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Unitary Representations of W Infinity Algebras

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Abstract

We study the irreducible unitary highest weight representations, which are obtained from free field realizations, of W infinity algebras (W_∞, $W_{1+\infty}$, $W_\infty^{1,1}$, W_∞^M, $W_{1+\infty}^N$, $W_\infty^{M,N}$) with central charges (2, 1, 3, $2M$, N, $2M+N$). The characters of these representations are computed.

We construct a new extended superalgebra $W_\infty^{M,N}$, whose bosonic sector is $W_\infty^M \oplus W_{1+\infty}^N$. Its representations obtained from a free field realization with central charge $2M+N$, are classified into two classes: continuous series and discrete series. For the former there exists a supersymmetry, but for the latter a supersymmetry exists only for $M=N$.

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1 Introduction

The conformal field theory (CFT) in two dimensional space-time has made great progress in close contact with both the string theory and various branches of mathematics. The Virasoro algebra plays a central role in CFT, and to construct models of CFT and extend this theory, one needs an extension of the Virasoro algebra. In view of this several extensions (superconformal algebras, W algebras, parafermions, etc.) have been studied. The notable example is Zamolodchikov's W_3 algebra and its W_N generalization (A_{N-1} type W algebra) containing fields of conformal weight (spin) $2, \cdots, N$ as conserved currents [1]. All of extended Virasoro algebras containing currents of spin > 2 have a non-linear property.

By taking an appropriate $N \to \infty$ limit of the W_N algebra, one can obtain a linear algebra with infinite number of fields. The first example is the w_∞ algebra [2], which can be interpreted as the algebra of area-preserving diffeomorphisms of two dimensional phase space. But w_∞ admits a central extension only in the Virasoro sector. By deforming w_∞, Pope, Romans and Shen constructed the W_∞ algebra [3], which admits central extension in all spin sectors. This is another large N limit of W_N. In addition, they constructed the $W_{1+\infty}$ algebra [4], which contains a spin 1 field too, and their super extension, super W_∞ algebra [5], whose bosonic sector is $W_\infty \oplus W_{1+\infty}$. Soon afterward Bakas and Kiritsis constructed the W_∞^M algebra [6], which is a $u(M)$ matrix version of W_∞, and Sano and the present author constructed the $\widehat{su}(N)$-$W_{1+\infty}$ algebra [7], which is an extension of $W_{1+\infty}$ and contains the $SU(N)$ current algebra. We will change the notation $\widehat{su}(N)$-$W_{1+\infty}$ to $W_{1+\infty}^N$. $W_{1+\infty}^N$ is to W_∞^N what $W_{1+\infty}$ is to W_∞ [8]. In this paper we will construct a new superalgebra $W_\infty^{M,N}$, whose bosonic sector is $W_\infty^M \oplus W_{1+\infty}^N$. In this notation, super W_∞ [5] and super $\widehat{su}(N)$-W_∞ [7] are $W_\infty^{1,1}$ and $W_\infty^{1,N}$ respectively.

Given an algebra which generates the symmetry of a model, it is important to develop its representation theory to find what fields appear in the model. The success of the representation theory of the Virasoro algebra is a good example [9]. In spite of difficulties due to the non-linearity, the minimal representations of the W_N algebra have been studied in detail by using the free field realization of Feigin-Fuchs type or the coset model of affine Lie algebras [1, 10]. Although W infinity algebras are linear algebras and their structure constants are explicitly known, their representation theories have been studied very poorly [11, 8]. In this paper we will initiate the study of representation theories of W infinity algebras and construct their unitary representations based on free field realizations.

Our methods to study representation theories are as follows. First we prepare a free field realization of the W infinity algebra and its Fock space. In the Fock space, we try finding all the highest weight states (HWS's) of the subalgebra (an affine Lie algebra or the Virasoro algebra), whose generators have the lowest spin. Next we check these states are also the HWS's of the whole W infinity algebra. To compute the characters of $W_{1+\infty}$ and $W^N_{1+\infty}$, we use the character formulas of the affine Lie algebras and the fact that the generators of $W_{1+\infty}$ and $W^N_{1+\infty}$ do not change the $U(1)$ charge. For W_∞ and W^M_∞, we use the facts that the generators of the W infinity algebras are represented in terms of bilinears of the free fields and they preserve a certain quantum number. For super cases, we use the results of bosonic cases and the spectral flow invariance. The Witten index is also computed.

The organization of this paper is as follows. In §2 we construct a new superalgebra $W^{M,N}_\infty$ and present its free field realization. In §3-5 the representations of W infinity algebras ($W_{1+\infty}$, $W^N_{1+\infty}$, W_∞, W^M_∞, $W^{1,1}_\infty$, $W^{M,N}_\infty$) are studied, and their characters are computed. In §6 we present a discussion.

2 Algebras and Free Field Realizations

We will define a new extended superalgebra $W^{M,N}_\infty$, whose bosonic sector is $W^M_\infty \oplus W^N_{1+\infty}$. Other W infinity algebras are contained in it as subalgebras. $W^{M,N}_\infty$ is generated by[1]

$$W^{i,(\alpha\beta)}(z), \qquad (i \geq -1; \alpha, \beta = 1, 2, \cdots, N), \tag{1}$$

$$\tilde{W}^{i,(ab)}(z), \qquad (i \geq 0; a, b = 1, 2, \cdots, M), \tag{2}$$

[1] As usual, the mode expansion of a field $A(z)$ of conformal weight h is $A(z) = \sum A_n z^{-n-h}$, where the sum is taken over $n \in \mathbb{Z} - h$ for the Neveu-Schwarz (NS) sector, $n \in \mathbb{Z}$ for the Ramond (R) sector.

$$G^{i,a\alpha}(z), \qquad (i \geq 0; a = 1, \cdots, M; \alpha = 1, \cdots, N), \tag{3}$$

$$\bar{G}^{i,a\alpha}(z), \qquad (i \geq 0; a = 1, \cdots, M; \alpha = 1, \cdots, N). \tag{4}$$

$W^{i,(\alpha\beta)}(z)$ and $\tilde{W}^{i,(ab)}(z)$ are bosonic fields with conformal spin $i+2$, and generate $W^N_{1+\infty}$ and W^M_∞ respectively. $W^N_{1+\infty}$ (W^M_∞) contains $W_{1+\infty}$ (W_∞) as a subalgebra, and their generators are

$$V^i(z) = \sum_{\alpha=1}^N W^{i,(\alpha\alpha)}(z), \quad \tilde{V}^i(z) = \sum_{a=1}^M \tilde{W}^{i,(aa)}(z). \tag{5}$$

$V^0(z)$ and $\tilde{V}^0(z)$ are the Virasoro generators with central charge c and $\tilde{c}$ respectively. $W^N_{1+\infty}$ contains the $U(N)$ current algebra, which is generated by

$$\widehat{su}(N)_k \quad : \quad \begin{cases} H^i(z) = J^{(ii)}(z) - J^{(i+1,i+1)}(z), \quad (i = 1, \cdots, N-1) \ \ \text{Cartan} \\ J^{(\alpha\beta)}(z), \quad (\alpha < \beta) \ \ \text{raising} \ ; \quad (\alpha > \beta) \ \ \text{lowering}, \end{cases} \tag{6}$$

$$\hat{u}(1)_K \quad : \quad J(z) = \sum_{\alpha=1}^N J^{(\alpha\alpha)}(z), \tag{7}$$

where $J^{(\alpha\beta)}(z) = -4qW^{-1,(\beta\alpha)}(z)$ [2]. k is the level of $\widehat{su}(N)$ and K stands for a normalization of $\hat{u}(1)$ ($[J_m, J_n] = Km\delta_{m+n,0}$). $G^{i,a\alpha}(z)$ and $\bar{G}^{i,a\alpha}(z)$ are fermionic fields with conformal spin $i + \frac{3}{2}$. $W^{i,(\alpha\beta)}(z)$ ($\tilde{W}^{i,(ab)}(z)$, $G^{i,a\alpha}(z)$, $\bar{G}^{i,a\alpha}(z)$) transform according to the adjoint (trivial, $\mathbf{N}$, $\bar{\mathbf{N}}$) representation of $su(N)$, and have $U(1)$ charge 0 $(0, 1, -1)$, respectively.

