

Type II Multi-indexed Little q -Jacobi and Little q -Laguerre Polynomials

Satoru Odake

Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan

Abstract

For the isospectral Darboux transformations of the discrete quantum mechanics with real shifts, there are two methods: type I and type II constructions. Based on the type I construction, the type I multi-indexed little q -Jacobi and little q -Laguerre orthogonal polynomials were presented in J.Phys. **A50** (2017) 165204. Based on the type II construction, we present the type II multi-indexed little q -Jacobi and little q -Laguerre orthogonal polynomials.

1 Introduction

The new type of orthogonal polynomials – exceptional and multi-indexed polynomials – [1]–[14] have the characteristic that they form a complete set of orthogonal basis in spite of the missing degrees, by which the restrictions of Bochner’s theorem [15] are avoided. They are constructed based on the polynomials in the Askey-scheme of hypergeometric orthogonal polynomials [16], which satisfy second order differential or difference equations. To study such orthogonal polynomials, the quantum mechanical formulation is very useful. We consider ordinary quantum mechanics (oQM) and two kinds of discrete quantum mechanics (dQM), dQM with pure imaginary shifts (idQM) and dQM with real shifts (rdQM) [17]. The Schrödinger equation for oQM is a differential equation and that for dQM is a difference equation. The coordinate x for oQM and idQM is continuous and that for rdQM is discrete.

The multi-indexed orthogonal polynomials are systematically constructed by the multi-step Darboux transformations for quantum mechanical systems. When the wavefunctions of the virtual states are used as seed solutions, the multi-step Darboux transformations give the case-(1) multi-indexed orthogonal polynomials. Here, the case-(1) is the case that the

set of missing degrees of the multi-indexed polynomials is $\{0, 1, \dots, \ell - 1\}$, and the case-(2) is otherwise. The quantum mechanical systems associated to the case-(1) multi-indexed orthogonal polynomials have shape invariance [10]–[14].

For oQM (Jacobi and Laguerre [10]) and idQM (Askey-Wilson and Wilson [12], continuous Hahn [14]) cases, there are two types of virtual states, type I and type II. For rdQM (finite: q -Racah and Racah [11], semi-infinite: Meixner, little q -Jacobi and little q -Laguerre [13]) cases, we considered one type of virtual state. In the first manuscript of [11] (arXiv:1203.5868v1), we considered two types of virtual states (type I and II) for Racah and q -Racah cases. But the multi-step Darboux transformations with these two types of virtual states as seed solutions give essentially the same multi-indexed polynomials, because these rdQM systems are finite systems. So we discussed only one type of virtual state in [11]. However, this situation may be different for semi-infinite systems. The purpose of this paper is to consider two types of virtual states for the semi-infinite rdQM systems and to obtain the type II multi-indexed little q -Jacobi and little q -Laguerre polynomials.

This paper is organized as follows. In section 2 the finite and semi-infinite rdQM systems are recapitulated and the multi-step Darboux transformations are discussed. There are two methods, the type I and type II constructions. Section 3 is the main part of the paper. Based on the type II construction, we obtain the case-(1) type II multi-indexed little q -Jacobi polynomials. Similarly, the case-(1) type II multi-indexed little q -Laguerre polynomials are obtained in section 4. Section 5 is for a summary and comments.

2 Darboux Transformations for rdQM

In this section we recapitulate the isospectral Darboux transformations for rdQM systems [11]. The first manuscript of [11] is arXiv:1203.5868v1 and we will cite it as [11](v1).

2.1 rdQM systems

The Hamiltonian \mathcal{H} of a finite rdQM system is a real symmetric matrix, and we consider a tri-diagonal one [18],

$$\mathcal{H} = (\mathcal{H}_{x,y})_{x,y \in \{0,1,\dots,N\}}, \quad (2.1)$$

$$\mathcal{H}_{x,y} = -\sqrt{B(x)D(x+1)}\delta_{x+1,y} - \sqrt{B(x-1)D(x)}\delta_{x-1,y} + (B(x) + D(x))\delta_{x,y}, \quad (2.2)$$

where the potential functions $B(x)$ and $D(x)$ are positive but vanish at the boundary,

$$\begin{aligned} B(x) &> 0 \quad (x = 0, 1, \dots, N-1), \quad B(N) = 0, \\ D(x) &> 0 \quad (x = 1, 2, \dots, N), \quad D(0) = 0. \end{aligned} \quad (2.3)$$

We write this \mathcal{H} as

$$\begin{aligned} \mathcal{H} &= -\sqrt{B(x)D(x+1)}e^\partial - \sqrt{B(x-1)D(x)}e^{-\partial} + B(x) + D(x) \\ &= -\sqrt{B(x)}e^\partial\sqrt{D(x)} - \sqrt{D(x)}e^{-\partial}\sqrt{B(x)} + B(x) + D(x), \end{aligned} \quad (2.4)$$

where $e^{\pm\partial}$ is a matrix whose (x, y) -element is $\delta_{x\pm 1, y}$, and $F(x)$ means a diagonal matrix $F(x) = \text{diag}(F(0), F(1), \dots, F(N))$. The Schrödinger equation of rdQM system is a matrix eigenvalue problem,

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n = 0, 1, \dots, N), \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \dots < \mathcal{E}_N, \quad (2.5)$$

where the eigenstate vector (eigenvector) is $\phi_n = (\phi_n(x))_{x \in \{0, 1, \dots, N\}}$ and the product of \mathcal{H} and ϕ_n is given by $\mathcal{H}\phi_n(x) \stackrel{\text{def}}{=} \sum_{y=0}^N \mathcal{H}_{x,y}\phi_n(y)$. The constant term of \mathcal{H} is chosen so that $\mathcal{E}_0 = 0$. We remark that the matrix notation $\mathcal{H}\phi_n(x)$ represent

$$\begin{aligned} \text{(a):} & \quad \mathcal{H}_{0,0}\phi_n(0) + \mathcal{H}_{0,1}\phi_n(1) \quad (x = 0), \\ \text{(b):} & \quad \mathcal{H}_{x,x-1}\phi_n(x-1) + \mathcal{H}_{x,x}\phi_n(x) + \mathcal{H}_{x,x+1}\phi_n(x+1) \quad (1 \leq x \leq N-1), \\ \text{(c):} & \quad \mathcal{H}_{N,N-1}\phi_n(N-1) + \mathcal{H}_{N,N}\phi_n(N) \quad (x = N), \end{aligned}$$

and the expressions (a) and (c) can be regarded as given by (b) because the boundary conditions $D(0) = 0$ and $B(N) = 0$ (2.3) give “ $\mathcal{H}_{0,-1}$ ” = 0 and “ $\mathcal{H}_{N,N+1}$ ” = 0. The inner product of two state vectors $f(x)$ and $g(x)$ is $(f, g) = \sum_{x=0}^N f(x)g(x)$ and the norm of $f(x)$ is $\|f\| = \sqrt{(f, f)}$. The orthogonality relations for $\phi_n(x)$ are

$$(\phi_n, \phi_m) = \frac{1}{d_n^2} \delta_{nm} \quad (n, m = 0, 1, \dots, N), \quad (2.6)$$

where d_n are constants.

The Hamiltonian (2.4) can be expressed in factorized form in two ways (type-(i) and type-(ii) factorizations) [11](v1) :

$$\mathcal{H} = \mathcal{A}^\dagger \mathcal{A} = \mathcal{A}^{(\text{ii})\dagger} \mathcal{A}^{(\text{ii})}, \quad (2.7)$$

$$\text{type-(i):} \quad \mathcal{A} = \sqrt{B(x)} - e^\partial \sqrt{D(x)}, \quad \mathcal{A}^\dagger = \sqrt{B(x)} - \sqrt{D(x)} e^{-\partial}, \quad (2.8)$$

$$\text{type-(ii)} : \mathcal{A}^{(\text{ii})} = \sqrt{D(x)} - e^{-\partial} \sqrt{B(x)}, \quad \mathcal{A}^{(\text{ii})\dagger} = \sqrt{D(x)} - \sqrt{B(x)} e^{\partial}. \quad (2.9)$$

We remark that since the factor $D(0)$ does not appear in \mathcal{A} and \mathcal{A}^\dagger , the boundary condition “ $D(0) = 0$ ” is automatically satisfied in the type-(i) factorization ($B(N) = 0$ is needed). Similarly, since the factor $B(N)$ does not appear in $\mathcal{A}^{(\text{ii})}$ and $\mathcal{A}^{(\text{ii})\dagger}$, the boundary condition “ $B(N) = 0$ ” is automatically satisfied in the type-(ii) factorization ($D(0) = 0$ is needed).

For the rdQM systems associated to the orthogonal polynomials in the Askey scheme, the eigenstate $\phi_n(x)$ ($n \in \mathbb{Z}_{\geq 0}$) has the following form [18],

$$\phi_n(x) = \phi_0(x) \check{P}_n(x), \quad \check{P}_n(x) \stackrel{\text{def}}{=} P_n(\eta(x)). \quad (2.10)$$

Here $\eta(x)$ is the sinusoidal coordinate [18] and $P_n(\eta)$ is a polynomial of degree n in η . We choose the normalization $\check{P}_n(0) = 1$ with $\eta(0) = 0$, and set $\check{P}_n(x) = 0$ ($n \in \mathbb{Z}_{< 0}$). The ground state $\phi_0(x)$ is characterized by $\mathcal{A}\phi_0(x) = 0$ (or $\mathcal{A}^{(\text{ii})}\phi_0(x) = 0$) and its explicit form is

$$\phi_0(x) = \sqrt{\prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)}}, \quad (2.11)$$

with the normalization $\phi_0(0) = 1$ (convention: $\prod_{k=n}^{n-1} * = 1$). In the concrete examples, $B(x)$ and $D(x)$ are rational functions of x or q^x , and the defining range of $\phi_0(x)$ can be extended to $x \in \mathbb{R}$ (e.g., $(a; q)_x$ is expressed as $(a; q)_x = (a; q)_\infty / (aq^x; q)_\infty$, which is defined for $x \in \mathbb{R}$). The similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ in terms of the ground state $\phi_0(x)$ is

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = B(x)(1 - e^\partial) + D(x)(1 - e^{-\partial}), \quad (2.12)$$

and (2.5) becomes

$$\tilde{\mathcal{H}}\check{P}_n(x) = \mathcal{E}_n\check{P}_n(x). \quad (2.13)$$

The semi-infinite rdQM systems, whose coordinate x takes values in $\mathbb{Z}_{\geq 0}$, can be obtained by taking $N \rightarrow \infty$ limit.

