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“Diophantine” and Factorisation Properties of Finite Orthogonal Polynomials in the Askey Scheme

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Abstract

A new interpretation and applications of the “Diophantine” and factorisation properties of *finite* orthogonal polynomials in the Askey scheme are explored. The corresponding twelve polynomials are the (q-)Racah, (dual, q-)Hahn, Krawtchouk and five types of q-Krawtchouk. These (q-)hypergeometric polynomials, defined only for the degrees of $0, 1, \ldots, N$, constitute the main part of the eigenvectors of $N + 1$-dimensional tri-diagonal real symmetric matrices, which correspond to the difference equations governing the polynomials. The *monic* versions of these polynomials all exhibit the “Diophantine” and factorisation properties at higher degrees than N. This simply means that these higher degree polynomials are zero-norm “eigenvectors” of the $N+1$-dimensional tri-diagonal real symmetric matrices. A new type of multi-indexed orthogonal polynomials belonging to these twelve polynomials could be introduced by using the higher degree polynomials as the seed solutions of the multiple Darboux transformations for the corresponding matrix eigenvalue problems. The shape-invariance properties of the simplest type of the multi-indexed polynomials are demonstrated. The explicit transformation formulas are presented.

1 Introduction

It is noted for some time that some of the best known orthogonal polynomials, the Laguerre, Jacobi, Wilson and Askey-Wilson, etc exhibit strange phenomena, called “Diophantine” and factorisation properties [1, 2], when some parameters are tuned at values which invalidate the orthogonality. In this paper we show that these properties are universally shared, without tuning parameters, by a not so well known group of orthogonal polynomials, the *monic* versions of finite polynomials in the Askey scheme [3, 4, 5, 6].

A simple interpretation or explanation is that the monic versions of these polynomials satisfy the same $(N+1)\times(N+1)$ matrix eigenvalue problem (2.49) which govern the polynomials (2.4)–(2.12). Therefore at higher $n = N+1+m$ ($m \in \mathbb{Z}_{\geq 0}$) degrees, the polynomials are zero-norm solutions, which means the “Diophantine” and factorisation properties.

This paper is organised as follows. Section two start with some typical examples of the "Diophantine" and factorisation properties together with the background description. The finite orthogonal polynomials in the Askey scheme, the main subject, are briefly introduced as the solutions of certain tri-diagonal real symmetric $(N+1)\times(N+1)$ matrix eigenvalue problems [7] in § 2.1. The explicit data of the twelve polynomials are displayed according to the five families of the sinusoidal coordinates in § 2.2. In § 2.3 we present the main **Theorem 2.1** stating that the "Diophantine" and factorisation is the consequence of the zero norm nature of the higher degree $N+1+m$ ($m\in\mathbb{Z}_{\geq 0}$) monic polynomials. Starting with the most generic q-Racah polynomial, the explicit expressions of the "Diophantine" properties and factorisation are displayed in § 2.3.1–§ 2.3.11. The general setting of the multi-indexed orthogonal polynomials generated by multiple Darboux transformations by using the zero-norm solutions is outlined in section three. The very special cases of the new multi-indexed polynomials constructed by using M contiguous lowest degree zero-norm solutions are detailed in section four. In these cases, the multi-indexed polynomials take the same form as the original with x and some parameters shifted, displaying the so-called shape invariance. The general transformation rules for the five families are listed in **Theorem 4.1** together with the explicit forms of the transformation rules of the polynomials in **Theorem 4.2**. Corresponding to the new type of shape invariance, the forward x-shift operators are introduced. It is shown in **Theorem 4.3** that the multiple applications of the forward x-shift operators reproduce the results of **Theorem 4.2**. The explicit forms of the corresponding backward x-shift operators are listed. The final section is for a summary and some comments.

2 "Diophantine" Properties and Factorisation

Many interesting examples of the "Diophantine" properties and factorisation of various orthogonal polynomials belonging to the Askey scheme are reported and explained by Calogero and collaborators [1] (to be cited as I) and Ismail and a coauthor [2] (to be cited as II). Here we list some typical examples, which are stated for the *monic* versions of the named polynomials: (typos in (II.3.10) and (II.4.4) are corrected)

Racah with $\alpha=-n$,

$$p_n(x;-n,\beta,\gamma,\delta)=\prod_{k=0}^{n-1}\big(x-k(k+\gamma+\delta+1)\big), \tag{I.3.29a}$$

Jacobi with $\alpha=-n$,

$$p_n(x;-n,\beta)=(x-1)^n, \tag{I.3.153}$$

Laguerre with $\alpha=-n$,

$$p_n(x;-n)=x^n, \tag{I.3.162}$$

Wilson with $t_3=1-m-t_4,\quad 0\le m\le n$,

$$p_n(x;t_1,t_2,1-m-t_4,t_4)=\prod_{k=0}^{m-1}\bigl(x+(t_4+k)^2\bigr)\cdot p_{n-m}(x;t_1,t_2,1-t_4,t_4+m), \tag{II.3.10}$$

Askey-Wilson with $t_3=q^{1-m}/t_4,\quad x\stackrel{\text{def}}{=}\cos\theta,\quad 0\le m\le n$,

$$\begin{aligned}p_n(x;t_1,t_2,q^{1-m}/t_4,t_4)&=(t_4e^{i\theta}\,;q)_m(t_4e^{-i\theta}\,;q)_m\,p_{n-m}(x;t_1,t_2,q/t_4,q^mt_4)\\&\quad\times(-1)^mt_4^{-m}q^{-\frac{1}{2}m(m-1)}2^{-m}.\end{aligned} \tag{II.4.4}$$

It should be stressed that in all these examples, some parameters are tuned to a degree number n, in a rather ad-hoc manner. Therefore the "Diophantine" and factorisation properties do not belong to the polynomials in general but only to the particular degree polynomial to which the parameters are tuned. Moreover, with those parameter assignments, the polynomials are no longer orthogonal with each other. This situation makes it difficult to find satisfactory interpretations of the "Diophantine" and factorisation properties and tends to give a wrong impression that they are of haphazard or unsystematic origin and having rather peripheral importance.

In this paper we present a different perspective and show that for a certain group of orthogonal polynomials in the Askey scheme, the "Diophantine" and factorisation properties are essential and inherent within the proper parameter ranges in which the orthogonality holds. Let us first introduce the general features of the polynomials belonging to this group.

2.1 Finite orthogonal polynomials of a discrete variable

The polynomials $\{\check{P}_n(x;N,\boldsymbol{\lambda})\}$ in the group are also called finite orthogonal polynomials of a discrete variable [8]. They are defined on a finite integer lattice $\mathcal{X}=\{0,1,\ldots,N\}$ satisfying

$$\sum_{x\in\mathcal{X}}w(x;N,\boldsymbol{\lambda})\check{P}_m(x;N,\boldsymbol{\lambda})\check{P}_n(x;N,\boldsymbol{\lambda})=\frac{\delta_{m\,n}}{d_n(N,\boldsymbol{\lambda})^2}\quad(m,n\in\mathcal{X}), \tag{2.1}$$

with a positive weight function $w(x;N,\boldsymbol{\lambda})>0$. Here $\boldsymbol{\lambda}$ stands for the set of parameters other than the lattice size N. The group consists of five families according to the type of the *sinusoidal coordinate* $\eta(x;\boldsymbol{\lambda})$,

$$\check{P}_n(x;N,\boldsymbol{\lambda})=P_n\big(\eta(x;\boldsymbol{\lambda});N,\boldsymbol{\lambda}\big), \tag{2.2}$$

in which $P_n(\eta(x;\boldsymbol{\lambda});N,\boldsymbol{\lambda})$ is a degree n polynomial in $\eta(x;\boldsymbol{\lambda})$. There are five types of the sinusoidal coordinates for the finite polynomials in the Askey scheme [7],

$$\begin{aligned}
&\text{(i)}:\ \ \eta(x)=x,\quad \text{(ii)}:\ \ \eta(x)=x(x+d),\quad \text{(iii)}:\ \ \eta(x)=1-q^x,\\
&\text{(iv)}:\ \ \eta(x)=q^{-x}-1,\quad \text{(v)}:\ \ \eta(x)=(q^{-x}-1)(1-dq^x),\\
&\eta(0)=0,\quad \eta(x)>0\ \ (x\in\mathcal{X}\backslash\{0\}).
\end{aligned}$$

In this paper we adopt the normalisation convention

$$\check{P}_n(0;N,\boldsymbol{\lambda})=P_n(0;N,\boldsymbol{\lambda})=1, \tag{2.3}$$

except for the monic polynomials to be introduced shortly.

As the other orthogonal polynomials in the Askey scheme [3, 4, 5, 6], the polynomials in this group satisfy second order difference equation

$$\begin{aligned}
&B(x;N,\boldsymbol{\lambda})\big(\check{P}_n(x;N,\boldsymbol{\lambda})-\check{P}_n(x+1;N,\boldsymbol{\lambda})\big)+D(x;N,\boldsymbol{\lambda})\big(\check{P}_n(x;N,\boldsymbol{\lambda})-\check{P}_n(x-1;N,\boldsymbol{\lambda})\big)\\
&=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_n(x;N,\boldsymbol{\lambda})\quad (x,n\in\mathcal{X}),
\end{aligned} \tag{2.4}$$

on top of the three term recurrence relations. Here the coefficients $B(x;N,\boldsymbol{\lambda})$ and $D(x;N,\boldsymbol{\lambda})$ are positive in $\mathcal{X}$ [7] and vanish only at the boundary of $\mathcal{X}$,

$$\begin{aligned}
&B(x;N,\boldsymbol{\lambda})>0\ \ (x\in\mathcal{X}\backslash\{N\}),\quad &&D(x;N,\boldsymbol{\lambda})>0\ \ (x\in\mathcal{X}\backslash\{0\}),\\
&B(N;N,\boldsymbol{\lambda})=0, &&D(0;N,\boldsymbol{\lambda})=0.
\end{aligned}$$

As shown explicitly in [7], the difference equation (2.4) can be rewritten as a matrix eigenvalue equation in terms of an $(N+1)\times(N+1)$ tri-diagonal matrix $\big(\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,y}\big)_{x,y\in\mathcal{X}}$,

$$\begin{aligned}
&\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})\check{P}_n(x;N,\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_n(x;N,\boldsymbol{\lambda}),\\
&\Big(\Longleftrightarrow \sum_{y\in\mathcal{X}}\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,y}\check{P}_n(y;N,\boldsymbol{\lambda})=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_n(x;N,\boldsymbol{\lambda})\Big), && (2.5)\\
&\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,x+1}=-B(x;N,\boldsymbol{\lambda}),\quad \widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,x-1}=-D(x;N,\boldsymbol{\lambda}),\\
&\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,x}=B(x;N,\boldsymbol{\lambda})+D(x;N,\boldsymbol{\lambda}),\quad \widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})_{x\,y}=0\ \ (|x-y|\geq 2), && (2.6)
\end{aligned}$$

with the eigenvalue $\mathcal{E}(n;\boldsymbol{\lambda})$. By a similarity transformation in terms of a positive diagonal matrix $\Phi(x;N,\boldsymbol{\lambda})$, the matrix $\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})$ is related to a *real symmetric tri-diagonal matrix* $\mathcal{H}(N,\boldsymbol{\lambda})$,

$$\mathcal{H} \stackrel{\text{def}}{=} \Phi\widetilde{\mathcal{H}}\Phi^{-1} \Leftrightarrow \widetilde{\mathcal{H}} = \Phi^{-1}\mathcal{H}\Phi \ \Leftrightarrow \mathcal{H}_{x\,y} = \phi_0(x)\widetilde{\mathcal{H}}_{x\,y}\phi_0(y)^{-1}, \tag{2.7}$$

$$\Phi_{x\,x} = \phi_0(x), \quad \Phi_{x\,y} = 0 \ \ (x \neq y). \tag{2.8}$$

In these formulas and hereafter the parameter dependence of $\widetilde{\mathcal{H}}$, $\mathcal{H}$, Φ, $\check{P}_n$, B, D, η, $\mathcal{E}$ etc is omitted occasionally for simplicity of presentation. Here a positive function $\phi_0(x)$ on $\mathcal{X}$ is introduced by the ratios of $B(x)$ and $D(x+1)$,

$$\phi_0(x) \stackrel{\text{def}}{=} \sqrt{\prod_{y=0}^{x-1}\frac{B(y)}{D(y+1)}} \Leftrightarrow \frac{\phi_0(x+1)}{\phi_0(x)} = \frac{\sqrt{B(x)}}{\sqrt{D(x+1)}} \ \ (x \in \mathcal{X}\backslash\{N\}), \tag{2.9}$$

where $\prod_{j=n}^{n-1} * \stackrel{\text{def}}{=} 1$ ($\Rightarrow \phi_0(0) = 1$). The tri-diagonal real symmetric matrix $\mathcal{H}$ (2.7), expressed explicitly as

$$\mathcal{H}_{x\,x+1} = -\sqrt{B(x)D(x+1)}, \quad \mathcal{H}_{x\,x-1} = -\sqrt{B(x-1)D(x)},$$
$$\mathcal{H}_{x\,x} = B(x) + D(x), \quad \mathcal{H}_{x\,y} = 0 \ \ (|x-y| \geq 2), \tag{2.10}$$

has the following eigenvectors

$$\phi_n(x) \stackrel{\text{def}}{=} \phi_0(x)\check{P}_n(x) \ \ (n \in \mathcal{X}), \quad \big(\phi_n(x)\big)_{x\in\mathcal{X}} \in \mathbb{R}^{N+1}, \tag{2.11}$$

$$\mathcal{H}\phi_n(x) = \sum_{y\in\mathcal{X}}\mathcal{H}_{x\,y}\phi_n(y) = \sum_{y\in\mathcal{X}}\phi_0(x)\widetilde{\mathcal{H}}_{x\,y}\check{P}_n(y) = \mathcal{E}(n)\phi_n(x) \ \ (n \in \mathcal{X}). \tag{2.12}$$