(Anti-)commutation relations of $W^{M,N}_\infty$ are given by[3]

$$\begin{aligned}
[W^{i,(\alpha\beta)}_m, W^{j,(\gamma\delta)}_n] &= \tfrac{1}{2} \sum_{r\geq -1} q^r g^{ij}_r(m,n)(\delta^{\gamma\beta} W^{i+j-r,(\alpha\delta)}_{m+n} + (-1)^r \delta^{\alpha\delta} W^{i+j-r,(\gamma\beta)}_{m+n}) \\
&\quad + \delta^{ij}\delta^{\alpha\delta}\delta^{\gamma\beta}\delta_{m+n,0} q^{2i} k_i(m),
\end{aligned} \tag{8}$$

$$\begin{aligned}
[\tilde{W}^{i,(ab)}_m, \tilde{W}^{j,(cd)}_n] &= \tfrac{1}{2} \sum_{r\geq -1} q^r \tilde{g}^{ij}_r(m,n)(\delta^{cb} \tilde{W}^{i+j-r,(ad)}_{m+n} + (-1)^r \delta^{ad} \tilde{W}^{i+j-r,(cb)}_{m+n}) \\
&\quad + \delta^{ij}\delta^{ad}\delta^{cb}\delta_{m+n,0} q^{2i} \tilde{k}_i(m),
\end{aligned} \tag{9}$$

$$[W^{i,(\alpha\beta)}_m, G^{j,c\gamma}_n] = \delta^{\alpha\gamma} \sum_{r\geq -1} q^r a^{ij}_r(m,n) G^{i+j-r,c\beta}_{m+n}, \tag{10}$$

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[2] This definition is slightly different from [7].

[3] q is a deformation parameter and we can take it to be an arbitrary non-zero constant(e.g., $q = \frac{1}{4}$). This q has nothing to do with $q = e^{2\pi i\tau}$ appearing in the characters.

$$
\begin{aligned}
[W_m^{i,(\alpha\beta)}, \bar{G}_n^{j,c\gamma}] &= \delta^{\beta\gamma} \sum_{r\geq -1} q^r (-1)^r a_r^{ij}(m,n) \bar{G}_{m+n}^{i+j-r,c\alpha}, && (11)\\
[\tilde{W}_m^{i,(ab)}, G_n^{j,c\gamma}] &= \delta^{bc} \sum_{r\geq -1} q^r \tilde{a}_r^{ij}(m,n) G_{m+n}^{i+j-r,a\gamma}, && (12)\\
[\tilde{W}_m^{i,(ab)}, \bar{G}_n^{j,c\gamma}] &= \delta^{ac} \sum_{r\geq -1} q^r (-1)^r \tilde{a}_r^{ij}(m,n) \bar{G}_{m+n}^{i+j-r,b\gamma}, && (13)\\
\{G_m^{i,a\alpha}, \bar{G}_n^{j,b\beta}\} &= \sum_{r\geq 0} q^r (\delta^{ab} b_r^{ij}(m,n) W_{m+n}^{i+j-r,(\beta\alpha)} + \delta^{\alpha\beta} \tilde{b}_r^{ij}(m,n) \tilde{W}_{m+n}^{i+j-r,(ab)}) \\
&\quad + \delta^{ij} \delta^{ab} \delta^{\alpha\beta} \delta_{m+n,0} q^{2i} \check{k}_i(m), && (14)\\
[W_m^{i,(\alpha\beta)}, \tilde{W}_n^{j,(ab)}] &= \{G_m^{i,a\alpha}, G_n^{j,b\beta}\} = \{\bar{G}_m^{i,a\alpha}, \bar{G}_n^{j,b\beta}\} = 0. && (15)
\end{aligned}
$$

The structure constants [3, 4, 5, 6, 7] are

$$
\begin{aligned}
g_r^{ij}(m,n) &= \tfrac{1}{2(r+1)!} \phi_r^{ij}(0,-\tfrac{1}{2}) N_r^{i,j}(m,n), && (16)\\
\tilde{g}_r^{ij}(m,n) &= \tfrac{1}{2(r+1)!} \phi_r^{ij}(0,0) N_r^{i,j}(m,n), && (17)\\
a_r^{ij}(m,n) &= \tfrac{(-1)^r}{4(r+2)!}((i+1)\phi_{r+1}^{ij}(0,0) - (i-r-1)\phi_{r+1}^{ij}(0,-\tfrac{1}{2})) N_r^{i,j-\frac{1}{2}}(m,n), && (18)\\
\tilde{a}_r^{ij}(m,n) &= \tfrac{-1}{4(r+2)!}((i-r)\phi_{r+1}^{ij}(0,0) - (i+2)\phi_{r+1}^{ij}(0,-\tfrac{1}{2})) N_r^{i,j-\frac{1}{2}}(m,n), && (19)\\
b_r^{ij}(m,n) &= \tfrac{(-1)^r 4}{r!}((i+j+2-r)\phi_r^{ij}(\tfrac{1}{2},-\tfrac{1}{4}) \\
&\qquad - (i+j+\tfrac{3}{2}-r)\phi_{r+1}^{ij}(\tfrac{1}{2},-\tfrac{1}{4})) N_{r-1}^{i-\frac{1}{2},j-\frac{1}{2}}(m,n), && (20)\\
\tilde{b}_r^{ij}(m,n) &= -\tfrac{4}{r!}((i+j+1-r)\phi_r^{ij}(\tfrac{1}{2},-\tfrac{1}{4}) \\
&\qquad - (i+j+\tfrac{3}{2}-r)\phi_{r+1}^{ij}(\tfrac{1}{2},-\tfrac{1}{4})) N_{r-1}^{i-\frac{1}{2},j-\frac{1}{2}}(m,n), && (21)
\end{aligned}
$$

and

$$
\begin{aligned}
N_r^{x,y}(m,n) &= \sum_{\ell=0}^{r+1} (-1)^\ell \binom{r+1}{\ell} [x+1+m]_{r+1-\ell} [x+1-m]_\ell \\
&\qquad \times [y+1+n]_\ell [y+1-n]_{r+1-\ell}, && (22)\\
\phi_r^{ij}(x,y) &= {}_4F_3\Big[\begin{array}{c} -\frac{1}{2}-x-2y, \frac{3}{2}-x+2y, -\frac{r+1}{2}+x, -\frac{r}{2}+x \\ -i-\frac{1}{2}, -j-\frac{1}{2}, i+j-r+\frac{5}{2} \end{array} ; 1 \Big], && (23)
\end{aligned}
$$

$$
{}_4F_3\Big[\begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{array} ; z \Big] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n (a_4)_n}{(b_1)_n (b_2)_n (b_3)_n} \frac{z^n}{n!}, \qquad (24)
$$

