DPSU-22-3

Another Type of Forward and Backward Shift Relations for Orthogonal Polynomials in the Askey Scheme

Satoru Odake

Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan

Abstract

The forward and backward shift relations are basic properties of the (basic) hypergeometric orthogonal polynomials in the Askey scheme (Jacobi, Askey-Wilson, q-Racah, big q-Jacobi etc.) and they are related to the factorization of the differential or difference operators. Based on other factorizations, we obtain another type of forward and backward shift relations. Essentially, these new shift relations shift only the parameters.

1 Introduction

The (basic) hypergeometric orthogonal polynomials in the Askey scheme satisfy second order differential or difference equations and the forward and backward shift relations are their basic properties [1, 2]. The orthogonal polynomials in the Askey scheme provide us with exactly solvable quantum mechanical models. Conversely, we can use the quantum mechanical formulation as a tool to investigate orthogonal polynomials [3]. For example, the forward and backward shift relations are a consequence of the shape invariance, and the multi-indexed orthogonal polynomials are found by using the quantum mechanical formulation. The Schrödinger equation is a second order differential equation for ordinary quantum mechanics (oQM) and a second order difference equation for discrete quantum mechanics (dQM). There are two types of dQM, dQM with pure imaginary shifts (idQM) and dQM with real shifts (rdQM) [3]. The coordinate x for oQM and idQM is continuous and that for rdQM is discrete.

The forward and backward shift relations are related to the factorization of the Hamiltonian. Recently another factorization of the Hamiltonian was found in a study of the state-adding Darboux transformations for the finite rdQM systems [4]. It gives another forward and backward shift relations for the orthogonal polynomials appearing in the finite rdQM systems (q-Racah etc.), which were called the forward and backward x-shift relations [4]. In this paper, we investigate whether such new factorization and forward and backward shift relations exist for other orthogonal polynomials. In addition to the finite rdQM systems (q-Racah etc.), we examine the oQM systems (Jacobi etc.), the idQM systems (Askey-Wilson etc.), the semi-infinite rdQM systems (q-Meixner etc.) and the rdQM systems with the Jackson integral type measure (big q-Jacobi etc.). We call the last category rdQMJ. The quantum mechanical formulation of the rdQMJ systems needs two component formalism [5]. We consider all the polynomials in chapter 9 and 14 of [2] and the dual quantum q-Krawtchouk polynomial.

This paper is organized as follows. The orthogonal polynomials in the Askey scheme and their second order differential or difference equations are recalled in section 2. The forward and backward shift relations are reviewed in section 3. Section 4 is the main part of this paper and new factorization and new forward and backward shift relations are presented. Section 5 is for a summary and comments. In Appendix A the data for § 4 are given.

2 Orthogonal Polynomials in the Askey Scheme

In this section we fix the notation and recall the second order differential or difference equations for the orthogonal polynomials in the Askey scheme [1, 2].

In our quantum mechanical formulation [3], the orthogonal polynomials in the Askey scheme are expressed as

$$\check{P}_n(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_n\bigl(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}\bigr)\ :\ \text{a polynomial of degree } n \text{ in } \eta(x;\boldsymbol{\lambda}) \tag{2.1}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\check{P}_{-1}(x;\boldsymbol{\lambda}) \stackrel{\text{def}}{=} P_{-1}(\eta(x;\boldsymbol{\lambda});\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 0$. Here x is a coordinate of quantum mechanical system and $\eta(x)$ is a sinusoidal coordinate [6], and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ are parameters, whose dependence is expressed as $f = f(\boldsymbol{\lambda})$ and $f(x) = f(x;\boldsymbol{\lambda})$. The parameter q is $0 < q < 1$ and $q^{\boldsymbol{\lambda}}$ stands for $q^{(\lambda_1,\lambda_2,\ldots)} = (q^{\lambda_1}, q^{\lambda_2}, \ldots)$, and we omit writing q-dependence.

We consider the following orthogonal polynomials, all the polynomials in chapter 9 and 14 of [2] and the dual quantum q-Krawtchouk polynomial: Hermite (He), Laguerre (L), Jacobi (J), Bessel (B), pseudo Jacobi (pJ), continuous Hahn (cH), Meixner-Pollaczek (MP), Wilson (W), continuous dual Hahn (cdH), Askey-Wilson (AW), continuous dual q-Hahn (cd$q$H), Al-Salam-Chihara (ASC), continuous big q-Hermite (cb$q$He), continuous q-Hermite (cqHe), continuous q-Jacobi (cqJ), continuous q-Laguerre (cqL), continuous q-Hahn (cqH), q-Meixner-Pollaczek (qMP), Hahn (H), Krawtchouk (K), Racah (R), dual Hahn (dH), dual quantum q-Krawtchouk (dq$q$K) (which is not treated in [2]), q-Hahn (qH), q-Krawtchouk (qK), quantum q-Krawtchouk (qqK), affine q-Krawtchouk (aqK), q-Racah (qR), dual q-Hahn (dqH), dual q-Krawtchouk (dqK), Meixner (M), Charlier (C), little q-Jacobi (lqJ), little q-Laguerre/Wall (lqL), q-Bessel (qB) (=alternative q-Charlier), q-Meixner (qM), Al-Salam-Carlitz II (ASCII), q-Charlier (qC), big q-Jacobi (bqJ), big q-Laguerre (bqL), Al-Salam-Carlitz I (ASCI), discrete q-Hermite I (dqHeI), discrete q-Hermite II (dqHeII), q-Laguerre (qL) and Stieltjes-Wigert (SW). Explicit expressions of various quantities ($\check{P}_n(x)$, $P_n(\eta)$, $\eta(x)$, $\varphi(x)$, $\mathcal{E}_n$, $\boldsymbol{\lambda}$, $\boldsymbol{\delta}$, κ, f_n, b_n, $c_1(\eta)$, $c_2(\eta)$, $c_{\mathcal{F}}$, $w(x)$, $V(x)$, $B(x)$, $D(x)$, $B^{\text{J}}(x)$, $D^{\text{J}}(x)$, f_n^{J}, b_n^{J}) are given in Appendix A of arXiv:2301.00678v1. These polynomials appear in quantum mechanical systems as follows:

$$\text{oQM}: \ \text{He}, \text{L}, \text{J}, \text{B}, \text{pJ}, \tag{2.2}$$

$$\text{idQM}: \ \text{cH}, \text{MP}, \text{W}, \text{cdH}, \text{AW}, \text{cd}q\text{H}, \text{ASC}, \text{cb}q\text{He}, \ \text{c}q\text{He}, \text{c}q\text{J}, \text{c}q\text{L}, \text{c}q\text{H}, q\text{MP}, \tag{2.3}$$

$$\text{rdQM (finite)}: \ \text{H}, \text{K}, \text{R}, \text{dH}, \text{dq}q\text{K}, q\text{H}, q\text{K}, \ \text{q}q\text{K}, \text{a}q\text{K}, q\text{R}, \text{d}q\text{H}, \text{d}q\text{K}, \tag{2.4}$$

$$\text{rdQM (semi-infinite)}: \ \text{M}, \text{C}, \text{l}q\text{J}, \text{l}q\text{L}, q\text{B}, \ q\text{M}, \text{ASCII}, q\text{C}, \tag{2.5}$$

$$\text{rdQMJ}: \ \text{b}q\text{J}, \text{b}q\text{L}, \text{ASCI}, \text{d}q\text{HeI}, \text{d}q\text{HeII}, \ q\text{L}, \text{SW}. \tag{2.6}$$

We comment that the oQM systems described by the Bessel and pseudo Jacobi polynomials are the Morse potential and the hyperbolic symmetric top II, respectively. We also comment on an infinite sum orthogonality relations for the Stieltjes-Wigert polynomial (parameter: $c>0$),

$$\text{SW}: \ \sum_{x=-\infty}^{\infty} c^x q^{\frac12 x(x+1)} P_n(cq^x) P_m(cq^x) = \delta_{nm}\, q^{-n}(q\,;q)_n(q,-cq,-c^{-1}\,;q)_\infty \quad (n,m\in\mathbb{Z}_{\geq 0}), \tag{2.7}$$

which are obtained from those for q-Laguerre polynomial by taking an appropriate limit. This (2.7) is not given in [2].

The Schrödinger equations of oQM and dQM systems are second order differential and difference equations, respectively. By the similarity transformation in terms of the ground state wavefunction, the similarity transformed Hamiltonian $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ is a second order differential or difference operator acting on the eigenpolynomials $\check{P}_n(x;\boldsymbol{\lambda})$ [3],

$$\text{oQM}:\ \ \widetilde{\mathcal{H}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\frac{d^2}{dx^2}-2\frac{dw(x;\boldsymbol{\lambda})}{dx}\frac{d}{dx}\ \Bigl(=-4c_2(\eta)\frac{d^2}{d\eta^2}-4c_1(\eta;\boldsymbol{\lambda})\frac{d}{d\eta}\Bigr),\tag{2.8}$$

$$\text{idQM}:\ \ \widetilde{\mathcal{H}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}V(x;\boldsymbol{\lambda})(e^{\gamma p}-1)+V^*(x;\boldsymbol{\lambda})(e^{-\gamma p}-1),\tag{2.9}$$

$$\text{rdQM}:\ \ \widetilde{\mathcal{H}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}B(x;\boldsymbol{\lambda})(1-e^{\partial})+D(x;\boldsymbol{\lambda})(1-e^{-\partial}).\tag{2.10}$$

For oQM, the coordinate x is a continuous variable and the Hamiltonian $\mathcal{H}(\boldsymbol{\lambda})$ is

$$\mathcal{H}(\boldsymbol{\lambda})=-\frac{d^2}{dx^2}+U(x;\boldsymbol{\lambda}),\quad U(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}\Bigl(\frac{dw(x;\boldsymbol{\lambda})}{dx}\Bigr)^2+\frac{d^2w(x;\boldsymbol{\lambda})}{dx^2}.\tag{2.11}$$

While the orthogonality relations of $\check{P}_n(x)$ for B and pJ cases hold only for a finite number of n, we consider all $n\in\mathbb{Z}_{\geq 0}$, because we consider only differential equations (or relations) in this paper. For idQM, the coordinate x is a continuous variable and the momentum p is $p=-i\frac{d}{dx}$, and γ is a real constant ($\gamma=1$ for non q-polynomial, $\gamma=\log q$ for q-polynomial). The operator $e^{\alpha p}$ (α: constant) is a shift operator, $e^{\alpha p}f(x)=f(x-i\alpha)$. The $*$-operation on an analytic function $f(x)=\sum_n a_nx^n$ ($a_n\in\mathbb{C}$) is defined by $f^*(x)=\sum_n a_n^*x^n$, in which a_n^* is the complex conjugation of a_n. For rdQM, the Schrödinger equation is a matrix eigenvalue problem. The similarity transformed Hamiltonian $\widetilde{\mathcal{H}}=(\widetilde{\mathcal{H}}_{x,y})$ is a matrix labeled by the coordinate x, which takes discrete values in $\{0,1,\ldots,N\}$ or $\mathbb{Z}_{\geq 0}$. In this paper, however, we treat x as a continuous variable $x\in\mathbb{R}$, because we only deal with difference equations (or relations). The operators $e^{\pm\partial}$ are shift operators $e^{\pm\partial}=e^{\pm\frac{d}{dx}}$, $e^{\pm\partial}f(x)=f(x\pm 1)$. We consider $\check{P}_n(x)$ with all $n\in\mathbb{Z}_{\geq 0}$ even for finite systems. We remark that the polynomials $\check{P}_n(x)$ in finite rdQM (2.4), whose orthogonality holds for $n=0,1,\ldots,N$, are ill-defined for $n>N$ due to the normalization condition $\check{P}_n(0)=P_n(0)=1$. So we should replace $\check{P}_n(x)$ ($n>N$) in finite rdQM with the monic version $\check{P}_n^{\text{monic}}(x;\boldsymbol{\lambda})\stackrel{\text{def}}{=}c_n(\boldsymbol{\lambda})^{-1}\check{P}_n(x;\boldsymbol{\lambda})$ ($c_n(\boldsymbol{\lambda})$: the coefficient of the highest degree term) in Theorem 1, 2.1 and 3.3 (with the replacements $f_n(\boldsymbol{\lambda})\to f_n^{\text{monic}}(\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda})^{-1}c_{n-1}(\boldsymbol{\lambda}+\boldsymbol{\delta})$, etc.).

For rdQMJ (the rdQM system with Jackson integral type measure such as the big q-Jacobi polynomial), its quantum mechanical formulation needs two component formalism with two sinusoidal coordinates $\eta^{(\pm)}(x;\boldsymbol{\lambda})$ [5]. Since only difference equations (or relations) are considered in this paper, we use η only (we do not use x) and treat η as a continuous variable $\eta\in\mathbb{R}$. The similarity transformed Hamiltonian $\widetilde{\mathcal{H}}^{\text{J}}(\boldsymbol{\lambda})$ is a second order difference operator acting on the eigenpolynomials $P_n(\eta;\boldsymbol{\lambda})$ [5],

$$\text{rdQMJ}:\ \ \widetilde{\mathcal{H}}^{\text{J}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}B^{\text{J}}(\eta;\boldsymbol{\lambda})(1-q^{\eta\frac{d}{d\eta}})+D^{\text{J}}(\eta;\boldsymbol{\lambda})(1-q^{-\eta\frac{d}{d\eta}}),\tag{2.12}$$

where the operators $q^{\pm\eta\frac{d}{d\eta}}$ are q-shift operators, $q^{\pm\eta\frac{d}{d\eta}}f(\eta)=f(q^{\pm 1}\eta)$.

