)–(A.3) have not been reported yet.

The general strategy is as follows. Apply formulas (A.1)-(A.3) with various arguments, e.g. $\eta(y-z)$, $\eta(z)$, etc. repeatedly to K(y,x) and reduce $\eta(y)^n$ to a degree *n* polynomial in $\eta(x)$ through various intermediaries including z (z_1 and z_2). This is guaranteed as the formulas (A.1)-(A.3) do not increase the powers of η . After the reduction, the remaining π 's are evaluated to 1 by the summation in y and z (z_1 and z_2). For each type of convolutions K, the structure of the reduction $\eta(y) \to \eta(x)$ is the same for the group having $\eta(x) = x$ *i.e.* (K), (C), (M) and (H). It is more involved for those having $\eta(x) = q^{-x} - 1$ *i.e.* (qH) and (qM). It is important to stress that K's for the semi-infinite Markov chains given in §4 have at least one π belonging to (K), (H) or (qH), so that (A.3) can be applied to extract $\eta(x)$. The triangularity also holds for these semi-infinite Markov chains.

Below we demonstrate the first step reduction $\eta(y) \to \eta(x)$ for type (i) convolution

$$K(y,x) = \sum_{z=0}^{\min(x,y)} \pi(y-z, N-z, \lambda_2) \pi(z, x, \lambda_1).$$
(A.4)

For $\eta(x) = q^{-x} - 1$, by using $\eta(y) = q^{-z}\eta(y-z) + \eta(z)$ and $q^{-z}\eta(N-z) = \eta(N) - \eta(z)$, we obtain

$$K(y, x)\eta(y) = \sum_{z=0}^{\min(x,y)} q^{-z}\eta(y-z)\pi(y-z, N-z, \lambda_2)\pi(z, x, \lambda_1) + \sum_{z=0}^{\min(x,y)} \pi(y-z, N-z, \lambda_2)\eta(z)\pi(z, x, \lambda_1) = \sum_{z=0}^{\min(x,y-1)} \beta_2 q^{-z}\eta(N-z)\pi(y-z-1, N-z-1, \lambda'_2)\pi(z, x, \lambda_1) + \sum_{z=1}^{\min(x,y)} \pi(y-z, N-z, \lambda_2)\beta_1\eta(x)\pi(z-1, x-1, \lambda'_1) = \beta_2\eta(N) \sum_{z=0}^{\min(x,y-1)} \pi(y-z-1, N-z-1, \lambda'_2)\pi(z, x, \lambda_1) - \beta_2\beta_1\eta(x) \sum_{z=1}^{\min(x,y-1)} \pi(y-z-1, N-z-1, \lambda'_2)\pi(z-1, x-1, \lambda'_1) + \beta_1\eta(x) \sum_{z=1}^{\min(x,y)} \pi(y-z, N-z, \lambda_2)\pi(z-1, x-1, \lambda'_1).$$
(A.5)

For $\eta(x) = x$, by using $\eta(y) = \eta(y - z) + \eta(z)$ and $\eta(N - z) = \eta(N) - \eta(z)$, the same result is obtained similarly. Summing (A.5) over y, we obtain

$$\sum_{y \in \mathcal{X}} K(y, x)\eta(y) = \beta_2 \eta(N) + \beta_1 (1 - \beta_2)\eta(x),$$

which is (2.10) for n = 1.

By a similar calculation, we obtain

(ii) $\eta(x) = x$:

$$K(y,x)\eta(y) = \beta_2\eta(N) \sum_{z=\max(0,x+y-N)}^{\min(x,y-1)} \pi(y-z-1, N-x-1, \lambda'_2)\pi(z, x, \lambda_1) + \eta(x) \Big(-\beta_2 \sum_{z=\max(0,x+y-N)}^{\min(x,y-1)} \pi(y-z-1, N-x-1, \lambda'_2)\pi(z, x, \lambda_1) + \beta_1 \sum_{z=\max(1,x+y-N)}^{\min(x,y)} \pi(y-z, N-x, \lambda_2)\pi(z-1, x-1, \lambda'_1) \Big),$$
(A.6)

(iii)
$$\eta(x) = x, q^{-x} - 1$$
:

$$K(y, x)\eta(y) = \beta_1\beta_2\eta(N) \sum_{z=\max(x+1,y)}^{N} \pi(y-1, z-1, \lambda'_2)\pi(z-x-1, N-x-1, \lambda'_1)$$

$$+ \beta_2\eta(x) \Big(-\beta_1 \sum_{z=\max(x+1,y)}^{N} \pi(y-1, z-1, \lambda'_2)\pi(z-x-1, N-x-1, \lambda'_1) + \sum_{z=\max(1,x,y)}^{N} \pi(y-1, z-1, \lambda'_2)\pi(z-x, N-x, \lambda_1) \Big), \quad (A.7)$$