The norms are all finite as $w(x) \stackrel{\text{def}}{=} \sum_{x\in\mathcal{X}}\phi_0(x)^2 < \infty$ for all $B(x)$ and $D(x)$ in the group. We arrive at the complete set of eigenvectors $\{\phi_n(x)\}$ $(n \in \mathcal{X})$, since the orthogonality of the eigenvectors (2.1) is guaranteed by the simpleness of the eigenvalues of the tri-diagonal matrices.

2.2 Polynomials data

Here we present the basic data of the polynomials in the group. For more details of the polynomials, consult [3]–[6]. The parametrisations of some polynomials are different from the conventional ones, see [7]. The polynomials are divided into five families according to the forms of the sinusoidal coordinate $\eta(x)$. It should be stressed that the explicit forms of the sinusoidal coordinates $\eta(x)$ are determined by the difference equations governing the polynomials [7]. The essential roles played by the sinusoidal coordinates $\eta(x)$ for the construction of exactly solvable matrix eigenvalue problems are explored in detail in [7]. The "Diophantine" properties are identical within the same family and the factorisation is universal for all the polynomials in this group. As mirror symmetry of the Krawtchouk polynomials is mentioned in the final paragraph of § 5, the relationship for the families $\eta(x) = x, x(x + d)$ are explicitly displayed. The finite polynomials of the Askey scheme are, when seen from behind the mirror ($x \to N - x$), still exactly solvable. Therefore they are expressed by the same or different polynomials with different parameters. As expected, mirror symmetries for $\eta(x) = x$ are simple and well known. Those for the q-polynomials are complicated and they are irrelevant for the main topic of this paper.

(i) Family with $\boldsymbol{\eta(x) = x}$

Two polynomials belong to this family, the Krawtchouk and Hahn polynomials. They are polynomials in x.

2.2.1 Krawtchouk (K)

The polynomial depends on one positive parameter $\boldsymbol{\lambda} = p$ $(0 < p < 1)$.

$$B(x; N, p) = p(N - x), \quad D(x; N, p) = (1 - p)x, \quad \mathcal{E}(n) = n, \quad \eta(x) = x, \tag{2.13}$$

$$\check{P}_n(x; N, p) = P_n(x; N, p) = {}_2F_1\Big(\begin{matrix} -n,\ -x \\ -N \end{matrix} \Big| \, p^{-1}\Big), \quad P_n(x; N, p) = P_x(n; N, p), \tag{2.14}$$

$$\check{P}_n(N - x; N, p) = (-1)^n (p^{-1} - 1)^n \check{P}_n(x; N, 1 - p) \ \ \text{(Mirror symmetry).} \tag{2.15}$$

2.2.2 Hahn (H)

The Hahn polynomial depends on two positive parameters $\boldsymbol{\lambda} = (a, b)$ $(a, b > 0)$.

$$B(x; N, \boldsymbol{\lambda}) = (x + a)(N - x), \quad D(x; N, \boldsymbol{\lambda}) = x(b + N - x), \tag{2.16}$$

$$\mathcal{E}(n; \boldsymbol{\lambda}) = n(n + a + b - 1), \quad \eta(x) = x,$$

$$\check{P}_n(x; N, \boldsymbol{\lambda}) = P_n(x; N, \boldsymbol{\lambda}) = {}_3F_2\Big(\begin{matrix} -n,\ n + a + b - 1,\ -x \\ a,\ -N \end{matrix} \Big| \, 1\Big), \tag{2.17}$$

$$\check{P}_n(N - x; N, a, b) = (-1)^n \frac{(b)_n}{(a)_n} \check{P}_n(x; N, b, a) \ \ \text{(Mirror symmetry).} \tag{2.18}$$

(ii) Family with $\boldsymbol{\eta(x) = x(x+d)}$

Two polynomials belong to this family, the Racah and dual Hahn polynomials. They are polynomials in $x(x+d)$.

2.2.3 Racah (R)

The Racah polynomial depends on three parameters $\boldsymbol{\lambda} = (b, c, d)$ ($0 < d < b - N$, $0 < c < 1 + d$) on top of the lattice size N.

$$B(x; N, \boldsymbol{\lambda}) = \frac{(x+b)(x+c)(x+d)(N-x)}{(2x+d)(2x+1+d)}, \tag{2.19}$$

$$D(x; N, \boldsymbol{\lambda}) = \frac{(b-d-x)(x+d-c)x(x+d+N)}{(2x-1+d)(2x+d)}, \tag{2.20}$$

$$\mathcal{E}(n; \boldsymbol{\lambda}) = n(n+\tilde{d}), \quad \eta(x; \boldsymbol{\lambda}) = x(x+d), \quad \tilde{d} \stackrel{\text{def}}{=} b + c - d - N - 1,$$

$$\check{P}_n(x; N, \boldsymbol{\lambda}) = P_n\big(\eta(x; \boldsymbol{\lambda}); N, \boldsymbol{\lambda}\big) = {}_4F_3\Big(\begin{matrix} -n,\ n+\tilde{d},\ -x,\ x+d \\ b,\ c,\ -N \end{matrix} \Big| \, 1\Big), \tag{2.21}$$

$$\check{P}_n(N-x; N, b, c, d) = \frac{(b', c')_n}{(b, c)_n} \check{P}_n(x; N, b', c', d') \ \ \text{(Mirror symmetry)},$$

$$b' \stackrel{\text{def}}{=} -N - d + b, \quad c' \stackrel{\text{def}}{=} -N - d + c, \quad d' \stackrel{\text{def}}{=} -2N - d. \tag{2.22}$$

2.2.4 dual Hahn (dH)

The dual Hahn polynomial depends on two positive parameters $\boldsymbol{\lambda} = (a, b)$ $(a, b > 0)$, and the parameter d is $d = a + b - 1$.

$$B(x; N, \boldsymbol{\lambda}) = \frac{(x+a)(x+a+b-1)(N-x)}{(2x-1+a+b)(2x+a+b)}, \tag{2.23}$$

$$D(x; N, \boldsymbol{\lambda}) = \frac{x(x+b-1)(x+a+b+N-1)}{(2x-2+a+b)(2x-1+a+b)}, \tag{2.24}$$

$$\mathcal{E}(n) = n, \quad \eta(x; \boldsymbol{\lambda}) = x(x+a+b-1),$$

$$\check{P}_n(x; N, \boldsymbol{\lambda}) = P_n\big(\eta(x; \boldsymbol{\lambda}); N, \boldsymbol{\lambda}\big) = {}_3F_2\Big(\begin{matrix} -n,\ x+a+b-1,\ -x \\ a,\ -N \end{matrix} \Big| \, 1\Big), \tag{2.25}$$

$$\check{P}_n(N-x; N, a, b) = \frac{(a')_n}{(a)_n} \check{P}_n(x; N, a', b') \ \ \text{(Mirror symmetry)},$$

$$a' \stackrel{\text{def}}{=} -N - b + 1, \quad b' \stackrel{\text{def}}{=} -N - a + 1. \tag{2.26}$$

The remaining three families belong to the q-hypergeometric polynomial category. The parameter q, $0 < q < 1$, dependence is not explicitly displayed.

(iii) Family with $\eta(x) = 1 - q^x$

2.2.5 dual quantum q-Krawtchouk (dq$q$K)

The only polynomial belonging to this family is the dual quantum q-Krawtchouk polynomial [7] depending on one positive parameter $\boldsymbol{\lambda} = p > q^{-N}$.

$$B(x;N,p) = p^{-1}q^{-x-N-1}(1-q^{N-x}), \quad D(x;N,p) = (q^{-x}-1)(1-p^{-1}q^{-x}), \tag{2.27}$$

$$\mathcal{E}(n) = q^{-n}-1, \quad \eta(x) = 1-q^x,$$

$$\check{P}_n(x;N,p) = P_n\big(\eta(x);N,p\big) = {}_2\phi_1\Big(\begin{matrix} q^{-n},\, q^{-x} \\ q^{-N} \end{matrix} \Big|\, q\,;pq^{x+1}\Big). \tag{2.28}$$

(iv) Family with $\eta(x) = q^{-x} - 1$

On top of the q-Hahn polynomial, three sibling polynomials belong to this family, the q-Krawtchouk, quantum q-Krawtchouk and affine q-Krawtchouk polynomials, all depending on one positive parameter $\boldsymbol{\lambda} = p > 0$, but the parameter ranges are different, as shown in each entry.

2.2.6 q-Hahn (qH)

The q-Hahn polynomial depends on two positive parameters $\boldsymbol{\lambda} = (a,b)$ $(0 < a,b < 1)$.

$$B(x;N,\boldsymbol{\lambda}) = (1-aq^x)(q^{x-N}-1), \quad D(x;N,\boldsymbol{\lambda}) = aq^{-1}(1-q^x)(q^{x-N}-b), \tag{2.29}$$

$$\mathcal{E}(n;\boldsymbol{\lambda}) = (q^{-n}-1)(1-abq^{n-1}), \quad \eta(x) = q^{-x}-1,$$

$$\check{P}_n(x;N,\boldsymbol{\lambda}) = P_n\big(\eta(x);N,\boldsymbol{\lambda}\big) = {}_3\phi_2\Big(\begin{matrix} q^{-n},\, abq^{n-1},\, q^{-x} \\ a,\, q^{-N} \end{matrix} \Big|\, q\,;q\Big). \tag{2.30}$$

2.2.7 q-Krawtchouk (qK)

$$B(x;N,p) = q^{x-N}-1, \quad D(x;N,p) = p(1-q^x), \tag{2.31}$$

$$\mathcal{E}(n;p) = (q^{-n}-1)(1+pq^n), \quad \eta(x) = q^{-x}-1, \quad p > 0,$$

$$\check{P}_n(x;N,p) = P_n\big(\eta(x);N,p\big) = {}_3\phi_2\Big(\begin{matrix} q^{-n},\, q^{-x},\, -pq^n \\ q^{-N},\, 0 \end{matrix} \Big|\, q\,;q\Big). \tag{2.32}$$

2.2.8 quantum q-Krawtchouk (qqK)

$$B(x;N,p) = p^{-1}q^x(q^{x-N}-1), \quad D(x;N,p) = (1-q^x)(1-p^{-1}q^{x-N-1}), \tag{2.33}$$

$$\mathcal{E}(n)=1-q^n,\quad \eta(x)=q^{-x}-1,\quad p>q^{-N},$$
$$\check{P}_n(x;N,p)=P_n\big(\eta(x);N,p\big)={}_2\phi_1\Big(\begin{matrix}q^{-n},\,q^{-x}\\ q^{-N}\end{matrix}\ \Big|\ q\,;pq^{n+1}\Big). \tag{2.34}$$

2.2.9 affine q-Krawtchouk (aqK)

$$B(x;N,p)=(q^{x-N}-1)(1-pq^{x+1}),\quad D(x;N,p)=pq^{x-N}(1-q^x), \tag{2.35}$$
$$\mathcal{E}(n)=q^{-n}-1,\quad \eta(x)=q^{-x}-1,\quad 0<p<q^{-1},$$
$$\check{P}_n(x;N,p)=P_n\big(\eta(x);N,p\big)={}_3\phi_2\Big(\begin{matrix}q^{-n},\,q^{-x},\,0\\ pq,\,q^{-N}\end{matrix}\ \Big|\ q\,;q\Big). \tag{2.36}$$

(v) Family with $\boldsymbol{\eta(x)=(q^{-x}-1)(1-dq^x)}$

Three polynomials belong to this family, the q-Racah, dual q-Hahn and dual q-Krawtchouk polynomials.