2.2 Multi-step Darboux Transformations

The property of the Darboux transformations depends on the choice of seed solutions. We consider the virtual states as seed ‘solutions’, which give isospectral deformations. There are two types of virtual states, the type I and type II [11](v1). For the construction of the multi-indexed (q -)Racah polynomials, these two types of virtual states give essentially the same polynomials. So the type II construction is omitted in [11].

Remark: We comment on [11](v1). Instead of $\mathbf{t}^{(\text{ex})}$ (in the last paragraph of § 3.4 of [11](v1)), we should consider $\mathbf{t}^{(\text{ex}')}$:

$$\mathbf{t}^{(\text{ex}')}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\lambda_1, \lambda_1 + \lambda_3 - \lambda_4, \lambda_1 + \lambda_2 - \lambda_4, 2\lambda_1 - \lambda_4), \quad (2.14)$$

with $\lambda_1 = -N$. Then we have

$$B'^{\text{II}}(x; \boldsymbol{\lambda}) = D'^{\text{I}}(N - x; \mathbf{t}^{(\text{ex}')}(\boldsymbol{\lambda})), \quad D'^{\text{II}}(x; \boldsymbol{\lambda}) = B'^{\text{I}}(N - x; \mathbf{t}^{(\text{ex}')}(\boldsymbol{\lambda})), \quad (2.15)$$

and the type I and II multi-indexed (q -)Racah polynomials are related:

$$\check{P}_{\mathcal{D},n}^{\text{II}}(x; \boldsymbol{\lambda}) = \check{P}_{\mathcal{D},n}^{\text{I}}(N - x; \mathbf{t}^{(\text{ex}')}(\boldsymbol{\lambda})). \quad (2.16)$$

Let us consider a finite rdQM system. We assume the existence of two rational functions $B'(x)$ and $D'(x)$ of x or q^x satisfying

$$B(x)D(x+1) = \alpha^2 B'(x)D'(x+1), \quad \alpha > 0, \quad (2.17)$$

$$B(x) + D(x) = \alpha(B'(x) + D'(x)) + \alpha', \quad \alpha' < 0, \quad (2.18)$$

where α and α' are constants. We impose on them the following conditions:

$$\begin{aligned} \text{type I: } & B'(x) > 0 \quad (x = 0, 1, \dots, N + L - 1), \\ & D'(x) > 0 \quad (x = 1, 2, \dots, N), \quad D'(0) = D'(N + 1) = 0, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \text{type II: } & D'(x) > 0 \quad (x = -L + 1, \dots, -1, 0, 1, \dots, N), \\ & B'(x) > 0 \quad (x = 0, 1, \dots, N - 1), \quad B'(N) = B'(-1) = 0, \end{aligned} \quad (2.20)$$

where L is a certain positive integer to be specified later. The function $\tilde{\phi}_0(x)$ is defined by

$$\tilde{\phi}_0(x) \stackrel{\text{def}}{=} \sqrt{\prod_{y=0}^{x-1} \frac{B'(y)}{D'(y+1)}} \quad (x = 0, 1, \dots, N), \quad (2.21)$$

with the normalization $\tilde{\phi}_0(0) = 1$. Like $\phi_0(x)$, the defining range of $\tilde{\phi}_0(x)$ can be extended to $x \in \mathbb{R}$. We define a function $\nu(x)$,

$$\nu(x) \stackrel{\text{def}}{=} \frac{\phi_0(x)}{\tilde{\phi}_0(x)} = \prod_{y=0}^{x-1} \frac{B(y)}{\alpha B'(y)} = \prod_{y=0}^{x-1} \frac{\alpha D'(y+1)}{D(y+1)} \quad (x = 0, 1, \dots, N), \quad (2.22)$$

whose defining range can also be extended to $x \in \mathbb{R}$. Note that $\nu^I(x) = 0$ for $x \in \mathbb{Z}_{\geq N+1}$ and $\nu^{II}(x) = 0$ for $x \in \mathbb{Z}_{\leq -1}$. The relations (2.17)–(2.18) imply the following relation between two Hamiltonians:

$$\mathcal{H} = \alpha \mathcal{H}' + \alpha', \quad (2.23)$$

$$\mathcal{H}' \stackrel{\text{def}}{=} -\sqrt{B'(x)} e^\partial \sqrt{D'(x)} - \sqrt{D'(x)} e^{-\partial} \sqrt{B'(x)} + B'(x) + D'(x). \quad (2.24)$$

We assume the existence of virtual state vectors $\tilde{\phi}_v(x)$ ($v \in \mathcal{V}$),

$$\tilde{\phi}_v(x) \stackrel{\text{def}}{=} \tilde{\phi}_0(x) \check{\xi}_v(x), \quad \check{\xi}_v(x) \stackrel{\text{def}}{=} \xi_v(\eta(x)). \quad (2.25)$$

Here \mathcal{V} is the index set of the virtual state vectors, and the virtual state polynomial $\xi_v(\eta)$ is a polynomial of degree v in η satisfying the difference equation (for $x \in \mathbb{R}$)

$$B'(x)(\check{\xi}_v(x) - \check{\xi}_v(x+1)) + D'(x)(\check{\xi}_v(x) - \check{\xi}_v(x-1)) = \mathcal{E}'_v \check{\xi}_v(x), \quad (2.26)$$

where \mathcal{E}'_v is a constant. We impose on \mathcal{E}'_v and $\check{\xi}_v(x)$ the following conditions:

$$\tilde{\mathcal{E}}_v \stackrel{\text{def}}{=} \alpha \mathcal{E}'_v + \alpha', \quad \tilde{\mathcal{E}}_v < 0, \quad (2.27)$$

$$\text{type I: } \check{\xi}_v(x) > 0 \quad (x = 0, 1, \dots, N, N+1), \quad (2.28)$$

$$\text{type II: } \check{\xi}_v(x) > 0 \quad (x = -1, 0, 1, \dots, N). \quad (2.29)$$

Since the matrix \mathcal{H} is a positive semi-definite real symmetric matrix, $\tilde{\mathcal{E}}_v$ being negative is a sufficient condition for the virtual state $\tilde{\phi}_v(x)$ to not be a true eigenvector, see (2.30)–(2.31). We choose the normalization $\check{\xi}_v(0) = 1$ for type I and $\check{\xi}_v(-1) = 1$ for type II (which is different from [11](v1)). Relations (2.23), (2.25) and (2.26) imply that virtual state vectors $\tilde{\phi}_v(x)$ are polynomial ‘solutions’ of the Schrödinger equation except for one end-point:

$$\text{type I: } \mathcal{H}\tilde{\phi}_v(x) = \tilde{\mathcal{E}}_v \tilde{\phi}_v(x) \quad (x = 0, 1, \dots, N-1), \quad \mathcal{H}\tilde{\phi}_v(x) \neq \tilde{\mathcal{E}}_v \tilde{\phi}_v(x) \quad (x = N), \quad (2.30)$$

$$\text{type II: } \mathcal{H}\tilde{\phi}_v(x) = \tilde{\mathcal{E}}_v \tilde{\phi}_v(x) \quad (x = 1, 2, \dots, N), \quad \mathcal{H}\tilde{\phi}_v(x) \neq \tilde{\mathcal{E}}_v \tilde{\phi}_v(x) \quad (x = 0), \quad (2.31)$$

due to $D'(0) = 0$ and $B'(N) > 0$ for type I (2.19), and $D'(0) > 0$ and $B'(N) = 0$ for type II (2.20).

For the Darboux transformation with the type I virtual state vector $\tilde{\phi}_v^I(x)$ (type II virtual state vector $\tilde{\phi}_v^{II}(x)$) as a seed solution, the type-(i) factorization (2.8) (type-(ii) factorization (2.9)) is used, respectively:

$$\text{type I: } \begin{cases} \mathcal{H} = \hat{\mathcal{A}}_v^\dagger \hat{\mathcal{A}}_v + \tilde{\mathcal{E}}_v^I \\ \hat{\mathcal{A}}_v \tilde{\phi}_v^I(x) = 0 \quad (x = 0, 1, \dots, N-1) \end{cases} \Rightarrow \begin{cases} \mathcal{H}^{\text{new}} \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v \hat{\mathcal{A}}_v^\dagger + \tilde{\mathcal{E}}_v^I \\ \phi_n^{\text{new}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v \phi_n(x), \end{cases}$$

$$\text{type II: } \begin{cases} \mathcal{H} = \hat{\mathcal{A}}_v^{(ii)\dagger} \hat{\mathcal{A}}_v^{(ii)} + \tilde{\mathcal{E}}_v^{\text{II}} \\ \hat{\mathcal{A}}_v^{(ii)} \tilde{\phi}_v^{\text{II}}(x) = 0 \quad (x = 1, 2, \dots, N) \end{cases} \Rightarrow \begin{cases} \mathcal{H}^{\text{new}} \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v^{(ii)} \hat{\mathcal{A}}_v^{(ii)\dagger} + \tilde{\mathcal{E}}_v^{\text{II}} \\ \phi_n^{\text{new}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v^{(ii)} \phi_n(x), \end{cases}$$

and new virtual state vectors are $\tilde{\phi}_v^{\text{Inew}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v \tilde{\phi}_v^{\text{I}}(x) + (\text{correction term at } x = N)$ and $\tilde{\phi}_v^{\text{IInew}}(x) \stackrel{\text{def}}{=} \hat{\mathcal{A}}_v^{(ii)} \tilde{\phi}_v^{\text{II}}(x) + (\text{correction term at } x = 0)$, see [11](v1) for explicit formulas. Here we explain why the type-(i) (type-(ii)) factorization is used for the type I (type II) virtual state vector, respectively. The matrices $\hat{\mathcal{A}}_v$ and $\hat{\mathcal{A}}_v^{(ii)}$ are given by $\hat{\mathcal{A}}_v \stackrel{\text{def}}{=} \sqrt{\hat{B}_v(x)} - e^\partial \sqrt{\hat{D}_v(x)}$ and $\hat{\mathcal{A}}_v^{(ii)} \stackrel{\text{def}}{=} \sqrt{\hat{D}_v(x)} - e^{-\partial} \sqrt{\hat{B}_v(x)}$ with $\hat{B}_v(x) \stackrel{\text{def}}{=} \alpha B'(x) \frac{\xi_v(x+1)}{\xi_v(x)}$ and $\hat{D}_v(x) \stackrel{\text{def}}{=} \alpha D'(x) \frac{\xi_v(x-1)}{\xi_v(x)}$, and they satisfy $\hat{B}_v(x) \hat{D}_v(x+1) = B(x) D(x+1)$ and $\hat{B}_v(x) + \hat{D}_v(x) + \tilde{\mathcal{E}}_v = B(x) + D(x)$. Since $(0, 0)$ -element of $\hat{\mathcal{A}}_v^\dagger \hat{\mathcal{A}}_v + \tilde{\mathcal{E}}_v$ is $\hat{B}_v(0) + \tilde{\mathcal{E}}_v$, we need the condition $\hat{D}_v(0) = 0$ for type-(i) factorization. Similary, since (N, N) -element of $\hat{\mathcal{A}}_v^{(ii)\dagger} \hat{\mathcal{A}}_v^{(ii)} + \tilde{\mathcal{E}}_v$ is $\hat{D}_v(N) + \tilde{\mathcal{E}}_v$, we need the condition $\hat{B}_v(N) = 0$ for type-(ii) factorization. Be careful not to confuse $(\mathcal{A}, \mathcal{A}^{(ii)})$ and $(\hat{\mathcal{A}}_v, \hat{\mathcal{A}}_v^{(ii)})$. We call the multi-step Darboux transformations with only the type I (type II) virtual state vectors as seed solutions type I (type II) construction. In the multi-step Darboux transformations, various quantities are neatly expressed in terms of Casoratians: W_C (for type I) and $W_C^{(-)}$ (for type II) [11](v1). The Casorati determinants of a set of n functions $\{f_j(x)\}$ are defined by