The orthogonal polynomials in the Askey scheme studied in this paper have the following property.

Theorem 1 *[1, 2] The polynomials in* (2.2)–(2.6) *satisfy the second order differential or difference equations for* $n\in\mathbb{Z}_{\geq 0}$*,*

$$\text{oQM, idQM, rdQM}: \ \widetilde{\mathcal{H}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}), \tag{2.13}$$

$$\text{rdQMJ}: \widetilde{\mathcal{H}}^{\text{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}). \tag{2.14}$$

We remark that the constant terms of $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}^{\text{J}}$ are chosen such that $\mathcal{E}_0=0$. For idQM, the relation (2.13) is invariant under the $*$-operation.

3 Forward and Backward Shift Relations

The similarity transformed Hamiltonians $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ (2.8)–(2.10) are factorized as

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})=\mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}), \tag{3.1}$$

where the forward and backward shift operators, $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$, are defined by [7, 8],

$$\text{oQM}: \ \mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}c_{\mathcal{F}}\Bigl(\frac{d\eta(x)}{dx}\Bigr)^{-1}\frac{d}{dx}\ \Bigl(=c_{\mathcal{F}}\frac{d}{d\eta}\Bigr), \tag{3.2}$$

$$\mathcal{B}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-c_{\mathcal{F}}^{-1}\Bigl(\frac{d\eta(x)}{dx}\frac{d}{dx}+4c_1\bigl(\eta(x);\boldsymbol{\lambda}\bigr)\Bigr)\ \Bigl(=-4c_{\mathcal{F}}^{-1}\Bigl(c_2(\eta)\frac{d}{d\eta}+c_1(\eta;\boldsymbol{\lambda})\Bigr)\Bigr), \tag{3.3}$$

$$\text{idQM}: \ \mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}i\varphi(x;\boldsymbol{\lambda})^{-1}(e^{\frac{\gamma}{2}p}-e^{-\frac{\gamma}{2}p}), \tag{3.4}$$

$$\mathcal{B}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-i\bigl(V(x;\boldsymbol{\lambda})e^{\frac{\gamma}{2}p}-V^*(x;\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}\bigr)\varphi(x;\boldsymbol{\lambda}), \tag{3.5}$$

$$\text{rdQM}: \ \mathcal{F}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}B(0;\boldsymbol{\lambda})\varphi(x;\boldsymbol{\lambda})^{-1}(1-e^{\partial}), \tag{3.6}$$

$$\mathcal{B}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{1}{B(0;\boldsymbol{\lambda})}\bigl(B(x;\boldsymbol{\lambda})-D(x;\boldsymbol{\lambda})e^{-\partial}\bigr)\varphi(x;\boldsymbol{\lambda}). \tag{3.7}$$

Since $w(x)$, $B(x)$, $D(x)$ and $V(x)$ satisfy

$$\Bigl(\frac{dw(x;\boldsymbol{\lambda})}{dx}\Bigr)^2-\frac{d^2w(x;\boldsymbol{\lambda})}{dx^2}=\Bigl(\frac{dw(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{dx}\Bigr)^2+\frac{d^2w(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}{dx^2}+\mathcal{E}_1(\boldsymbol{\lambda}), \tag{3.8}$$

$$\frac{V(x-i\frac{\gamma}{2};\boldsymbol{\lambda})}{V(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}=\kappa\frac{\varphi(x;\boldsymbol{\lambda})}{\varphi(x-i\gamma;\boldsymbol{\lambda})}, \tag{3.9}$$

$$V(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})+V^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})=\kappa\bigl(V(x;\boldsymbol{\lambda}+\boldsymbol{\delta})+V^*(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\bigr)-\mathcal{E}_1(\boldsymbol{\lambda}), \tag{3.10}$$

$$\frac{B(x+1;\boldsymbol{\lambda})}{B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}=\kappa\frac{\varphi(x;\boldsymbol{\lambda})}{\varphi(x+1;\boldsymbol{\lambda})},\quad \frac{D(x;\boldsymbol{\lambda})}{D(x;\boldsymbol{\lambda}+\boldsymbol{\delta})}=\kappa\frac{\varphi(x;\boldsymbol{\lambda})}{\varphi(x-1;\boldsymbol{\lambda})},\tag{3.11}$$

$$B(x;\boldsymbol{\lambda})+D(x+1;\boldsymbol{\lambda})=\kappa\big(B(x;\boldsymbol{\lambda}+\boldsymbol{\delta})+D(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)+\mathcal{E}_1(\boldsymbol{\lambda}),\tag{3.12}$$

we obtain

$$\mathcal{F}(\boldsymbol{\lambda})\mathcal{B}(\boldsymbol{\lambda})=\kappa\mathcal{B}(\boldsymbol{\lambda}+\boldsymbol{\delta})\mathcal{F}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}),\tag{3.13}$$

which is the (similarity transformed) shape invariance condition. The constants f_n and b_n satisfy

$$\mathcal{E}_n(\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq 0}),\tag{3.14}$$

and the energy eigenvalues $\mathcal{E}_n$ satisfy

$$\mathcal{E}_{n+1}(\boldsymbol{\lambda})=\kappa\mathcal{E}_n(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq 0}).\tag{3.15}$$

Note that we have $f_n(\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})$ and $b_n(\boldsymbol{\lambda})=1$ for rdQM due to our normalization $\check{P}_n(0)=P_n(0)=1$. Corresponding to the factorization (3.1), the shape invariance combined with the Crum's theorem give the following relations [7, 8, 3].

Theorem 2.1 *[2] For the polynomials in* (2.2)–(2.5)*, the forward and backward shift relations hold:*

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda})=f_n(\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})\quad(n\in\mathbb{Z}_{\geq 0}),\tag{3.16}$$

$$\mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x;\boldsymbol{\lambda}+\boldsymbol{\delta})=b_{n-1}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq 1}).\tag{3.17}$$

For idQM, the relations (3.16)–(3.17) are invariant under the $*$-operation.

The similarity transformed Hamiltonians $\widetilde{\mathcal{H}}^{\mathrm{J}}(\boldsymbol{\lambda})$ (2.12) are factorized as

$$\widetilde{\mathcal{H}}^{\mathrm{J}}(\boldsymbol{\lambda})=\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda})\mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda}).\tag{3.18}$$

Here the forward and backward shift operators, $\mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda})$ and $\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda})$, are defined by [5]

$$\text{rdQMJ}:\ \ \mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{A}{q\eta}(1-q^{\eta\frac{d}{d\eta}}),\tag{3.19}$$

$$\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\big(B^{\mathrm{J}}(\eta;\boldsymbol{\lambda})-D^{\mathrm{J}}(\eta;\boldsymbol{\lambda})q^{-\eta\frac{d}{d\eta}}\big)\frac{q\eta}{A},\tag{3.20}$$

where the constant A is given by

$$A=\begin{cases}-D^{\mathrm{J}}(1;\boldsymbol{\lambda}) & :\mathrm{b}q\mathrm{J},\mathrm{b}q\mathrm{L},q\mathrm{L},\mathrm{SW}\\ -a & :\mathrm{ASCI}\\ 1 & :\mathrm{d}q\mathrm{HeI}\\ q & :\mathrm{d}q\mathrm{HeII}\end{cases}.\tag{3.21}$$

We can show that $B^{\mathrm{J}}(\eta)$ and $D^{\mathrm{J}}(\eta)$ satisfy

$$qB^{\mathrm{J}}(qr^{-1}\eta;\boldsymbol{\lambda})=\kappa B^{\mathrm{J}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}),\quad q^{-1}D^{\mathrm{J}}(r^{-1}\eta;\boldsymbol{\lambda})=\kappa D^{\mathrm{J}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}),\tag{3.22}$$

$$B^{\mathrm{J}}(r^{-1}\eta;\boldsymbol{\lambda})+D^{\mathrm{J}}(qr^{-1}\eta;\boldsymbol{\lambda})=\kappa\big(B^{\mathrm{J}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})+D^{\mathrm{J}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\big)+\mathcal{E}_1(\boldsymbol{\lambda}),\tag{3.23}$$

where r is given by

$$r=\begin{cases} q & :\mathrm{b}q\mathrm{J},\mathrm{b}q\mathrm{L},\mathrm{d}q\mathrm{HeII},q\mathrm{L}\\ 1 & :\mathrm{ASCI},\mathrm{d}q\mathrm{HeI}\\ q^2 & :\mathrm{SW}\end{cases}.\tag{3.24}$$

Therefore we obtain

$$\mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda})\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda})\Big|_{\eta\to r^{-1}\eta}=\kappa\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda}+\boldsymbol{\delta})\mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda}+\boldsymbol{\delta})+\mathcal{E}_1(\boldsymbol{\lambda}),\tag{3.25}$$

which is the (similarity transformed) shape invariance condition. The constants f_n^{J} and b_n^{J} satisfy

$$\mathcal{E}_n(\boldsymbol{\lambda})=f_n^{\mathrm{J}}(\boldsymbol{\lambda})b_{n-1}^{\mathrm{J}}(\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq0}),\tag{3.26}$$

and the energy eigenvalues $\mathcal{E}_n$ satisfy (3.15). Note that we have $f_n^{\mathrm{J}}(\boldsymbol{\lambda})=\mathcal{E}_n(\boldsymbol{\lambda})$ and $b_n^{\mathrm{J}}(\boldsymbol{\lambda})=1$ due to our normalization of $P_n(\eta)$. Corresponding to the factorization (3.18), we have the following relations [5].

Theorem 2.2 *[2] For the polynomials in* (2.6)*, the forward and backward shift relations hold:*

$$\mathcal{F}^{\mathrm{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda})=f_n^{\mathrm{J}}(\boldsymbol{\lambda})P_{n-1}(r\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})\quad(n\in\mathbb{Z}_{\geq0}),\tag{3.27}$$

$$\mathcal{B}^{\mathrm{J}}(\boldsymbol{\lambda})P_{n-1}(r\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})=b_{n-1}^{\mathrm{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq1}).\tag{3.28}$$

4 New Forward and Backward Shift Relations

In this section, based on other factorizations of $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}^{\mathrm{J}}$, we present another type of forward and backward shift relations.

4.1 Polynomials in oQM systems

For the oQM systems described by the polynomials (2.2) (except He and B), let us define the operators $\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})$. For J case, they are given by

$$(\mathrm{a}):\ \tilde{\mathcal{F}}(\boldsymbol{\lambda})\stackrel{\mathrm{def}}{=}\frac{1}{2}\tan x\frac{d}{dx}+g-\frac{1}{2}\ \Big(=-(1-\eta)\frac{d}{d\eta}+g-\frac{1}{2}\Big),\tag{4.1}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\frac{1}{2}\cot x\frac{d}{dx} + h + \frac{1}{2}\ \Big(= (1+\eta)\frac{d}{d\eta} + h + \frac{1}{2}\Big), \tag{4.2}$$

$$\text{(b)}:\ \tilde{\mathcal{F}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -\frac{1}{2}\cot x\frac{d}{dx} + h - \frac{1}{2}\ \Big(= (1+\eta)\frac{d}{d\eta} + h - \frac{1}{2}\Big), \tag{4.3}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{2}\tan x\frac{d}{dx} + g + \frac{1}{2}\ \Big(= -(1-\eta)\frac{d}{d\eta} + g + \frac{1}{2}\Big), \tag{4.4}$$

and the constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$ are given by

$$\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda}) = n + g - \tfrac{1}{2},\quad \tilde{b}_n(\boldsymbol{\lambda}) = n + h + \tfrac{1}{2},\quad \bar{\boldsymbol{\delta}} = (1,-1), \tag{4.5}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda}) = n + h - \tfrac{1}{2},\quad \tilde{b}_n(\boldsymbol{\lambda}) = n + g + \tfrac{1}{2},\quad \bar{\boldsymbol{\delta}} = (-1,1). \tag{4.6}$$

For L and pJ cases, see Appendix A.1.

Then we can show that

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda}) = 4\big(\tilde{\mathcal{B}}(\boldsymbol{\lambda})\tilde{\mathcal{F}}(\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})\big), \tag{4.7}$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = 4\big(\tilde{f}_n(\boldsymbol{\lambda})\tilde{b}_n(\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})\big)\quad (n \in \mathbb{Z}_{\geq 0}). \tag{4.8}$$

Corresponding to this factorization (4.7), the following relations are obtained by direct calculation.

Theorem 3.1 *For the polynomials in* (2.2) *(except* He *and* B*), the following forward and backward shift relations hold for $n \in \mathbb{Z}_{\geq 0}$,*

$$\tilde{\mathcal{F}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) = \tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda} - \bar{\boldsymbol{\delta}}), \tag{4.9}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda} - \bar{\boldsymbol{\delta}}) = \tilde{b}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}). \tag{4.10}$$

We think that these identities (4.9)–(4.10) may be known formulas but this interpretation is new.

Remark 1.1 Two formulas with $\bar{\boldsymbol{\delta}}$ and $-\bar{\boldsymbol{\delta}}$ are equivalent by interchanging $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{B}}$, e.g. (4.9) and (4.10) for L (b) agree with (4.10) and (4.9) for L (a) with the replacement $g \to g+1$, respectively. For He and B, we do not have new factorization (4.7) and new forward and backward shift relations (4.9)–(4.10).