(iv) $\eta(x) = x, q^{-x} - 1$:

$$\begin{split} K(y,x)\eta(y) \\ &= \beta_2\beta_3\eta(N) \sum_{z_2=0}^{\min(x,y-1)} \pi(z_2,x,\boldsymbol{\lambda}_1) \sum_{z_1=\max(x+1,y)}^{N} \pi(y-z_2-1,z_1-z_2-1,\boldsymbol{\lambda}'_3) \\ &\times \pi(z_1-x-1,N-x-1,\boldsymbol{\lambda}'_2) \\ &+ \eta(x) \Big(-\beta_2\beta_3 \sum_{z_2=0}^{\min(x,y-1)} \pi(z_2,x,\boldsymbol{\lambda}_1) \sum_{z_1=\max(x+1,y)}^{N} \pi(y-z_2-1,z_1-z_2-1,\boldsymbol{\lambda}'_3) \\ &\times \pi(z_1-x-1,N-x-1,\boldsymbol{\lambda}'_2) \\ &+ \beta_1 \sum_{z_2=1}^{\min(x,y)} \pi(z_2-1,x-1,\boldsymbol{\lambda}'_1) \sum_{z_1=\max(x,y)}^{N} \pi(y-z_2,z_1-z_2,\boldsymbol{\lambda}_3)\pi(z_1-x,N-x,\boldsymbol{\lambda}_2) \\ &+ \beta_3 \sum_{z_2=0}^{\min(x,y-1)} \pi(z_2,x,\boldsymbol{\lambda}_1) \sum_{z_1=\max(x-1,y-1)}^{N-1} \pi(y-1-z_2,z_1-z_2,\boldsymbol{\lambda}'_3)\pi(z_1+1-x,N-x,\boldsymbol{\lambda}_2) \\ &- \beta_1\beta_3 \sum_{z_2=0}^{\min(x-1,y-2)} \pi(z_2,x-1,\boldsymbol{\lambda}'_1) \sum_{z_1=\max(x-2,y-2)}^{N-2} \pi(y-2-z_2,z_1-z_2,\boldsymbol{\lambda}'_3) \\ &\times \pi(z_1+2-x,N-x,\boldsymbol{\lambda}_2) \Big), \end{split}$$

(v) $\eta(x) = x$:

$$K(y, x)\eta(y) = \beta_2\beta_3\eta(N) \sum_{z_2=0}^{\min(x,y-1)} \pi(z_2, x, \lambda_1) \sum_{z_1=x+y-z_2}^N \pi(y-z_2-1, z_1-x-1, \lambda'_3) \times \pi(z_1-x-1, N-x-1, \lambda'_2) + \eta(x) \Big(-\beta_2\beta_3 \sum_{z_2=0}^{\min(x,y-1)} \pi(z_2, x, \lambda_1) \sum_{z_1=x+y-z_2}^N \pi(y-z_2-1, z_1-x-1, \lambda'_3)\Big)$$

$$\times \pi(z_1 - x - 1, N - x - 1, \lambda'_2)$$
 (A.9)

+
$$\beta_1 \sum_{z_2=1}^{\min(x,y)} \pi(z_2-1,x-1,\lambda_1') \sum_{z_1=x+y-z_2}^N \pi(y-z_2,z_1-x,\lambda_3)\pi(z_1-x,N-x,\lambda_2) \Big).$$

Since each summation in (A.5)–(A.9) has the same structure as the original K(y, x) with shifted arguments and parameters, the next step $(K(y, x)\eta(y)^2)$ and further steps $(K(y, x)\eta(y)^3, \ldots)$ go almost parallel with the help of formulas, like

$$\begin{split} \eta(x) &= x: \quad \eta(y) = \eta(y-i) + \eta(i), \quad \eta(x-j) = \eta(x) + \eta(-j), \\ \eta(x) &= q^{-x} - 1: \quad \eta(y) = q^{-i}\eta(y-i) + \eta(i), \quad \eta(x-j) = q^{j}\eta(x) + \eta(-j), \end{split}$$

and summation over y gives (2.10).

For type (ii) and (v) with $\eta(x) = q^{-x} - 1$, we obtain $\sum_{y \in \mathcal{X}} K(y, x) \eta(y) = \alpha_1 + \alpha_2 \eta(x) + \alpha_3 \eta(N-x)$ (α_i : constant), and the triangularity (2.10) does not hold for these cases.

For the semi-infinite Markov chains given in §4 and the examples with $\eta(x) = 1 - q^x$ in §5.1.1, similar proof of triangularity holds. Obtaining the explicit form of the coefficients $\{a_{nm}\}$ in (2.10) is not necessary. One only has to convince oneself that the triangularity holds. The eigenvalues are easily obtained by the formula (2.21) in **Theorem 2.2**. Since triangularity is the consequence of (A.1)–(A.3), it is quite natural to expect that it also holds for convolutions other than type (i)–(v).

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