2.2.10 q-Racah (qR)

The q-Racah polynomial depends on three parameters $\boldsymbol{\lambda}=(b,c,d)$ $(0<bq^{-N}<d<1,$ $qd<c<1,\ \tilde{d}\stackrel{\text{def}}{=}bcd^{-1}q^{-N-1})$.

$$B(x;N,\boldsymbol{\lambda})=\frac{(1-bq^x)(1-cq^x)(1-dq^x)(q^{x-N}-1)}{(1-dq^{2x})(1-dq^{2x+1})}, \tag{2.37}$$
$$D(x;N,\boldsymbol{\lambda})=\tilde{d}\,\frac{(b^{-1}dq^x-1)(1-c^{-1}dq^x)(1-q^x)(1-dq^{x+N})}{(1-dq^{2x-1})(1-dq^{2x})}, \tag{2.38}$$
$$\mathcal{E}(n;\boldsymbol{\lambda})=(q^{-n}-1)(1-\tilde{d}q^n),\quad \eta(x;\boldsymbol{\lambda})=(q^{-x}-1)(1-dq^x),$$
$$\check{P}_n(x;N,\boldsymbol{\lambda})=P_n\big(\eta(x;\boldsymbol{\lambda});N,\boldsymbol{\lambda}\big)={}_4\phi_3\Big(\begin{matrix}q^{-n},\,\tilde{d}q^n,\,q^{-x},\,dq^x\\ b,\,c,\,q^{-N}\end{matrix}\ \Big|\ q\,;q\Big). \tag{2.39}$$

2.2.11 dual q-Hahn (dqH)

The dual q-Hahn polynomial depends on two parameters $\boldsymbol{\lambda}=(a,b)$ $(0<a,b<1)$, and the parameter d is $d=abq^{-1}$.

$$B(x;N,\boldsymbol{\lambda})=\frac{(1-aq^x)(1-abq^{x-1})(q^{x-N}-1)}{(1-abq^{2x-1})(1-abq^{2x})}, \tag{2.40}$$
$$D(x;N,\boldsymbol{\lambda})=aq^{x-N-1}\frac{(1-q^x)(1-bq^{x-1})(1-abq^{x+N-1})}{(1-abq^{2x-2})(1-abq^{2x-1})}, \tag{2.41}$$
$$\mathcal{E}(n)=q^{-n}-1,\quad \eta(x;\boldsymbol{\lambda})=(q^{-x}-1)(1-abq^{x-1}),$$
$$\check{P}_n(x;N,\boldsymbol{\lambda})=P_n\big(\eta(x;\boldsymbol{\lambda});N,\boldsymbol{\lambda}\big)={}_3\phi_2\Big(\begin{matrix}q^{-n},\,abq^{x-1},\,q^{-x}\\ a,\,q^{-N}\end{matrix}\ \Big|\ q\,;q\Big). \tag{2.42}$$

2.2.12 dual q-Krawtchouk (dqK)

Similar to the q-Krawtchouk, the dual q-Krawtchouk polynomial depends on one parameter $\boldsymbol{\lambda} = p > 0$, and the parameter d is $d = -p$.

$$B(x;N,p) = \frac{(q^{x-N}-1)(1+pq^x)}{(1+pq^{2x})(1+pq^{2x+1})}, \quad D(x;N,p) = pq^{2x-N-1}\frac{(1-q^x)(1+pq^{x+N})}{(1+pq^{2x-1})(1+pq^{2x})}, \tag{2.43}$$

$$\mathcal{E}(n) = q^{-n}-1, \quad \eta(x;p) = (q^{-x}-1)(1+pq^x),$$

$$\check{P}_n(x;N,p) = P_n\big(\eta(x;p);N,p\big) = {}_3\phi_2\Big(\begin{matrix} q^{-n},\, q^{-x},\, -pq^x \\ q^{-N},\, 0 \end{matrix}\ \Big|\ q\,;q\Big). \tag{2.44}$$

2.3 Zero norm = "Diophantine" and factorisation

The polynomials listed above § 2.2.1–§ 2.2.12 are *finite* polynomials, since their degree n is limited in $\mathcal{X} \ni n$ due to the presence of the lower indices $-N$ or q^{-N} in the (q-)hypergeometric expressions, ${}_{r+1}F_r$, ${}_{r+1}\phi_r$, of these polynomials, (2.14), (2.17), (2.21), (2.25), (2.28), (2.30), (2.32), (2.34), (2.36), (2.39), (2.42) and (2.44). For non-negative integer $m \geq 0$, the polynomial $\check{P}_{N+1+m}(x;N,\boldsymbol{\lambda})$ is ill-defined due the presence of the factors $(-N)_{N+1+k} = 0$ and $(q^{-N}\,;q)_{N+1+k} = 0$, $0 \leq k \leq m$ in the denominator of the (q-)hypergeometric series expansion.

The situation is drastically changed by the introduction of the monic version of each polynomial in § 2.2,

$$\check{P}_n^{\text{monic}}(x;N,\boldsymbol{\lambda}) = \frac{1}{c_n(N,\boldsymbol{\lambda})}\check{P}_n(x;N,\boldsymbol{\lambda}), \tag{2.45}$$

where $c_n(N,\boldsymbol{\lambda})$ is the coefficient of the highest degree term $\eta(x;\boldsymbol{\lambda})^n$. It is well defined for all degrees $n = N+1+m$ ($m \in \mathbb{Z}_{\geq 0}$) and it satisfies the same difference equation as (2.4), (2.5)

$$\begin{aligned} &B(x;N,\boldsymbol{\lambda})\big(\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}) - \check{P}_{N+1+m}^{\text{monic}}(x+1;N,\boldsymbol{\lambda})\big) \\ &\quad + D(x;N,\boldsymbol{\lambda})\big(\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}) - \check{P}_{N+1+m}^{\text{monic}}(x-1;N,\boldsymbol{\lambda})\big) \\ &= \mathcal{E}(N+1+m;\boldsymbol{\lambda})\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}) \quad (x \in \mathcal{X}), \end{aligned} \tag{2.46}$$

$$\Longrightarrow \widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}) = \mathcal{E}(N+1+m;\boldsymbol{\lambda})\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}). \tag{2.47}$$

This also means that the corresponding monic vector

$$\phi_{N+1+m}^{\text{monic}}(x) \stackrel{\text{def}}{=} \phi_0(x)\check{P}_{N+1+m}^{\text{monic}}(x) \quad (m \in \mathbb{Z}_{\geq 0}), \tag{2.48}$$

satisfies the eigenvalue equation of the tri-diagonal real symmetric matrix $\mathcal{H}$,

$$\mathcal{H}\phi_{N+1+m}^{\text{monic}}(x) = \mathcal{E}(N+1+m;\boldsymbol{\lambda})\phi_{N+1+m}^{\text{monic}}(x) \quad (m \in \mathbb{Z}_{\geq 0}). \tag{2.49}$$

If $\check{P}^{\text{monic}}_{N+1+m}(x)$ takes a non-vanishing value at some point in $\mathcal{X}$, $\phi^{\text{monic}}_{N+1+m}(x)$ becomes an eigenvector of $\mathcal{H}$, which contradicts the complete set of eigenvectors $\{\phi_n(x)\}$ ($n \in \mathcal{X}$), (2.12). Thus we arrive at

Theorem 2.1 *For all the finite polynomials listed in § 2.2, the "Diophantine" property and factorisation hold,*

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda}) = \Lambda(x;N,\boldsymbol{\lambda})\check{Q}_m(x;N,\boldsymbol{\lambda}) \quad (m \in \mathbb{Z}_{\geq 0}), \tag{2.50}$$

$$\Lambda(x;N,\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \prod_{k=0}^{N}\bigl(\eta(x;\boldsymbol{\lambda}) - \eta(k;\boldsymbol{\lambda})\bigr), \tag{2.51}$$

in which $\check{Q}_m(x;N,\boldsymbol{\lambda})$ is a monic degree m polynomial in $\eta(x;\boldsymbol{\lambda})$.

Of course, "Diophantine" property and factorisation can be demonstrated for each polynomial by direct calculation of the (q-)hypergeometric series expansion. Below we show the explicit derivation of the "Diophantine" and factorisation property for the q-Racah polynomial, the most generic member of the group.

Since the coefficient of the highest degree of $\check{P}_n(x\,;N,\boldsymbol{\lambda})$ (2.39) is

$$c_n(N,\boldsymbol{\lambda}) = \frac{(\tilde{d}q^n\,;q)_n}{(b,c,q^{-N}\,;q)_n}, \tag{2.52}$$

the monic q-Racah polynomial is

$$\check{P}^{\text{monic}}_n(x;N,\boldsymbol{\lambda}) = \frac{1}{c_n(N,\boldsymbol{\lambda})}\check{P}_n(x;N,\boldsymbol{\lambda}) = \sum_{k=0}^{n}\frac{(bq^k,cq^k,q^{k-N}\,;q)_{n-k}}{(\tilde{d}q^{n+k}\,;q)_{n-k}}\frac{(q^{-n},q^{-x},dq^x\,;q)_k}{(q\,;q)_k}q^k. \tag{2.53}$$

At $n = N+1+m$ ($m \in \mathbb{Z}_{\geq 0}$), $(q^{k-N}\,;q)_{n-k}$ vanishes for $k = 0,1,\ldots,N$, and the k summation is reduced to $\sum_{k=N+1}^{N+1+m}$. By changing $k = N+1+l$, we obtain

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda}) = \sum_{l=0}^{m}\frac{(bq^{N+1+l},cq^{N+1+l},q^{l+1}\,;q)_{m-l}}{(\tilde{d}q^{2(N+1)+m+l}\,;q)_{m-l}}\frac{(q^{-N-1-m},q^{-x},dq^x\,;q)_{N+1+l}}{(q\,;q)_{N+1+l}}q^{N+1+l}.$$

From the explicit form of $\eta(x;\boldsymbol{\lambda})$

$$\eta(x;\boldsymbol{\lambda}) - \eta(k;\boldsymbol{\lambda}) = -q^{-k}(1-q^{-x+k})(1-dq^{x+k}),$$

we obtain

$$\Lambda(x;N,\boldsymbol{\lambda}) = \prod_{k=0}^{N}\bigl(\eta(x;\boldsymbol{\lambda}) - \eta(k;\boldsymbol{\lambda})\bigr) = (-1)^{N+1}q^{-\binom{N+1}{2}}(q^{-x},dq^x\,;q)_{N+1}. \tag{2.54}$$

By using the basic properties of the q-shifted factorial,

$$(a\,;q)_{n+k}=(a\,;q)_n(aq^n\,;q)_k,\quad (a\,;q)_n=(-a)^nq^{\binom{n}{2}}(a^{-1}q^{1-n}\,;q)_n, \tag{2.55}$$

we obtain

$$\frac{(q^{-N-1-m},q^{-x},dq^x\,;q)_{N+1+l}}{(q\,;q)_{N+1+l}}=\frac{(q^{-N-1-m},q^{-x},dq^x\,;q)_{N+1}}{(q\,;q)_l}\frac{(q^{-m},q^{-x+N+1},dq^{x+N+1}\,;q)_l}{(q^{l+1}\,;q)_{N+1}}$$

and

$$\begin{aligned}\frac{(q^{-N-1-m}\,;q)_{N+1}}{(q^{l+1}\,;q)_{N+1}}&=\frac{(-q^{-N-1-m})^{N+1}q^{\binom{N+1}{2}}(q^{m+1}\,;q)_{N+1}}{(q^{l+1}\,;q)_{N+1}}\\&=(-q^{-N-1-m})^{N+1}q^{\binom{N+1}{2}}\frac{(q^{N+l+2}\,;q)_{m-l}}{(q^{l+1}\,;q)_{m-l}}.\end{aligned}$$

These lead to

$$\begin{aligned}\check{P}^{\text{monic}}_{N+1+m}(x;\boldsymbol{\lambda})&=\Lambda(x;N,\boldsymbol{\lambda})\cdot q^{-(N+1)m}\\&\quad\times\sum_{l=0}^{m}\frac{(bq^{N+1+l},cq^{N+1+l},qq^{N+1+l}\,;q)_{m-l}}{(\tilde{d}q^{2(N+1)+m+l}\,;q)_{m-l}}\frac{(q^{-m},q^{-x+N+1},dq^{x+N+1}\,;q)_l}{(q\,;q)_l}q^l\\&=\Lambda(x;N,\boldsymbol{\lambda})\cdot q^{-(N+1)m}\check{P}^{\text{monic}}_m(x-N-1;-N-2,\boldsymbol{\lambda}').\end{aligned}\tag{2.56}$$

in which the shifted parameters are

$$\boldsymbol{\lambda}'=(bq^{N+1},cq^{N+1},dq^{2(N+1)}). \tag{2.57}$$

Below we show the "Diophantine" and factorisation property for the other polynomials in the group.