$$W_C[f_1, f_2, \dots, f_n](x) \stackrel{\text{def}}{=} \det \left(f_k(x+j-1) \right)_{1 \leq j, k \leq n}, \quad (2.32)$$

$$\begin{aligned} W_C^{(-)}[f_1, f_2, \dots, f_n](x) &\stackrel{\text{def}}{=} \det \left(f_k(x-j+1) \right)_{1 \leq j, k \leq n} \\ &= (-1)^{\binom{n}{2}} W_C[f_1, f_2, \dots, f_n](x-n+1), \end{aligned} \quad (2.33)$$

(for $n = 0$, we set $W_C[\cdot](x) = W_C^{(-)}[\cdot](x) = 1$). The auxiliary functions $\varphi(x)$, $\varphi_M(x)$ and $\varphi_M^{(-)}(x)$ ($M \in \mathbb{Z}_{\geq 0}$) are defined by [18, 19] [11](v1)

$$\varphi(x) \stackrel{\text{def}}{=} \frac{\eta(x+1) - \eta(x)}{\eta(1)}, \quad (2.34)$$

$$\varphi_M(x) \stackrel{\text{def}}{=} \prod_{1 \leq j < k \leq M} \frac{\eta(x+k-1) - \eta(x+j-1)}{\eta(k-j)}, \quad (2.35)$$

$$\varphi_M^{(-)}(x) \stackrel{\text{def}}{=} \prod_{1 \leq j < k \leq M} \frac{\eta(x-j+1) - \eta(x-k+1)}{\eta(k-j)} = \varphi_M(x-M+1), \quad (2.36)$$

and $\varphi_0(x) = \varphi_1(x) = \varphi_0^{(-)}(x) = \varphi_1^{(-)}(x) = 1$.

Let us consider the M -step Darboux transformations with virtual state vectors $\tilde{\phi}_v(x)$ ($v \in \mathcal{D}$) as seed solutions. Here \mathcal{D} is

$$\mathcal{D} = \{d_1, d_2, \dots, d_M\} \quad (1 \leq d_1 < d_2 < \dots < d_M; d_j \in \mathcal{V}), \quad (2.37)$$

and we assume $|\mathcal{V}| \geq M$ and $L \geq M$. Although this notation d_j conflicts with the notation of the normalization constant d_n in (2.6), we think this does not cause any confusion because the latter appears as $\frac{1}{d_n^2} \delta_{nm}$. The Hamiltonian of the deformed system, the Schrödinger equation and the orthogonality relations are given by [11]

$$\begin{aligned} \mathcal{H}_{\mathcal{D}} &= (\mathcal{H}_{\mathcal{D}x,y})_{x,y \in \{0,1,\dots,N\}} \\ &\stackrel{\text{def}}{=} -\sqrt{B_{\mathcal{D}}(x)} e^{\partial} \sqrt{D_{\mathcal{D}}(x)} - \sqrt{D_{\mathcal{D}}(x)} e^{-\partial} \sqrt{B_{\mathcal{D}}(x)} + B_{\mathcal{D}}(x) + D_{\mathcal{D}}(x) \end{aligned} \quad (2.38)$$

$$= \mathcal{A}_{\mathcal{D}}^{\dagger} \mathcal{A}_{\mathcal{D}}, \quad \mathcal{A}_{\mathcal{D}} \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x)} - e^{\partial} \sqrt{D_{\mathcal{D}}(x)}, \quad \mathcal{A}_{\mathcal{D}}^{\dagger} \stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(x)} - \sqrt{D_{\mathcal{D}}(x)} e^{-\partial}, \quad (2.39)$$

$$\mathcal{H}_{\mathcal{D}} \phi_{\mathcal{D}n}(x) = \mathcal{E}_n \phi_{\mathcal{D}n}(x) \quad (n = 0, 1, \dots, N), \quad (2.40)$$

$$(\phi_{\mathcal{D}n}, \phi_{\mathcal{D}m}) = \prod_{j=1}^M (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}) \cdot \frac{1}{d_n^2} \delta_{nm} \quad (n, m = 0, 1, \dots, N). \quad (2.41)$$

The semi-infinite systems are obtained by taking $N \rightarrow \infty$ limit. We present explicit forms of $B_{\mathcal{D}}(x)$, $D_{\mathcal{D}}(x)$ and $\phi_{\mathcal{D}n}(x)$ for the semi-infinite systems in §2.2.1–2.2.2. Only the final results are given here. For intermediate steps, see [11, 13], [11](v1).

2.2.1 type I construction for semi-infinite systems

In the type I construction for semi-infinite systems, the virtual state vectors $\tilde{\phi}_v(x)$ are now solutions of the Schrödinger equation (2.30) (with $N \rightarrow \infty$), but they have infinite norms. Explicit forms of $B_{\mathcal{D}}(x)$, $D_{\mathcal{D}}(x)$ and $\phi_{\mathcal{D}n}(x)$ are

$$B_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \alpha B'(x+M) \frac{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x)}{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x+1)} \frac{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x+1)}{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x)}, \quad (2.42)$$

$$D_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \alpha D'(x) \frac{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x+1)}{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x)} \frac{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x-1)}{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x)}, \quad (2.43)$$

$$\phi_{\mathcal{D}n}(x) \stackrel{\text{def}}{=} \frac{(-1)^M \sqrt{\prod_{j=1}^M \alpha B'(x+j-1)} \tilde{\phi}_0(x) W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu \check{P}_n](x)}{\sqrt{W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x) W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x+1)}}. \quad (2.44)$$

The Casoratian $W_{\mathcal{C}}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x)$ has definite sign for $x \in \mathbb{Z}_{\geq 0}$ (see [13] for proof). The potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ are positive: $B_{\mathcal{D}}(x) > 0$ ($x \in \mathbb{Z}_{\geq 0}$), $D_{\mathcal{D}}(x) > 0$ ($x \in \mathbb{Z}_{\geq 1}$) and $D_{\mathcal{D}}(0) = 0$.

2.2.2 type II construction for semi-infinite systems

In the type II construction for semi-infinite systems, the virtual state vectors $\tilde{\phi}_v(x)$ satisfy the Schrödinger equation except for the end-point $x = 0$, (2.31) (with $N \rightarrow \infty$). Explicit

forms of $B_{\mathcal{D}}(x)$, $D_{\mathcal{D}}(x)$ and $\phi_{\mathcal{D}n}(x)$ are

$$B_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \alpha B'(x) \frac{W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x-1) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x+1)}{W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x)}, \quad (2.45)$$

$$D_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \alpha D'(x-M) \frac{W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x-1)}{W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x-1) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu](x)}, \quad (2.46)$$

$$\phi_{\mathcal{D}n}(x) \stackrel{\text{def}}{=} \frac{(-1)^M \sqrt{\prod_{j=1}^M \alpha D'(x-j+1)} \tilde{\phi}_0(x) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu \check{P}_n](x)}{\sqrt{W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x) W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x-1)}}. \quad (2.47)$$

The Casoratian $W_{\mathbb{C}}^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x)$ has definite sign for $x \in \mathbb{Z}_{\geq -1}$ (by similar proof as type I). The potential functions $B_{\mathcal{D}}(x)$ and $D_{\mathcal{D}}(x)$ are positive: $B_{\mathcal{D}}(x) > 0$ ($x \in \mathbb{Z}_{\geq 0}$), $D_{\mathcal{D}}(x) > 0$ ($x \in \mathbb{Z}_{\geq 1}$) and $D_{\mathcal{D}}(0) = 0$. In §3 and §4, we will denote $\phi_{\mathcal{D}n}(x)$ in (2.47) as $\phi_{\mathcal{D}n}^{\text{gen}}(x)$.

3 Multi-indexed Little q -Jacobi polynomials

In this section we present the case-(1) multi-indexed little q -Jacobi polynomials, especially type II polynomials. Various quantities depend on a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ and q ($0 < q < 1$), and $q^{\boldsymbol{\lambda}}$ stands for $q^{(\lambda_1, \lambda_2, \dots)} = (q^{\lambda_1}, q^{\lambda_2}, \dots)$. Their dependence is expressed as $f = f(\boldsymbol{\lambda})$ and $f(x) = f(x; \boldsymbol{\lambda})$, but q -dependence is suppressed.