4.2 Polynomials in idQM systems

For the idQM systems described by the polynomials (2.3), let us define the operators $\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})$ as follows:

$$\tilde{\mathcal{F}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} V_1(x + i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} + V_1^*(x - i\tfrac{\gamma}{2};\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}, \tag{4.11}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} V_2(x;\boldsymbol{\lambda})e^{\frac{\gamma}{2}p} + V_2^*(x;\boldsymbol{\lambda})e^{-\frac{\gamma}{2}p}, \tag{4.12}$$

where the potential functions $V_1(x;\boldsymbol{\lambda})$ and $V_2(x;\boldsymbol{\lambda})$ satisfy

$$V(x;\boldsymbol{\lambda}) = V_1(x;\boldsymbol{\lambda})V_2(x;\boldsymbol{\lambda}). \tag{4.13}$$

For AW case, their explicit forms are given by

$$\begin{aligned}
&\text{Assume } \{a_j^*, a_k^*\} = \{a_j, a_k\} \text{ (as a set) and set } \{l, m\} = \{1,2,3,4\}\backslash\{j,k\},\\
&V_i(x;\boldsymbol{\lambda}) = V_i^{(j,k)}(x;\boldsymbol{\lambda}) \ \ (i=1,2),\\
&V_1(x;\boldsymbol{\lambda}) = \frac{(1-a_je^{ix})(1-a_ke^{ix})}{1-qe^{2ix}}, \ \ V_2(x;\boldsymbol{\lambda}) = \frac{(1-a_le^{ix})(1-a_me^{ix})}{1-e^{2ix}},
\end{aligned} \tag{4.14}$$

and the constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$ as given by

$$\begin{aligned}
&\tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_ja_kq^{n-1}), \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_la_mq^n),\\
&(\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = \tfrac{1}{2}, \ \ (\bar{\boldsymbol{\delta}})_l = (\bar{\boldsymbol{\delta}})_m = -\tfrac{1}{2}.
\end{aligned} \tag{4.15}$$

For other cases, see Appendix A.2.

Then we can show that $V_1(x)$ and $V_2(x)$ satisfy

$$V_1(x+i\gamma;\boldsymbol{\lambda})V_2^*(x;\boldsymbol{\lambda}) + V_1^*(x-i\gamma;\boldsymbol{\lambda})V_2(x;\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}) = -V(x;\boldsymbol{\lambda}) - V^*(x;\boldsymbol{\lambda}), \tag{4.16}$$

and the constants $\tilde{f}_n$ and $\tilde{b}_n$ satisfy

$$\mathcal{E}_n(\boldsymbol{\lambda}) = \tilde{f}_n(\boldsymbol{\lambda})\tilde{b}_n(\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}) \ \ (n \in \mathbb{Z}_{\geq 0}). \tag{4.17}$$

The relation (4.16) gives other factorizations of $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ (2.9),

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda}) = \tilde{\mathcal{B}}(\boldsymbol{\lambda})\tilde{\mathcal{F}}(\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}). \tag{4.18}$$

Corresponding to this factorization (4.18), we obtain the following relations.

Theorem 3.2 *For the polynomials in* (2.3)*, the following forward and backward shift relations hold for* $n \in \mathbb{Z}_{\geq 0}$*,*

$$\tilde{\mathcal{F}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) = \tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \tag{4.19}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) = \tilde{b}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}). \tag{4.20}$$

Proof: It is sufficient to show (4.19), because (2.13) and (4.17)–(4.19) imply (4.20). Taking AW (4.14) with $(j,k)=(1,2)$ as an example, let us prove (4.19). It is shown by direct calculation:

$$
\begin{aligned}
&\tilde{\mathcal{F}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda})\\
&=\frac{(1-a_1q^{-\frac{1}{2}}e^{ix})(1-a_2q^{-\frac{1}{2}}e^{ix})}{1-e^{2ix}}\frac{(a_1a_2,a_1a_3,a_1a_4\,;q)_n}{a_1^n}\\
&\quad\times{}_4\phi_3\Bigl(\begin{matrix}q^{-n},\ a_1a_2a_3a_4q^{n-1},\ a_1q^{\frac{1}{2}}e^{ix},\ a_1q^{-\frac{1}{2}}e^{-ix}\\ a_1a_2,\ a_1a_3,\ a_1a_4\end{matrix}\Bigm| q\,;q\Bigr)\\
&\quad+\frac{(1-a_1q^{-\frac{1}{2}}e^{-ix})(1-a_2q^{-\frac{1}{2}}e^{-ix})}{1-e^{-2ix}}\frac{(a_1a_2,a_1a_3,a_1a_4\,;q)_n}{a_1^n}\\
&\quad\times{}_4\phi_3\Bigl(\begin{matrix}q^{-n},\ a_1a_2a_3a_4q^{n-1},\ a_1q^{-\frac{1}{2}}e^{ix},\ a_1q^{\frac{1}{2}}e^{-ix}\\ a_1a_2,\ a_1a_3,\ a_1a_4\end{matrix}\Bigm| q\,;q\Bigr)\\
&=\frac{(a_1a_2,a_1a_3,a_1a_4\,;q)_n}{a_1^n(1-e^{2ix})}\sum_{k=0}^n\frac{(q^{-n},a_1a_2a_3a_4q^{n-1},a_1q^{-\frac{1}{2}}e^{ix},a_1q^{-\frac{1}{2}}e^{-ix}\,;q)_k}{(a_1a_2,a_1a_3,a_1a_4\,;q)_k}\frac{q^k}{(q\,;q)_k}\\
&\quad\times\bigl((1-a_1e^{ix}q^{k-\frac{1}{2}})(1-a_2q^{-\frac{1}{2}}e^{ix})-e^{2ix}(1-a_1e^{-ix}q^{k-\frac{1}{2}})(1-a_2q^{-\frac{1}{2}}e^{-ix})\bigr)\\
&=\frac{(a_1a_2,a_1a_3,a_1a_4\,;q)_n}{a_1^n(1-e^{2ix})}\sum_{k=0}^n\frac{(q^{-n},a_1a_2a_3a_4q^{n-1},a_1q^{-\frac{1}{2}}e^{ix},a_1q^{-\frac{1}{2}}e^{-ix}\,;q)_k}{(a_1a_2,a_1a_3,a_1a_4\,;q)_k}\frac{q^k}{(q\,;q)_k}\\
&\quad\times(1-a_1a_2q^{k-1})(1-e^{2ix})\\
&=\frac{(a_1a_2,a_1a_3,a_1a_4\,;q)_n}{a_1^n}\sum_{k=0}^n(1-a_1a_2q^{-1})\frac{(q^{-n},a_1a_2a_3a_4q^{n-1},a_1q^{-\frac{1}{2}}e^{ix},a_1q^{-\frac{1}{2}}e^{-ix}\,;q)_k}{(a_1a_2q^{-1},a_1a_3,a_1a_4\,;q)_k}\frac{q^k}{(q\,;q)_k}\\
&=q^{-\frac{n}{2}}(1-a_1a_2q^{n-1})\\
&\quad\times\frac{(a_1a_2q^{-1},a_1a_3,a_1a_4\,;q)_n}{(a_1q^{-\frac{1}{2}})^n}\sum_{k=0}^n\frac{(q^{-n},a_1a_2a_3a_4q^{n-1},a_1q^{-\frac{1}{2}}e^{ix},a_1q^{-\frac{1}{2}}e^{-ix}\,;q)_k}{(a_1a_2q^{-1},a_1a_3,a_1a_4\,;q)_k}\frac{q^k}{(q\,;q)_k}\\
&=\tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}).
\end{aligned}
$$

The other cases are proved in the same way. □

Remark 2.1 Two formulas with $\bar{\boldsymbol{\delta}}$ and $-\bar{\boldsymbol{\delta}}$ are equivalent by interchanging $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{B}}$, e.g. (4.19) and (4.20) for cH (b) agree with (4.20) and (4.19) for cH (a) with the replacements $a_1\to a_1+\frac{1}{2}$ and $a_2\to a_2-\frac{1}{2}$, respectively.

Remark 2.2 The relations (4.19)–(4.20) are invariant under the $*$-operation. In contrast to the x-shift relations studied in [4] (see Theorem 3.3), the coordinate x is not shifted, and only the parameters $\boldsymbol{\lambda}$ are shifted. We choose the operators $\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})$ (4.11)–(4.12) to respect this $*$-operation invariance. See also Remark 3.3.

Remark 2.3 We can show that

$$
\begin{aligned}
&V_1(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})V_2(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})=V_1(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})V_2(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),\\
&V_1(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})V_2^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})+V_1^*(x-i\tfrac{\gamma}{2};\boldsymbol{\lambda})V_2(x+i\tfrac{\gamma}{2};\boldsymbol{\lambda})-\tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})\\
&\ =V_1(x+i\gamma;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})V_2^*(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})+V_1^*(x-i\gamma;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})V_2(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})-\tilde{f}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),
\end{aligned}
\tag{4.21}
$$

which imply

$$
\tilde{\mathcal{F}}(\boldsymbol{\lambda})\tilde{\mathcal{B}}(\boldsymbol{\lambda})-\tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})=\tilde{\mathcal{B}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{\mathcal{F}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})-\tilde{f}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}). \tag{4.22}
$$

4.3 Polynomials in rdQM systems

For the rdQM systems described by the polynomials (2.4)–(2.5) (except C and qB), let us define the operators $\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})$ as follows:

$$
\tilde{\mathcal{F}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}D_1(x+1;\boldsymbol{\lambda})+B_1(x;\boldsymbol{\lambda})e^{\partial}, \tag{4.23}
$$

$$
\tilde{\mathcal{B}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}B_2(x;\boldsymbol{\lambda})+D_2(x;\boldsymbol{\lambda})e^{-\partial}, \tag{4.24}
$$

where the potential functions $B_1(x;\boldsymbol{\lambda}), B_2(x;\boldsymbol{\lambda}), D_1(x;\boldsymbol{\lambda})$ and $D_2(x;\boldsymbol{\lambda})$ satisfy

$$
B(x;\boldsymbol{\lambda})=B_1(x;\boldsymbol{\lambda})B_2(x;\boldsymbol{\lambda}),\quad D(x;\boldsymbol{\lambda})=D_1(x;\boldsymbol{\lambda})D_2(x;\boldsymbol{\lambda}). \tag{4.25}
$$

For qR case, their explicit forms are given by

$$
\begin{aligned}
\text{(a)}:\ \ &B_1(x;\boldsymbol{\lambda})=-\frac{(1-q^{x-N})(1-dq^x)}{(q^{-N-1}-1)(1-dq^{2x+1})},\quad B_2(x;\boldsymbol{\lambda})=\frac{(q^{-N-1}-1)(1-bq^x)(1-cq^x)}{1-dq^{2x}},\\
&D_1(x;\boldsymbol{\lambda})=\frac{(1-dq^{x+N})(1-q^x)}{(1-q^{N+1})(1-dq^{2x-1})},\ D_2(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-q^{N+1})(1-b^{-1}dq^x)(1-c^{-1}dq^x)}{1-dq^{2x}},
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
\text{(b)}:\ \ &B_1(x;\boldsymbol{\lambda})=\frac{(1-bq^x)(1-dq^x)}{(1-bq^{-1})(1-dq^{2x+1})},\quad B_2(x;\boldsymbol{\lambda})=-\frac{(1-bq^{-1})(1-q^{x-N})(1-cq^x)}{1-dq^{2x}},\\
&D_1(x;\boldsymbol{\lambda})=\frac{(1-b^{-1}dq^x)(1-q^x)}{(1-b^{-1}q)(1-dq^{2x-1})},\ D_2(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-b^{-1}q)(1-dq^{x+N})(1-c^{-1}dq^x)}{1-dq^{2x}},
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
\text{(c)}:\ \ &B_1(x;\boldsymbol{\lambda})=\frac{(1-cq^x)(1-dq^x)}{(1-cq^{-1})(1-dq^{2x+1})},\quad B_2(x;\boldsymbol{\lambda})=-\frac{(1-cq^{-1})(1-q^{x-N})(1-bq^x)}{1-dq^{2x}},\\
&D_1(x;\boldsymbol{\lambda})=\frac{(1-c^{-1}dq^x)(1-q^x)}{(1-c^{-1}q)(1-dq^{2x-1})},\ D_2(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-c^{-1}q)(1-dq^{x+N})(1-b^{-1}dq^x)}{1-dq^{2x}},
\end{aligned}
\tag{4.28}
$$