(i) Family with $\eta(x)=x$

This family possesses the true Diophantine property as all the zeros of $\Lambda(x)$ are integers. Miki, Tsujimoto and Vinet showed the Diophantine and factorisation property of the Krawtchouk polynomial by explicit calculations in the seminal paper [9].

2.3.1 Krawtchouk (K)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,p)=\Lambda(x;N)\check{P}^{\text{monic}}_m(x-N-1;-N-2,p),\quad c_n(N,p)=\frac{1}{(-N)_n\,p^n}, \tag{2.58}$$

$$\check{P}^{\text{monic}}_m(x-N-1;-N-2,p)=\sum_{k=0}^{m}(N+2+k)_{m-k}\frac{(-m,-x+N+1)_k}{k!}p^{m-k}. \tag{2.59}$$

2.3.2 Hahn (H)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda})=\Lambda(x;N)\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}'), \tag{2.60}$$

$$\boldsymbol{\lambda}'=(a+N+1,b+N+1),\quad c_n(N,\boldsymbol{\lambda})=\frac{(n+a+b-1)_n}{(a,-N)_n}, \tag{2.61}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}')=\sum_{k=0}^{m}\frac{(a+N+1+k,N+2+k)_{m-k}}{(m+a+b+2N+1+k)_{m-k}}\frac{(-m,-x+N+1)_k}{k!}. \tag{2.62}$$

(ii) Family with $\boldsymbol{\eta(x)=x(x+d)}$

2.3.3 Racah (R)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda})=\Lambda(x;N,\boldsymbol{\lambda})\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}'), \tag{2.63}$$

$$\boldsymbol{\lambda}'=\big(b+N+1,c+N+1,d+2(N+1)\big),\quad c_n(N,\boldsymbol{\lambda})=\frac{(\tilde{d}+n)_n}{(b,c,-N)_n}, \tag{2.64}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}')=\sum_{k=0}^{m}\frac{(b+N+1+k,c+N+1+k,N+2+k)_{m-k}}{(\tilde{d}+2N+2+m+k)_{m-k}}\times\frac{(-m,-x+N+1,x+N+1+d)_k}{k!}. \tag{2.65}$$

2.3.4 dual Hahn (dH)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda})=\Lambda(x;N,\boldsymbol{\lambda})\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}'), \tag{2.66}$$

$$\boldsymbol{\lambda}'=\big(a+N+1,b+N+1\big),\quad c_n(N,\boldsymbol{\lambda})=\frac{1}{(a,-N)_n}, \tag{2.67}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}')=\sum_{k=0}^{m}(a+N+1+k,N+2+k)_{m-k}\times\frac{(-m,-x+N+1,x+a+b+N)_k}{k!}. \tag{2.68}$$

(iii) Family with $\boldsymbol{\eta(x)=1-q^x}$

2.3.5 dual quantum q-Krawtchouk (dq$q$K)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,p)=\Lambda(x;N)\,q^{(N+1)m}\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,p'), \tag{2.69}$$

$$p' = pq^{N+1}, \quad c_n(N,p) = \frac{p^n q^{-\frac{1}{2}n(n-1)}}{(q^{-N}\,;q)_n}, \tag{2.70}$$

$$\check{P}_m^{\text{monic}}(x-N-1;-N-2,p') = \sum_{k=0}^{m} (q^{N+2+k}\,;q)_{m-k} \frac{(q^{-m},q^{-x+N+1}\,;q)_k}{(q\,;q)_k} \times p^{k-m} q^{kx+k+\frac{1}{2}m(m-1)-m(N+1)}. \tag{2.71}$$

(iv) Family with $\boldsymbol{\eta(x) = q^{-x} - 1}$

2.3.6 q-Hahn (qH)

$$\check{P}_{N+1+m}^{\text{monic}}(x;N,\boldsymbol{\lambda}) = \Lambda(x;N)\,q^{-(N+1)m}\check{P}_m^{\text{monic}}(x-N-1;-N-2,\boldsymbol{\lambda}'), \tag{2.72}$$

$$\boldsymbol{\lambda}' = (aq^{N+1}, bq^{N+1}), \quad c_n(N,\boldsymbol{\lambda}) = \frac{(abq^{n-1}\,;q)_n}{(a,q^{-N}\,;q)_n}, \tag{2.73}$$

$$\check{P}_m^{\text{monic}}(x-N-1;-N-2,\boldsymbol{\lambda}') = \sum_{k=0}^{m} \frac{(aq^{N+1+k},q^{N+2+k}\,;q)_{m-k}}{(abq^{m+2N+1+k}\,;q)_{m-k}} \frac{(q^{-m},q^{-x+N+1}\,;q)_k}{(q\,;q)_k} q^k. \tag{2.74}$$

2.3.7 q-Krawtchouk (qK)

$$\check{P}_{N+1+m}^{\text{monic}}(x;N,p) = \Lambda(x;N)\,q^{-(N+1)m}\check{P}_m^{\text{monic}}(x-N-1;-N-2,p'), \tag{2.75}$$

$$p' = pq^{2(N+1)}, \quad c_n(N,p) = \frac{(-pq^n\,;q)_n}{(q^{-N}\,;q)_n}, \tag{2.76}$$

$$\check{P}_m^{\text{monic}}(x-N-1;-N-2,p') = \sum_{k=0}^{m} \frac{(q^{N+2+k}\,;q)_{m-k}}{(-pq^{2(N+1)+m+k}\,;q)_{m-k}} \frac{(q^{-m},q^{-x+N+1}\,;q)_k}{(q\,;q)_k} q^k. \tag{2.77}$$

2.3.8 quantum q-Krawtchouk (qqK)

$$\check{P}_{N+1+m}^{\text{monic}}(x;N,p) = \Lambda(x;N)\,q^{-(N+1)m}\check{P}_m^{\text{monic}}(x-N-1;-N-2,p'), \tag{2.78}$$

$$p' = pq^{N+1}, \quad c_n(N,p) = \frac{p^n q^{n^2}}{(q^{-N}\,;q)_n}, \tag{2.79}$$

$$\check{P}_m^{\text{monic}}(x-N-1;-N-2,p') = \sum_{k=0}^{m} (q^{N+2+k}\,;q)_{m-k} \frac{(q^{-m},q^{-x+N+1}\,;q)_k}{(q\,;q)_k} \times p^{k-m} q^{(N+m+2)k-m(m+N+1)}. \tag{2.80}$$

2.3.9 affine q-Krawtchouk (aqK)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,p)=\Lambda(x;N)\,q^{-(N+1)m}\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,p'), \tag{2.81}$$

$$p'=pq^{N+1},\quad c_n(N,p)=\frac{1}{(pq,q^{-N}\,;q)_n}, \tag{2.82}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,p')=\sum_{k=0}^{m}(pq^{N+2+k},q^{N+2+k}\,;q)_{m-k}\frac{(q^{-m},q^{-x+N+1}\,;q)_k}{(q\,;q)_k}q^k. \tag{2.83}$$

(v) Family with $\eta(x)=(q^{-x}-1)(1-dq^{x})$

2.3.10 dual q-Hahn (dqH)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,\boldsymbol{\lambda})=\Lambda(x;N,\boldsymbol{\lambda})\,q^{-(N+1)m}\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}'), \tag{2.84}$$

$$\boldsymbol{\lambda}'=(aq^{N+1},bq^{N+1}),\quad c_n(N,\boldsymbol{\lambda})=\frac{1}{(a,q^{-N}\,;q)_n}, \tag{2.85}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,\boldsymbol{\lambda}')=\sum_{k=0}^{m}(aq^{N+1+k},q^{N+2+k}\,;q)_{m-k}\frac{(q^{-m},abq^{x+N},q^{-x+N+1}\,;q)_k}{(q\,;q)_k}q^k. \tag{2.86}$$

2.3.11 dual q-Krawtchouk (dqK)

$$\check{P}^{\text{monic}}_{N+1+m}(x;N,p)=\Lambda(x;N,p)\,q^{-(N+1)m}\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,p'), \tag{2.87}$$

$$p'=p^{2(N+1)},\quad c_n(N,p)=\frac{1}{(q^{-N}\,;q)_n}, \tag{2.88}$$

$$\check{P}^{\text{monic}}_{m}(x-N-1;-N-2,p')=\sum_{k=0}^{m}(q^{N+2+k}\,;q)_{m-k}\frac{(q^{-m},q^{-x+N+1},-pq^{x+N+1}\,;q)_k}{(q\,;q)_k}q^k. \tag{2.89}$$

2.4 Factorisation of multi-indexed and Krein-Adler systems

As shown in § 2.1, the "Diophantine" property and factorisation are a consequence of a general fact that the finite orthogonal polynomials in the Askey scheme are the eigenvectors of certain real symmetric matrices $\mathcal{H}$ (2.10). Certain generalisation of the finite orthogonal polynomials in the Askey scheme are known for some time. They are generated by multiple Darboux transformations [10] of these polynomials by adopting certain seed solutions. When the polynomial themselves are chosen as the seed solutions [11], the obtained polynomials are called Krein-Adler [12, 13] polynomials. When certain virtual state vectors are used as the seed solutions, the new polynomials are called, for example, multi-indexed (q-)Racah polynomials, etc [14]. By construction, *i.e.* by appropriate choices of the seed solutions and the parameter ranges, these new types of orthogonal polynomials are the eigenvectors of certain real symmetric matrices. Therefore, when $\check{P}_n(x)$ is changed to $\check{P}^{\text{monic}}_{N+1+m}(x)$ in (3.17), the new types of orthogonal polynomials have the "Diophantine" property and factorisation.

As will be discussed in the subsequent two sections, a new type of multi-indexed polynomials can be generated by using $\check{P}^{\text{monic}}_{N+1+m}(x)$ as seed solutions for the multiple Darboux transformations. If the resulting orthogonality measures are positive definite, which is the case for those introduced in § 4, such systems have also the "Diophantine" property and factorisation.

3 New Multi-indexed Polynomials, General Setting

One possible application of the new polynomials satisfying the "Diophantine" and factorisation properties (2.50) is the generation of new multi-indexed orthogonal polynomials. It is well known that certain infinite norm solutions of the Schrödinger equations with the radial oscillator and Pöschl-Teller potentials are used as seed solutions for multiple Darboux transformations to generate multi-indexed Laguerre and Jacobi polynomials [15, 16, 17, 18]. Likewise the non-eigenvector solutions of the tri-diagonal real symmetric matrix $\mathcal{H}$ eigenvalue problem (2.12) are used as seed solutions to generate the multi-indexed versions of the (q-)Racah polynomials [14] etc through the difference equation analogues of the multiple Darboux transformations. Now we have plenty of zero norm solutions

$$\tilde{\phi}_m(x) = \phi^{\text{monic}}_{N+1+m}(x) = \phi_0(x)\check{P}^{\text{monic}}_{N+1+m}(x; N, \boldsymbol{\lambda}) \quad (m \in \mathbb{Z}_{\geq 0}) \tag{3.1}$$

$$= \phi_0(x)\Lambda(x; N, \boldsymbol{\lambda})\check{Q}_m(x; N, \boldsymbol{\lambda}), \quad \big(\tilde{\phi}_m(x)\big)_{x\in\mathcal{X}} \in \mathbb{R}^{N+1}, \tag{3.2}$$

corresponding to (2.56), (2.58), (2.60), (2.63), (2.66), (2.69), (2.72), (2.75), (2.78), (2.81), (2.84) and (2.87), which could be used as seed solutions to generate the multi-indexed versions of the twelve polynomials listed in § 2.2.1–§ 2.2.12.