3.1 Original system

Let us present the basic data of little q -Jacobi rdQM system. The standard little q -Jacobi polynomial $p_n(q^x; a, b|q) = {}_2\phi_1\left(q^{-n}, abq^{n+1} \middle| q; q^{x+1}\right)$ [16] does not satisfy our normalization $\check{P}_n(0) = 1$. We change the parametrization slightly from the standard one, $(a, b)^{\text{standard}} = (aq^{-1}, bq^{-1})$ [20]. The basic data are as follows [18, 21]:

$$q^{\boldsymbol{\lambda}} = (a, b), \quad \boldsymbol{\delta} = (1, 1), \quad \kappa = q^{-1}, \quad 0 < a < 1, \quad b < 1, \quad (3.1)$$

$$B(x; \boldsymbol{\lambda}) = aq^{-1}(q^{-x} - b), \quad D(x) = q^{-x} - 1, \quad (3.2)$$

$$\mathcal{E}_n(\boldsymbol{\lambda}) = (q^{-n} - 1)(1 - abq^{n-1}), \quad \eta(x) = 1 - q^x, \quad \varphi(x) = q^x, \quad (3.3)$$

$$\begin{aligned} \check{P}_n(x; \boldsymbol{\lambda}) &= {}_3\phi_1\left(q^{-n}, abq^{n-1}, q^{-x} \middle| q; a^{-1}q^{x+1}\right) = c'_n(\boldsymbol{\lambda}) p_n(1 - \eta(x); aq^{-1}, bq^{-1}|q) \\ &= c'_n(\boldsymbol{\lambda}) {}_2\phi_1\left(q^{-n}, abq^{n-1} \middle| q; q^{x+1}\right), \quad c'_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-a)^{-n} q^{-\binom{n}{2}} \frac{(a; q)_n}{(b; q)_n}, \end{aligned} \quad (3.4)$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = c_n(\boldsymbol{\lambda}) \eta(x)^n + \text{lower degree terms}, \quad c_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-a)^{-n} q^{-n(n-1)} \frac{(abq^{n-1}; q)_n}{(b; q)_n}, \quad (3.5)$$

$$\phi_0(x; \boldsymbol{\lambda})^2 = \frac{(b; q)_x}{(q; q)_x} a^x = \frac{(b, q^x; q)_\infty}{(bq^x, q; q)_\infty} a^x, \quad \phi_0(x; \boldsymbol{\lambda}) > 0, \quad \phi_0(0; \boldsymbol{\lambda}) = 1, \quad (3.6)$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{(b, ab; q)_n a^n q^{n(n-1)}}{(a, q; q)_n} \frac{1 - abq^{2n-1}}{1 - abq^{n-1}} \times \frac{(a; q)_\infty}{(ab; q)_\infty}, \quad d_n(\boldsymbol{\lambda}) > 0, \quad (3.7)$$

and $\check{P}_n(x; \boldsymbol{\lambda})$ satisfies

$$\check{P}_n(0; \boldsymbol{\lambda}) = 1, \quad \check{P}_n(\infty; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \check{P}_n(x; \boldsymbol{\lambda}) = c'_n(\boldsymbol{\lambda}). \quad (3.8)$$

Note that the most right hand side of $\phi_0(x; \boldsymbol{\lambda})^2$ (3.6) is defined for $x \in \mathbb{R}$. This rdQM system is shape invariant,

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \kappa \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \quad (3.9)$$

As a consequence of the shape invariance combined with the Crum's theorem and the normalization, we obtain [17]

$$\mathcal{A}(\boldsymbol{\lambda})\phi_n(x; \boldsymbol{\lambda}) = \frac{\mathcal{E}_n(\boldsymbol{\lambda})}{\sqrt{B(0; \boldsymbol{\lambda})}} \phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.10)$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger \phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \sqrt{B(0; \boldsymbol{\lambda})} \phi_n(x; \boldsymbol{\lambda}) \quad (n \in \mathbb{Z}_{\geq 1}). \quad (3.11)$$

These relations give the forward and backward shift relations:

$$\mathcal{F}(\boldsymbol{\lambda})\check{P}_n(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.12)$$

$$\mathcal{B}(\boldsymbol{\lambda})\check{P}_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \check{P}_n(x; \boldsymbol{\lambda}), \quad (n \in \mathbb{Z}_{\geq 1}), \quad (3.13)$$

where the forward and backward shift operators are

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{B(0; \boldsymbol{\lambda})} \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) = B(0; \boldsymbol{\lambda})\varphi(x)^{-1}(1 - e^\partial), \quad (3.14)$$

$$\begin{aligned} \mathcal{B}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{B(0; \boldsymbol{\lambda})}} \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^\dagger \circ \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ &= B(0; \boldsymbol{\lambda})^{-1}(B(x; \boldsymbol{\lambda}) - D(x)e^{-\partial})\varphi(x). \end{aligned} \quad (3.15)$$

The similarity transformed Hamiltonian (2.12) is expressed as $\tilde{\mathcal{H}}(\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda})$. The auxiliary functions $\varphi_M(x)$ (2.35) and $\varphi_M^{(-)}(x)$ (2.36) become

$$\varphi_M(x) = q^{\binom{M}{2}x + \binom{M}{3}}, \quad \varphi_M^{(-)}(x) = q^{\binom{M}{2}x - \frac{1}{6}M(M-1)(2M-1)}. \quad (3.16)$$

3.2 Type I polynomials

The potential functions $B'(x)$ and $D'(x)$ and the virtual state polynomials $\check{\xi}_v(x)$ are given by [13]

$$B'^I(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} B(x; \mathbf{t}^I(\boldsymbol{\lambda})), \quad D'^I(x) \stackrel{\text{def}}{=} D(x), \quad (3.17)$$

$$\check{\xi}_v^I(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{P}_v(x; \mathfrak{t}^I(\boldsymbol{\lambda})), \quad (3.18)$$

where the twist operation \mathfrak{t} and the shift $\tilde{\boldsymbol{\delta}}$ are (remark: $(a, b)^{\text{standard}}$ are used in [13])

$$\mathfrak{t}^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (2 - \lambda_1, \lambda_2), \quad \text{namely } q^{\mathfrak{t}^I(\boldsymbol{\lambda})} = (a^{-1}q^2, b), \quad (3.19)$$

$$\tilde{\boldsymbol{\delta}}^I \stackrel{\text{def}}{=} (-1, 1), \quad \mathfrak{t}^I(\boldsymbol{\lambda}) + u\boldsymbol{\delta} = \mathfrak{t}^I(\boldsymbol{\lambda} + u\tilde{\boldsymbol{\delta}}^I) \quad (\forall u \in \mathbb{R}), \quad (3.20)$$

with $\alpha^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} aq^{-1}$ and $\alpha'^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(1 - aq^{-1})(1 - b)$. The parameter range is $0 < a < q^{1+d_M}$ and $b < 1$. Various formulas for the type I multi-indexed little q -Jacobi polynomials are presented in [13].

3.3 Type II polynomials

The twist operation \mathfrak{t} and the shift $\tilde{\boldsymbol{\delta}}$ are defined by

$$\mathfrak{t}^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (\lambda_1, 2 - \lambda_2), \quad \text{namely } q^{\mathfrak{t}^{\text{II}}(\boldsymbol{\lambda})} = (a, b^{-1}q^2), \quad (3.21)$$

$$\tilde{\boldsymbol{\delta}}^{\text{II}} \stackrel{\text{def}}{=} (1, -1), \quad \mathfrak{t}^{\text{II}}(\boldsymbol{\lambda}) + u\boldsymbol{\delta} = \mathfrak{t}^{\text{II}}(\boldsymbol{\lambda} + u\tilde{\boldsymbol{\delta}}^{\text{II}}) \quad (\forall u \in \mathbb{R}). \quad (3.22)$$

Without using this twist operation, let us define the potential functions and the virtual state polynomials. The potential functions $B'(x)$ and $D'(x)$ are given by

$$B'^{\text{II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} ab^{-1}q(q^{-x-1} - 1), \quad D'^{\text{II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} b^{-1}q^{1-x} - 1, \quad (3.23)$$

which satisfy the conditions (2.17)–(2.18) with

$$\alpha^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} bq^{-1}, \quad \alpha'^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(1 - a)(1 - bq^{-1}). \quad (3.24)$$

For (2.20) (with $N \rightarrow \infty$), we take $L = M$ and assume $0 < b < q^M$. The virtual state polynomial $\check{\xi}_v^{\text{II}}(x)$ ($v \in \mathbb{Z}_{\geq 0}$) is given by

$$\check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{c}_v^{\text{II}}(\boldsymbol{\lambda}) {}_2\phi_1 \left(\begin{matrix} q^{-v}, ab^{-1}q^{v+1} \\ a \end{matrix} \middle| q; bq^x \right), \quad \check{c}_v^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{(a; q)_v}{(bq^{-v-1}; q)_v}, \quad (3.25)$$

which satisfies

$$\check{\xi}_v^{\text{II}}(-1; \boldsymbol{\lambda}) = 1, \quad \check{\xi}_v^{\text{II}}(\infty; \boldsymbol{\lambda}) \left(\stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda}) \right) = \check{c}_v^{\text{II}}(\boldsymbol{\lambda}), \quad (3.26)$$

and

$$\check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda}) = \check{c}_v^{\text{II}}(\boldsymbol{\lambda})\eta(x)^v + \text{lower degree terms}, \quad \check{c}_v^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} b^v q^{-\binom{v+1}{2}} \frac{(ab^{-1}q^{v+1}; q)_v}{(bq^{-v-1}; q)_v}. \quad (3.27)$$

As mentioned below (3.22), $\check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda})$ is defined without using \mathfrak{t}^{II} , and we remark that $\check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda}) \not\propto \check{P}_v(x; \mathfrak{t}^{\text{II}}(\boldsymbol{\lambda}))$ in contrast to the type I case (3.18). For simplicity of presentation, the superscript II is omitted in the following.

The virtual state polynomial $\check{\xi}_v(x)$ satisfies the difference equation (for $x \in \mathbb{R}$) (2.26) with

$$\mathcal{E}'_v(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{E}_v(\mathfrak{t}(\boldsymbol{\lambda})) = (q^{-v} - 1)(1 - ab^{-1}q^{v+1}). \quad (3.28)$$

Proof: Let us consider the following function $f_v(z)$ ($v \in \mathbb{Z}_{\geq 0}$),

$$f_v(z) = {}_2\phi_1\left(\begin{matrix} q^{-v}, ab^{-1}q^{v+1} \\ a \end{matrix} \middle| q; bz\right) = \sum_{k=0}^v a_k z^k, \quad a_k = \frac{(q^{-v}, ab^{-1}q^{v+1}; q)_k}{(a; q)_k} \frac{b^k}{(q; q)_k},$$

$$a_k = \frac{(1 - q^{-v+k-1})(1 - ab^{-1}q^{v+k})}{1 - aq^{k-1}} \frac{b}{1 - q^k} a_{k-1} \quad (1 \leq k \leq v).$$

This $f_v(z)$ satisfies the q -difference equation

$$ab^{-1}(1 - qz)(f_v(z) - f_v(qz)) + (b^{-1}q - z)(f_v(z) - f_v(q^{-1}z)) = (q^{-v} - 1)(1 - ab^{-1}q^{v+1})zf_v(z),$$

which is shown by comparing the coefficients of z^k terms of both sides ($k = 0, 1 \leq k \leq v$ and $k = v + 1$). By substituting $z = q^x$ and $f_v(z) = \check{c}'_v{}^{-1}\check{\xi}_v(x)$ into this q -difference equation, we obtain (2.26). \square

From (2.29) (with $N \rightarrow \infty$), the virtual state polynomials should satisfy $\check{\xi}_v(x) > 0$ ($x \in \mathbb{Z}_{\geq -1}$). Let us check this condition. Note that $\check{c}'_v > 0$ for $a < 1$ and $b < q^{v+1}$. Since $\check{\xi}_v(-1) = 1 > 0$, we consider $x \in \mathbb{Z}_{\geq 0}$. By using the identity ((1.13.17) in [16])