$$\text{(d)}:\ B_1(x;\boldsymbol{\lambda})=-\frac{(1-c)(1-q^{x-N})(1-bq^x)}{1-dq^{2x+1}},\quad B_2(x;\boldsymbol{\lambda})=\frac{(1-cq^x)(1-dq^x)}{(1-c)(1-dq^{2x})},\tag{4.29}$$
$$D_1(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-c^{-1})(1-dq^{x+N})(1-b^{-1}dq^x)}{1-dq^{2x-1}},\quad D_2(x;\boldsymbol{\lambda})=\frac{(1-c^{-1}dq^x)(1-q^x)}{(1-c^{-1})(1-dq^{2x})},$$
$$\text{(e)}:\ B_1(x;\boldsymbol{\lambda})=-\frac{(1-b)(1-q^{x-N})(1-cq^x)}{1-dq^{2x+1}},\quad B_2(x;\boldsymbol{\lambda})=\frac{(1-bq^x)(1-dq^x)}{(1-b)(1-dq^{2x})},\tag{4.30}$$
$$D_1(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-b^{-1})(1-dq^{x+N})(1-c^{-1}dq^x)}{1-dq^{2x-1}},\quad D_2(x;\boldsymbol{\lambda})=\frac{(1-b^{-1}dq^x)(1-q^x)}{(1-b^{-1})(1-dq^{2x})},$$
$$\text{(f)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(q^{-N}-1)(1-bq^x)(1-cq^x)}{1-dq^{2x+1}},\quad B_2(x;\boldsymbol{\lambda})=-\frac{(1-q^{x-N})(1-dq^x)}{(q^{-N}-1)(1-dq^{2x})},\tag{4.31}$$
$$D_1(x;\boldsymbol{\lambda})=-\tilde{d}\,\frac{(1-q^N)(1-b^{-1}dq^x)(1-c^{-1}dq^x)}{1-dq^{2x-1}},\quad D_2(x;\boldsymbol{\lambda})=\frac{(1-dq^{x+N})(1-q^x)}{(1-q^N)(1-dq^{2x})},$$

and the constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$ are given by

$$\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\ \bar{\boldsymbol{\delta}}=(1,0,0,1),\tag{4.32}$$
$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-bq^{n-1})(1-cd^{-1}q^{n-N}),\ \bar{\boldsymbol{\delta}}=(0,1,0,1),\tag{4.33}$$
$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-cq^{n-1})(1-bd^{-1}q^{n-N}),\ \bar{\boldsymbol{\delta}}=(0,0,1,1),\tag{4.34}$$
$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-cq^n)(1-bd^{-1}q^{n-N-1}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \bar{\boldsymbol{\delta}}=(0,0,-1,-1),\tag{4.35}$$
$$\text{(e)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-bq^n)(1-cd^{-1}q^{n-N-1}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \bar{\boldsymbol{\delta}}=(0,-1,0,-1),\tag{4.36}$$
$$\text{(f)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \bar{\boldsymbol{\delta}}=(-1,0,0,-1).\tag{4.37}$$

For other cases, see Appendix A.3.

Then we can show that $B_1(x)$, $B_2(x)$, $D_1(x)$ and $D_2(x)$ satisfy

$$B_1(x-1;\boldsymbol{\lambda})D_2(x;\boldsymbol{\lambda})+D_1(x+1;\boldsymbol{\lambda})B_2(x;\boldsymbol{\lambda})-\tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})=-B(x;\boldsymbol{\lambda})-D(x;\boldsymbol{\lambda}),\tag{4.38}$$

and the constants $\tilde{f}_n$ and $\tilde{b}_n$ satisfy

$$\mathcal{E}_n(\boldsymbol{\lambda})=\tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda})-\tilde{f}_n(\boldsymbol{\lambda})\tilde{b}_n(\boldsymbol{\lambda})\ \ (n\in\mathbb{Z}_{\geq 0}).\tag{4.39}$$

The relation (4.38) gives another factorization of $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ (2.10),

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda})=-\tilde{\mathcal{B}}(\boldsymbol{\lambda})\tilde{\mathcal{F}}(\boldsymbol{\lambda})+\tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}).\tag{4.40}$$

Corresponding to this factorization (4.40), we obtain the following relations.

Theorem 3.3 *For the polynomials in* (2.4)–(2.5) *(except* C *and* qB*), the following forward and backward shift relations hold for* $n \in \mathbb{Z}_{\geq 0}$*,*

$$\tilde{\mathcal{F}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) = \tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x+s;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \tag{4.41}$$

$$\tilde{\mathcal{B}}(\boldsymbol{\lambda})\check{P}_n(x+s;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) = \tilde{b}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}), \tag{4.42}$$

where s *is given by*

$$s = \begin{cases} 1 & : \mathrm{H\,(a)(b), K\,(a), R\,(a)(b)(c), dH\,(a)(b)(c), dq}q\mathrm{K\,(a)(b)}, q\mathrm{H\,(a)(b)}, \\ & \quad q\mathrm{K\,(a), q}q\mathrm{K\,(a)(b), a}q\mathrm{K\,(a)(b)}, q\mathrm{R\,(a)(b)(c), d}q\mathrm{H\,(a)(b)(c)}, \\ & \quad \mathrm{d}q\mathrm{K\,(a)(b), M\,(a), l}q\mathrm{J\,(a), l}q\mathrm{L\,(a)}, q\mathrm{M\,(a)(b), ASCII\,(a)}, q\mathrm{C\,(a)} \\ 0 & : \textit{others} \end{cases}. \tag{4.43}$$

Proof: It is sufficient to show (4.41), because (2.13) and (4.39)–(4.41) imply (4.42). Taking qR (a) (4.26) as an example, let us prove (4.41). It is shown by direct calculation:

$$\begin{aligned}
&\tilde{\mathcal{F}}(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}) \\
&= \frac{(1-a^{-1}dq^{x+1})(1-q^{x+1})}{(1-a^{-1}q)(1-dq^{2x+1})}\,{}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ abcd^{-1}q^{n-1},\ q^{-x},\ dq^{x}}{a,\ b,\ c}\Bigm|\,q\,;q\Bigr) \\
&\quad - \frac{(1-aq^{x})(1-dq^{x})}{(aq^{-1}-1)(1-dq^{2x+1})}\,{}_4\phi_3\Bigl(\genfrac{}{}{0pt}{}{q^{-n},\ abcd^{-1}q^{n-1},\ q^{-x-1},\ dq^{x+1}}{a,\ b,\ c}\Bigm|\,q\,;q\Bigr) \\
&= \frac{1}{(1-aq^{-1})(1-dq^{2x+1})}\sum_{k=0}^{n}\frac{(q^{-n},abcd^{-1}q^{n-1},q^{-x-1},dq^{x}\,;q)_k}{(a,b,c\,;q)_k}\frac{q^k}{(q\,;q)_k} \\
&\qquad \times\bigl(-aq^{-1}(1-a^{-1}dq^{x+1})(-q^{x+1})(1-q^{-x+k-1})+(1-aq^{x})(1-dq^{x+k})\bigr) \\
&= \frac{1}{(1-aq^{-1})(1-dq^{2x+1})}\sum_{k=0}^{n}\frac{(q^{-n},abcd^{-1}q^{n-1},q^{-x-1},dq^{x}\,;q)_k}{(a,b,c\,;q)_k}\frac{q^k}{(q\,;q)_k}(1-aq^{k-1})(1-dq^{2x+1}) \\
&= \sum_{k=0}^{n}\frac{(q^{-n},abcd^{-1}q^{n-1},q^{-x-1},dq^{x}\,;q)_k}{(aq^{-1},b,c\,;q)_k}\frac{q^k}{(q\,;q)_k} \\
&= \tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x+s;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}).
\end{aligned}$$

The other cases are proved in the same way. □

Remark 3.1 Two formulas with $\bar{\boldsymbol{\delta}}$ and $-\bar{\boldsymbol{\delta}}$ are equivalent by interchanging $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{B}}$, e.g. (4.41) and (4.42) for H (c) agree with (4.42) and (4.41) for H (b) with the replacements $a \to a+1$ and $b \to b-1$, respectively. For C and qB, we do not have new factorization (4.40) and new forward and backward shift relations (4.41)–(4.42).

Remark 3.2 The relations (4.41)–(4.42) for twelve cases ((a) of H, K, R, dH, dq$q$K, qH, qK, qqK, aqK, qR, dqH, dqK, which have $\tilde{f}_n = 1$, $\tilde{b}_n(\boldsymbol{\lambda}) = \mathcal{E}_{N+1}(\boldsymbol{\lambda}) - \mathcal{E}_n(\boldsymbol{\lambda})$, $s = 1$ and

$D_1(0;\boldsymbol{\lambda}) = B_1(N;\boldsymbol{\lambda}) = 0$) were given in [4] and they were called forward and backward x-shift relations. By considering $e^{-\partial}\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})e^{\partial}$, the above results (4.40) and (4.41)–(4.42) with $s=1$ are rewritten as

$$\widetilde{\mathcal{H}}(\boldsymbol{\lambda}) = -\big(\tilde{\mathcal{B}}(\boldsymbol{\lambda})e^{\partial}\big)\big(e^{-\partial}\tilde{\mathcal{F}}(\boldsymbol{\lambda})\big) + \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}), \tag{4.44}$$

$$\big(e^{-\partial}\tilde{\mathcal{F}}(\boldsymbol{\lambda})\big)\check{P}_n(x;\boldsymbol{\lambda}) = \tilde{f}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \tag{4.45}$$

$$\big(\tilde{\mathcal{B}}(\boldsymbol{\lambda})e^{\partial}\big)\check{P}_n(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) = \tilde{b}_n(\boldsymbol{\lambda})\check{P}_n(x;\boldsymbol{\lambda}). \tag{4.46}$$

That is, x is not shifted. As an identity of polynomial, the x-shift is not essential. However, this x-shift has important implications in the state-adding Darboux transformation for the finite rdQM systems [4, 9].

Remark 3.3 AW and qR polynomials are related as [2]

$$\begin{aligned} &e^{ix^{\text{AW}}} = d^{\frac12}q^{x^{q\text{R}}}, \quad (a_1,a_2,a_3,a_4) = (ad^{-\frac12}, bd^{-\frac12}, cd^{-\frac12}, d^{\frac12}),\\ &\check{P}_n^{\text{AW}}(x^{\text{AW}};\boldsymbol{\lambda}^{\text{AW}}) = d^{-\frac{n}{2}}(a,b,c\,;q)_n\check{P}_n^{q\text{R}}(x^{q\text{R}};\boldsymbol{\lambda}^{q\text{R}}). \end{aligned} \tag{4.47}$$

For the $(j,k)=(1,4)$ case in (4.14), the operators $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{B}}$ for AW are related to those for qR (a) (4.26) as

$$\begin{aligned} e^{\frac{\gamma}{2}p}\tilde{\mathcal{F}}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}}) &= -(q^{-N-1}-1)\tilde{\mathcal{F}}^{q\text{R}}(\boldsymbol{\lambda}^{q\text{R}}),\\ \tilde{\mathcal{B}}^{\text{AW}}(\boldsymbol{\lambda}^{\text{AW}})e^{-\frac{\gamma}{2}p} &= (q^{-N-1}-1)^{-1}\tilde{\mathcal{B}}^{q\text{R}}(\boldsymbol{\lambda}^{q\text{R}}). \end{aligned} \tag{4.48}$$

These extra factors $e^{\pm\frac{\gamma}{2}p}$ give the property in Remark 2.2. Similarly AW with $(j,k)=(2,4)$, $(3,4)$, $(1,2)$, $(1,3)$ and $(2,4)$ cases correspond to qR (b), (c), (d), (e) and (f), respectively.

Remark 3.4 We can show that

$$\begin{aligned} &B_1(x-s;\boldsymbol{\lambda})B_2(x-s+1;\boldsymbol{\lambda}) = B(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \quad D_1(x-s+1;\boldsymbol{\lambda})D_2(x-s;\boldsymbol{\lambda}) = D(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),\\ &B_1(x-s;\boldsymbol{\lambda})D_2(x-s+1;\boldsymbol{\lambda}) + D_1(x-s+1;\boldsymbol{\lambda})B_2(x-s;\boldsymbol{\lambda}) - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}) \qquad (4.49)\\ &\ = B_1(x-1;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})D_2(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) + D_1(x+1;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})B_2(x;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) - \tilde{f}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \end{aligned}$$

which imply

$$\tilde{\mathcal{F}}(\boldsymbol{\lambda})\tilde{\mathcal{B}}(\boldsymbol{\lambda})\Big|_{x\to x-s} - \tilde{f}_0(\boldsymbol{\lambda})\tilde{b}_0(\boldsymbol{\lambda}) = \tilde{\mathcal{B}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{\mathcal{F}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) - \tilde{f}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}). \tag{4.50}$$

4.4 Polynomials in rdQMJ systems

For the rdQMJ systems described by the polynomials (2.6) (except dqHeI, dqHeII and SW), let us define the operators $\tilde{\mathcal{F}}^{\mathrm{J}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}^{\mathrm{J}}(\boldsymbol{\lambda})$ as follows:

$$\tilde{\mathcal{F}}^{\mathrm{J}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} D_1^{\mathrm{J}}(q\eta;\boldsymbol{\lambda}) + B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda})q^{\eta\frac{d}{d\eta}}, \tag{4.51}$$

$$\tilde{\mathcal{B}}^{\mathrm{J}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) + D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda})q^{-\eta\frac{d}{d\eta}}, \tag{4.52}$$

where the potential functions $B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}), B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}), D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda})$ and $D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda})$ satisfy

$$B^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda})B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}), \quad D^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda})D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}). \tag{4.53}$$