where $[x]_n = x(x-1)\cdots(x-n+1)$, $[x]_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$, $(x)_0 = 1$ and $\binom{x}{n} = [x]_n/n!$. Since $g_r^{ij} = b_r^{ij} = 0$ for $i-j-r < -1$ and $\tilde{g}_r^{ij} = a_r^{ij} = \tilde{a}_r^{ij} = \tilde{b}_r^{ij} = 0$

for $i-j-r<0$, the summations over r are finite sums and the algebra closes. The central terms are

$$k_i(m)=k_i\prod_{j=-i-1}^{i+1}(m+j)\ , \qquad k_i=\frac{2^{2i-2}((i+1)!)^2}{(2i+1)!!(2i+3)!!}k, \tag{25}$$

$$\tilde{k}_i(m)=\tilde{k}_i\prod_{j=-i-1}^{i+1}(m+j)\ , \qquad \tilde{k}_i=\frac{2^{2i-3}i!(i+2)!}{(2i+1)!!(2i+3)!!}\tilde{k}, \tag{26}$$

$$\check{k}_i(m)=\check{k}_i\prod_{j=-i-1}^{i}(m+j+\frac{1}{2})\ , \qquad \check{k}_i=\frac{2^{2i}i!(i+1)!}{3((2i+1)!!)^2}\check{k}. \tag{27}$$

In the case of $W_\infty^{M,N}$, the Jacobi identity requires

$$K=Nk\ , \quad c=Nk\ , \quad \tilde{c}=M\tilde{k}\ , \quad \tilde{k}=2k\ , \quad \check{k}=3k. \tag{28}$$

Since the level k of $\widehat{su}(N)$ $(N>1)$ is a positive integer for unitary representations, central charges c and $\tilde{c}$ must be multiples of N and $2M$ respectively. (Anti-)commutation relations of $W_\infty^{M,N}$ are consistent with the hermiticity properties of the generators:

$$W_n^{i,(\alpha\beta)\dagger}=W_{-n}^{i,(\beta\alpha)},\ \tilde{W}_n^{i,(ab)\dagger}=\tilde{W}_{-n}^{i,(ba)},\ G_n^{i,a\alpha\dagger}=\bar{G}_{-n}^{i,a\alpha}. \tag{29}$$

The Cartan subalgebra of $W_\infty^{M,N}$ is generated by

$$W_0^{i,(\alpha\alpha)},\quad \tilde{W}_0^{i,(aa)}. \tag{30}$$

The HWS of $W_\infty^{M,N}$ in the NS sector is defined by

$$\begin{cases} A_n|\mathrm{hws}\rangle=0, & (n>0; A=W,\tilde{W},G,\bar{G}) \\ W_0^{i,(\alpha\beta)}|\mathrm{hws}\rangle=0, & (\alpha>\beta) \\ \tilde{W}_0^{i,(ab)}|\mathrm{hws}\rangle=0, & (a>b), \end{cases} \tag{31}$$

and, in the R sector, we require one more condition:

$$G_0^{i,a\alpha,R}|\mathrm{hws}\rangle^R=0. \tag{32}$$

The HWS's of other W infinity algebras are defined in a similar way.

Since $W_\infty^{M,N}$ contains a current algebra, there exists an automorphism, so called spectral flow [12]. Namely (anti-)commutation relations are invariant under the transformations of the generators. Explicit forms of the transformation rules are essentially the same as $W_\infty^{1,N}$ [7]. Due to this property, representations in the R sector and those in the NS sector have one-to-one correspondence. We define the representations in the R sector as those mapped from the NS sector by the spectral flow with $\eta = \frac{1}{2}$. Then we can show $|\text{hws}\rangle^R = |\text{hws}\rangle^{NS}$, because

$$\left\{ \begin{array}{ll} W_n^{i\prime} = \sum_{j=-1}^{i} (\text{coeff.}) \cdot W_n^j + (\text{coeff.}) \cdot \delta_{n0}, & G_n^{i\prime} = \sum_{j=0}^{i} (\text{coeff.}) \cdot G_{n+\frac{1}{2}}^j \\ \tilde{W}_n^{i\prime} = \sum_{j=0}^{i} (\text{coeff.}) \cdot \tilde{W}_n^j + (\text{coeff.}) \cdot \delta_{n0}, & \bar{G}_n^{i\prime} = \sum_{j=0}^{i} (\text{coeff.}) \cdot \bar{G}_{n-\frac{1}{2}}^j . \end{array} \right. \tag{33}$$

$W_\infty^{M,N}$ with level $k=1$ is realized by N complex free fermions $\psi^\alpha(z) = \sum_n \psi_n^\alpha z^{-n-\frac{1}{2}}$ $(\alpha = 1, \cdots, N)$ and M complex free bosons $i\partial\varphi^a(z) = \sum_n \alpha_n^a z^{-n-1}$ $(a = 1, \cdots, M)$. Operator product expansions of the free fields are

$$\bar{\psi}^\alpha(z)\psi^\beta(w) \sim \frac{\delta^{\alpha\beta}}{z-w}, \quad i\partial\bar{\varphi}^a(z) i\partial\varphi^b(w) \sim \frac{\delta^{ab}}{(z-w)^2}. \tag{34}$$

Generators of $W_\infty^{M,N}$ are represented in terms of bilinears of the free fields [4, 5, 6, 7] :

$$W^{j,(\alpha\beta)}(z) = \frac{2^{j-1}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j+1} (-1)^r \binom{j+1}{r}^2 (\partial^{j+1-r}\bar{\psi}^\alpha \partial^r \psi^\beta)(z), \tag{35}$$

$$\tilde{W}^{j,(ab)}(z) = \frac{2^{j-1}(j+2)!}{(2j+1)!!} q^j \sum_{r=0}^{j} \frac{(-1)^r}{j+1} \binom{j+1}{r} \binom{j+1}{r+1} (\partial^{j-r} i\partial\bar{\varphi}^a \partial^r i\partial\varphi^b)(z), \tag{36}$$

$$G^{j,a\alpha}(z) = \frac{2^{j+\frac{1}{2}}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j} (-1)^r \binom{j+1}{r} \binom{j}{r} (\partial^{j-r} i\partial\bar{\varphi}^a \partial^r \psi^\alpha)(z), \tag{37}$$

$$\bar{G}^{j,a\alpha}(z) = \frac{2^{j+\frac{1}{2}}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j} (-1)^{j+r} \binom{j+1}{r} \binom{j}{r} (\partial^{j-r} i\partial\varphi^a \partial^r \bar{\psi}^\alpha)(z), \tag{38}$$

where the normal ordered product of two fields $A(z)$ and $B(z)$ is defined by $(AB)(z) = \oint_z \frac{dx}{2\pi i} \frac{1}{x-z} A(x)B(z)$. One of the methods to obtain the general level k realization is to prepare k copies of the above realization, because $W_\infty^{M,N}$ is linear. The spectral flow transformation rules of the generators with a parameter η are easily derived from those of the free fields:

$$\psi^{\alpha\prime}(z) = z^\eta \psi^\alpha(z),\ \bar{\psi}^{\alpha\prime}(z) = z^{-\eta}\bar{\psi}^\alpha(z),\ \varphi^{a\prime}(z) = \varphi^a(z),\ \bar{\varphi}^{a\prime}(z) = \bar{\varphi}^a(z), \tag{39}$$

because eq. (34) are invariant under this transformation. The transformation rules that will be needed later are

$$\begin{aligned} W_n^{-1,(\alpha\beta)\prime} &= W_n^{-1,(\alpha\beta)} - \delta^{\alpha\beta}\delta_{n0}\tfrac{1}{4q}\eta, \\ W_n^{0,(\alpha\beta)\prime} &= W_n^{0,(\alpha\beta)} - 4q\eta W_n^{-1,(\alpha\beta)} + \delta^{\alpha\beta}\delta_{n0}\tfrac{1}{2}\eta^2, \\ G_n^{0,a\alpha\prime} &= G_{n+\eta}^{0,a\alpha}, \quad \bar{G}_n^{0,a\alpha\prime} = \bar{G}_{n-\eta}^{0,a\alpha}, \quad \tilde{W}_n^{0,(ab)\prime} = \tilde{W}_n^{0,(ab)}. \end{aligned} \tag{40}$$