Let us recapitulate the basic formulas of the multiple Darboux transformation of the original eigenvectors $\{\phi_n(x)\}$ (2.11) by using the zero-norm vectors $\{\tilde{\phi}_m(x)\}$ (3.1) as seed solutions. These formulas apply for each of the finite polynomials listed in § 2.2. We employ the formulas reported in § 2.3 of [19].

We choose M distinct zero-norm seed solutions

$$\{\tilde{\phi}_{m_1}(x), \tilde{\phi}_{m_2}(x), \ldots, \tilde{\phi}_{m_M}(x)\}, \quad 0 \le m_1 < m_2 < \cdots < m_M, \tag{3.3}$$

which correspond to the index set

$$\mathcal{D} = \{m_1, \ldots, m_M\}. \tag{3.4}$$

The deformed set of eigenvectors $\{\bar{\phi}_{\mathcal{D},n}\}$ $(n \in \mathcal{X})$ satisfy the difference equation

$$\mathcal{H}_{\mathcal{D}}\bar{\phi}_{\mathcal{D},n}(x) = \mathcal{E}(n)\bar{\phi}_{\mathcal{D},n}(x), \tag{3.5}$$

$$\mathcal{H}_{\mathcal{D}} \stackrel{\text{def}}{=} \mathcal{A}_{\mathcal{D}}^{\dagger}\mathcal{A}_{\mathcal{D}}, \quad \mathcal{A}_{\mathcal{D}} \stackrel{\text{def}}{=} \sqrt{\bar{B}(x)} - e^{\partial}\sqrt{\bar{D}(x)}, \quad \mathcal{A}_{\mathcal{D}}^{\dagger} \stackrel{\text{def}}{=} \sqrt{\bar{B}(x)} - \sqrt{\bar{D}(x)}e^{-\partial}, \tag{3.6}$$

$$(\bar{\phi}_{\mathcal{D},n}, \bar{\phi}_{\mathcal{D},\ell}) = \prod_{j=1}^{M}\bigl(\mathcal{E}(n) - \mathcal{E}(N+1+m_j)\bigr) \cdot \frac{\delta_{n\,\ell}}{d_n^2}, \tag{3.7}$$

in which $(\cdot,\cdot)$ is the inner product of two eigenvectors. In this particular case, $(\psi, \varphi) = \sum_{x=0}^{N+M} \psi(x)\varphi(x)$. In these formulas e^{∂} $(\partial \stackrel{\text{def}}{=} \frac{d}{dx})$ is a finite shift operator acting on smooth functions,

$$e^{\partial} f(x) = f(x+1), \quad (e^{\partial})^j = e^{j\partial}, \quad e^{j\partial} f(x) = f(x+j) \quad (j \in \mathbb{Z}). \tag{3.8}$$

As an operator it passes a smooth function $g(x)$ as

$$e^{j\partial} g(x) = g(x+j)e^{j\partial}.$$

Here $\bar{B}(x)$, $\bar{D}(x)$ and $\bar{\phi}_{\mathcal{D},n}(x)$ are expressed in terms of the Casoratians involving the seed solutions $\{\tilde{\phi}_m\}$, eigenvector ϕ_n and the groundstate eigenvector ϕ_0,

$$\mathrm{W}_{\mathrm{C}}[f_1, \ldots, f_n](x) \stackrel{\text{def}}{=} \det\Bigl(f_k(x+j-1)\Bigr)_{1 \le j,k \le n}, \tag{3.9}$$

$$\bar{B}(x) = \sqrt{B(x+M)D(x+M+1)}\,\frac{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}](x)}{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}](x+1)} \times \frac{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}, \phi_0](x+1)}{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}, \phi_0](x)}, \tag{3.10}$$

$$\bar{D}(x) = \sqrt{B(x-1)D(x)}\,\frac{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}](x+1)}{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}](x)} \times \frac{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}, \phi_0](x-1)}{\mathrm{W}_{\mathrm{C}}[\tilde{\phi}_{m_1}, \ldots, \tilde{\phi}_{m_M}, \phi_0](x)}, \tag{3.11}$$

$$F(x) \stackrel{\text{def}}{=} \frac{\sqrt{\prod_{k=1}^{M} B(x+k-1)D(x+k)}}{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x)\,\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x+1)},$$
$$\bar{\phi}_{\mathcal{D},n}(x) = (-1)^M \sqrt{F(x)}\,\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_n](x). \tag{3.12}$$

By using the Casoratian identity

$$\mathrm{W_C}[gf_1, gf_2, \ldots, gf_n](x) = \prod_{k=0}^{n-1} g(x+k)\cdot \mathrm{W_C}[f_1,f_2,\ldots,f_n](x), \tag{3.13}$$

some formulas are simplified,

$$\frac{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x)}{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x+1)} = \frac{\phi_0(x)\Lambda(x)}{\phi_0(x+M)\Lambda(x+M)}\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)},$$
$$\frac{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x+1)}{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x)} = \frac{\phi_0(x+M)\Lambda(x+M)}{\phi_0(x)\Lambda(x)}\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)},$$
$$\frac{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_0](x+1)}{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_0](x)} = \frac{\phi_0(x+M+1)\Lambda(x+M+1)}{\phi_0(x)\Lambda(x)} \times \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)},$$
$$\frac{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_0](x-1)}{\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_0](x)} = \frac{\phi_0(x-1)\Lambda(x-1)}{\phi_0(x+M)\Lambda(x+M)}\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x-1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)},$$
$$\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M},\phi_n](x) = \prod_{k=0}^{M}\phi_0(x+k)\Lambda(x+k)\cdot \mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}\check{P}_n](x).$$

These lead to

$$\begin{aligned}\bar{B}(x) &= \sqrt{B(x+M)D(x+M+1)}\,\frac{\phi_0(x+M+1)\Lambda(x+M+1)}{\phi_0(x+M)\Lambda(x+M)}\\ &\quad\times \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}\,\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)}\\ &= B(x+M)\frac{\Lambda(x+M+1)}{\Lambda(x+M)}\\ &\quad\times \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}\,\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)},\\ \bar{D}(x) &= \sqrt{B(x-1)D(x)}\,\frac{\phi_0(x-1)\Lambda(x-1)}{\phi_0(x)\Lambda(x)}\\ &\quad\times \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)}\,\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x-1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)}\\ &= D(x)\frac{\Lambda(x-1)}{\Lambda(x)}\end{aligned} \tag{3.14}$$

$$\times \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)} \frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x-1)}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}](x)}. \tag{3.15}$$

Furthermore, we have

$$\begin{aligned}
&\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x)\,\mathrm{W_C}[\tilde{\phi}_{m_1},\ldots,\tilde{\phi}_{m_M}](x+1)\\
&= \big(\phi_0(x)\phi_0(x+M)\Lambda(x)\Lambda(x+M)\big)^{-1}\prod_{k=0}^{M}\phi_0(x+k)^2\Lambda(x+k)^2\\
&\quad\times \mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1),\\
&\sqrt{\prod_{k=1}^{M}B(x+k-1)D(x+k)}\ \phi_0(x)\phi_0(x+M) = \phi_0(x)^2\prod_{k=1}^{M}B(x+k-1).
\end{aligned}$$

In terms of these we arrive at

$$\begin{aligned}
F(x) &= \prod_{k=1}^{M}B(x+k-1)\cdot\frac{\Lambda(x+M)}{\Lambda(x)}\frac{1}{\prod_{k=1}^{M}\phi_0(x+k)^2\Lambda(x+k)^2}\\
&\quad\times\frac{1}{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\bar{\phi}_{\mathcal{D},n}(x) &= (-1)^M\sqrt{\prod_{k=1}^{M}B(x+k-1)}\ \phi_0(x)\sqrt{\Lambda(x)\Lambda(x+M)}\\
&\quad\times\frac{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M},\Lambda^{-1}\check{P}_n](x)}{\sqrt{\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x)\mathrm{W_C}[\check{Q}_{m_1},\ldots,\check{Q}_{m_M}](x+1)}}.
\end{aligned} \tag{3.17}$$

For each type of the polynomials, the multi-indexed polynomials are identified by removing various kinematical factors from $\bar{\phi}_{\mathcal{D},n}$. As shown by (3.7), the orthogonality is built in. After the identification and the orthogonality, securing the positivity of the orthogonality measures is essential. It must be verified for each polynomial, for each choice of the set $\mathcal{D}$ of the zero-norm seed solutions and for the appropriate ranges of the involved parameters. The situation is much more complicated than the multi-indexed Laguerre and Jacobi polynomials [18] and (q-)Racah polynomials [14], etc. In these established cases, the seed solutions can be chosen to have a definite sign, and that is closely related to the positive definite orthogonality measures. We hope these detailed tasks for each type of polynomials will be carried out in future. A few examples of single-indexed exceptional Krawtchouk polynomials are reported in [9] and their "Diophantine" property and factorisation are discussed.

4 Shape-invariant Cases

Here we report on multi-indexed polynomials corresponding to a very special choice of the seed solutions, M contiguous lowest degree zero-norm solutions,

$$\mathcal{D} = \{0, 1, \dots, M-1\}. \tag{4.1}$$

For the Schrödinger equations, for which the Laguerre and Jacobi polynomials are the eigen polynomials, the multiple Darboux transformations by using M contiguous lowest degree eigenfunctions produce remarkable effects [10, 12, 13], that is, the resulting polynomial is the same as the original with the degree decreased by M and the parameters shifted by M multiplied by a numerical factor,

$$\begin{aligned}
\text{Laguerre}: &\quad L_n^{(\alpha)}\big(\eta(x)\big) \to (\text{const}) \times L_{n-M}^{(\alpha+M)}\big(\eta(x)\big) \qquad (n \geq M),\\
\text{Jacobi}: &\quad P_n^{(\alpha,\beta)}\big(\eta(x)\big) \to (\text{const}) \times P_{n-M}^{(\alpha+M,\beta+M)}\big(\eta(x)\big) \quad (n \geq M).
\end{aligned}$$

Under the transformations, the polynomials keep their identity with shifted parameters and the positivity of the orthogonality measure unchanged. This phenomenon is called shape-invariance [20]. Shape-invariance also holds for all the polynomials listed in § 2.2 and the others in the Askey scheme. It is called the forward shift relation [5]. For example

$$\begin{aligned}
\text{Racah}: &\quad \check{P}_n(x; N, b, c, d) \to (\text{const}) \times \check{P}_{n-M}(x; N-M, b+M, c+M, d+M) \quad (n \geq M),\\
q\text{-Racah}: &\quad \check{P}_n(x; N, b, c, d) \to (\text{const}) \times \check{P}_{n-M}(x; N-M, bq^M, cq^M, dq^M) \qquad (n \geq M).
\end{aligned}$$

For more details, see [7].