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q; z\right) = (bc^{-1}q^{-n}z; q)_n {}_3\phi_2\left(\begin{matrix} q^{-n}, b^{-1}c, 0 \\ c, b^{-1}cqz^{-1} \end{matrix} \middle| q; q\right) \quad (n \in \mathbb{Z}_{\geq 0}),$$

$\check{\xi}_v(x)$ is rewritten as

$$\begin{aligned} \check{\xi}_v(x) &= \check{c}'_v(q^{x+1}; q)_v {}_3\phi_2\left(\begin{matrix} q^{-v}, bq^{-v-1}, 0 \\ a, q^{-v-x} \end{matrix} \middle| q; q\right) \\ &= \check{c}'_v(q^{x+1}; q)_v \sum_{k=0}^v \frac{(q^{-v}, bq^{-v-1}; q)_k}{(a, q^{-v-x}; q)_k} \frac{q^k}{(q; q)_k} \\ &= \check{c}'_v(q^{x+1}; q)_v \sum_{k=0}^v \frac{(q^{v-k+1}, bq^{-v-1}; q)_k}{(a, q^{x+v-k+1}; q)_k} \frac{q^{k(x+1)}}{(q; q)_k}, \end{aligned} \quad (3.29)$$

and each k -th term of the sum are positive for $a < 1$ and $b < q^{v+1}$. Therefore we obtain $\check{\xi}_v(x) > 0$ ($x \in \mathbb{Z}_{\geq -1}$) for $a < 1$ and $b < q^{v+1}$.

In the following, we assume the following parameter range:

$$0 < a < 1, \quad 0 < b < q^{1+d_M}, \quad (3.30)$$

which will be extended in the last paragraph of this subsection. The functions $\tilde{\phi}_0(x)$ (> 0) (2.21) and $\nu(x)$ (2.22) become

$$\tilde{\phi}_0(x; \boldsymbol{\lambda})^2 = \frac{(q; q)_x}{(b; q)_x} a^x = \frac{(bq^x, q; q)_\infty}{(b, q^{x+1}; q)_\infty} a^x, \quad (3.31)$$

$$\nu(x; \boldsymbol{\lambda}) = \frac{(b; q)_x}{(q; q)_x} = \frac{(b, q^{x+1}; q)_\infty}{(bq^x, q; q)_\infty}, \quad (3.32)$$

and the virtual state energy $\tilde{\mathcal{E}}_v$ (2.27) becomes

$$\tilde{\mathcal{E}}_v(\boldsymbol{\lambda}) = -(1 - aq^v)(1 - bq^{-1-v}). \quad (3.33)$$

The virtual state vectors $\tilde{\phi}_v(x)$ (2.25) satisfy the Schrödinger equation except for the endpoint $x = 0$, (2.31) (with $N \rightarrow \infty$).

We define the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})$ and the multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})$ ($n \in \mathbb{Z}_{\geq 0}$) as follows:

$$\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})^{-1} \varphi_M^{(-)}(x)^{-1} \mathbb{W}_C^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}](x; \boldsymbol{\lambda}), \quad (3.34)$$

$$\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})^{-1} \varphi_{M+1}^{(-)}(x)^{-1} \nu(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})^{-1} \mathbb{W}_C^{(-)}[\check{\xi}_{d_1}, \dots, \check{\xi}_{d_M}, \nu\check{P}_n](x; \boldsymbol{\lambda}) \quad (3.35)$$

$$= \mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})^{-1} \varphi_{M+1}^{(-)}(x)^{-1} \times \begin{vmatrix} \check{\xi}_{d_1}(x_1) & \cdots & \check{\xi}_{d_M}(x_1) & r_1(x_1)\check{P}_n(x_1) \\ \check{\xi}_{d_1}(x_2) & \cdots & \check{\xi}_{d_M}(x_2) & r_2(x_2)\check{P}_n(x_2) \\ \vdots & \cdots & \vdots & \vdots \\ \check{\xi}_{d_1}(x_{M+1}) & \cdots & \check{\xi}_{d_M}(x_{M+1}) & r_{M+1}(x_{M+1})\check{P}_n(x_{M+1}) \end{vmatrix}, \quad (3.36)$$

where $x_j \stackrel{\text{def}}{=} x - j + 1$ and $r_j(x) = r_j(x; \boldsymbol{\lambda}, M)$ ($1 \leq j \leq M + 1$) are given by

$$r_j(x - j + 1; \boldsymbol{\lambda}, M) \stackrel{\text{def}}{=} \frac{\nu(x - j + 1; \boldsymbol{\lambda})}{\nu(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})} = \frac{(bq^{-M+x}; q)_{M-j+1} (q^{x-j+2}; q)_{j-1}}{(bq^{-M}; q)_M}, \quad (3.37)$$

and the constants $\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda})$ are given by

$$\mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{1}{\varphi_M^{(-)}(-1)} \prod_{1 \leq j < k \leq M} \frac{\tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda}) - \tilde{\mathcal{E}}_{d_k}(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda}) D'(-j; \boldsymbol{\lambda})}, \quad (3.38)$$

$$\mathcal{C}_{\mathcal{D},n}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-1)^M q^{\binom{M+1}{2}} \mathcal{C}_{\mathcal{D}}(\boldsymbol{\lambda}). \quad (3.39)$$

They are polynomials in $\eta(x)$,

$$\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{\Xi}_{\mathcal{D}}(\eta(x); \boldsymbol{\lambda}), \quad \deg \Xi_{\mathcal{D}}(\eta) = \ell_{\mathcal{D}}, \quad (3.40)$$

$$\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{P}_{\mathcal{D},n}(\eta(x); \boldsymbol{\lambda}), \quad \deg P_{\mathcal{D},n}(\eta) = \ell_{\mathcal{D}} + n, \quad (3.41)$$

where $\ell_{\mathcal{D}}$ is

$$\ell_{\mathcal{D}} \stackrel{\text{def}}{=} \sum_{j=1}^M d_j - \frac{1}{2}M(M-1). \quad (3.42)$$

Their normalizations are

$$\check{\Xi}_{\mathcal{D}}(-1; \boldsymbol{\lambda}) = 1, \quad \check{P}_{\mathcal{D},n}(0; \boldsymbol{\lambda}) = 1, \quad (3.43)$$

and we set $\check{P}_{\mathcal{D},n}(x) = 0$ ($n \in \mathbb{Z}_{<0}$). The coefficients of the highest degree terms are

$$\begin{aligned} \check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}) &= c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})\eta(x)^{\ell_{\mathcal{D}}} + \text{lower degree terms}, \\ c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) &= \prod_{j=1}^M \frac{\check{c}_{d_j}(\boldsymbol{\lambda})}{\check{c}_{j-1}(\boldsymbol{\lambda})} \cdot \prod_{1 \leq j < k \leq M} \frac{bq^{-1} - aq^{j-1+k-1}}{bq^{-1} - aq^{d_j+d_k}}, \end{aligned} \quad (3.44)$$

$$\begin{aligned} \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) &= c_{\mathcal{D},n}^P(\boldsymbol{\lambda})\eta(x)^{\ell_{\mathcal{D}}+n} + \text{lower degree terms}, \\ c_{\mathcal{D},n}^P(\boldsymbol{\lambda}) &= c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda})q^{-nM} \prod_{j=1}^M \frac{1 - bq^{n-d_j-1}}{1 - bq^{-j}}. \end{aligned} \quad (3.45)$$

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ is positive for $x \in \mathbb{Z}_{\geq -1}$ (see the remark below (2.47)). The multi-indexed orthogonal polynomial $P_{\mathcal{D},n}(\eta)$ has n zeros in the physical region $0 \leq \eta < 1$ ($\Leftrightarrow x \in \mathbb{R}_{\geq 0}$), which interlace the $n+1$ zeros of $P_{\mathcal{D},n+1}(\eta)$ in the physical region, and $\ell_{\mathcal{D}}$ zeros in the unphysical region $\eta \in \mathbb{C} \setminus [0, 1)$. This property can be verified by numerical calculation. The lowest degree multi-indexed orthogonal polynomial is related to the denominator polynomial as

$$\check{P}_{\mathcal{D},0}(x; \boldsymbol{\lambda}) = \check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta}). \quad (3.46)$$

The deformed potential functions $B_{\mathcal{D}}(x)$ (2.45) and $D_{\mathcal{D}}(x)$ (2.46) and the eigenvectors $\phi_{\mathcal{D}_n}^{\text{gen}}(x)$ (2.47) become

$$B_{\mathcal{D}}(x; \boldsymbol{\lambda}) = B(x; \boldsymbol{\lambda} + M\check{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})} \frac{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (3.47)$$

$$D_{\mathcal{D}}(x; \boldsymbol{\lambda}) = D(x) \frac{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})} \frac{\check{\Xi}_{\mathcal{D}}(x-2; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})}, \quad (3.48)$$

$$\phi_{\mathcal{D}_n}^{\text{gen}}(x; \boldsymbol{\lambda}) = \sqrt{(bq^{-M}; q)_M} \frac{\phi_0(x; \boldsymbol{\lambda} + M\check{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}} \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}). \quad (3.49)$$

We define the eigenvectors $\phi_{\mathcal{D}n}(x)$ as

$$\phi_{\mathcal{D}n}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}), \quad \phi_{\mathcal{D}n}(0; \boldsymbol{\lambda}) = 1, \quad (3.50)$$

$$\psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \sqrt{\check{\Xi}_{\mathcal{D}}(0; \boldsymbol{\lambda})} \frac{\phi_0(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}}, \quad \psi_{\mathcal{D}}(0; \boldsymbol{\lambda}) = 1. \quad (3.51)$$

Note that the formula (2.11) gives

$$\phi_{\mathcal{D}0}(x; \boldsymbol{\lambda}) = \sqrt{\prod_{y=0}^{x-1} \frac{B_{\mathcal{D}}(y; \boldsymbol{\lambda})}{D_{\mathcal{D}}(y+1; \boldsymbol{\lambda})}} = \sqrt{\check{\Xi}_{\mathcal{D}}(0; \boldsymbol{\lambda})} \frac{\phi_0(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}} \check{P}_{\mathcal{D},0}(x; \boldsymbol{\lambda}),$$

where (3.43) and (3.46) are used. The orthogonality relations for $\phi_{\mathcal{D}n}^{\text{gen}}(x)$ (2.41) (with $N \rightarrow \infty$) give those for $\check{P}_{\mathcal{D},n}(x)$,

$$\sum_{x=0}^{\infty} \frac{\phi_0(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})^2}{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})} \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \check{P}_{\mathcal{D},m}(x; \boldsymbol{\lambda}) = \frac{\delta_{nm}}{d_n(\boldsymbol{\lambda})^2 \tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2} \quad (n, m \in \mathbb{Z}_{\geq 0}), \quad (3.52)$$

where $\tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})$ (> 0) is given by

$$\tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2 \stackrel{\text{def}}{=} \kappa \binom{M}{2} \prod_{j=1}^M \frac{\alpha(\boldsymbol{\lambda}) D'(0; \boldsymbol{\lambda} + (j-1)\tilde{\boldsymbol{\delta}})}{\mathcal{E}_n(\boldsymbol{\lambda}) - \tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda})} = \frac{(bq^{-M}; q)_M}{\prod_{j=1}^M (\mathcal{E}_n(\boldsymbol{\lambda}) - \tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda}))}. \quad (3.53)$$