The vacuum states of the fermion and boson Fock spaces, $|0\rangle$ and $|\vec{p},\vec{\bar{p}}\rangle$, are defined as usual: $\psi^\alpha_m|0\rangle = \bar{\psi}^\alpha_n|0\rangle = \alpha^a_n|\vec{p},\vec{\bar{p}}\rangle = \bar{\alpha}^a_n|\vec{p},\vec{\bar{p}}\rangle = 0$, $(m \geq 0, n > 0)$, $\alpha^a_0|\vec{p},\vec{\bar{p}}\rangle = p_a|\vec{p},\vec{\bar{p}}\rangle$, $\bar{\alpha}^a_0|\vec{p},\vec{\bar{p}}\rangle = \bar{p}_a|\vec{p},\vec{\bar{p}}\rangle$. Hermiticity properties of the generators eq. (29) are satisfied by those of the free fields ($\psi^{\alpha\dagger}_n = \bar{\psi}^\alpha_{-n}$, $\alpha^{a\dagger}_n = \bar{\alpha}^a_{-n}$). In the following we take $p^*_a = \bar{p}_a$, so that the unitarity of the representations is manifest.

3 Representations of $W_{1+\infty}$ and $W^N_{1+\infty}$

We first consider the representations of $W_{1+\infty}$ with $c = 1$ realized by one complex free fermion. We remark that the Virasoro generator $V^0(z)$ agrees with the Sugawara form of $\hat{u}(1)$ current. For each integer n, we can find the HWS of the subalgebra $\hat{u}(1)$ contained in the fermion Fock space, and we denote them as

$$|n\rangle \stackrel{\text{def}}{=} \begin{cases} \psi_{-\frac{1}{2}}\psi_{-\frac{3}{2}}\cdots\psi_{-n+\frac{1}{2}}|0\rangle & n \geq 1 \\ |0\rangle & n = 0 \\ \bar{\psi}_{-\frac{1}{2}}\bar{\psi}_{-\frac{3}{2}}\cdots\bar{\psi}_{n+\frac{1}{2}}|0\rangle & n \leq -1. \end{cases} \tag{41}$$

These states are well known in Sato theory [13]. We can check that $|n\rangle$ is not only the HWS of $\hat{u}(1)$ but also the HWS of $W_{1+\infty}$. The conformal weight h_n and $U(1)$ charge Q_n of $|n\rangle$ are

$$h_n = \frac{1}{2}n^2, \quad Q_n = n. \tag{42}$$

Although the eigenvalues of the higher-spin generators are easily calculated, we omit them here.

Since the dependence on the eigenvalues of higher-spin generators are very complicated, we consider the characters which count conformal weight and $U(1)$ charge only:

$$ch^{W_{1+\infty}}(\theta,\tau) \stackrel{\text{def}}{=} \mathrm{tr} q^{V^0_0 - \frac{1}{24}} z^{J_0}, \tag{43}$$

where $q = e^{2\pi i\tau}$ ($\mathrm{Im}\tau > 0$) and $z = e^{i\theta}$. Since $W_{1+\infty}$ contains $\hat{u}(1)$ as a subalgebra, the representation of $W_{1+\infty}$ has more states than one of $\hat{u}(1)$. On the other hand, the representation of $W_{1+\infty}$ has less states than the Fock space with the fixed $U(1)$ charge, because generators of $W_{1+\infty}$ do not change $U(1)$ charge. These statements are expressed in terms of characters as follows:

$$\chi_n^{\hat{u}(1)_1}(\theta,\tau) \leq ch_n^{W_{1+\infty}}(\theta,\tau) \leq z^n \chi_n^{Fock}(\tau), \tag{44}$$

where $A \leq B$ means $B-A$ is a q-series with non-negative coefficients. In general, the character formula of $\hat{u}(1)_K$ with $U(1)$ charge Q is

$$\chi_Q^{\hat{u}(1)_K}(\theta,\tau) \stackrel{\text{def}}{=} \mathrm{tr} q^{L_0-\frac{1}{24}} z^{J_0} = \frac{1}{\eta(\tau)} q^{\frac{1}{2K}Q^2} z^Q, \tag{45}$$

where $\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$. On the other hands, the generating function of $\chi_n^{Fock}(\tau)$ is

$$\sum_{n\in\mathbb{Z}} z^n \chi_n^{Fock}(\tau) = \mathrm{tr}_{Fock} q^{V_0^0-\frac{1}{24}} z^{J_0} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}}). \tag{46}$$

Due to the Jacobi's triple product identity, we have $\chi_n^{\hat{u}(1)_1}(\theta,\tau) = z^n\chi_n^{Fock}(\tau)$. Therefore we obtain the character formula of $W_{1+\infty}$,

$$ch_n^{W_{1+\infty}}(\theta,\tau) = \chi_n^{\hat{u}(1)_1}(\theta,\tau). \tag{47}$$

Next we consider the representations of $W_{1+\infty}^N$ with $c=N$ realized by N complex free fermions. We use the same techniques as $W_{1+\infty}$ case. We remark that, in the case of level $k=1$, the Virasoro generator $V^0(z)$ agrees with the sum of the Sugawara form of $\hat{u}(1)_N$ and $\widehat{su}(N)_1$ [14]. For each integer n, there exists the HWS of $\widehat{su}(N)_1$, and we denote them as

$$|n\rangle \stackrel{\text{def}}{=} \begin{cases} \prod_{j=1}^{m}(\prod_{\alpha=1}^{N}\psi^{\alpha}_{-j+\frac{1}{2}})\cdot\prod_{\alpha=1}^{a}\psi^{\alpha}_{-m-\frac{1}{2}}|0\rangle & n\geq 1 \\ |0\rangle & n=0 \\ \prod_{j=1}^{-m-1}(\prod_{\alpha=1}^{N}\bar{\psi}^{N+1-\alpha}_{-j+\frac{1}{2}})\cdot\prod_{\alpha=1}^{N-a}\bar{\psi}^{N+1-\alpha}_{m+\frac{1}{2}}|0\rangle & n\leq -1, \end{cases} \tag{48}$$

where we express n as $n=Nm+a$, $(m\in\mathbb{Z}; a=0,1,\cdots,N-1)$. $|n\rangle$ is the HWS of the a-th rank antisymmetric representation of $\widehat{su}(N)_1$. The state $|n\rangle$ is also the HWS of $W_{1+\infty}^N$. The conformal weight h_n and $U(1)$ charge Q_n are

$$h_n = \frac{1}{2N}n^2 + \frac{a(N-a)}{2N}, \quad Q_n = n. \tag{49}$$

The first and second factors of h_n are contributions from $\hat{u}(1)_N$ and $\widehat{su}(N)_1$ respectively.