We will demonstrate similar effects for the present case of using M contiguous lowest zero-norm states (4.1) for the finite orthogonal polynomials. The explicit formulas of the transformation of $\bar{B}$ (3.14), $\bar{D}$ (3.15) and $\bar{\phi}_{\mathcal{D},n}$ (3.17), depend on the type of the sinusoidal coordinates $\eta(x)$. For three families (i), (iii) and (iv), in which the sinusoidal coordinate contains no parameter other than q, (i) $\eta(x) = x$, (iii) $\eta(x) = 1 - q^x$ and (iv) $\eta(x) = q^{-x} - 1$. For the other two families (ii) and (v), in which $\eta(x)$ contains d, (ii) $\eta(x) = x(x+d)$, (v) $\eta(x) = (q^{-x} - 1)(1 - dq^x)$, the parameter d shifts in the formulas. They are summarised in the following

Theorem 4.1 *The results of the multiple Darboux transformation are*

$$\textit{Family}\ (\text{i}), (\text{iv}): \qquad B(x; N, \boldsymbol{\lambda}) \to B(x+M; N+M, \boldsymbol{\lambda}), \tag{4.2}$$

$$
\begin{aligned}
D(x;N,\boldsymbol{\lambda}) &\to D(x+M;N+M,\boldsymbol{\lambda}), && (4.3)\\
\check{P}_n(x;N,\boldsymbol{\lambda}) &\to (\text{const})\times\check{P}_n(x+M;N+M,\boldsymbol{\lambda}), && (4.4)\\
\textit{Family}\ (\text{ii}):\quad B(x;N,d,\bar{\boldsymbol{\lambda}}) &\to B(x+M;N+M,d-M,\bar{\boldsymbol{\lambda}}), && (4.5)\\
D(x;N,d,\bar{\boldsymbol{\lambda}}) &\to D(x+M;N+M,d-M,\bar{\boldsymbol{\lambda}}), && (4.6)\\
\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}) &\to (\text{const})\times\check{P}_n(x+M;N+M,d-M,\bar{\boldsymbol{\lambda}}), && (4.7)\\
\textit{Family}\ (\text{iii}):\quad B(x;N,p) &\to B(x+M;N+M,pq^{-M}), && (4.8)\\
D(x;N,p) &\to D(x+M;N+M,pq^{-M}), && (4.9)\\
\check{P}_n(x;N,p) &\to (\text{const})\times\check{P}_n(x+M;N+M,pq^{-M}), && (4.10)\\
\textit{Family}\ (\text{v}):\quad B(x;N,d,\bar{\boldsymbol{\lambda}}) &\to B(x+M;N+M,dq^{-M},\bar{\boldsymbol{\lambda}}), && (4.11)\\
D(x;N,d,\bar{\boldsymbol{\lambda}}) &\to D(x+M;N+M,dq^{-M},\bar{\boldsymbol{\lambda}}), && (4.12)\\
\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}) &\to (\text{const})\times\check{P}_n(x+M;N+M,dq^{-M},\bar{\boldsymbol{\lambda}}), && (4.13)
\end{aligned}
$$

in which $n\in\mathcal{X}$ *and* $\bar{\boldsymbol{\lambda}}$ *stands for the parameters other than* N *and* d*. The positivity of the orthogonality measures is unchanged for the formulas* (4.2)–(4.4)*. The positivity also holds for the formulas* (4.5)–(4.13) *for an appropriate range of the parameter* p *or* d*.*

In the rest of this section we present the outline of the derivation of the formulas (4.2)–(4.13).

In order to obtain the explicit forms for $\bar{B}(x)$ (3.14), $\bar{D}(x)$ (3.15) and $\bar{\phi}_{\mathcal{D},n}$ (3.17) we need to evaluate only four formulas

$$
\begin{aligned}
&\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1}](x),\quad \Lambda(x)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}](x),\\
&\Lambda(x+M)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}](x),\quad \Lambda(x)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}\check{P}_n](x).
\end{aligned}
$$

The other components are obtained by shifting $x\to x\pm1$. As shown in **Theorem 2.1**, $\check{Q}_m(x;N,\boldsymbol{\lambda})$ is a monic degree m polynomial in $\eta(x;\boldsymbol{\lambda})$. Therefore these formulas are simplified by sweeping the Casoratian determinants successively,

$$
\begin{aligned}
\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1}](x) &= \mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1}](x), && (4.14)\\
\Lambda(x)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}](x) &= \Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x), && (4.15)\\
\Lambda(x+M)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}](x) &= \Lambda(x+M)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x), && (4.16)\\
\Lambda(x)\mathrm{W_C}[1,\check{Q}_1,\ldots,\check{Q}_{M-1},\Lambda^{-1}\check{P}_n](x) &= \Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}\check{P}_n](x). && (4.17)
\end{aligned}
$$

It should be stressed that the explicit expressions of $\check{Q}_m(x)$ listed in § 2.2.1–§ 2.2.12 are not necessary. Now that the r.h.s. of (4.14) is explicitly known as it is a Vandermonde determinant. Likewise the Casoratian in (4.15) and (4.16) are easily evaluated by expanding along the $M+1$-st column. The $(j, M+1)$ components $\Lambda(x)/\Lambda(x+j-1)$ and $\Lambda(x+M)/\Lambda(x+j-1)$ have simple expressions, for example $\Lambda(x)/\Lambda(x+j-1) = (x-N)_{j-1}/(x+1)_{j-1}$ for $\eta(x) = x$ and the other factors are closely related to the Vandermonde determinant. It is important to stress that these quantities depend on N and $\eta(x;\boldsymbol{\lambda})$ only, except for the one (4.17) containing $\check{P}_n$.

Now we list the explicit forms of the expressions of (4.14)–(4.17) for each family (i)–(v). By using the following constants,

$$c(M) \stackrel{\text{def}}{=} \prod_{k=1}^{M-1} k!, \quad c(N,M) \stackrel{\text{def}}{=} (-1)^M \,(N+1)_M\, c(M), \tag{4.18}$$

$$c_q(M) \stackrel{\text{def}}{=} q^{-\frac{1}{6}M(M-1)(2M-1)} \prod_{k=1}^{M-1} (q\,;q)_k, \tag{4.19}$$

$$c_q(N,M) \stackrel{\text{def}}{=} (-1)^M q^{-\frac{1}{2}M(M-1)} (q^{N+1}\,;q)_M\, c_q(M), \tag{4.20}$$

$$\binom{M}{j} = \frac{M!}{j!\,(M-j)!}, \quad \begin{bmatrix} M \\ j \end{bmatrix} \stackrel{\text{def}}{=} \frac{(q\,;q)_M}{(q\,;q)_j (q\,;q)_{M-j}}, \tag{4.21}$$

they are given as follows.

Family (i) :

$$\mathrm{W_C}[1, x, x^2, \ldots, x^{M-1}](x) = c(M), \tag{4.22}$$

$$\Lambda(x)\mathrm{W_C}[1, x, x^2, \ldots, x^{M-1}, \Lambda^{-1}](x) = \frac{c(N,M)}{(x+1)_M}, \tag{4.23}$$

$$\Lambda(x+M)\mathrm{W_C}[1, x, x^2, \ldots, x^{M-1}, \Lambda^{-1}](x) = \frac{c(N,M)}{(x-N)_M}, \tag{4.24}$$

$$\begin{aligned} &\Lambda(x)\mathrm{W_C}[1, x, x^2, \ldots, x^{M-1}, \Lambda^{-1}\check{P}_n](x) \\ &= \frac{(-1)^M c(M)}{(x+1)_M} \sum_{j=0}^{M} (-1)^j \binom{M}{j} (x+1+j)_{M-j} (x-N)_j \check{P}_n(x+j), \end{aligned} \tag{4.25}$$

Family (ii) :

$$\mathrm{W_C}[1, \eta, \eta^2, \ldots, \eta^{M-1}](x) = c(M) \prod_{k=1}^{M-1} (2x+k+d)_k, \tag{4.26}$$

$$\Lambda(x)\mathrm{W_C}[1, \eta, \eta^2, \ldots, \eta^{M-1}, \Lambda^{-1}](x) = \frac{c(N,M) \prod_{k=1}^{M} (2x+k+d)_k}{(x+1, x+N+1+d)_M}, \tag{4.27}$$

$$\Lambda(x+M)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c(N,M)\prod_{k=1}^{M}(2x+k+d)_k}{(x-N,x+d)_M}, \tag{4.28}$$

$$\begin{aligned}&\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}\check{P}_n](x)\\&=\frac{(-1)^M c(M)\prod_{k=1}^{M-2}(2x+2+k+d)_k}{(x+1,x+N+1+d)_M}\sum_{j=0}^{M}(-1)^j\binom{M}{j}\mathcal{T}(x,M,j,d)\\&\qquad\times(x+1+j,x+1+j+N+d)_{M-j}(x-N,x+d)_j\check{P}_n(x+j),\end{aligned} \tag{4.29}$$

$$\begin{aligned}\mathcal{T}(x,M,j,d)&\stackrel{\text{def}}{=}(2x+M+1+j+d)_{M-j-1}(2x+1+d)_{j-1}\\&\qquad\times\begin{cases}1 & : j=0,M\\(2x+2j+d) & : \text{otherwise}\end{cases},\end{aligned} \tag{4.30}$$

Family (iii) :

$$\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1}](x)=c_q(M)\,q^{\frac12 M(M-1)x}q^{\frac12 M(M-1)^2}, \tag{4.31}$$

$$\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{\frac12 M(M+1)x}q^{\frac12 M(M^2-2N-1)}}{(q^{x+1}\,;q)_M}, \tag{4.32}$$

$$\Lambda(x+M)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{\frac12 M(M+1)x}q^{\frac12 M(M^2-2N-1)}}{(q^{x-N}\,;q)_M}, \tag{4.33}$$

$$\begin{aligned}&\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}\check{P}_n](x)\\&=\frac{(-1)^M c_q(M)q^{\frac12 M(M-1)x}q^{\frac12 M^2(M-1)-MN}}{(q^{x+1}\,q)_M}\sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12 j(j+1)+M(N-j)}\\&\qquad\times(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\check{P}_n(x+j),\end{aligned} \tag{4.34}$$

Family (iv) :

$$\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1}](x)=c_q(M)\,q^{-\frac12 M(M-1)x}, \tag{4.35}$$

$$\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{-\frac12 M(M-1)x}}{(q^{x+1}\,;q)_M}, \tag{4.36}$$

$$\Lambda(x+M)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{-\frac12 M(M-1)x}}{q^{M(N+1)}(q^{x-N}\,;q)_M}, \tag{4.37}$$

$$\begin{aligned}&\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}\check{P}_n](x)\\&=\frac{(-1)^M c_q(M)\,q^{-\frac12 M(M-1)x}q^{-\frac12 M(M-1)}}{(q^{x+1}\,;q)_M}\sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12 j(j+1)+Nj}\\&\qquad\times(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\check{P}_n(x+j),\end{aligned} \tag{4.38}$$

Family (v) :

$$\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1}](x)=c_q(M)\,q^{-\frac12 M(M-1)x}\prod_{k=1}^{M-1}(dq^{2x+k}\,;q)_k, \tag{4.39}$$

$$\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{-\frac12M(M-1)x}\prod_{k=1}^M(dq^{2x+k}\,;q)_k}{(q^{x+1},dq^{x+1+N}\,;q)_M},\tag{4.40}$$

$$\Lambda(x+M)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}](x)=\frac{c_q(N,M)\,q^{-\frac12M(M-1)x}\prod_{k=1}^M(dq^{2x+k}\,;q)_k}{q^{M(N+1)}(q^{x-N},dq^{x}\,;q)_M},\tag{4.41}$$

$$\begin{aligned}&\Lambda(x)\mathrm{W_C}[1,\eta,\eta^2,\ldots,\eta^{M-1},\Lambda^{-1}\check{P}_n](x)\\&=\frac{(-1)^Mc_q(M)\,q^{-\frac12M(M-1)x}q^{-\frac12M(M-1)}\prod_{k=1}^{M-2}(dq^{2x+2+k}\,;q)_k}{(q^{x+1},dq^{x+N+1}\,;q)_M}\sum_{j=0}^M(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12j(j+1)+Nj}\\&\quad\times\mathcal{T}_q(x,M,j,d)(q^{x+1+j},dq^{x+1+j+N}\,;q)_{M-j}(q^{x-N},dq^{x}\,;q)_j\check{P}_n(x+j),\end{aligned}\tag{4.42}$$

$$\mathcal{T}_q(x,M,j,d)\stackrel{\text{def}}{=}(dq^{2x+M+1+j}\,;q)_{M-j-1}(dq^{2x+1}\,;q)_{j-1}\times\left\{\begin{array}{cl}1&:j=0,M\\(1-dq^{2x+2j})&:\text{otherwise}\end{array}\right..\tag{4.43}$$