The Hamiltonian of the deformed system is (2.38) (with $N \rightarrow \infty$),

$$\begin{aligned} \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) &= (\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda})_{x,y})_{x,y \in \mathbb{Z}_{\geq 0}} = \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \\ &\stackrel{\text{def}}{=} -\sqrt{B_{\mathcal{D}}(x; \boldsymbol{\lambda})} e^\partial \sqrt{D_{\mathcal{D}}(x; \boldsymbol{\lambda})} - \sqrt{D_{\mathcal{D}}(x; \boldsymbol{\lambda})} e^{-\partial} \sqrt{B_{\mathcal{D}}(x; \boldsymbol{\lambda})} + B_{\mathcal{D}}(x; \boldsymbol{\lambda}) + D_{\mathcal{D}}(x; \boldsymbol{\lambda}). \end{aligned} \quad (3.54)$$

The eigenvectors $\phi_{\mathcal{D}n}(x)$ (3.50) satisfy the Schrödinger equation,

$$\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D}n}(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda}) \phi_{\mathcal{D}n}(x; \boldsymbol{\lambda}) \quad (n \in \mathbb{Z}_{\geq 0}). \quad (3.55)$$

The similarity transformed Hamiltonian is defined by

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \\ &= B(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}}) \frac{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})} \left(\frac{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})} - e^\partial \right) \\ &\quad + D(x) \frac{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})} \left(\frac{\check{\Xi}_{\mathcal{D}}(x-2; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})} - e^{-\partial} \right), \end{aligned} \quad (3.56)$$

and the multi-indexed orthogonal polynomials $\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})$ are its eigenpolynomials:

$$\tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \quad (n \in \mathbb{Z}_{\geq 0}). \quad (3.57)$$

The shape invariance of the original system (3.9) is inherited by the deformed systems:

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger = \kappa\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}_1(\boldsymbol{\lambda}). \quad (3.58)$$

As a consequence of the shape invariance and the normalization, we obtain

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})\phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}) = \frac{\mathcal{E}_n(\boldsymbol{\lambda})}{\sqrt{B_{\mathcal{D}}(0; \boldsymbol{\lambda})}}\phi_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.59)$$

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger\phi_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \sqrt{B_{\mathcal{D}}(0; \boldsymbol{\lambda})}\phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \quad (n \in \mathbb{Z}_{\geq 1}). \quad (3.60)$$

These relations give the forward and backward shift relations:

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) = \mathcal{E}_n(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (n \in \mathbb{Z}_{\geq 0}), \quad (3.61)$$

$$\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\check{P}_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \quad (n \in \mathbb{Z}_{\geq 1}), \quad (3.62)$$

where the forward and backward shift operators are

$$\begin{aligned} \mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \sqrt{B_{\mathcal{D}}(0; \boldsymbol{\lambda})}\psi_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \\ &= \frac{B(0; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})}{\varphi(x)\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})} \left(\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) - \check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})e^{\theta} \right), \end{aligned} \quad (3.63)$$

$$\begin{aligned} \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \frac{1}{\sqrt{B_{\mathcal{D}}(0; \boldsymbol{\lambda})}}\psi_{\mathcal{D}}(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger \circ \psi_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \\ &= \frac{1}{B(0; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda} + \boldsymbol{\delta})} \\ &\quad \times \left(B(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda}) - D(x)\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})e^{-\theta} \right) \varphi(x). \end{aligned} \quad (3.64)$$

The similarity transformed Hamiltonian (3.56) is expressed as $\tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) = \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})$.

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ and the multi-indexed polynomials $\check{P}_{\mathcal{D},n}(x)$ are normalized as (3.43). Their values at $x = \infty$ are given by (cf. (3.8) and (3.26))

$$\check{\Xi}_{\mathcal{D}}(\infty; \boldsymbol{\lambda}) \left(\stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda}) \right) = \prod_{j=1}^M \frac{\check{c}'_{d_j}(\boldsymbol{\lambda})}{\check{c}'_{j-1}(\boldsymbol{\lambda})}, \quad (3.65)$$

$$\check{P}_{\mathcal{D},n}(\infty; \boldsymbol{\lambda}) \left(\stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \right) = \prod_{j=1}^M \frac{\check{c}'_{d_j}(\boldsymbol{\lambda})}{\check{c}'_{j-1}(\boldsymbol{\lambda})} \cdot \prod_{j=1}^M \frac{\mathcal{E}_n(\boldsymbol{\lambda}) - \tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda})}{-\tilde{\mathcal{E}}_{j-1}(\boldsymbol{\lambda})} \cdot c'_n(\boldsymbol{\lambda}). \quad (3.66)$$

In (2.37), we have assumed the order $d_1 < d_2 < \dots < d_M$ (standard order). Under the permutations of d_j 's, the denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ (3.34) and the multi-indexed polynomial $\check{P}_{\mathcal{D},n}(x)$ (3.35) may change their sign, but the deformed potential functions $B_{\mathcal{D}}(x)$ (3.47) and $D_{\mathcal{D}}(x)$ (3.48) are invariant. So the deformed Hamiltonian $\mathcal{H}_{\mathcal{D}}$ (3.54) does not depend on the order of d_j 's. Setting one of d_j to 0, for example $d_M = 0$, we obtain the following relation between M -indexed polynomial and $(M - 1)$ -indexed polynomial,

$$\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \Big|_{d_M=0} = \check{P}_{\mathcal{D}',n}(x; \boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}}), \quad \mathcal{D}' = \{d_1 - 1, d_2 - 1, \dots, d_{M-1} - 1\}. \quad (3.67)$$

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ behaves similarly. This is why we have restricted $d_j \geq 1$.

We have assumed the parameter range (3.30). This range is needed in the intermediate Darboux transformations, but may be extended in the final results of the deformed system. There is one more reason why the range may be extended. Following our previous papers [11, 13], we have used the symmetry (2.23) and imposed positivity on $B'(x)$, $D'(x)$ and α , respectively. However, $B'(x)$ and $D'(x)$ always appear in combination with α . So, by introducing the following new quantities,

$$\begin{aligned} B'^{\text{new}}(x) &\stackrel{\text{def}}{=} \alpha B'(x), & D'^{\text{new}}(x) &\stackrel{\text{def}}{=} \alpha D'(x), & \mathcal{E}'_{\text{v}}{}^{\text{new}} &\stackrel{\text{def}}{=} \alpha \mathcal{E}'_{\text{v}}, \\ \mathcal{H}'^{\text{new}} &\stackrel{\text{def}}{=} -\sqrt{B'^{\text{new}}(x)} e^{\partial} \sqrt{D'^{\text{new}}(x)} - \sqrt{D'^{\text{new}}(x)} e^{-\partial} \sqrt{B'^{\text{new}}(x)} + B'^{\text{new}}(x) + D'^{\text{new}}(x), \end{aligned} \quad (3.68)$$

the symmetry (2.23) is expressed as $\mathcal{H} = \mathcal{H}'^{\text{new}} + \alpha'$ and the positivity condition is imposed on $B'^{\text{new}}(x)$ and $D'^{\text{new}}(x)$. The virtual state energy (2.27) is $\tilde{\mathcal{E}}_{\text{v}} = \mathcal{E}'_{\text{v}}{}^{\text{new}} + \alpha'$ and it should be negative for $\text{v} \in \mathcal{D}$ and $\text{v} = 0$ ($\alpha' = \tilde{\mathcal{E}}_0$). By considering these, the parameter range is extended to

$$0 < a < 1, \quad b < q^{1+d_M}. \quad (3.69)$$

3.4 Examples

For illustration, let us write down the type I and II single-indexed little q -Jacobi polynomials. For $\mathcal{D} = \{d\}$, they are given by

$$\check{P}_{\{d\},n}^{\text{I}}(x) = \frac{(1-b)q^n}{(1-aq^{n-d-1})(1-bq^{n+d})} (\check{\xi}_d^{\text{I}}(x+1)\check{P}_n(x) - aq^{-1}\check{\xi}_d^{\text{I}}(x)\check{P}_n(x+1)), \quad (3.70)$$

$$\check{P}_{\{d\},n}^{\text{II}}(x) = \frac{q^{-x}}{1-bq^{-1}} ((1-bq^{x-1})\check{\xi}_d^{\text{II}}(x-1)\check{P}_n(x) - (1-q^x)\check{\xi}_d^{\text{II}}(x)\check{P}_n(x-1)). \quad (3.71)$$

The type I and II multi-indexed polynomials are different polynomials, except for $\mathcal{D} = \{1\}$, in which case the two polynomials are related as $\check{P}_{\{1\},n}^{\text{I}}(x; \boldsymbol{\lambda} - \tilde{\boldsymbol{\delta}}^{\text{I}}) = \check{P}_{\{1\},n}^{\text{II}}(x; \boldsymbol{\lambda} - \tilde{\boldsymbol{\delta}}^{\text{II}})$. For example, for $\mathcal{D} = \{2\}$ and $n = 0, 1$, we have

$$\begin{aligned} \check{P}_{\{2\},0}^{\text{I}}(x) &= \frac{1}{(1-bq)(1-bq^2)} \left((1-aq^{-1})(1-aq^{-2}) + (1+q)(1-aq^{-2})(a-bq^3)q^{x-2} \right. \\ &\quad \left. + (a-bq^3)(a-bq^4)q^{2x-4} \right), \\ \check{P}_{\{2\},1}^{\text{I}}(x) &= \frac{q}{a(1-b)(1-bq)(1-bq^3)} \left(-(1-a)(1-aq^{-1})(1-aq^{-3}) \right. \\ &\quad \left. - (1-a)(1-aq^{-3})((1+q)(a-bq^3) - q^2(1-ab))q^{x-2} \right. \\ &\quad \left. + (1-aq^{-3})(a-bq^3)((1+q)(1-abq) - a+bq^2)q^{2x-2} \right. \\ &\quad \left. + (1-ab)(a-bq^3)(a-bq^4)q^{3x-4} \right), \\ \check{P}_{\{2\},0}^{\text{II}}(x) &= \frac{1}{(1-bq^{-1})(1-bq^{-2})} \left((1-aq)(1-aq^2) - (1+q)(1-aq^2)(b-aq^3)q^{x-2} \right. \\ &\quad \left. + (b-aq^3)(b-aq^4)q^{2x-3} \right), \\ \check{P}_{\{2\},1}^{\text{II}}(x) &= \frac{q^{-1}}{a(1-b)(1-bq^{-1})(1-bq^{-3})} \left(-(1-a)(1-aq)(1-aq^3) \right. \\ &\quad \left. + (1-a)(1-aq^3)((1+q)(b-aq^3) + q^2(1-ab))q^{x-2} \right. \\ &\quad \left. - (1-aq^3)(b-aq^3)((1+q)(q-ab) + b-aq^2)q^{2x-3} \right. \\ &\quad \left. + (1-ab)(b-aq^3)(b-aq^4)q^{3x-3} \right). \end{aligned}$$

We remark that $\check{P}_{\{2\},n}^{\text{I}}(x; \boldsymbol{\lambda})|_{x \rightarrow -x, q \rightarrow q^{-1}} = \check{P}_{\{2\},n}^{\text{II}}(x; \boldsymbol{\lambda})$ holds for $n = 0, 1$, but does not hold for $n \geq 2$.