Neglecting the dependence on the eigenvalues of higher-spin generators, we consider the characters which count conformal weight, $U(1)$ charge and eigenvalues of $SU(N)$,

$$ch^{W^N_{1+\infty}}(\theta,\vec{\theta},\tau) \stackrel{\rm def}{=} {\rm tr} q^{V^0_0-\frac{N}{24}} e^{i\theta J_0} e^{i\vec{\theta}\cdot\vec{H}_0}, \tag{50}$$

where $\vec{\theta}=\sum_{i=1}^{N-1}\theta_i\vec{\alpha}_i$ and $\vec{\alpha}_i$ is a simple root of $su(N)$. $W^N_{1+\infty}$ contains $\hat{u}(1)_N \oplus \widehat{su}(N)_1$ as a subalgebra, and generators of $W^N_{1+\infty}$ do not change $U(1)$ charge. Therefore, the similar argument as $W_{1+\infty}$ case shows that the character formula of $W^N_{1+\infty}$ is given by

$$ch^{W^N_{1+\infty}}_n(\theta,\vec{\theta},\tau) = \chi^{\hat{u}(1)_N}_n(\theta,\tau)\chi^{\widehat{su}(N)_1}_a(\vec{\theta},\tau), \tag{51}$$

where $a\equiv n ({\rm mod}\ N), 0\leq a\leq N-1$. The character of $\hat{u}(1)$ is given by eq. (45), and the character formula of $\widehat{su}(N)_1$ is given by [15]

$$\chi^{\widehat{su}(N)_1}_a(\vec{\theta},\tau) \stackrel{\rm def}{=} {\rm tr} q^{L_0-\frac{N-1}{24}} e^{i\vec{\theta}\cdot\vec{H}_0} = \frac{1}{\eta(\tau)^{N-1}} \sum_{\vec{M}\in\Lambda_R} q^{\frac{1}{2}(\vec{M}+\vec{\Lambda}_a)^2} e^{i\vec{\theta}\cdot(\vec{M}+\vec{\Lambda}_a)}, \tag{52}$$

where Λ_R is a root lattice of $su(N)$, and $\vec{\Lambda}_i$ $(1\leq i\leq N-1)$ is a fundamental weight of $su(N)$ and $\vec{\Lambda}_0=\vec{0}$. To show eq. (51), we need the identity:

$$\begin{aligned} {\rm tr}_{Fock} q^{V^0_0-\frac{N}{24}} e^{i\theta J_0} e^{i\vec{\theta}\cdot\vec{H}_0} &= q^{-\frac{N}{24}} \prod_{j=1}^{N}\prod_{n=1}^{\infty}(1+z z_{j-1}^{-1} z_j q^{n-\frac{1}{2}})(1+z^{-1} z_{j-1} z_j^{-1} q^{n-\frac{1}{2}}) \\ &= \sum_{n\in\mathbb{Z}} \chi^{\hat{u}(1)_N}_n(\theta,\tau)\chi^{\widehat{su}(N)_1}_a(\vec{\theta},\tau), \end{aligned} \tag{53}$$

where $z_j=e^{i\theta_j}$, $z_0=z_N=1$, and $a\equiv n ({\rm mod}\ N)$. This identity is proved by the Jacobi's triple product identity.

4 Representations of W_∞ and W^M_∞

We first consider the representations of W_∞ with $\tilde{c}=2$ realized by one complex free boson. In the boson Fock space, the HWS's of the Virasoro generator $\tilde{V}^0(z)$ are classified into two classes: continuous series, whose momentum can be changed continuously, and discrete series, whose state exists for each integer n,

$$|p,\bar{p}\rangle, \quad (p\neq 0), \tag{54}$$

$$|n\rangle \stackrel{\rm def}{=} \begin{cases} (\alpha_{-1})^n|0,0\rangle & n\geq 1 \\ |0,0\rangle & n=0 \\ (\bar{\alpha}_{-1})^{-n}|0,0\rangle & n\leq -1. \end{cases} \tag{55}$$

We can check that these states are the HWS's of W_∞. Their conformal weights are

$$h_{p\bar{p}} = |p|^2, \quad h_n = |n|. \tag{56}$$

Neglecting the dependence on the eigenvalues of higher-spin generators, we consider the characters which count conformal weight only,

$$ch^{W_\infty}(\tau) \stackrel{\mathrm{def}}{=} \mathrm{tr} q^{\tilde{V}_0^0 - \frac{2}{24}}. \tag{57}$$

Since $\tilde{V}_n^i$ contains the terms $\bar{\alpha}_0 \alpha_n$ and $\bar{\alpha}_n \alpha_0$, and the momentum p is non-zero for the continuous series, the set of generators $\tilde{V}_n^i$ is identified with the set of oscillators α_n, $\bar{\alpha}_n$. Therefore the character formula of the continuous series of W_∞ is

$$ch_{p\bar{p}}^{W_\infty}(\tau) = \frac{q^{|p|^2}}{\eta(\tau)^2}. \tag{58}$$

This result was first derived by Bakas and Kiritsis, using the Z_∞ parafermion [11].

For the discrete series, $\bar{\alpha}_0 \alpha_n$ and $\bar{\alpha}_n \alpha_0$ are acting on the state as 0. So, the number of the states of the discrete series is less than one of the continuous series. Let us define the quantum number B as (number of oscillators without $^-$) $-$ (number of oscillators with $^-$). Then $B|n\rangle = n|n\rangle$, and generators of W_∞ do not change B on $|n\rangle$. By taking the appropriate linear combinations of $\tilde{V}_n^i$, we can obtain all the oscillators of the form $\bar{\alpha}_n \alpha_m$. From these two facts, the generating function of the characters of the discrete series is

$$\sum_{n \in \mathbb{Z}} t^n ch_n^{W_\infty}(\tau) = \mathrm{tr}_{Fock} q^{\tilde{V}_0^0 - \frac{2}{24}} t^B = \frac{q^{-\frac{2}{24}}}{\prod_{n=1}^\infty (1 - tq^n)(1 - t^{-1} q^n)}. \tag{59}$$

From this, the character formula of the discrete series of W_∞ is expressed as[4]

$$\begin{aligned} ch_n^{W_\infty}(\tau) &= \frac{1}{2\eta(\tau)^2} \sum_{m \in \mathbb{Z}} sign(m)(-1)^m q^{mn - \frac{1}{8}} (q^{\frac{1}{2}(m+\frac{1}{2})^2} - q^{\frac{1}{2}(m-\frac{1}{2})^2}) \\ &= \frac{1}{\eta(\tau)^2} \sum_{m=1}^\infty (-1)^m q^{\frac{1}{2}m(m-1)+mn} (q^m - 1). \end{aligned} \tag{60}$$

The relation between continuous and discrete series is

$$\lim_{p \to 0} ch_{p\bar{p}}^{W_\infty}(\tau) = \sum_{n \in \mathbb{Z}} ch_n^{W_\infty}(\tau). \tag{61}$$

[4] Recently Bakas and Kiritsis have constructed the non-linear deformation of W_∞ based on the $SL(2,\mathbb{R})_k/U(1)$ coset model, and investigated its characters, which include the characters of W_∞ in the large k limit [16].