For bqJ case, their explicit forms are given by

$$\begin{aligned}
\text{(a)}:\ \ B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= \frac{\eta^{-1}a(1-\eta)}{1-a}, \ \ B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = (1-a)\eta^{-1}q(b\eta-c),\\
D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= \frac{\eta^{-1}(aq-\eta)}{a-1}, \ \ D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = (a-1)\eta^{-1}(\eta-cq),
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
\text{(b)}:\ \ B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= \frac{\eta^{-1}(1-\eta)}{c^{-1}-1}, \ \ B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = (c^{-1}-1)\eta^{-1}aq(b\eta-c),\\
D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= \frac{\eta^{-1}(\eta-cq)}{1-c}, \ \ D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = (1-c)\eta^{-1}(aq-\eta),
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
\text{(c)}:\ \ B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= (c^{-1}q^{-1}-1)\eta^{-1}aq(b\eta-c), \ \ B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = \frac{\eta^{-1}(1-\eta)}{c^{-1}q^{-1}-1},\\
D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= (1-cq)\eta^{-1}(aq-\eta), \ \ D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = \frac{\eta^{-1}(\eta-cq)}{1-cq},
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
\text{(d)}:\ \ B_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= (1-aq)\eta^{-1}(b\eta-c), \ \ B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = \frac{\eta^{-1}aq(1-\eta)}{1-aq},\\
D_1^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) &= (aq-1)\eta^{-1}(\eta-cq), \ \ D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda}) = \frac{\eta^{-1}(aq-\eta)}{aq-1},
\end{aligned} \tag{4.57}$$

and the constants $\tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda})$, $\tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$ are given by

$$\text{(a)}:\ \ \tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = 1, \ \ \tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = -q^{-n}(1-aq^n)(1-bq^{n+1}), \ \ \bar{\boldsymbol{\delta}} = (1,-1,0), \tag{4.58}$$

$$\text{(b)}:\ \ \tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = 1, \ \ \tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = -q^{-n}(1-cq^n)(1-abc^{-1}q^{n+1}), \ \ \bar{\boldsymbol{\delta}} = (0,0,1), \tag{4.59}$$

$$\text{(c)}:\ \ \tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = -q^{-n}(1-cq^{n+1})(1-abc^{-1}q^n), \ \ \tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = 1, \ \ \bar{\boldsymbol{\delta}} = (0,0,-1), \tag{4.60}$$

$$\text{(d)}:\ \ \tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = -q^{-n}(1-aq^{n+1})(1-bq^n), \ \ \tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda}) = 1, \ \ \bar{\boldsymbol{\delta}} = (-1,1,0). \tag{4.61}$$

For other cases, see Appendix A.4.

Then we can show that $B_1^{\mathrm{J}}(\eta)$, $B_2^{\mathrm{J}}(\eta)$, $D_1^{\mathrm{J}}(\eta)$ and $D_2^{\mathrm{J}}(\eta)$ satisfy

$$B_1^{\mathrm{J}}(q^{-1}\eta;\boldsymbol{\lambda})D_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda})+D_1^{\mathrm{J}}(q\eta;\boldsymbol{\lambda})B_2^{\mathrm{J}}(\eta;\boldsymbol{\lambda})-\tilde{f}_0^{\mathrm{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\mathrm{J}}(\boldsymbol{\lambda})=-B^{\mathrm{J}}(\eta;\boldsymbol{\lambda})-D^{\mathrm{J}}(\eta;\boldsymbol{\lambda}), \tag{4.62}$$

and the constants $\tilde{f}_n^{\mathrm{J}}$ and $\tilde{b}_n^{\mathrm{J}}$ satisfy

$$\mathcal{E}_n(\boldsymbol{\lambda})=\tilde{f}_0^{\mathrm{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\mathrm{J}}(\boldsymbol{\lambda})-\tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda})\tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda})\quad(n\in\mathbb{Z}_{\geq 0}). \tag{4.63}$$

The relations (4.62) give other factorizations of $\widetilde{\mathcal{H}}(\boldsymbol{\lambda})$ (2.12),

$$\widetilde{\mathcal{H}}^{\mathrm{J}}(\boldsymbol{\lambda})=-\tilde{\mathcal{B}}^{\mathrm{J}}(\boldsymbol{\lambda})\tilde{\mathcal{F}}^{\mathrm{J}}(\boldsymbol{\lambda})+\tilde{f}_0^{\mathrm{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\mathrm{J}}(\boldsymbol{\lambda}). \tag{4.64}$$

Corresponding to this factorization (4.64), we obtain the following relations.

Theorem 3.4 *For the polynomials in* (2.6) *(except* dqHeI, dqHeII *and* SW*), the following forward and backward shift relations hold for $n\in\mathbb{Z}_{\geq 0}$,*

$$\tilde{\mathcal{F}}^{\mathrm{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda})=\tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda})P_n(r'\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \tag{4.65}$$

$$\tilde{\mathcal{B}}^{\mathrm{J}}(\boldsymbol{\lambda})P_n(r'\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})=\tilde{b}_n^{\mathrm{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}), \tag{4.66}$$

where r' is given by

$$r'=\begin{cases} q & :\mathrm{b}q\mathrm{J}\,(\mathrm{c})(\mathrm{d}),\,\mathrm{b}q\mathrm{L}\,(\mathrm{c})(\mathrm{d}),\,\mathrm{ASCI}\,(\mathrm{a}),\,q\mathrm{L}\,(\mathrm{b})\\ 1 & :\mathrm{b}q\mathrm{J}\,(\mathrm{a})(\mathrm{b}),\,\mathrm{b}q\mathrm{L}\,(\mathrm{a})(\mathrm{b}),\,\mathrm{ASCI}\,(\mathrm{b}),\,q\mathrm{L}\,(\mathrm{a})\end{cases}. \tag{4.67}$$

Proof: It is sufficient to show (4.65), because (2.14) and (4.63)–(4.65) imply (4.66). Taking bqJ (a) (4.54) as an example, let us prove (4.65). It is shown by direct calculation:

$$\begin{aligned}
&\tilde{\mathcal{F}}^{\mathrm{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda})\\
&=\frac{\eta^{-1}(a-\eta)}{a-1}{}_3\phi_2\Bigl(\begin{matrix}q^{-n},\,abq^{n+1},\,\eta\\ aq,\,cq\end{matrix}\Bigm|q\,;q\Bigr)+\frac{\eta^{-1}a(1-\eta)}{1-a}{}_3\phi_2\Bigl(\begin{matrix}q^{-n},\,abq^{n+1},\,q\eta\\ aq,\,cq\end{matrix}\Bigm|q\,;q\Bigr)\\
&=\frac{\eta^{-1}}{1-a}\sum_{k=0}^{n}\frac{(q^{-n},abq^{n+1},\eta\,;q)_k}{(aq,cq\,;q)_k}\frac{q^k}{(q\,;q)_k}\bigl(-(a-\eta)+a(1-\eta q^k)\bigr)\\
&=\frac{\eta^{-1}}{1-a}\sum_{k=0}^{n}\frac{(q^{-n},abq^{n+1},\eta\,;q)_k}{(aq,cq\,;q)_k}\frac{q^k}{(q\,;q)_k}(1-aq^k)\eta\\
&=\sum_{k=0}^{n}\frac{(q^{-n},abq^{n+1},\eta\,;q)_k}{(a,cq\,;q)_k}\frac{q^k}{(q\,;q)_k}\\
&=\tilde{f}_n^{\mathrm{J}}(\boldsymbol{\lambda})P_n(r'\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}).
\end{aligned}$$

The other cases are proved in the same way. □

Remark 4.1 Two formulas with $\bar{\boldsymbol{\delta}}$ and $-\bar{\boldsymbol{\delta}}$ are equivalent by interchanging $\tilde{\mathcal{F}}^{\text{J}}$ and $\tilde{\mathcal{B}}^{\text{J}}$, e.g. (4.65) and (4.66) for bqJ (d) agree with (4.66) and (4.65) for bqJ (a) with the replacements $a \to aq$ and $b \to bq^{-1}$, respectively. For dqHeI, dqHeII and SW, we do not have new factorization (4.64) and new forward and backward shift relations (4.65)–(4.66).

Remark 4.2 As in Remark 3.2, by considering $q^{-\eta\frac{d}{d\eta}}\tilde{\mathcal{F}}^{\text{J}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}^{\text{J}}(\boldsymbol{\lambda})q^{\eta\frac{d}{d\eta}}$, the above results (4.64) and (4.65)–(4.66) with $r' = q$ are rewritten as

$$\widetilde{\mathcal{H}}^{\text{J}}(\boldsymbol{\lambda}) = -\big(\tilde{\mathcal{B}}^{\text{J}}(\boldsymbol{\lambda})q^{\eta\frac{d}{d\eta}}\big)\big(q^{-\eta\frac{d}{d\eta}}\tilde{\mathcal{F}}^{\text{J}}(\boldsymbol{\lambda})\big) + \tilde{f}_0^{\text{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\text{J}}(\boldsymbol{\lambda}), \tag{4.68}$$

$$\big(q^{-\eta\frac{d}{d\eta}}\tilde{\mathcal{F}}^{\text{J}}(\boldsymbol{\lambda})\big)P_n(\eta;\boldsymbol{\lambda}) = \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}), \tag{4.69}$$

$$\big(\tilde{\mathcal{B}}^{\text{J}}(\boldsymbol{\lambda})q^{\eta\frac{d}{d\eta}}\big)P_n(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) = \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})P_n(\eta;\boldsymbol{\lambda}). \tag{4.70}$$

That is, η is not q-shifted. As an identity of polynomial, the q-shift of η is not essential.

Remark 4.3 We can show that

$$\begin{aligned}
&B_1^{\text{J}}(r'^{-1}\eta;\boldsymbol{\lambda})B_2^{\text{J}}(qr'^{-1}\eta;\boldsymbol{\lambda}) = B^{\text{J}}(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),\quad D_1^{\text{J}}(qr'^{-1}\eta;\boldsymbol{\lambda})D_2^{\text{J}}(r'^{-1}\eta;\boldsymbol{\lambda}) = D^{\text{J}}(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),\\
&B_1^{\text{J}}(r'^{-1}\eta;\boldsymbol{\lambda})D_2^{\text{J}}(qr'^{-1}\eta;\boldsymbol{\lambda}) + D_1^{\text{J}}(qr'^{-1}\eta;\boldsymbol{\lambda})B_2^{\text{J}}(r'^{-1}\eta;\boldsymbol{\lambda}) - \tilde{f}_0^{\text{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\text{J}}(\boldsymbol{\lambda})\\
&\ = B_1^{\text{J}}(q^{-1}\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})D_2^{\text{J}}(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) + D_1^{\text{J}}(q\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})B_2^{\text{J}}(\eta;\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) - \tilde{f}_0^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}),
\end{aligned} \tag{4.71}$$

which imply

$$\tilde{\mathcal{F}}^{\text{J}}(\boldsymbol{\lambda})\tilde{\mathcal{B}}^{\text{J}}(\boldsymbol{\lambda})\,\Big|_{\eta\to r'^{-1}\eta} - \tilde{f}_0^{\text{J}}(\boldsymbol{\lambda})\tilde{b}_0^{\text{J}}(\boldsymbol{\lambda}) = \tilde{\mathcal{B}}^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{\mathcal{F}}^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}) - \tilde{f}_0^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}})\tilde{b}_0^{\text{J}}(\boldsymbol{\lambda}-\bar{\boldsymbol{\delta}}). \tag{4.72}$$

5 Summary and Comments

The orthogonal polynomials in the Askey scheme satisfy second order differential or difference equations (Theorem 1) and we study them by using quantum mechanical formulation (oQM, idQM, rdQM, rdQMJ). The forward and backward shift relations are their basic properties (Theorem 2.1, 2.2), in which the degree n and the parameters $\boldsymbol{\lambda}$ are shifted. They are based on the factorizations of the differential or difference operators $\widetilde{\mathcal{H}}$ (3.1) and $\widetilde{\mathcal{H}}^{\text{J}}$ (3.18). Motivated by the recently found forward and backward x-shift relations [4], in which the coordinate x and parameters $\boldsymbol{\lambda}$ are shifted, we have tried to find new forward and backward relations. We have found new factorizations of $\widetilde{\mathcal{H}}$ (4.7), (4.18), (4.40) and $\widetilde{\mathcal{H}}^{\text{J}}$ (4.64), and based on them, we have obtained another type of forward and backward shift relations (Theorem 3.1, 3.2, 3.3, 3.4). In these new forward and backward shift relations except for some cases of rdQM and rdQMJ, only the parameters $\boldsymbol{\lambda}$ are shifted. As an identity of polynomial, the x-shift (or q-shift of η) is not essential (Remark 3.2, 4.2).

The forward and backward shift relations are related to the shape invariance property of quantum mechanical systems [7, 8, 3, 5]. It is an interesting problem to investigate the quantum mechanical implications of the new forward and backward shift relations obtained in this paper (cf. Remark 2.3, 3.4, 4.3). Especially the twelve finite rdQM cases in Remark 3.2 are interesting. In these cases, the x-shift has important implications related to the state-adding Darboux transformations [4, 9]. We will report this topic elsewhere.

The case-(1) multi-indexed orthogonal polynomials are constructed for R, qR, W, AW, M, lqJ, lqL, cH and MP, and they have shape invariant property, namely, satisfy the forward and backward shift relations like Theorem 2.1. It is an interesting problem to investigate whether these multi-indexed polynomials satisfy new forward and backward shift relations such as Theorem 3.2 and 3.3.

Acknowledgements

This work is supported by JSPS KAKENHI Grant Number JP19K03667.

A Data for § 4

We give the data for the new forward and backward shift relations in § 4.