Based on these results, the transformed functions $\bar{B}$ (3.14) and $\bar{D}$ (3.15) are calculated for each family:

$$\text{Family (i)}:\quad\bar{B}(x)=B(x+M)\,\frac{(x-N)_M}{(x+1-N)_M}=B(x+M)\,\frac{N-x}{N-x-M},\tag{4.44}$$

$$\bar{D}(x)=D(x)\,\frac{(x+1)_M}{(x)_M}=D(x)\,\frac{x+M}{x},\tag{4.45}$$

$$\text{Family (ii)}:\quad\bar{B}(x)=B(x+M)\,\frac{(N-x)(x+d)(2x+2M+d)_2}{(N-x-M)(x+d+M)(2x+M+d)_2},\tag{4.46}$$

$$\bar{D}(x)=D(x)\,\frac{(x+M)(x+N+M+d)(2x-1+d)_2}{x(x+N+d)(2x-1+M+d)_2},\tag{4.47}$$

$$\text{Family (iii)}:\quad\bar{B}(x)=B(x+M)\,\frac{q^M(q^{x-N}\,;q)_M}{(q^{x+1-N}\,;q)_M}=B(x+M)\,\frac{q^M(1-q^{x-N})}{1-q^{x+M-N}},\tag{4.48}$$

$$\bar{D}(x)=D(x)\,\frac{q^{-M}(q^{x+1}\,;q)_M}{(q^{x}\,;q)_M}=D(x)\,\frac{q^{-M}(1-q^{x+M})}{1-q^x},\tag{4.49}$$

$$\text{Family (iv)}:\quad\bar{B}(x)=B(x+M)\,\frac{(q^{x-N}\,;q)_M}{(q^{x+1-N}\,;q)_M}=B(x+M)\,\frac{1-q^{x-N}}{1-q^{x+M-N}},\tag{4.50}$$

$$\bar{D}(x)=D(x)\,\frac{(q^{x+1}\,;q)_M}{(q^{x}\,;q)_M}=D(x)\,\frac{1-q^{x+M}}{1-q^x},\tag{4.51}$$

$$\text{Family (v)}:\quad\bar{B}(x)=B(x+M)\,\frac{(1-q^{x-N})(1-dq^x)(dq^{2x+2M}\,;q)_2}{(1-q^{x+M-N})(1-dq^{x+M})(dq^{2x+M}\,;q)_2},\tag{4.52}$$

$$\bar{D}(x)=D(x)\,\frac{(1-q^{x+M})(1-dq^{x+M+N})(dq^{2x-1}\,;q)_2}{(1-q^x)(1-dq^{x+N})(dq^{2x+M-1}\,;q)_2}.\tag{4.53}$$

It is straightforward to verify **Theorem 4.1** for the transformations of $B(x)$ and $D(x)$ for the families (i)–(v) by consulting the explicit expressions of $B(x)$ and $D(x)$ of each polynomial listed in § 2.2. The transformation rules of the polynomials (4.4), (4.7), (4.10), and (4.13) are the direct consequences of those for $\bar{B}$ and $\bar{D}$. We list the explicit forms of the transformations of the polynomials of degree $n \in \mathcal{X}$ in the following:

Theorem 4.2

$$\begin{aligned}
\textit{Family (i)}: \quad & \sum_{j=0}^{M}(-1)^j\binom{M}{j}(x+1+j)_{M-j}(x-N)_j\check{P}_n(x+j;N,\boldsymbol{\lambda}) \\
& = (N+1)_M\,\check{P}_n(x+M;N+M,\boldsymbol{\lambda}), \qquad (4.54)\\
\textit{Family (ii)}: \quad & \sum_{j=0}^{M}(-1)^j\binom{M}{j}\mathcal{T}(x,M,j,d)(x+1+j,x+1+j+N+d)_{M-j} \\
& \quad \times (x-N,x+d)_j\,\check{P}_n(x+j;N,d,\bar{\boldsymbol{\lambda}}) \\
& = (N+1)_M(2x+1+d)_{2M-1}\,\check{P}_n(x+M;N+M,d-M,\bar{\boldsymbol{\lambda}}), \qquad (4.55)\\
\textit{Family (iii)}: \quad & \sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)+M(N-j)}(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\check{P}_n(x+j;N,p) \\
& = (q^{N+1}\,;q)_M q^{Mx}\,\check{P}_n(x+M;N+M,pq^{-M}), \qquad (4.56)\\
\textit{Family (iv)}: \quad & \sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)+Nj}(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\check{P}_n(x+j;N,\boldsymbol{\lambda}) \\
& = (q^{N+1}\,;q)_M\,\check{P}_n(x+M;N+M,\boldsymbol{\lambda}), \qquad (4.57)\\
\textit{Family (v)}: \quad & \sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\ j\end{bmatrix}q^{\frac{1}{2}j(j+1)+Nj}\mathcal{T}_q(x,M,j,d)(q^{x+1+j},dq^{x+1+j+N}\,;q)_{M-j} \\
& \quad \times (q^{x-N},dq^{x}\,;q)_j\,\check{P}_n(x+j;N,d,\bar{\boldsymbol{\lambda}}) \\
& = (q^{N+1}\,;q)_M(dq^{2x+1}\,;q)_{2M-1}\,\check{P}_n(x+M;N+M,dq^{-M},\bar{\boldsymbol{\lambda}}). \qquad (4.58)
\end{aligned}$$

These are polynomial relations and they are valid for $x \in \mathbb{C}$.

The summations in l.h.s. are taken from (4.25), (4.29), (4.34), (4.38) and (4.42). The factors in r.h.s. are determined by setting $n = 0$. A consistency check of the formulas can be done by setting $x = -M$. Again, it is straightforward to verify these transformation formulas for the families (i)–(v) by consulting the explicit expressions of the polynomials. Since the size parameter of the matrix is now $N + M$, there are additional members of the orthogonal polynomials. They are simply $\{\check{P}_{N+1+m}(x + M, N + M, \boldsymbol{\lambda})\}$ $(0 \le m \le M)$, for family (i), etc. This concludes the proof of **Theorem 4.1**.

It is interesting and instructive to scrutinise **Theorem 4.2** from a different perspective. The formulas (4.54)–(4.58) contain $\sum_{j=0}^{M}$, $\binom{M}{j}$ or $\left[\begin{smallmatrix}M\\ j\end{smallmatrix}\right]$ and subscripts $M - j$ and j, showing a semblance to binomial expansions. Let us introduce operators $\tilde{\mathcal{F}}$ to realise one step transformation for each family. Rewriting **Theorem 4.2** for $M=1$ explicitly gives

$$\text{(i)}: \ \tilde{\mathcal{F}}(x,N) \stackrel{\text{def}}{=} \frac{1}{N+1}\big(x+1+(N-x)e^{\partial}\big), \tag{4.59}$$
$$\tilde{\mathcal{F}}(x,N)\check{P}_n(x;N,\boldsymbol{\lambda}) = \check{P}_n(x+1;N+1,\boldsymbol{\lambda}),$$
$$\text{(ii)}: \ \tilde{\mathcal{F}}(x,N,d) \stackrel{\text{def}}{=} \frac{1}{(N+1)(2x+1+d)}\big((x+1)(x+1+N+d)+(N-x)(x+d)e^{\partial}\big), \tag{4.60}$$
$$\tilde{\mathcal{F}}(x,N,d)\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}) = \check{P}_n(x+1;N+1,d-1,\bar{\boldsymbol{\lambda}}),$$
$$\text{(iii)}: \ \tilde{\mathcal{F}}(x,N) \stackrel{\text{def}}{=} \frac{q^{N-x}}{1-q^{N+1}}\big(1-q^{x+1}+(q^{x-N}-1)e^{\partial}\big), \tag{4.61}$$
$$\tilde{\mathcal{F}}(x,N)\check{P}_n(x;N,p) = \check{P}_n(x+1;N+1,pq^{-1}),$$
$$\text{(iv)}: \ \tilde{\mathcal{F}}(x,N) \stackrel{\text{def}}{=} \frac{1}{1-q^{N+1}}\big(1-q^{x+1}+q^{N+1}(q^{x-N}-1)e^{\partial}\big), \tag{4.62}$$
$$\tilde{\mathcal{F}}(x,N)\check{P}_n(x;N,\boldsymbol{\lambda}) = \check{P}_n(x+1;N+1,\boldsymbol{\lambda}),$$
$$\text{(v)}: \ \tilde{\mathcal{F}}(x,N,d) \stackrel{\text{def}}{=} \frac{1}{(1-q^{N+1})(1-dq^{2x+1})} \times \big((1-q^{x+1})(1-dq^{x+1+N})+q^{N+1}(q^{x-N}-1)(1-dq^{x})e^{\partial}\big), \tag{4.63}$$
$$\tilde{\mathcal{F}}(x,N,d)\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}) = \check{P}_n(x+1;N+1,dq^{-1},\bar{\boldsymbol{\lambda}}).$$

Here e^{∂} is the finite shift operator (3.8). Let us tentatively call $\tilde{\mathcal{F}}$ a forward x-shift operator. Each $\tilde{\mathcal{F}}$ has two terms, one is an ordinary function and the other is a function times e^{∂}. Since the polynomial is mapped from $\check{P}_n(x;N,\boldsymbol{\lambda})$ to $\check{P}_n(x+1;N+1,\boldsymbol{\lambda})$, for family (i), the next steps are obviously

$$\text{(i)}: \ \tilde{\mathcal{F}}(x+1,N+1)\tilde{\mathcal{F}}(x,N)\check{P}_n(x;N,\boldsymbol{\lambda}) = \check{P}_n(x+2;N+2,\boldsymbol{\lambda}),$$
$$\tilde{\mathcal{F}}(x+2,N+2)\tilde{\mathcal{F}}(x+1,N+1)\tilde{\mathcal{F}}(x,N)\check{P}_n(x;N,\boldsymbol{\lambda}) = \check{P}_n(x+3;N+3,\boldsymbol{\lambda}),$$
$$\vdots,$$

leading to

$$\text{(i), (iv)}: \ \prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k)\cdot\check{P}_n(x;N,\boldsymbol{\lambda}) = \check{P}_n(x+M;N+M,\boldsymbol{\lambda}), \tag{4.64}$$
$$\text{(ii)}: \ \prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k,d-k)\cdot\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}) = \check{P}_n(x+M;N+M,d-M,\bar{\boldsymbol{\lambda}}), \tag{4.65}$$

$$\text{(iii)}:\qquad \prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k)\cdot\check{P}_n(x;N,p)=\check{P}_n(x+M;N+M,pq^{-M}),\qquad(4.66)$$

$$\text{(v)}:\quad \prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k,dq^{-k})\cdot\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}})=\check{P}_n(x+M;N+M,dq^{-M},\bar{\boldsymbol{\lambda}}).\quad(4.67)$$

Here we use the convention to express the ordered product

$$\prod_{j=1}^{\overleftarrow{n}}a_j\stackrel{\text{def}}{=}a_n\cdots a_2a_1.\qquad(4.68)$$

Thus we arrive at the following

Theorem 4.3 *The formulas representing the special cases of the multi-indexed polynomials* (4.54)–(4.58) *are factorised,*

$$\text{(i)}:\quad \prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k)=\frac{1}{(N+1)_M}\sum_{j=0}^{M}(-1)^j\binom{M}{j}(x+1+j)_{M-j}(x-N)_j\,e^{j\partial},\quad(4.69)$$

$$\begin{aligned}\text{(ii)}:\quad &\prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k,d-k)\\&=\frac{1}{(N+1)_M(2x+1+d)_{2M-1}}\sum_{j=0}^{M}(-1)^j\binom{M}{j}\mathcal{T}(x,M,j,d)\\&\qquad\times(x+1+j,x+1+j+N+d)_{M-j}(x-N,x+d)_j\,e^{j\partial},\end{aligned}\qquad(4.70)$$

$$\begin{aligned}\text{(iii)}:\quad &\prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k)\\&=\frac{q^{-Mx}}{(q^{N+1}\,;q)_M}\sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12 j(j+1)+M(N-j)}(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\,e^{j\partial},\end{aligned}\qquad(4.71)$$

$$\begin{aligned}\text{(iv)}:\quad &\prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k)\\&=\frac{1}{(q^{N+1}\,;q)_M}\sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12 j(j+1)+Nj}(q^{x+1+j}\,;q)_{M-j}(q^{x-N}\,;q)_j\,e^{j\partial},\end{aligned}\qquad(4.72)$$

$$\begin{aligned}\text{(v)}:\quad &\prod_{k=0}^{\overleftarrow{M-1}}\tilde{\mathcal{F}}(x+k,N+k,dq^{-k})\\&=\frac{1}{(q^{N+1}\,;q)_M(dq^{2x+1}\,;q)_{2M-1}}\sum_{j=0}^{M}(-1)^j\begin{bmatrix}M\\j\end{bmatrix}q^{\frac12 j(j+1)+Nj}\mathcal{T}_q(x,M,j,d)\end{aligned}$$

$$\times\,(q^{x+1+j},dq^{x+1+j+N}\,;q)_{M-j}(q^{x-N},dq^{x}\,;q)_j\,e^{j\partial}. \tag{4.73}$$

By expanding the l.h.s and collecting terms containing $e^{j\partial}$ gives r.h.s.