Next we consider the representations of W_∞^M with $\tilde{c} = 2M$ realized by M complex free bosons. Since the argument is the same as W_∞ case, we present the results only. The HWS's of W_∞^M are

$$|\vec{p}, \vec{\bar{p}}\rangle, \quad \vec{p} = (0, \cdots, 0, p_a, 0, \cdots, 0), \quad p_a \neq 0, \tag{62}$$

$$|n\rangle \stackrel{\text{def}}{=} \begin{cases} (\alpha^1_{-1})^n |\vec{0}, \vec{0}\rangle & n \geq 1 \\ |\vec{0}, \vec{0}\rangle & n = 0 \\ (\bar{\alpha}^1_{-1})^{-n} |\vec{0}, \vec{0}\rangle & n \leq -1. \end{cases} \tag{63}$$

The degeneracy of the ground states are 1 and $\binom{M+|n|-1}{|n|}$ respectively. The conformal weights are

$$h_{\vec{p}\vec{\bar{p}}} = |\vec{p}|^2, \quad h_n = |n|. \tag{64}$$

The characters which count conformal weight only, are defined by

$$ch^{W_\infty^M}(\tau) \stackrel{\text{def}}{=} \text{tr} q^{\tilde{V}^0_0 - \frac{2M}{24}}. \tag{65}$$

The character formula of the continuous series of W_∞^M is

$$ch^{W_\infty^M}_{\vec{p}\vec{\bar{p}}}(\tau) = \frac{q^{|\vec{p}|^2}}{\eta(\tau)^{2M}}. \tag{66}$$

The generating function of the character formulas of the discrete series is

$$\sum_{n \in \mathbb{Z}} t^n ch^{W_\infty^M}_n(\tau) = \left(\frac{q^{-\frac{2}{24}}}{\prod_{n=1}^\infty (1 - tq^n)(1 - t^{-1}q^n)} \right)^M. \tag{67}$$

The relation between continuous and discrete series is

$$\lim_{\vec{p} \to \vec{0}} ch^{W_\infty^M}_{\vec{p}\vec{\bar{p}}}(\tau) = \sum_{n \in \mathbb{Z}} ch^{W_\infty^M}_n(\tau). \tag{68}$$

5 Representations of $W_\infty^{1,1}$ and $W_\infty^{M,N}$

We first consider the representations of $W_\infty^{1,1}$ with $\tilde{c} = 2$, $c = 1$ realized by one pair of complex free boson and fermion. Each state in the Fock space is expressed as a linear combination of $|\tilde{*}\rangle \otimes |*\rangle$, where the first and second factors are states in the boson and fermion Fock spaces respectively. $W_\infty^{1,1}$ contains W_∞ and $W_{1+\infty}$ as subalgebras, and the generators of W_∞ and $W_{1+\infty}$ are expressed by a boson and a fermion respectively. Therefore $|\tilde{*}\rangle$ and $|*\rangle$ are the HWS's of W_∞ and $W_{1+\infty}$ respectively. By using the results obtained in the previous sections, we find the HWS's of $W_\infty^{1,1}$, continuous series and discrete series:

$$|p,\bar{p}\rangle \otimes |0\rangle, \tag{69}$$

$$|n\rangle \stackrel{\rm def}{=} \begin{cases} |n-1\rangle \otimes |1\rangle & n \geq 1 \\ |0\rangle \otimes |0\rangle & n = 0 \\ |n+1\rangle \otimes |-1\rangle & n \leq -1. \end{cases} \tag{70}$$

The first and second factors of the tensor product are eqs. (54,55) and eq. (41) respectively. Their conformal weight h and $U(1)$ charge Q are

$$h_{p\bar{p}} = |p|^2, \quad Q_{p\bar{p}} = 0, \tag{71}$$

$$(h_n, Q_n) = \begin{cases} (n-\frac{1}{2}, 1) & n \geq 1 \\ (0,0) & n = 0 \\ (-n-\frac{1}{2}, -1) & n \leq -1. \end{cases} \tag{72}$$

Neglecting the dependence on the eigenvalues of higher-spin generators, we consider the characters which count conformal weight and $U(1)$ charge only,

$$ch^{W_\infty^{1,1}}(\theta,\tau) \stackrel{\rm def}{=} {\rm tr} q^{\tilde{V}_0^0 - \frac{2}{24} + V_0^0 - \frac{1}{24}} e^{i\theta J_0}. \tag{73}$$

$\tilde{V}_n^i$ contains the terms $\bar{\alpha}_0 \alpha_n$ and $\bar{\alpha}_n \alpha_0$, and G_n^i and $\bar{G}_n^i$ contain the terms $\bar{\alpha}_0 \psi_n$ and $\alpha_0 \bar{\psi}_n$. For the continuous series, the momentum p is non-zero, so the set of generators of $W_\infty^{1,1}$ is identified with the set of oscillators ψ_n, $\bar{\psi}_n$, α_n, $\bar{\alpha}_n$. Therefore the character of the continuous series is

$$\begin{aligned} ch_{p\bar{p}}^{W_\infty^{1,1}}(\theta,\tau) &= \frac{q^{|p|^2}}{\eta(\tau)^2} q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}}) & (74) \\ &= ch_{p\bar{p}}^{W_\infty}(\tau) \sum_{n\in\mathbb{Z}} ch_n^{W_{1+\infty}}(\theta,\tau) & (75) \\ &= ch_{p\bar{p}}^{W_\infty}(\tau) f_{1,0}(\theta,\tau), & (76) \end{aligned}$$

where we define $f_{K,Q}(\theta,\tau)$ as

$$f_{K,Q}(\theta,\tau) \stackrel{\rm def}{=} \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{K}{2}(n+\frac{Q}{K})^2} z^{K(n+\frac{Q}{K})}. \tag{77}$$

For the discrete series, $B|n\rangle = n|n\rangle$, and the generators of $W_\infty^{1,1}$ do not change B on $|n\rangle$. By taking appropriate linear combinations of generators of $W_\infty^{1,1}$, we obtain all the oscillators of the form $\bar{\psi}_n\psi_m$, $\bar{\alpha}_n\psi_m$, $\alpha_n\bar{\psi}_m$, $\bar{\alpha}_n\alpha_m$. From these two facts, the generating function of the characters of the discrete series is

$$\sum_{n\in\mathbb{Z}} t^n ch_n^{W_\infty^{1,1}}(\theta,\tau) = \frac{q^{-\frac{2}{24}}}{\prod_{n=1}^\infty (1-tq^n)(1-t^{-1}q^n)} \cdot q^{-\frac{1}{24}} \prod_{n=1}^\infty (1+tzq^{n-\frac{1}{2}})(1+t^{-1}z^{-1}q^{n-\frac{1}{2}}). \tag{78}$$

From this equation, the character formula of the discrete series is

$$ch_n^{W_\infty^{1,1}}(\theta,\tau) = \sum_{\ell\in\mathbb{Z}} ch_{n-\ell}^{W_\infty}(\tau) ch_\ell^{W_{1+\infty}}(\theta,\tau). \tag{79}$$

The relation between continuous and discrete series is

$$\lim_{p\to 0} ch_{p\bar{p}}^{W_\infty^{1,1}}(\theta,\tau) = \sum_{n\in\mathbb{Z}} ch_n^{W_\infty^{1,1}}(\theta,\tau). \tag{80}$$

Since $W_\infty^{1,1}$ is a superalgebra, we must consider the R sector also. By using the spectral flow, the character of the R sector is expressed in terms of the character of the NS sector:

$$ch^{W_\infty^{1,1},R}(\theta,\tau) = q^{\frac{3}{24}} z^{\frac{3}{6}} ch^{W_\infty^{1,1}}(\theta+\pi\tau,\tau). \tag{81}$$

Explicitly they are

$$ch_{p\bar{p}}^{W_\infty^{1,1},R}(\theta,\tau) = ch_{p\bar{p}}^{W_\infty}(\tau) f_{1,\frac{1}{2}}(\theta,\tau), \tag{82}$$
$$ch_n^{W_\infty^{1,1},R}(\theta,\tau) = \sum_{\ell\in\mathbb{Z}} ch_{n-\ell}^{W_\infty}(\tau)\chi_{\ell+\frac{1}{2}}^{\hat{u}(1)_1}(\theta,\tau). \tag{83}$$

The conformal weight h^R and $U(1)$ charge Q^R in the R sector are

$$h_{p\bar{p}}^R = \frac{1}{8} + |p|^2, \quad Q_{p\bar{p}}^R = \frac{1}{2} \tag{84}$$

$$(h_n^R, Q_n^R) = \begin{cases} (\frac{1}{8}+n, \frac{3}{2}) & n \geq 1 \\ (\frac{1}{8}, \frac{1}{2}) & n = 0 \\ (\frac{1}{8}-(n+1), -\frac{1}{2}) & n \leq -1. \end{cases} \tag{85}$$

The ground states are singlets for $n = 0, -1$, and doublets for others.