A.1 Data for § 4.1

We present explicit forms of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{B}}$, $\tilde{f}_n$, $\tilde{b}_n$ and $\bar{\boldsymbol{\delta}}$ in § 4.1.

Operators $\tilde{\mathcal{F}}(\boldsymbol{\lambda})$ and $\tilde{\mathcal{B}}(\boldsymbol{\lambda})$:

$$\text{L}:\text{(a)}:\ \tilde{\mathcal{F}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{1}{2}x\frac{d}{dx}+g-\frac{1}{2}\ \Bigl(=\eta\frac{d}{d\eta}+g-\frac{1}{2}\Bigr),\ \ \tilde{\mathcal{B}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\frac{1}{2}\frac{1}{x}\frac{d}{dx}+1\ \Bigl(=-\frac{d}{d\eta}+1\Bigr),\tag{A.1}$$

$$\text{(b)}:\ \tilde{\mathcal{F}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}-\frac{1}{2}\frac{1}{x}\frac{d}{dx}+1\ \Bigl(=-\frac{d}{d\eta}+1\Bigr),\ \ \tilde{\mathcal{B}}(\boldsymbol{\lambda})\stackrel{\text{def}}{=}\frac{1}{2}x\frac{d}{dx}+g+\frac{1}{2}\ \Bigl(=\eta\frac{d}{d\eta}+g+\frac{1}{2}\Bigr),\tag{A.2}$$

$$\begin{aligned}\text{pJ}:\text{(a)}:\ \tilde{\mathcal{F}}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\Bigl(\tanh x+\frac{i}{\cosh x}\Bigr)\frac{d}{dx}-h-\frac{1}{2}-i\mu\ \Bigl(=(\eta+i)\frac{d}{d\eta}-h-\frac{1}{2}-i\mu\Bigr),\\ \tilde{\mathcal{B}}(\boldsymbol{\lambda})&\stackrel{\text{def}}{=}\Bigl(\tanh x-\frac{i}{\cosh x}\Bigr)\frac{d}{dx}-h+\frac{1}{2}+i\mu\ \Bigl(=(\eta-i)\frac{d}{d\eta}-h+\frac{1}{2}+i\mu\Bigr),\end{aligned}\tag{A.3}$$

$$
\begin{aligned}
\text{(b)}:\ \ \tilde{\mathcal{F}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \Bigl(\tanh x - \frac{i}{\cosh x}\Bigr)\frac{d}{dx} - h - \frac{1}{2} + i\mu\ \ \Bigl(= (\eta - i)\frac{d}{d\eta} - h - \frac{1}{2} + i\mu\Bigr),\\
\tilde{\mathcal{B}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \Bigl(\tanh x + \frac{i}{\cosh x}\Bigr)\frac{d}{dx} - h + \frac{1}{2} - i\mu\ \ \Bigl(= (\eta + i)\frac{d}{d\eta} - h + \frac{1}{2} - i\mu\Bigr).
\end{aligned}
\tag{A.4}
$$

Constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$:

$$
\text{L}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda}) = n + g - \tfrac{1}{2},\ \ \tilde{b}_n(\boldsymbol{\lambda}) = 1,\ \ \bar{\boldsymbol{\delta}} = 1, \tag{A.5}
$$

$$
\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda}) = 1,\ \ \tilde{b}_n(\boldsymbol{\lambda}) = n + g + \tfrac{1}{2},\ \ \bar{\boldsymbol{\delta}} = -1, \tag{A.6}
$$

$$
\text{pJ}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda}) = n - h - \tfrac{1}{2} - i\mu,\ \ \tilde{b}_n(\boldsymbol{\lambda}) = n - h + \tfrac{1}{2} + i\mu,\ \ \bar{\boldsymbol{\delta}} = (0, i), \tag{A.7}
$$

$$
\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda}) = n - h - \tfrac{1}{2} + i\mu,\ \ \tilde{b}_n(\boldsymbol{\lambda}) = n - h + \tfrac{1}{2} - i\mu,\ \ \bar{\boldsymbol{\delta}} = (0. - i). \tag{A.8}
$$

We remark that the second components of $\bar{\boldsymbol{\delta}}$ for pJ are unphysical values.

A.2 Data for § 4.2

We present explicit forms of $V_1(x)$, $\tilde{f}_n$, $\tilde{b}_n$ and $\bar{\boldsymbol{\delta}}$ in § 4.2. The potential function $V_2(x)$ can be obtained from (4.13).

Potential functions $V_1(x;\boldsymbol{\lambda})$:

$$
\text{cH}:\text{(a)}:\ \ V_1(x;\boldsymbol{\lambda}) = a_1 + ix,\quad \text{(b)}:\ \ V_1(x;\boldsymbol{\lambda}) = a_2 + ix, \tag{A.9}
$$

$$
\text{MP}:\text{(a)}:\ \ V_1(x;\boldsymbol{\lambda}) = a + ix,\quad \text{(b)}:\ \ V_1(x;\boldsymbol{\lambda}) = e^{i(\frac{\pi}{2}-\phi)}, \tag{A.10}
$$

$$
\begin{aligned}
&\text{W}:\ \ \text{Assume } \{a_j^*, a_k^*\} = \{a_j, a_k\} \text{ (as a set) and set } \{l, m\} = \{1,2,3,4\}\backslash\{j,k\},\\
&\qquad V_i(x;\boldsymbol{\lambda}) = V_i^{(j,k)}(x;\boldsymbol{\lambda})\ \ (i = 1,2),\quad V_1(x;\boldsymbol{\lambda}) = \frac{(a_j + ix)(a_k + ix)}{2ix + 1},
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
&\text{cdH}:\ \ \text{Assume } \{a_j^*, a_k^*\} = \{a_j, a_k\} \text{ (as a set) and set } \{l\} = \{1,2,3\}\backslash\{j,k\},\\
&\qquad V_i(x;\boldsymbol{\lambda}) = V_i^{(j,k)}(x;\boldsymbol{\lambda})\ \ (i = 1,2),\\
&\qquad \text{(a)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{(a_j + ix)(a_k + ix)}{2ix + 1},\quad \text{(b)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{a_l + ix}{2ix + 1},
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
&\text{cd}q\text{H}:\ \ \text{Assume } \{a_j^*, a_k^*\} = \{a_j, a_k\} \text{ (as a set) and set } \{l\} = \{1,2,3\}\backslash\{j,k\},\\
&\qquad V_i(x;\boldsymbol{\lambda}) = V_i^{(j,k)}(x;\boldsymbol{\lambda})\ \ (i = 1,2),\\
&\qquad \text{(a)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{(1 - a_j e^{ix})(1 - a_k e^{ix})}{1 - qe^{2ix}},\quad \text{(b)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{1 - a_l e^{ix}}{1 - qe^{2ix}},
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
&\text{ASC}:\ \ \text{Assume } a_1, a_2 \in \mathbb{R} \text{ for (b) and (c)},\\
&\qquad \text{(a)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{(1 - a_1 e^{ix})(1 - a_2 e^{ix})}{1 - qe^{2ix}},\quad \text{(b)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{1 - a_1 e^{ix}}{1 - qe^{2ix}},
\end{aligned}
\tag{A.14}
$$

$$
\text{(c)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{1 - a_2 e^{ix}}{1 - qe^{2ix}},\quad \text{(d)}:\ \ V_1(x;\boldsymbol{\lambda}) = \frac{1}{1 - qe^{2ix}}, \tag{A.15}
$$

$$\text{cb}q\text{He}: \text{(a)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{1-ae^{ix}}{1-qe^{2ix}}, \quad \text{(b)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{1}{1-qe^{2ix}}, \tag{A.16}$$

$$\text{c}q\text{He}: \ V_1(x;\boldsymbol{\lambda}) = \frac{1}{1-qe^{2ix}}, \tag{A.17}$$

$$\text{c}q\text{J}: \text{(a)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1-q^{\frac12(\alpha+\frac12)}e^{ix})(1-q^{\frac12(\alpha+\frac32)}e^{ix})}{1-qe^{2ix}}, \tag{A.18}$$

$$\text{(b)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1+q^{\frac12(\beta+\frac12)}e^{ix})(1+q^{\frac12(\beta+\frac32)}e^{ix})}{1-qe^{2ix}}, \tag{A.19}$$

$$\text{c}q\text{L}: \text{(a)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1-q^{\frac12(\alpha+\frac12)}e^{ix})(1-q^{\frac12(\alpha+\frac32)}e^{ix})}{1-qe^{2ix}}, \quad \text{(b)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{1}{1-qe^{2ix}}, \tag{A.20}$$

$$\text{c}q\text{H}: \text{(a)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1-a_1e^{2i\phi}e^{ix})(1-a_1^*e^{ix})}{1-qe^{2i\phi}e^{2ix}}, \tag{A.21}$$

$$\text{(b)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1-a_2e^{2i\phi}e^{ix})(1-a_2^*e^{ix})}{1-qe^{2i\phi}e^{2ix}}, \tag{A.22}$$

$$q\text{MP}: \text{(a)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{(1-ae^{2i\phi}e^{ix})(1-ae^{ix})}{1-qe^{2i\phi}e^{2ix}}, \quad \text{(b)}: \ V_1(x;\boldsymbol{\lambda}) = \frac{1}{1-qe^{2i\phi}e^{2ix}}. \tag{A.23}$$

Constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$:

$$\text{cH}: \text{(a)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = a_1+a_1^*+n-1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = a_2+a_2^*+n, \ \ \bar{\boldsymbol{\delta}} = (\tfrac12,-\tfrac12), \tag{A.24}$$

$$\text{(b)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = a_2+a_2^*+n-1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = a_1+a_1^*+n, \ \ \bar{\boldsymbol{\delta}} = (-\tfrac12,\tfrac12), \tag{A.25}$$

$$\text{MP}: \text{(a)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = 2a+n-1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = 2\sin\phi, \ \ \bar{\boldsymbol{\delta}} = (\tfrac12,0), \tag{A.26}$$

$$\text{(b)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = 2\sin\phi, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = 2a+n, \ \ \bar{\boldsymbol{\delta}} = (-\tfrac12,0), \tag{A.27}$$

$$\begin{aligned}\text{W}: \ &\tilde{f}_n(\boldsymbol{\lambda}) = a_j+a_k+n-1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = a_l+a_m+n,\\ &(\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = \tfrac12, \ \ (\bar{\boldsymbol{\delta}})_l = (\bar{\boldsymbol{\delta}})_m = -\tfrac12,\end{aligned} \tag{A.28}$$

$$\text{cdH}: \text{(a)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = a_j+a_k+n-1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = 1, \ \ (\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = \tfrac12, \ \ (\bar{\boldsymbol{\delta}})_l = -\tfrac12, \tag{A.29}$$

$$\text{(b)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = 1, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = a_j+a_k+n, \ \ (\bar{\boldsymbol{\delta}})_l = \tfrac12, \ \ (\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = -\tfrac12, \tag{A.30}$$

$$\text{cd}q\text{H}: \text{(a)}: \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_ja_kq^{n-1}), \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ (\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = \tfrac12, \ (\bar{\boldsymbol{\delta}})_l = -\tfrac12, \tag{A.31}$$

$$\text{(b)}: \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_ja_kq^{n}), \ (\bar{\boldsymbol{\delta}})_l = \tfrac12, \ (\bar{\boldsymbol{\delta}})_j = (\bar{\boldsymbol{\delta}})_k = -\tfrac12, \tag{A.32}$$

$$\text{ASC}: \text{(a)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_1a_2q^{n-1}), \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \bar{\boldsymbol{\delta}} = (\tfrac12,\tfrac12), \tag{A.33}$$

$$\text{(b)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \bar{\boldsymbol{\delta}} = (\tfrac12,-\tfrac12), \tag{A.34}$$

$$\text{(c)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \bar{\boldsymbol{\delta}} = (-\tfrac12,\tfrac12), \tag{A.35}$$

$$\text{(d)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}(1-a_1a_2q^{n}), \ \ \bar{\boldsymbol{\delta}} = (-\tfrac12,-\tfrac12), \tag{A.36}$$

$$\text{cb}q\text{He}: \text{(a)}: \ \tilde{f}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \tilde{b}_n(\boldsymbol{\lambda}) = q^{-\frac{n}{2}}, \ \ \bar{\boldsymbol{\delta}} = \tfrac12, \tag{A.37}$$

$$(\text{b}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \bar{\boldsymbol{\delta}}=-\tfrac{1}{2}, \tag{A.38}$$

$$c q\text{He}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \bar{\boldsymbol{\delta}}:\text{none}, \tag{A.39}$$

$$c q\text{J}:(\text{a}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=1-q^{\alpha+n},\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-n}(1-q^{\beta+n+1}),\ \ \bar{\boldsymbol{\delta}}=(1,-1), \tag{A.40}$$

$$(\text{b}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-n}(1-q^{\beta+n}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1-q^{\alpha+n+1},\ \ \bar{\boldsymbol{\delta}}=(-1,1), \tag{A.41}$$

$$c q\text{L}:(\text{a}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=1-q^{\alpha+n},\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-n},\ \ \bar{\boldsymbol{\delta}}=1, \tag{A.42}$$

$$(\text{b}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-n},\ \ \tilde{b}_n(\boldsymbol{\lambda})=1-q^{\alpha+n+1},\ \ \bar{\boldsymbol{\delta}}=-1, \tag{A.43}$$

$$c q\text{H}:(\text{a}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a_1a_1^*q^{n-1}),\ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a_2a_2^*q^{n}),\ \bar{\boldsymbol{\delta}}=(\tfrac{1}{2},-\tfrac{1}{2},0), \tag{A.44}$$

$$(\text{b}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a_2a_2^*q^{n-1}),\ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a_1a_1^*q^{n}),\ \bar{\boldsymbol{\delta}}=(-\tfrac{1}{2},\tfrac{1}{2},0), \tag{A.45}$$

$$q\text{MP}:(\text{a}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a^2q^{n-1}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \bar{\boldsymbol{\delta}}=(\tfrac{1}{2},0), \tag{A.46}$$

$$(\text{b}):\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}},\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^{-\frac{n}{2}}(1-a^2q^{n}),\ \ \bar{\boldsymbol{\delta}}=(-\tfrac{1}{2},0). \tag{A.47}$$

A.3 Data for § 4.3

We present explicit forms of $B_1(x)$, $D_1(x)$, $\tilde{f}_n$, $\tilde{b}_n$ and $\bar{\boldsymbol{\delta}}$ in § 4.3. The potential functions $B_2(x)$ and $D_2(x)$ can be obtained from (4.25).