These formulas can be demonstrated by straightforward induction. For example, for (i),

$$\begin{aligned}
&\overleftarrow{\prod_{k=0}^{M}}\tilde{\mathcal{F}}(x+k,N+k)=\tilde{\mathcal{F}}(x+M,N+M)\overleftarrow{\prod_{k=0}^{M-1}}\tilde{\mathcal{F}}(x+k,N+k)\\
&=\frac{1}{(N+1)_{M+1}}\sum_{j=0}^{M}\Bigl((-1)^j\binom{M}{j}(x+1+j)_{M+1-j}(x-N)_j\,e^{j\partial}\\
&\qquad\qquad -(-1)^j\binom{M}{j}(x+2+j)_{M-j}(x-N)_{j+1}\,e^{(j+1)\partial}\Bigr)\\
&=\frac{1}{(N+1)_{M+1}}\sum_{j=0}^{M+1}(-1)^j\binom{M+1}{j}(x+1+j)_{M+1-j}(x-N)_j\,e^{j\partial}.
\end{aligned}$$

Here we used Pascal' triangle relation

$$\binom{M}{j}+\binom{M}{j-1}=\binom{M+1}{j}. \tag{4.74}$$

The q-analogues of the above relation

$$\begin{bmatrix}M\\ j\end{bmatrix}q^j+\begin{bmatrix}M\\ j-1\end{bmatrix}=\begin{bmatrix}M+1\\ j\end{bmatrix},\quad \begin{bmatrix}M\\ j\end{bmatrix}+\begin{bmatrix}M\\ j-1\end{bmatrix}q^{M+1-j}=\begin{bmatrix}M+1\\ j\end{bmatrix} \tag{4.75}$$

are used in the inductions for (iii)–(v) and the relations

$$j\binom{M}{j}-(M+1-j)\binom{M}{j-1}=0,\quad (1-q^j)\begin{bmatrix}M\\ j\end{bmatrix}-(1-q^{M+1-j})\begin{bmatrix}M\\ j-1\end{bmatrix}=0 \tag{4.76}$$

are used for (ii) and (v).

It is well known that the difference operator $\widetilde{\mathcal{H}}$ (2.5), (2.6) is factorised into the forward $\mathcal{F}$ and backward $\mathcal{B}$ shift operators [5, 7],

$$\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})=B(x;N,\boldsymbol{\lambda})(1-e^{\partial})+D(x;N,\boldsymbol{\lambda})(1-e^{-\partial})=\mathcal{B}(N,\boldsymbol{\lambda})\mathcal{F}(N,\boldsymbol{\lambda}), \tag{4.77}$$

which is an expression of the shape invariance. For example, for the Racah,

$$\begin{aligned}
\text{Racah}:\quad &\mathcal{F}(N,\boldsymbol{\lambda})=\frac{Nbc}{2x+d+1}(1-e^{\partial}),\\
&\mathcal{B}(N,\boldsymbol{\lambda})=\frac{1}{Nbc}\bigl(B(x;N,\boldsymbol{\lambda})-D(x;N,\boldsymbol{\lambda})e^{-\partial}\bigr)(2x+d+1),
\end{aligned} \tag{4.78}$$

$$\mathcal{F}(N,\boldsymbol{\lambda})\check{P}_n(x;N,b,c,d)=\mathcal{E}(n;\boldsymbol{\lambda})\check{P}_{n-1}(x;N-1,b+1,c+1,d+1),$$
$$\mathcal{B}(N,\boldsymbol{\lambda})\check{P}_{n-1}(x;N-1,b+1,c+1,d+1)=\check{P}_n(x;N,b,c,d). \tag{4.79}$$

These $\mathcal{F}$ and $\mathcal{B}$ change the degree n but keep x unchanged, in contrast with $\tilde{\mathcal{F}}$, which changes x and keeps the degree n unchanged. In a similar fashion as above (4.77), reflecting the new type of shape invariance, $\widetilde{\mathcal{H}}-\mathcal{E}(N+1)$ is factorised by the forward and backward x-shift operators,

$$\widetilde{\mathcal{H}}(N,\boldsymbol{\lambda})-\mathcal{E}(N+1;\boldsymbol{\lambda})=-\tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\tilde{\mathcal{F}}(x,N,\boldsymbol{\lambda}), \tag{4.80}$$

as the zero norm solutions start at degree $N+1$. The explicit forms of the operator $\tilde{\mathcal{B}}$ are

$$\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(N+1)\big(p+(1-p)e^{-\partial}\big), \tag{4.81}$$
$$\text{H}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(N+1)\big(x+a+(b+N-x)e^{-\partial}\big), \tag{4.82}$$
$$\text{R}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(N+1)\Big(\frac{(x+b)(x+c)}{2x+d}+\frac{(b-d-x)(x+d-c)}{2x+d}e^{-\partial}\Big), \tag{4.83}$$
$$\text{dH}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(N+1)\Big(\frac{x+a}{2x-1+a+b}+\frac{x+b-1}{2x-1+a+b}e^{-\partial}\Big), \tag{4.84}$$
$$\text{dq}q\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(1-q^{N+1})\big(p^{-1}q^{-x-N-1}+q^{-N-1}(1-p^{-1}q^{-x})e^{-\partial}\big), \tag{4.85}$$
$$q\text{H}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(1-q^{N+1})\big(q^{-N-1}(1-aq^{x})+aq^{-1}(q^{x-N}-b)e^{-\partial}\big), \tag{4.86}$$
$$q\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(1-q^{N+1})\big(q^{-N-1}+p\,e^{-\partial}\big), \tag{4.87}$$
$$\text{q}q\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(1-q^{N+1})\big(p^{-1}q^{x-N-1}+(1-p^{-1}q^{x-N-1})e^{-\partial}\big), \tag{4.88}$$
$$\text{a}q\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(1-q^{N+1})\big(q^{-N-1}(1-pq^{x+1})+pq^{x-N}e^{-\partial}\big), \tag{4.89}$$
$$q\text{R}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(1-q^{N+1})\Big(\frac{q^{-N-1}(1-bq^{x})(1-cq^{x})}{1-dq^{2x}}+\tilde{d}\,\frac{(b^{-1}dq^{x}-1)(1-c^{-1}dq^{x})}{1-dq^{2x}}e^{-\partial}\Big), \tag{4.90}$$
$$\text{d}q\text{H}:\ \ \tilde{\mathcal{B}}(x,N,\boldsymbol{\lambda})\stackrel{\text{def}}{=}(1-q^{N+1})\Big(\frac{q^{-N-1}(1-aq^{x})}{1-abq^{2x-1}}+\frac{aq^{x-N-1}(1-bq^{x-1})}{1-abq^{2x-1}}e^{-\partial}\Big), \tag{4.91}$$
$$\text{d}q\text{K}:\ \ \tilde{\mathcal{B}}(x,N,p)\stackrel{\text{def}}{=}(1-q^{N+1})\Big(\frac{q^{-N-1}}{1+pq^{2x}}+\frac{pq^{2x-N-1}}{1+pq^{2x}}e^{-\partial}\Big). \tag{4.92}$$

The action of these operators on the polynomials is as follows:

$$\text{(i), (iv)}:\ \ \tilde{\mathcal{B}}(x,N)\check{P}_n(x+1;N+1,\boldsymbol{\lambda})=\big(\mathcal{E}(N+1;\boldsymbol{\lambda})-\mathcal{E}(n;\boldsymbol{\lambda})\big)\check{P}_n(x;N,\boldsymbol{\lambda}), \tag{4.93}$$
$$\text{(ii)}:\ \tilde{\mathcal{B}}(x,N,d,\bar{\boldsymbol{\lambda}})\check{P}_n(x+1;N+1,d-1,\bar{\boldsymbol{\lambda}})=\big(\mathcal{E}(N+1;\boldsymbol{\lambda})-\mathcal{E}(n;\boldsymbol{\lambda})\big)\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}), \tag{4.94}$$
$$\text{(iii)}:\ \ \tilde{\mathcal{B}}(x,N,p)\check{P}_n(x+1;N+1,pq^{-1})=\big(\mathcal{E}(N+1)-\mathcal{E}(n)\big)\check{P}_n(x;N,p), \tag{4.95}$$
$$\text{(v)}:\ \tilde{\mathcal{B}}(x,N,d,\bar{\boldsymbol{\lambda}})\check{P}_n(x+1;N+1,dq^{-1},\bar{\boldsymbol{\lambda}})=\big(\mathcal{E}(N+1;\boldsymbol{\lambda})-\mathcal{E}(n;\boldsymbol{\lambda})\big)\check{P}_n(x;N,d,\bar{\boldsymbol{\lambda}}). \tag{4.96}$$

5 Summary and Comments

The reported "Diophantine" and factorisation properties of the Jacobi, Laguerre etc [1], Wilson and Askey-Wilson polynomials etc [2] are very interesting but rather puzzling, as these properties require the parameter ranges in which the orthogonality does not hold. In this paper we show, after the seminal work of [9], that these properties are shared by all the finite polynomials in the Askey scheme in the conventional orthogonality parameter ranges. All the *monic* higher degree polynomials $\{\check{P}^{\text{monic}}_{N+1+m}(x)\}$ $(m \in \mathbb{Z}_{\geq 0})$ in this group show the "Diophantine" and factorisation properties **Theorem 2.1** (2.50), (2.51). Here N is the maximal degree of the corresponding non-monic orthogonal polynomials. A simple and intuitive explanation is that these higher degree monic polynomials are the zero-norm solution of certain tri-diagonal real symmetric matrix $\mathcal{H}$ (2.10). The explicit expressions of the "Diophantine" and factorisation for the twelve polynomials are presented in § 2.3. These higher degree monic polynomials can be used as seed solutions for generating new types multi-indexed polynomials based on the twelve finite orthogonal polynomials. In order to obtain genuine multi-indexed orthogonal polynomials, correct choices of the seed solutions and the parameter ranges are essential. A simplest choice of M contiguous lowest degree $\{\check{P}^{\text{monic}}_{N+1+m}(x)\}$ $(m = 0, 1, \ldots, M-1)$ generate proper multi-indexed orthogonal polynomials as shown in § 4 for each of the twelve polynomials listed in § 2.2.

In paper I [1] Calogero and coauthors reported "Diophantine" and factorisation properties of some finite polynomials, the Racah, Hahn, dual Hahn and Krawtchouk. In these cases, the phenomena occur only at the particular degree with which the parameters are tuned. In 'Remarks' of paper II [2], Chen and Ismail mentioned a possible explanation of the "Diophantine" and factorisation properties of the Wilson polynomial as the occurrence of certain discrete masses in the orthogonality measure when some of the parameters are negative. It is a good challenge to provide a perspective in which their explanation and ours could be unified, since the Wilson and Askey-Wilson polynomials are the most general ones in the Askey scheme.

A few remarks on the results of Miki-Tsujimoto-Vinet paper [9]. They reported four types of single-indexed exceptional Krawtchouk polynomials. Among them, the first one, using the polynomial itself as the seed solution, is the well-known one called Krein-Adler polynomials, as briefly mentioned in § 2.4. The second type is the $M = 1$ case of the multi-indexed polynomials discussed in § 3. The remaining two types exist only for the Krawtchouk polynomials. They are obtained from the first and second types by using the mirror symmetry (2.15) of the Krawtchouk. Another polynomial in the same $\eta(x) = x$ family, the Hahn polynomial, has similar mirror symmetry (2.18).

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