In order to study whether a supersymmetry exist or not, and if it exists, whether it is broken or unbroken, we define the Witten index:

$$Index \stackrel{\text{def}}{=} \text{tr}_R q^{\tilde{V}_0^0 + V_0^0 - h^R}(-1)^F, \tag{86}$$

where F is the fermion number, and trace is taken over the representation space of the R sector. By using the property of the spectral flow and the fact that the fermion numbers of the generators of $W_\infty^{1,1}$ agree with their $U(1)$ charges, the Witten index is expressed as follows:

$$Index = q^{\frac{3}{24}-h-\frac{1}{2}Q} ch^{W_\infty^{1,1}}(\pi+\pi\tau,\tau). \tag{87}$$

For representations $n=0,-1$, the Witten indices are

$$Index_0 = 1, \quad Index_{-1} = -1. \tag{88}$$

For other representations, the Witten index vanishes. Therefore there exists a ($N=2$) supersymmetry for all representations and it is broken for $n=0,-1$.

Next we consider the representations of $W_\infty^{M,N}$ with $\tilde{c}=2M$, $c=N$ realized by M complex free bosons and N complex free fermions. Since the argument is the same as $W_\infty^{1,1}$ case, we present the results only. The HWS's of $W_\infty^{M,N}$ are

$$|\vec{p},\vec{\bar{p}}\rangle \otimes |0\rangle, \tag{89}$$

$$|n\rangle \stackrel{\text{def}}{=} \begin{cases} |n-N\rangle \otimes |N\rangle & n \geq N \\ |0\rangle \otimes |n\rangle & -N < n < N \\ |n+N\rangle \otimes |-N\rangle & n \leq -N. \end{cases} \tag{90}$$

The first and second factors of the tensor product are given by eqs. (62,63) and eq. (48) respectively. Their conformal weight h and $U(1)$ charge Q are

$$h_{\vec{p}\vec{\bar{p}}} = |\vec{p}|^2, \quad Q_{\vec{p}\vec{\bar{p}}} = 0, \tag{91}$$

$$(h_n, Q_n) = \begin{cases} (n-\frac{1}{2}N, N) & n \geq N \\ (\frac{1}{2}|n|, n) & -N < n < N \\ (-n-\frac{1}{2}N, -N) & n \leq -N. \end{cases} \tag{92}$$

Neglecting the dependence on the eigenvalues of higher-spin generators, we consider the characters which count conformal weight, $U(1)$ charge and eigenvalues of $SU(N)$,

$$ch^{W_\infty^{M,N}}(\theta,\vec{\theta},\tau) \stackrel{\text{def}}{=} \mathrm{tr} q^{\tilde{V}_0^0-\frac{2M}{24}+V_0^0-\frac{N}{24}} e^{i\theta J_0} e^{i\vec{\theta}\cdot\vec{H}_0}. \tag{93}$$

The character formula of the continuous series is

$$ch_{\vec{p}\vec{\bar{p}}}^{W_\infty^{M,N}}(\theta,\vec{\theta},\tau) = \frac{q^{|\vec{p}|^2}}{\eta(\tau)^{2M}} q^{-\frac{N}{24}} \prod_{j=1}^{N}\prod_{n=1}^{\infty}(1+zz_{j-1}^{-1}z_j q^{n-\frac{1}{2}})(1+z^{-1}z_{j-1}z_j^{-1}q^{n-\frac{1}{2}}) \tag{94}$$

$$= ch^{W^M_\infty}_{\vec{p}\vec{\bar{p}}}(\tau)\sum_{n\in\mathbb{Z}} ch^{W^N_{1+\infty}}_n(\theta,\vec{\theta},\tau) \tag{95}$$

$$= ch^{W^M_\infty}_{\vec{p}\vec{\bar{p}}}(\tau)\sum_{a=0}^{N-1} f_{N,a}(\theta,\tau)\chi_a^{\widehat{su}(N)_1}(\vec{\theta},\tau). \tag{96}$$

The generating function of the characters of the discrete series is

$$\sum_{n\in\mathbb{Z}} t^n ch^{W^{M,N}_\infty}_n(\theta,\vec{\theta},\tau) = \left(\frac{q^{-\frac{2}{24}}}{\prod_{n=1}^\infty(1-tq^n)(1-t^{-1}q^n)}\right)^M \times q^{-\frac{N}{24}}\prod_{j=1}^N\prod_{n=1}^\infty(1+tzz_{j-1}^{-1}z_jq^{n-\frac{1}{2}})(1+t^{-1}z^{-1}z_{j-1}z_j^{-1}q^{n-\frac{1}{2}}). \tag{97}$$

From this, the character formula of the discrete series is

$$ch^{W^{M,N}_\infty}_n(\theta,\vec{\theta},\tau) = \sum_{\ell\in\mathbb{Z}} ch^{W^M_\infty}_{n-\ell}(\tau)ch^{W^N_{1+\infty}}_\ell(\theta,\vec{\theta},\tau). \tag{98}$$

The relation between continuous and discrete series is

$$\lim_{\vec{p}\to\vec{0}} ch^{W^{M,N}_\infty}_{\vec{p}\vec{\bar{p}}}(\theta,\vec{\theta},\tau) = \sum_{n\in\mathbb{Z}} ch^{W^{M,N}_\infty}_n(\theta,\vec{\theta},\tau). \tag{99}$$

By using the spectral flow, the character of the R sector is expressed in terms of the character of the NS sector:

$$ch^{W^{M,N}_\infty,R}(\theta,\vec{\theta},\tau) = q^{\frac{3N}{24}}z^{\frac{3N}{6}}ch^{W^{M,N}_\infty}(\theta+\pi\tau,\vec{\theta},\tau). \tag{100}$$

Explicitly they are

$$ch^{W^{M,N}_\infty,R}_{\vec{p},\vec{\bar{p}}}(\theta,\vec{\theta},\tau) = ch^{W^M_\infty}_{\vec{p},\vec{\bar{p}}}(\tau)\sum_{a=0}^{N-1} f_{N,a+\frac{N}{2}}(\theta,\tau)\chi_a^{\widehat{su}(N)_1}(\vec{\theta},\tau) \tag{101}$$

$$ch^{W^{M,N}_\infty,R}_n(\theta,\vec{\theta},\tau) = \sum_{\ell\in\mathbb{Z}} ch^{W^M_\infty}_{n-\ell}(\tau)\chi^{\hat{u}(1)_N}_{\ell+\frac{N}{2}}(\theta,\tau)\chi_a^{\widehat{su}(N)_1}(\vec{\theta},\tau), \tag{102}$$

where, in the second equation, $a\equiv\ell \pmod N$. The conformal weight h^R, $U(1)$ charge Q^R and the degeneracy of the ground states in the R sector are

$$h^R_{\vec{p}\vec{\bar{p}}} = \frac{1}{8}N+|\vec{p}|^2,\quad Q^R_{\vec{p}\vec{\bar{p}}} = \frac{1}{2}N,\quad \text{degeneracy} = \sum_{a=0}^N \binom{N}{a} = 2^N, \tag{103}$$