Potential functions $B_1(x;\boldsymbol{\lambda})$ and $D_1(x;\boldsymbol{\lambda})$:

$$\text{H}:(\text{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{N-x}{N+1},\ \ D_1(x;\boldsymbol{\lambda})=\frac{x}{N+1}, \tag{A.48}$$

$$(\text{b}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{x+a}{a-1},\ \ D_1(x;\boldsymbol{\lambda})=\frac{x}{1-a}, \tag{A.49}$$

$$(\text{c}):\ \ B_1(x;\boldsymbol{\lambda})=a(N-x),\ \ D_1(x;\boldsymbol{\lambda})=-a(b+N-x), \tag{A.50}$$

$$(\text{d}):\ \ B_1(x;\boldsymbol{\lambda})=N(x+a),\ \ D_1(x;\boldsymbol{\lambda})=N(b+N-x), \tag{A.51}$$

$$\text{K}:(\text{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{N-x}{N+1},\ \ D_1(x;\boldsymbol{\lambda})=\frac{x}{N+1}, \tag{A.52}$$

$$(\text{b}):\ \ B_1(x;\boldsymbol{\lambda})=Np,\ \ D_1(x;\boldsymbol{\lambda})=N(1-p), \tag{A.53}$$

$$\text{R}:(\text{a}):\ \ B_1(x;\boldsymbol{\lambda})=-\frac{(x-N)(x+d)}{(N+1)(2x+1+d)},\ \ D_1(x;\boldsymbol{\lambda})=\frac{(x+d+N)x}{(N+1)(2x-1+d)}, \tag{A.54}$$

$$(\text{b}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{(x+b)(x+d)}{(b-1)(2x+1+d)},\ \ D_1(x;\boldsymbol{\lambda})=-\frac{(x+d-b)x}{(b-1)(2x-1+d)}, \tag{A.55}$$

$$(\text{c}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{(x+c)(x+d)}{(c-1)(2x+1+d)},\ \ D_1(x;\boldsymbol{\lambda})=-\frac{(x+d-c)x}{(c-1)(2x-1+d)}, \tag{A.56}$$

$$(\text{d}):\ \ B_1(x;\boldsymbol{\lambda})=-\frac{c(x-N)(x+b)}{2x+1+d},\ \ D_1(x;\boldsymbol{\lambda})=\frac{c(x+d+N)(x+d-b)}{2x-1+d}, \tag{A.57}$$

$$(\text{e}):\ \ B_1(x;\boldsymbol{\lambda})=-\frac{b(x-N)(x+c)}{2x+1+d},\ \ D_1(x;\boldsymbol{\lambda})=\frac{b(x+d+N)(x+d-c)}{2x-1+d}, \tag{A.58}$$

$$(\mathrm{f}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{N(x+b)(x+c)}{2x+1+d},\quad D_1(x;\boldsymbol{\lambda})=-\frac{N(x+d-b)(x+d-c)}{2x-1+d},\tag{A.59}$$

$$\mathrm{dH}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{(x+a+b-1)(N-x)}{(N+1)(2x+a+b)},\quad D_1(x;\boldsymbol{\lambda})=\frac{x(x+a+b+N-1)}{(N+1)(2x-2+a+b)},\tag{A.60}$$

$$(\mathrm{b}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{(x+a)(x+a+b-1)}{(a-1)(2x+a+b)},\quad D_1(x;\boldsymbol{\lambda})=\frac{x(x+b-1)}{(1-a)(2x-2+a+b)},\tag{A.61}$$

$$(\mathrm{c}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{x+a+b-1}{2x+a+b},\quad D_1(x;\boldsymbol{\lambda})=\frac{x}{2x-2+a+b},\tag{A.62}$$

$$(\mathrm{d}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{(x+a)(N-x)}{2x+a+b},\quad D_1(x;\boldsymbol{\lambda})=\frac{(x+b-1)(x+a+b+N-1)}{2x-2+a+b},\tag{A.63}$$

$$(\mathrm{e}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{a(N-x)}{2x+a+b},\quad D_1(x;\boldsymbol{\lambda})=-\frac{a(x+a+b+N-1)}{2x-2+a+b},\tag{A.64}$$

$$(\mathrm{f}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{N(x+a)}{2x+a+b},\quad D_1(x;\boldsymbol{\lambda})=\frac{N(x+b-1)}{2x-2+a+b},\tag{A.65}$$

$$\mathrm{dq}q\mathrm{K}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{q^{-N-1}(1-q^{N-x})}{q^{-N-1}-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{q^{-x}-1}{q^{-N-1}-1},\tag{A.66}$$

$$(\mathrm{b}):\ \ B_1(x;\boldsymbol{\lambda})=q^{-x-1},\quad D_1(x;\boldsymbol{\lambda})=-(q^{-x}-1),\tag{A.67}$$

$$(\mathrm{c}):\ \ B_1(x;\boldsymbol{\lambda})=p^{-1}q^{-N-1}(1-q^{N-x}),\quad D_1(x;\boldsymbol{\lambda})=-(1-p^{-1}q^{-x}),\tag{A.68}$$

$$(\mathrm{d}):\ \ B_1(x;\boldsymbol{\lambda})=(1-q^N)p^{-1}q^{-x-N-1},\quad D_1(x;\boldsymbol{\lambda})=(q^{-N}-1)(1-p^{-1}q^{-x}),\tag{A.69}$$

$$q\mathrm{H}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{q^{x-N}-1}{q^{-N-1}-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{1-q^x}{1-q^{N+1}},\tag{A.70}$$

$$(\mathrm{b}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{1-aq^x}{1-aq^{-1}},\quad D_1(x;\boldsymbol{\lambda})=\frac{aq^{-1}(1-q^x)}{aq^{-1}-1},\tag{A.71}$$

$$(\mathrm{c}):\ \ B_1(x;\boldsymbol{\lambda})=(1-a)(q^{x-N}-1),\quad D_1(x;\boldsymbol{\lambda})=(a-1)q^{-1}(q^{x-N}-b),\tag{A.72}$$

$$(\mathrm{d}):\ \ B_1(x;\boldsymbol{\lambda})=(q^{-N}-1)(1-aq^x),\quad D_1(x;\boldsymbol{\lambda})=(1-q^N)aq^{-1}(q^{x-N}-b),\tag{A.73}$$

$$q\mathrm{K}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{q^{x-N}-1}{q^{-N-1}-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{1-q^x}{1-q^{N+1}},\tag{A.74}$$

$$(\mathrm{b}):\ \ B_1(x;\boldsymbol{\lambda})=q^{-N}-1,\quad D_1(x;\boldsymbol{\lambda})=(1-q^N)p,\tag{A.75}$$

$$\mathrm{q}q\mathrm{K}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{q^{x-N}-1}{q^{-N-1}-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{1-q^x}{1-q^{N+1}},\tag{A.76}$$

$$(\mathrm{b}):\ \ B_1(x;\boldsymbol{\lambda})=qq^x,\quad D_1(x;\boldsymbol{\lambda})=1-q^x,\tag{A.77}$$

$$(\mathrm{c}):\ \ B_1(x;\boldsymbol{\lambda})=p^{-1}(q^{x-N}-1),\quad D_1(x;\boldsymbol{\lambda})=1-p^{-1}q^{x-N-1},\tag{A.78}$$

$$(\mathrm{d}):\ \ B_1(x;\boldsymbol{\lambda})=(q^{-N}-1)p^{-1}q^x,\quad D_1(x;\boldsymbol{\lambda})=(1-q^N)(1-p^{-1}q^{x-N-1}),\tag{A.79}$$

$$\mathrm{a}q\mathrm{K}:(\mathrm{a}):\ \ B_1(x;\boldsymbol{\lambda})=\frac{q^{x-N}-1}{q^{-N-1}-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{1-q^x}{1-q^{N+1}},\tag{A.80}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=\frac{1-pq^{x+1}}{1-p},\quad D_1(x;\boldsymbol{\lambda})=\frac{p(1-q^x)}{p-1},\tag{A.81}$$

$$\text{(c)}:\ B_1(x;\boldsymbol{\lambda})=(1-pq)(q^{x-N}-1),\quad D_1(x;\boldsymbol{\lambda})=(pq-1)q^{x-N-1},\tag{A.82}$$

$$\text{(d)}:\ B_1(x;\boldsymbol{\lambda})=(q^{-N}-1)(1-pq^{x+1}),\quad D_1(x;\boldsymbol{\lambda})=(q^{-N}-1)pq^x,\tag{A.83}$$

$$\mathrm{d}q\mathrm{H}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(q^{x-N}-1)(1-abq^{x-1})}{(q^{-N-1}-1)(1-abq^{2x})},\quad D_1(x;\boldsymbol{\lambda})=\frac{(1-q^x)(1-abq^{x+N-1})}{(1-q^{N+1})(1-abq^{2x-2})},\tag{A.84}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(1-aq^x)(1-abq^{x-1})}{(1-aq^{-1})(1-abq^{2x})},\quad D_1(x;\boldsymbol{\lambda})=\frac{(1-q^x)(1-bq^{x-1})}{(1-a^{-1}q)(1-abq^{2x-2})},\tag{A.85}$$

$$\text{(c)}:\ B_1(x;\boldsymbol{\lambda})=\frac{1-abq^{x-1}}{1-abq^{2x}},\quad D_1(x;\boldsymbol{\lambda})=\frac{bq^{-2}aq^x(1-q^x)}{1-abq^{2x-2}},\tag{A.86}$$

$$\text{(d)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(q^{x-N}-1)(1-aq^x)}{1-abq^{2x}},\quad D_1(x;\boldsymbol{\lambda})=\frac{q^{-N}(1-abq^{x+N-1})(1-bq^{x-1})}{b(1-abq^{2x-2})},\tag{A.87}$$

$$\text{(e)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(1-a)(q^{x-N}-1)}{1-abq^{2x}},\quad D_1(x;\boldsymbol{\lambda})=\frac{(a-1)q^{x-N-1}(1-abq^{x+N-1})}{1-abq^{2x-2}},\tag{A.88}$$

$$\text{(f)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(q^{-N}-1)(1-aq^x)}{1-abq^{2x}},\quad D_1(x;\boldsymbol{\lambda})=\frac{(1-q^N)aq^{x-N-1}(1-bq^{x-1})}{1-abq^{2x-2}},\tag{A.89}$$

$$\mathrm{d}q\mathrm{K}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=\frac{(q^{x-N}-1)(1+pq^x)}{(q^{-N-1}-1)(1+pq^{2x+1})},\quad D_1(x;\boldsymbol{\lambda})=\frac{(1-q^x)(1+pq^{x+N})}{(1-q^{N+1})(1+pq^{2x-1})},\tag{A.90}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=\frac{1+pq^x}{1+pq^{2x+1}},\quad D_1(x;\boldsymbol{\lambda})=-\frac{pq^{x-1}(1-q^x)}{1+pq^{2x-1}},\tag{A.91}$$

$$\text{(c)}:\ B_1(x;\boldsymbol{\lambda})=\frac{q^{x-N}-1}{1+pq^{2x+1}},\quad D_1(x;\boldsymbol{\lambda})=-\frac{q^{x-N-1}(1+pq^{x+N})}{1+pq^{2x-1}},\tag{A.92}$$

$$\text{(d)}:\ B_1(x;\boldsymbol{\lambda})=\frac{q^{-N}-1}{1+pq^{2x+1}},\quad D_1(x;\boldsymbol{\lambda})=\frac{(1-q^N)pq^{2x-N-1}}{1+pq^{2x-1}},\tag{A.93}$$

$$\mathrm{M}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=\frac{x+\beta}{\beta-1},\quad D_1(x;\boldsymbol{\lambda})=\frac{x}{1-\beta},\tag{A.94}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=\frac{\beta c}{1-c},\quad D_1(x;\boldsymbol{\lambda})=-\frac{\beta}{1-c},\tag{A.95}$$

$$\mathrm{l}q\mathrm{J}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=\frac{q^{-1}(q^{-x}-b)}{1-bq^{-1}},\quad D_1(x;\boldsymbol{\lambda})=\frac{q^{-x}-1}{bq^{-1}-1},\tag{A.96}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=(1-b)aq^{-1},\quad D_1(x;\boldsymbol{\lambda})=b-1,\tag{A.97}$$

$$\mathrm{l}q\mathrm{L}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=q^{-x-1},\quad D_1(x;\boldsymbol{\lambda})=-(q^{-x}-1),\tag{A.98}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=aq^{-1},\quad D_1(x;\boldsymbol{\lambda})=-1,\tag{A.99}$$

$$q\mathrm{M}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=q^{x+1},\quad D_1(x;\boldsymbol{\lambda})=1-q^x, \tag{A.100}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=\frac{1-bq^{x+1}}{1-b},\quad D_1(x;\boldsymbol{\lambda})=\frac{1-q^x}{1-b^{-1}}, \tag{A.101}$$

$$\text{(c)}:\ B_1(x;\boldsymbol{\lambda})=(1-bq)cq^x,\quad D_1(x;\boldsymbol{\lambda})=(1-b^{-1}q^{-1})(1+bcq^x), \tag{A.102}$$

$$\text{(d)}:\ B_1(x;\boldsymbol{\lambda})=c(1-bq^{x+1}),\quad D_1(x;\boldsymbol{\lambda})=1+bcq^x, \tag{A.103}$$

$$\mathrm{ASCII}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=q^{x+1},\quad D_1(x;\boldsymbol{\lambda})=1-q^x, \tag{A.104}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=aq^{x+1},\quad D_1(x;\boldsymbol{\lambda})=1-aq^x, \tag{A.105}$$

$$q\mathrm{C}:\text{(a)}:\ B_1(x;\boldsymbol{\lambda})=q^{x+1},\quad D_1(x;\boldsymbol{\lambda})=1-q^x, \tag{A.106}$$

$$\text{(b)}:\ B_1(x;\boldsymbol{\lambda})=a,\quad D_1(x;\boldsymbol{\lambda})=1. \tag{A.107}$$