4 Multi-indexed Little q -Laguerre polynomials

In this section we present the case-(1) multi-indexed little q -Laguerre polynomials, especially type II polynomials.

4.1 Original system

Let us present the basic data of little q -Laguerre rdQM system. The standard little q -Laguerre polynomial $p_n(q^x; a|q) = {}_2\phi_1\left(\begin{smallmatrix} q^{-n}, 0 \\ aq \end{smallmatrix} \middle| q; q^{x+1}\right)$ [16] does not satisfy our normalization $\check{P}_n(0) = 1$. The little q -Laguerre system is obtained from little q -Jacobi system by setting $b = 0$. We change the parametrization slightly from the standard one, $a^{\text{standard}} = aq^{-1}$. The

basic data are as follows [18, 21]:

$$q^\lambda = a, \quad \delta = 1, \quad \kappa = q^{-1}, \quad 0 < a < 1, \quad (4.1)$$

$$B(x; \boldsymbol{\lambda}) = aq^{-x-1}, \quad D(x) = q^{-x} - 1, \quad (4.2)$$

$$\mathcal{E}_n = q^{-n} - 1, \quad \eta(x) = 1 - q^x, \quad \varphi(x) = q^x, \quad (4.3)$$

$$\begin{aligned} \check{P}_n(x; \boldsymbol{\lambda}) &= {}_2\phi_0\left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix} \middle| q; a^{-1}q^{x+1}\right) = c'_n(\boldsymbol{\lambda}) p_n(1 - \eta(x); aq^{-1}|q) \\ &= c'_n(\boldsymbol{\lambda}) {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ a \end{matrix} \middle| q; q^{x+1}\right), \quad c'_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-a)^{-n} q^{-\binom{n}{2}} (a; q)_n, \end{aligned} \quad (4.4)$$

$$\check{P}_n(x; \boldsymbol{\lambda}) = c_n(\boldsymbol{\lambda}) \eta(x)^n + \text{lower degree terms}, \quad c_n(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-a)^{-n} q^{-n(n-1)}, \quad (4.5)$$

$$\phi_0(x; \boldsymbol{\lambda})^2 = \frac{a^x}{(q; q)_x} = \frac{(q^x; q)_\infty}{(q; q)_\infty} a^x, \quad \phi_0(x; \boldsymbol{\lambda}) > 0, \quad \phi_0(0; \boldsymbol{\lambda}) = 1, \quad (4.6)$$

$$d_n(\boldsymbol{\lambda})^2 = \frac{a^n q^{n(n-1)}}{(a, q; q)_n} \times (a; q)_\infty, \quad d_n(\boldsymbol{\lambda}) > 0, \quad (4.7)$$

and $\check{P}_n(0; \boldsymbol{\lambda})$ satisfies (3.8). The little q -Laguerre system has shape invariance (3.9) and the formulas (3.10)–(3.16) hold.

4.2 Type I polynomials

The potential functions $B'(x)$ and $D'(x)$ and the virtual state polynomials $\check{\xi}_v(x)$ are given by (3.17)–(3.18), where the twist operation \mathfrak{t} and the shift $\check{\delta}$ are (remark: a^{standard} is used in [13])

$$\mathfrak{t}^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} 2 - \lambda_1, \quad \text{namely } q^{\mathfrak{t}^I(\boldsymbol{\lambda})} = a^{-1}q^2, \quad (4.8)$$

$$\check{\delta}^I \stackrel{\text{def}}{=} -1, \quad \mathfrak{t}^I(\boldsymbol{\lambda}) + u\delta = \mathfrak{t}^I(\boldsymbol{\lambda} + u\check{\delta}^I) \quad (\forall u \in \mathbb{R}), \quad (4.9)$$

with $\alpha^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} aq^{-1}$ and $\alpha'^I(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(1 - aq^{-1})$. The parameter range is $0 < a < q^{1+d_M}$. Various formulas for the type I multi-indexed little q -Laguerre polynomials are presented in [13], which are obtained from those for the type I multi-indexed little q -Jacobi polynomials by setting $b = 0$.

4.3 Type II polynomials

The potential functions $B'(x)$ and $D'(x)$ for little q -Jacobi system (3.23) diverge in the $b \rightarrow 0$ limit. But $B'^{\text{new}}(x)$ and $D'^{\text{new}}(x)$ (3.68) have well-defined $b \rightarrow 0$ limits, because α (3.24) vanishes. So we define $B'^{\text{new}}(x)$ and $D'^{\text{new}}(x)$ as follows:

$$B'^{\text{new II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} a(q^{-x-1} - 1), \quad D'^{\text{new II}}(x) \stackrel{\text{def}}{=} q^{-x}. \quad (4.10)$$

The constant α' and the shift $\tilde{\delta}$ are defined by

$$\alpha'^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -(1-a), \quad \tilde{\delta}^{\text{II}} \stackrel{\text{def}}{=} 1. \quad (4.11)$$

By taking $b \rightarrow 0$ limit of (3.25), the virtual state polynomial $\check{\xi}_v(x)$ ($v \in \mathbb{Z}_{\geq 0}$) is given by

$$\check{\xi}_v^{\text{II}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \check{c}_v^{\text{II}}(\boldsymbol{\lambda}) {}_1\phi_1\left(\begin{matrix} q^{-v} \\ a \end{matrix} \middle| q; aq^{x+v+1}\right), \quad \check{c}_v^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (a; q)_v, \quad (4.12)$$

which satisfies (3.26) and (3.27) with

$$\check{c}_v^{\text{II}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} (-a)^v q^{v^2}. \quad (4.13)$$

For simplicity of presentation, the superscript II is omitted in the following.

The virtual state polynomial $\check{\xi}_v(x)$ satisfies the difference equation (for $x \in \mathbb{R}$) (2.26) (with superscript “new”) with

$$\mathcal{E}'^{\text{new}}_v(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} -a(1-q^v). \quad (4.14)$$

It is positive $\check{\xi}_v(x) > 0$ ($x \in \mathbb{Z}_{\geq -1}$) for $0 < a < 1$.

In the following, we assume the following parameter range:

$$0 < a < 1. \quad (4.15)$$

The functions $\check{\phi}_0(x)$ (> 0) (2.21) and $\nu(x)$ (2.22) become

$$\check{\phi}_0(x; \boldsymbol{\lambda})^2 = (q; q)_x a^x = \frac{(q; q)_\infty}{(q^{x+1}; q)_\infty} a^x, \quad (4.16)$$

$$\nu(x; \boldsymbol{\lambda}) = \frac{1}{(q; q)_x} = \frac{(q^{x+1}; q)_\infty}{(q; q)_\infty}, \quad (4.17)$$

and the virtual state energy $\check{\mathcal{E}}_v$ (2.27) becomes

$$\check{\mathcal{E}}_v(\boldsymbol{\lambda}) = -(1-aq^v). \quad (4.18)$$

The virtual state vectors $\check{\phi}_v(x)$ (2.25) satisfy the Schrödinger equation except for the end-point $x = 0$, (2.31) (with $N \rightarrow \infty$).

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})$ and the multi-indexed orthogonal polynomial $\check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda})$ ($n \in \mathbb{Z}_{\geq 0}$) are defined by (3.34)–(3.39) ($\alpha(\boldsymbol{\lambda})D'(-j; \boldsymbol{\lambda})$ is replaced by $D'^{\text{new}}(-j; \boldsymbol{\lambda})$ in (3.38)) with

$$r_j(x-j+1; \boldsymbol{\lambda}, M) = (q^{x-j+2}; q)_{j-1}. \quad (4.19)$$

They are polynomials in $\eta(x)$, (3.40)–(3.42), and their normalizations are (3.43). The coefficients of the highest degree terms are (3.44)–(3.45) with

$$c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda}) = \prod_{j=1}^M \frac{\tilde{c}_{d_j}(\boldsymbol{\lambda})}{\tilde{c}_{j-1}(\boldsymbol{\lambda})} \cdot q^{-(M-1)\ell_{\mathcal{D}}}, \quad (4.20)$$

$$c_{\mathcal{D},n}^P(\boldsymbol{\lambda}) = c_{\mathcal{D}}^{\Xi}(\boldsymbol{\lambda})c_n(\boldsymbol{\lambda})q^{-nM}. \quad (4.21)$$

The lowest degree multi-indexed orthogonal polynomial and the denominator polynomial are related as (3.46). In the end, the type II multi-indexed little q -Laguerre polynomials are obtained from the type II multi-indexed little q -Jacobi polynomials by taking $b \rightarrow 0$ limit.