$(h^R_n, Q^R_n, \text{degeneracy})$

$$
= \begin{cases} (\frac{1}{8}N + n, \frac{3}{2}N, \sum_{a=0}^{N} \binom{N}{a} \binom{M+n-a-1}{n-a}) & n \geq N \\ (\frac{1}{8}N + n, \frac{1}{2}N + n, \sum_{a=0}^{n} \binom{N}{a} \binom{M+n-a-1}{n-a}) & 0 < n < N \\ (\frac{1}{8}N, \frac{1}{2}N + n, \binom{N}{N+n}) & -N \leq n \leq 0 \\ (\frac{1}{8}N - (N+n), -\frac{1}{2}N, \sum_{a=2N+n}^{N} \binom{N}{a} \binom{M-n-2N+a-1}{-n-2N+a}) & -2N < n < -N \\ (\frac{1}{8}N - (N+n), -\frac{1}{2}N, \sum_{a=0}^{N} \binom{N}{a} \binom{M-n-2N+a-1}{-n-2N+a}) & n \leq -2N. \end{cases} \quad (104)
$$

By using the property of the spectral flow and the fact that the fermion numbers of the generators of $W_{\infty}^{M,N}$ agree with their $U(1)$ charges, the Witten index eq. (86) is expressed as

$$
Index = q^{\frac{N+2M}{24} - h - \frac{1}{2}Q} ch^{W_{\infty}^{M,N}}(\pi + \pi\tau, \vec{0}, \tau). \quad (105)
$$

For the continuous series, the Witten index vanishes for all M, N. Therefore, for the continuous series, there exists a supersymmetry, and it is unbroken.

In the case of the discrete series with $M \neq N$, the Witten index is not a number but a q-series. Namely, at excited state, the number of bosonic states does not agree with one of fermionic states. Therefore a supersymmetry does not exist in the discrete series of $W_{\infty}^{M,N}$ $(M \neq N)$. In the case of the discrete series with $M = N$, the Witten index is just a number. So a ($2N^2$ extended) supersymmetry exists. For representations n $(-N \leq n \leq 0)$, the Witten index is

$$
Index_n = (-1)^n \binom{N}{N+n}, \quad (106)
$$

and a supersymmetry is broken. For other representations, the Witten index vanishes and a supersymmetry is unbroken.

6 Discussion

In this paper we have studied the irreducible unitary highest weight representations of W infinity algebras, which are obtained from free field realizations, and derived their character formulas. We have also constructed a new superalgebra $W_{\infty}^{M,N}$, whose bosonic sector is $W_{\infty}^{M} \oplus W_{1+\infty}^{N}$. Its representations obtained from a free field realization are classified into two classes, continuous and discrete. There exists a supersymmetry in the continuous series, whereas a supersymmetry exists only for $M = N$ in the discrete series. This is expected from the counting of the bosonic and fermionic degrees of freedom of the generators:

$$\begin{array}{ccccccccc} W^{i-1,(\alpha\beta)}(z) & & \tilde{W}^{i,(ab)}(z) & & G^{i,a\alpha}(z) & & \bar{G}^{i,a\alpha}(z) & & \\ N^2 & + & M^2 & - & MN & - & MN & = & (M-N)^2. \end{array} \tag{107}$$

Perhaps a supersymmetry in the continuous series for $M \neq N$ may be an accidental one.

The representations with higher central charge and the realization independent representations are future subjects. There are two difficulties in developing the realization independent representation theories of W infinity algebras. One is that there are infinite number of fields. The other is the complicated dependence on the eigenvalues of higher-spin generators. For example, no one has succeeded in computing even the level 1 Kac determinant.

Although our representation theory is a restricted one, we have obtained the character formulas. It is interesting to apply these character formulas to models with W infinity symmetry, for example [17], integrable non-linear differential equation systems such as the KP hierarchy and the Toda hierarchy, four dimensional self-dual gravity [18], Virasoro (W) constraints on the partition function of the multi-matrix model [19], and W infinity gravity [20, 21]. The modular properties of the characters are also subjects of future research.

Finally, we mention the anomaly-free conditions. In refs.[22, 23], the anomaly-free conditions for W_∞, $W_{1+\infty}$ and $W_\infty^{1,1}$ are considered by the BRS formalism and ζ function regularization, and it is shown that their critical central charges are -2, 0 and -3 respectively. Similar calculations have been done for $W_{1+\infty}^N$ and $W_\infty^{1,N}$ by considering the ghost realizations of them, and their critical central charges are 0 and $-2-N$ respectively [8]. For $W_\infty^{M,N}$, we obtain $k_{ghost} = M$, and the critical central charge is

$$(\tilde{c}+c)_{critical} = -(\tilde{c}+c)_{ghost} = -2M^2 - MN. \tag{108}$$

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References

[1] A.B. Zamolodchikov, Theor. Math. Phys. **63** (1986) 1205;
V.A. Fateev and S.L. Lykyanov, Int. J. Mod. Phys. **A3** (1988) 507.

[2] I. Bakas, Phys. Lett. **B242** (1989) 57.

[3] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B236** (1990) 173.

[4] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B242** (1990) 401; **B245** (1990) 72; Nucl. Phys. **B339** (1990) 191.

[5] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, Phys. Lett. **B245** (1990) 447.

[6] I. Bakas and E. Kiritsis, Mod. Phys. Lett. **A5** (1990) 2039.

[7] S. Odake and T. Sano, Phys. Lett. **B258** (1991) 369.

[8] S. Odake and T. Sano, "Comments on the $W_{1+\infty}$ and Super W_∞ Algebras with $SU(N)$ Symmetry", in KEK Proceedings 91-6(1991)80-94.

[9] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575.

[10] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. **B304** (1988) 348;371.

[11] I. Bakas and E. Kiritsis, Nucl. Phys. **B343** (1990) 185, Erratum **B350** (1991) 512; Prog. Theor. Phys. Suppl. No. 102 (1990) 15.

[12] A. Schwimmer and N. Seiberg, Phys. Lett. **B184** (1987) 191.

[13] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, in "Nonlinear Integrable Systems - Classical Theory and Quantum Theory", edited by T. Miwa and M. Jimbo, World Scientific 1983.

[14] P. Goddard, W. Nahm and D. Olive, Phys. Lett. **B160** (1985) 111.

[15] P. Goddard and D. Olive, Int. J. Mod. Phys. **A1** (1986) 303.

[16] I. Bakas and E. Kiritsis, “Beyond the Large N Limit: Non-Linear W_∞ as Symmetry of the $SL(2,\mathbf{R})/U(1)$ Coset Model”, Sep. 1991, UCB-PTH-91/44.

[17] I. Bakas and E. Kiritsis, Int. J. Mod. Phys. **A6** (1991) 2871, and references there in.

[18] Q.-H. Park, Phys. Lett. **B236** (1990) 429; **B238** (1990) 287.

[19] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. **A6** (1991) 1385; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B348** (1991) 435.

[20] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen and K.S. Stelle, Phys. Lett. **B243** (1990) 350; E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, Mod. Phys. Lett. **A5** (1990) 1957.

[21] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Phys. Lett. **B251** (1990) 355.

[22] K. Yamagishi, Phys. Lett. **B266** (1991) 370.

[23] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B254** (1991) 401.