Constants $\tilde{f}_n(\boldsymbol{\lambda})$, $\tilde{b}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$:

$$\mathrm{H}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,0,-1), \tag{A.108}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-(n+a-1)(n+b),\quad \bar{\boldsymbol{\delta}}=(1,-1,0), \tag{A.109}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-(n+a)(n+b-1),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,1,0), \tag{A.110}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,0,1), \tag{A.111}$$

$$\mathrm{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,-1), \tag{A.112}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,1), \tag{A.113}$$

$$\mathrm{R}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\ \ \bar{\boldsymbol{\delta}}=(1,0,0,1), \tag{A.114}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-(n+b-1)(n+c-d-N),\ \ \bar{\boldsymbol{\delta}}=(0,1,0,1), \tag{A.115}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-(n+c-1)(n+b-d-N),\ \ \bar{\boldsymbol{\delta}}=(0,0,1,1), \tag{A.116}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-(n+c)(n+b-d-N-1),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(0,0,-1,-1), \tag{A.117}$$

$$\text{(e)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-(n+b)(n+c-d-N-1),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(0,-1,0,-1), \tag{A.118}$$

$$\text{(f)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(-1,0,0,-1), \tag{A.119}$$

$$\mathrm{dH}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,1,-1), \tag{A.120}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-(n+a-1),\quad \bar{\boldsymbol{\delta}}=(1,0,0), \tag{A.121}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-(n-b-N+1),\quad \bar{\boldsymbol{\delta}}=(0,1,0), \tag{A.122}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-(n-b-N),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,-1,0), \tag{A.123}$$

$$\text{(e)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-(n+a),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0,0), \tag{A.124}$$

$$\text{(f)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,-1,1), \tag{A.125}$$

$$\mathrm{dq}q\mathrm{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(1,-1), \tag{A.126}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-p^{-1}q^{n-N}),\quad \bar{\boldsymbol{\delta}}=(1,0),\tag{A.127}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-p^{-1}q^{n-N-1}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0),\tag{A.128}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,1),\tag{A.129}$$

$$q\text{H}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,0,-1),\tag{A.130}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^{n-1})(1-bq^n),\quad \bar{\boldsymbol{\delta}}=(1,-1,0),\tag{A.131}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^n)(1-bq^{n-1}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,1,0),\tag{A.132}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,0,1),\tag{A.133}$$

$$q\text{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,-1),\tag{A.134}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,1),\tag{A.135}$$

$$\text{q}q\text{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,-1),\tag{A.136}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-p^{-1}q^{-1}(1-pq^{n+1}),\quad \bar{\boldsymbol{\delta}}=(-1,0),\tag{A.137}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-p^{-1}(1-pq^n)\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(1,0),\tag{A.138}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,1),\tag{A.139}$$

$$\text{a}q\text{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,-1),\tag{A.140}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-pq^n),\quad \bar{\boldsymbol{\delta}}=(1,0),\tag{A.141}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-pq^{n+1}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0),\tag{A.142}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,1),\tag{A.143}$$

$$\text{d}q\text{H}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(0,1,-1),\tag{A.144}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^{n-1}),\quad \bar{\boldsymbol{\delta}}=(1,0,0),\tag{A.145}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-b^{-1}q^{n-N+1}),\quad \bar{\boldsymbol{\delta}}=(0,1,0),\tag{A.146}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-b^{-1}q^{n-N}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,-1,0),\tag{A.147}$$

$$\text{(e)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^n),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0,0),\tag{A.148}$$

$$\text{(f)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,-1,1),\tag{A.149}$$

$$\text{d}q\text{K}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=\mathcal{E}_{N+1}(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \bar{\boldsymbol{\delta}}=(1,-1),\tag{A.150}$$

$$\text{(b)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n},\quad \bar{\boldsymbol{\delta}}=(1,0),\tag{A.151}$$

$$\text{(c)}:\ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n},\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0),\tag{A.152}$$

$$\text{(d)}:\ \tilde{f}_n(\boldsymbol{\lambda})=\mathcal{E}_N(\boldsymbol{\lambda})-\mathcal{E}_n(\boldsymbol{\lambda}),\quad \tilde{b}_n(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,1),\tag{A.153}$$

$$\text{M}:\text{(a)}:\ \tilde{f}_n(\boldsymbol{\lambda})=1,\quad \tilde{b}_n(\boldsymbol{\lambda})=-(n+\beta-1),\quad \bar{\boldsymbol{\delta}}=(1,0),\tag{A.154}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=-(n+\beta),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(-1,0), \tag{A.155}$$

$$\text{l}q\text{J}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^n)(1-bq^{n-1}),\ \ \bar{\boldsymbol{\delta}}=(-1,1), \tag{A.156}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^{n-1})(1-bq^n),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(1,-1), \tag{A.157}$$

$$\text{l}q\text{L}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^n),\ \ \bar{\boldsymbol{\delta}}=-1, \tag{A.158}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^{n-1}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=1, \tag{A.159}$$

$$q\text{M}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^n+cq^{-1},\ \ \bar{\boldsymbol{\delta}}=(0,1), \tag{A.160}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=-b^{-1}(1-bq^n),\ \ \bar{\boldsymbol{\delta}}=(1,0), \tag{A.161}$$

$$\text{(c)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=-b^{-1}q^{-1}(1-bq^{n+1}),\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(-1,0), \tag{A.162}$$

$$\text{(d)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^n+c,\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=(0,-1), \tag{A.163}$$

$$\text{ASCII}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^n,\ \ \bar{\boldsymbol{\delta}}=1, \tag{A.164}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^n,\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=-1, \tag{A.165}$$

$$q\text{C}:\text{(a)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}_n(\boldsymbol{\lambda})=q^n+aq^{-1},\ \ \bar{\boldsymbol{\delta}}=1, \tag{A.166}$$

$$\text{(b)}:\ \ \tilde{f}_n(\boldsymbol{\lambda})=q^n+a,\ \ \tilde{b}_n(\boldsymbol{\lambda})=1,\ \ \bar{\boldsymbol{\delta}}=-1. \tag{A.167}$$

A.4 Data for § 4.4

We present explicit forms of $B^{\text{J}}_1(\eta)$, $D^{\text{J}}_1(\eta)$, $\tilde{f}_n$, $\tilde{b}_n$ and $\bar{\boldsymbol{\delta}}$ in § 4.4. The potential functions $B^{\text{J}}_2(\eta)$ and $D^{\text{J}}_2(\eta)$ can be obtained from (4.53).

Potential functions $B^{\text{J}}_1(\eta;\boldsymbol{\lambda})$ and $D^{\text{J}}_1(\eta;\boldsymbol{\lambda})$:

$$\text{b}q\text{L}:\text{(a)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\frac{\eta^{-1}a(1-\eta)}{1-a},\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\frac{\eta^{-1}(aq-\eta)}{a-1}, \tag{A.168}$$

$$\text{(b)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\frac{\eta^{-1}b(1-\eta)}{1-b},\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\frac{\eta^{-1}(\eta-bq)}{1-b}, \tag{A.169}$$

$$\text{(c)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=(bq-1)\eta^{-1}a,\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=(1-bq)\eta^{-1}(aq-\eta), \tag{A.170}$$

$$\text{(d)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=(aq-1)\eta^{-1}b,\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=(aq-1)\eta^{-1}(\eta-bq), \tag{A.171}$$

$$\text{ASCI}:\text{(a)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\eta^{-1}q^{-1},\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=-\eta^{-1}(1-\eta), \tag{A.172}$$

$$\text{(b)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=-\eta^{-1}aq^{-1},\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=-\eta^{-1}(\eta-a), \tag{A.173}$$

$$q\text{L}:\text{(a)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=\eta^{-1}(1+\eta),\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=-\eta^{-1}q, \tag{A.174}$$

$$\text{(b)}:\ \ B^{\text{J}}_1(\eta;\boldsymbol{\lambda})=1,\ \ D^{\text{J}}_1(\eta;\boldsymbol{\lambda})=-a^{-1}. \tag{A.175}$$

Constants $\tilde{f}^{\text{J}}_n(\boldsymbol{\lambda})$, $\tilde{b}^{\text{J}}_n(\boldsymbol{\lambda})$ and $\bar{\boldsymbol{\delta}}$:

$$\text{b}q\text{L}:\text{(a)}:\ \ \tilde{f}^{\text{J}}_n(\boldsymbol{\lambda})=1,\ \ \tilde{b}^{\text{J}}_n(\boldsymbol{\lambda})=-q^{-n}(1-aq^n),\ \ \bar{\boldsymbol{\delta}}=(1,0), \tag{A.176}$$

$$\text{(b)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=-q^{-n}(1-bq^n),\quad \bar{\boldsymbol{\delta}}=(0,1), \tag{A.177}$$

$$\text{(c)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=-q^{-n}(1-bq^{n+1}),\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(0,-1), \tag{A.178}$$

$$\text{(d)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=-q^{-n}(1-aq^{n+1}),\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=(-1,0), \tag{A.179}$$

$$\text{ASCI}: \text{(a)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=-q^{-n},\quad \bar{\boldsymbol{\delta}}=-1, \tag{A.180}$$

$$\text{(b)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=-q^{-n},\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=1, \tag{A.181}$$

$$q\text{L}: \text{(a)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=-a^{-1}q^{-1}(1-aq^{n+1}),\quad \bar{\boldsymbol{\delta}}=-1, \tag{A.182}$$

$$\text{(b)}: \quad \tilde{f}_n^{\text{J}}(\boldsymbol{\lambda})=-a^{-1}(1-aq^n),\quad \tilde{b}_n^{\text{J}}(\boldsymbol{\lambda})=1,\quad \bar{\boldsymbol{\delta}}=1. \tag{A.183}$$

References

[1] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, vol. 98 of Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge (2005).

[2] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues,* Springer-Verlag Berlin-Heidelberg (2010).

[3] S. Odake and R. Sasaki, "Discrete quantum mechanics," (Topical Review) J. Phys. **A44** (2011) 353001 (47pp), `arXiv:1104.0473[math-ph]`. Typo in (2.132), $c_1(\eta,\boldsymbol{\lambda})$ for H : $-\frac{1}{2} \Rightarrow -\frac{\eta}{2}$.

[4] S. Odake and R. Sasaki, "“Diophantine” and Factorisation Properties of Finite Orthogonal Polynomials in the Askey Scheme," `arXiv:2207.14479[math.CA]`.

[5] S. Odake and R. Sasaki, "Orthogonal Polynomials from Hermitian Matrices II," J. Math. Phys. **59** (2018) 013504 (42pp), `arXiv:1604.00714[math.CA]`.

[6] S. Odake and R. Sasaki, "Unified theory of annihilation-creation operators for solvable ('discrete') quantum mechanics," J. Math. Phys. **47** (2006) 102102 (33pp), `arXiv:quant-ph/0605215`.

[7] S. Odake and R. Sasaki, "Orthogonal Polynomials from Hermitian Matrices," J. Math. Phys. **49** (2008) 053503 (43pp), `arXiv:0712.4106[math.CA]`. (For the dual q-Meixner and dual q-Charlier polynomials, see [5].)

[8] S. Odake and R. Sasaki, “Exactly solvable ‘discrete’ quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states,” Prog. Theor. Phys. **119** (2008) 663-700, `arXiv:0802.1075[quant-ph]`.

[9] S. Odake, “New Finite Type Multi-Indexed Orthogonal Polynomials Obtained From State-Adding Darboux Transformations,” `arXiv:2209.12353[math-ph]`.