The deformed potential functions $B_{\mathcal{D}}(x)$ (2.45) and $D_{\mathcal{D}}(x)$ (2.46) become (3.47)–(3.48), and the eigenvectors $\phi_{\mathcal{D}n}^{\text{gen}}(x)$ (2.47) become

$$\phi_{\mathcal{D}n}^{\text{gen}}(x; \boldsymbol{\lambda}) = \frac{\phi_0(x; \boldsymbol{\lambda} + M\tilde{\boldsymbol{\delta}})}{\sqrt{\check{\Xi}_{\mathcal{D}}(x; \boldsymbol{\lambda})\check{\Xi}_{\mathcal{D}}(x-1; \boldsymbol{\lambda})}} \check{P}_{\mathcal{D},n}(x; \boldsymbol{\lambda}). \quad (4.22)$$

We define the eigenvectors $\phi_{\mathcal{D}n}(x)$ as (3.50)–(3.51). The orthogonality relations for $\phi_{\mathcal{D}n}^{\text{gen}}(x)$ (2.41) (with $N \rightarrow \infty$) give those for $\check{P}_{\mathcal{D},n}(x)$, (3.52) with

$$\tilde{d}_{\mathcal{D},n}(\boldsymbol{\lambda})^2 = \frac{1}{\prod_{j=1}^M (\mathcal{E}_n - \tilde{\mathcal{E}}_{d_j}(\boldsymbol{\lambda}))}. \quad (4.23)$$

The Hamiltonian of the deformed system, the Schrödinger equation and their similarity transformed versions are given by (3.54)–(3.57). The shape invariance of the original system (3.9) is inherited by the deformed systems (3.58). As its consequence, we have relations (3.59)–(3.60) and the forward and backward shift relations (3.61)–(3.64).

The denominator polynomial $\check{\Xi}_{\mathcal{D}}(x)$ and the multi-indexed polynomials $\check{P}_{\mathcal{D},n}(x)$ are normalized as (3.43). Their values at $x = \infty$ are given by (3.65)–(3.66). The reason for restricting $d_j \geq 1$ is the same as for little q -Jacobi case, (3.67).

5 Summary and Comments

We have reconsidered the multi-step Darboux transformations with the virtual states as seed solutions for rdQM systems. There are two types of virtual states vectors, type I and type II. For finite rdQM systems such as q -Racah and Racah cases, the multi-step Darboux transformations with these two types of virtual states as seed solutions give essentially the same multi-indexed polynomials. On the other hand, for semi-infinite rdQM systems such as

little q -Jacobi and little q -Laguerre cases, they give different multi-indexed polynomials. By constructing the type II virtual state vectors explicitly, we obtain the case-(1) type II multi-indexed little q -Jacobi and little q -Laguerre orthogonal polynomials, which satisfy second order difference equations. The deformed rdQM systems have shape invariance and the multi-indexed polynomial satisfy the forward and backward shift relations. It is an interesting problem to study other semi-infinite rdQM systems such as Meixner and q -Meixner cases.

In our previous studies [10]–[14] (except for the type I Laguerre), the virtual states are obtained from the eigenstates by twisting the parameters (For the type I Laguerre, the virtual states are obtained from the eigenstates by replacing x with ix). In the cases of the type II little q -Jacobi and little q -Laguerre, the situation is different. For the type II little q -Jacobi case in §3.3, the twist operation is defined by (3.21), but it is used only in (3.28). The potential functions $B'(x)$ and $D'(x)$ (3.23) and the virtual state polynomial $\check{\xi}_v(x)$ (3.25) are not obtained from $B(x)$, $D(x)$ and $\check{P}_n(x)$ by twisting the parameters. This is also the case for the type II little q -Laguerre in §4.3.

The little q -Jacobi (Laguerre) polynomials reduce to the Jacobi (Laguerre) polynomials in the $q \rightarrow 1$ limit. Similarly the multi-indexed little q -Jacobi (Laguerre) polynomials reduce to the multi-indexed Jacobi (Laguerre) polynomials in the $q \rightarrow 1$ limit. The type I (II) multi-indexed little q -Jacobi polynomials ($\boldsymbol{\lambda}^{lqJ} = (g + \frac{1}{2}, h + \frac{1}{2})$) reduce to the multi-indexed Jacobi polynomials ($\boldsymbol{\lambda}^J = (g, h)$) with only type II (I) indices. For the virtual states in oQM, see [22]. The reason for the exchange of type I and type II is the coordinate correspondence, $q^{x^{lqJ}+1} \leftrightarrow \frac{1}{2}(1 - \cos 2x^J)$. The minimum and maximum values of x^{lqJ} , $x_{\min}^{lqJ} = 0$ and $x_{\max}^{lqJ} = \infty$, correspond to the maximum and minimum values of x^J , $x_{\max}^J = \frac{1}{2}\pi$ and $x_{\min}^J = 0$, respectively. Similarly the type I (II) multi-indexed little q -Laguerre polynomials ($\boldsymbol{\lambda}^{lqL} = g + \frac{1}{2}$) reduce to the multi-indexed Laguerre polynomials ($\boldsymbol{\lambda}^L = g$) with only type II (I) indices in the $q \rightarrow 1$ limit. We have no twist operation for the type II little q -Laguerre and the type I Laguerre. For the multi-indexed Jacobi and Laguerre polynomials, it is possible to use type I and II indices at the same time [10]. It is a challenging problem to study whether mixed use of type I and II indices is possible for the multi-indexed little q -Jacobi and little q -Laguerre polynomials. In the type I (II) construction, various quantities are expressed in terms of the Casoratian W_C (2.32) ($W_C^{(-)}$ (2.33)), in which x is shifted to $+$ ($-$) direction, respectively. Since x is shifted in the opposite direction, using both W_C and $W_C^{(-)}$ is not a good combination. In the above (2.32), we mention that the type-(ii) (type-(i)) factorization

for the type I (type II) virtual state vectors is impossible, respectively. But this is the case for finite N . For $N \rightarrow \infty$ case, the type-(ii) factorization for the type I virtual state vectors is possible. This would be a hint for mixed use of type I and II indices.

Acknowledgements

I thank the support by Course of Physics, Department of Science.

References

- [1] D. Gómez-Ullate, N. Kamran and R. Milson, “An extended class of orthogonal polynomials defined by a Sturm-Liouville problem,” *J. Math. Anal. Appl.* **359** (2009) 352-367, [arXiv:0807.3939 \[math-ph\]](#).
- [2] S. Odake and R. Sasaki, “Infinitely many shape invariant potentials and new orthogonal polynomials,” *Phys. Lett.* **B679** (2009) 414-417, [arXiv:0906.0142 \[math-ph\]](#).
- [3] S. Odake and R. Sasaki, “Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials,” *Phys. Lett.* **B684** (2010) 173-176, [arXiv:0911.3442 \[math-ph\]](#).
- [4] S. Odake and R. Sasaki, “Infinitely many shape invariant discrete quantum mechanical systems and new exceptional orthogonal polynomials related to the Wilson and Askey-Wilson polynomials,” *Phys. Lett.* **B682** (2009) 130-136, [arXiv:0909.3668 \[math-ph\]](#).
- [5] S. Odake and R. Sasaki, “Exceptional (X_ℓ) (q)-Racah polynomials,” *Prog. Theor. Phys.* **125** (2011) 851-870, [arXiv:1102.0812 \[math-ph\]](#).
- [6] D. Gómez-Ullate, N. Kamran and R. Milson, “Two-step Darboux transformations and exceptional Laguerre polynomials,” *J. Math. Anal. Appl.* **387** (2012) 410-418, [arXiv:1103.5724 \[math-ph\]](#).
- [7] D. Gómez-Ullate, Y. Grandati and R. Milson, “Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials,” *J. Phys.* **A47** (2014) 015203 (27pp), [arXiv:1306.5143 \[math-ph\]](#).
- [8] A. J. Durán, “Exceptional Meixner and Laguerre orthogonal polynomials,” *J. Approx. Theory* **184** (2014) 176-208, [arXiv:1310.4658 \[math.CA\]](#).

- [9] A. J. Durán, “Exceptional Hahn and Jacobi orthogonal polynomials,” J. Approx. Theory **214** (2017) 9-48, [arXiv:1510.02579 \[math.CA\]](#).
- [10] S. Odake and R. Sasaki, “Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials,” Phys. Lett. **B702** (2011) 164-170, [arXiv:1105.0508 \[math-ph\]](#). (Remark: $\tilde{\delta}^I$ and $\tilde{\delta}^{II}$ in this paper are changed to $-\tilde{\delta}^I$ and $-\tilde{\delta}^{II}$ in the later references.)
- [11] S. Odake and R. Sasaki, “Multi-indexed (q -)Racah polynomials,” J. Phys. **A 45** (2012) 385201 (21pp), [arXiv:1203.5868 \[math-ph\]](#).
- [12] S. Odake and R. Sasaki, “Multi-indexed Wilson and Askey-Wilson polynomials,” J. Phys. **A46** (2013) 045204 (22pp), [arXiv:1207.5584 \[math-ph\]](#).
- [13] S. Odake and R. Sasaki, “Multi-indexed Meixner and Little q -Jacobi (Laguerre) Polynomials,” J. Phys. **A50** (2017) 165204 (23pp), [arXiv:1610.09854 \[math.CA\]](#).
- [14] S. Odake, “Exactly Solvable Discrete Quantum Mechanical Systems and Multi-indexed Orthogonal Polynomials of the Continuous Hahn and Meixner-Pollaczek Types,” Prog. Theor. Exp. Phys. **2019** (2019) 123A01 (20pp), [arXiv:1907.12218 \[math-ph\]](#).
- [15] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, vol. 98 of Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge (2005).
- [16] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q -analogues*, Springer-Verlag Berlin-Heidelberg (2010).
- [17] S. Odake and R. Sasaki, “Discrete quantum mechanics,” (Topical Review) J. Phys. **A44** (2011) 353001 (47pp), [arXiv:1104.0473 \[math-ph\]](#). (Typo in (2.132), $c_1(\eta, \lambda)$ for $H : -\frac{1}{2} \Rightarrow -\frac{\eta}{2}$.)
- [18] S. Odake and R. Sasaki, “Orthogonal Polynomials from Hermitian Matrices,” J. Math. Phys. **49** (2008) 053503 (43pp), [arXiv:0712.4106 \[math.CA\]](#). (For the dual q -Meixner and dual q -Charlier polynomials, see [21].)
- [19] S. Odake and R. Sasaki, “Dual Christoffel transformations,” Prog. Theor. Phys. **126** (2011) 1-34, [arXiv:1101.5468 \[math-ph\]](#).

- [20] S. Odake and R. Sasaki, “Markov Chains Generated by Convolutions of Orthogonality Measures,” *J. Phys. A: Math. Theor.* **55** (2022) 275201 (42pp), [arXiv:2106.04082](#) [[math.PR](#)].
- [21] S. Odake and R. Sasaki, “Orthogonal Polynomials from Hermitian Matrices II,” *J. Math. Phys.* **59** (2018) 013504 (42pp), [arXiv:1604.00714](#) [[math.CA](#)].
- [22] S. Odake and R. Sasaki, “Krein-Adler transformations for shape-invariant potentials and pseudo virtual states,” *J. Phys.* **A46** (2013) 245201 (24pp), [arXiv:1212.6595](#) [[math-ph](#)].