YITP/K-1016

May 1993

Heisenberg realization for $U_q(sl_n)$ on the flag manifold

Hidetoshi AWATA[1], Masatoshi NOUMI[2] and Satoru ODAKE[3]

[1] *Yukawa Institute for Theoretical Physics*
Kyoto University, Kyoto 606, Japan

[2] *Department of Mathematical Sciences, University of Tokyo*
Komaba 3-8-1, Meguro-ku, Tokyo 153, Japan

[3] *Department of Physics, Faculty of Liberal Arts*
Shinshu University, Matsumoto 390, Japan

Abstract

We give the Heisenberg realization for the quantum algebra $U_q(sl_n)$, which is written by the q-difference operator on the flag manifold. We construct it from the action of $U_q(sl_n)$ on the q-symmetric algebra $\mathcal{A}_q(Mat_n)$ by the Borel-Weil like approach. Our realization is applicable to the construction of the free field realization for the $U_q(\widehat{sl_n})$ [AOS].

hep-th/9306010

[1] e-mail address : awata@ps1.yukawa.kyoto-u.ac.jp
[2] e-mail address : noumi@tansei.cc.u-tokyo.ac.jp
[3] e-mail address : odake@jpnyitp.yukawa.kyoto-u.ac.jp

1.Introduction

Recently, the quantum Knizhnik-Zamolodchikov equations (q-KZ eq.) [Sm, FR] have been analyzed [M1, R]. This q-KZ equations are important both for physics and mathematics by the relationship with 2-dimensional integrable theories [Sm, DFJMN], quantum affine Lie algebras and elliptic R-matrices [FR, DJO].

To solve the classical ($q = 1$) KZ equations, an important and powerful tools were the free field realization for the affine Lie algebra $\widehat{\mathcal{G}}$ [W, FF] and the Heisenberg realization for the corresponding Lie algebra $\mathcal{G}$ which is written by the differential operator on the flag manifold [SV, ATY, FM]. Even in the quantum case ($q \neq 1$), for example for the algebra $U_q(\widehat{sl_2})$, the Heisenberg realization and the free field realization [FJ, M2, ABG, Sh] are also important for the analysis of the q-KZ equation [JMMN, KQS, M3]. We expect that this situation is the same for other quantum affine Lie algebras.

The aim of this paper is to construct the Heisenberg realization for the quantum algebra $U_q(sl_n)$. In the forthcoming paper [AOS], the free field realization for the quantum affine algebra $U_q(\widehat{sl_n})$ will be constructed by using this Heisenberg realization.

2. Quantum algebra $U_q(sl_n)$

§ **2.1.** First we fix some notations. The algebra $U_q(sl_n)$ is generated by e_i, f_i and invertible k_i $(1 \leq i \leq n-1)$ with relations

$$k_i e_j k_i^{-1} = q^{A_{ij}} e_j, \qquad \sum_{m=0}^{1-A_{ij}} (-1)^m \begin{bmatrix} 1-A_{ij} \\ m \end{bmatrix} e_i^{1-A_{ij}-m} e_j e_i^m = 0,$$

$$k_i f_j k_i^{-1} = q^{-A_{ij}} f_j,$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \qquad \sum_{m=0}^{1-A_{ij}} (-1)^m \begin{bmatrix} 1-A_{ij} \\ m \end{bmatrix} f_i^{1-A_{ij}-m} f_j f_i^m = 0,$$

where $q \in \mathbf{C}$, $(A_{ij})_{1\leq i,j\leq n-1}$ is the Cartan matrix such that $A_{ij} = 2\delta_{ij} - \delta_{i\,j+1} - \delta_{i\,j-1}$, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] = [n]!/[n-m]![m]!$ and $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The algebra $U_q(sl_n)$ is a Hopf algebra with the comultiplication Δ

$$\Delta(k_i) = k_i \otimes k_i, \qquad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,$$

the antipode S such that $S(k_i)=k_i^{-1}$, $S(e_i)=-k_i^{-1}e_i$, $S(f_i)=-f_ik_i$ and the co-unit ϵ such that $\epsilon(k_i)=1$, $\epsilon(e_i)=0$, $\epsilon(f_i)=0$.

§ **2.2.** Let M_λ be the Verma module over $U_q(sl_n)$ generated by the highest weight vector $|\lambda\rangle$ such that $e_i|\lambda\rangle=0$, $k_i|\lambda\rangle=q^{\lambda_i}|\lambda\rangle$ with $\lambda_i\in\mathbf{C}$. The dual module M_λ^* is generated by $\langle\lambda|$ which satisfies $\langle\lambda|f_i=0$, $\langle\lambda|k_i=q^{\lambda_i}\langle\lambda|$. The bilinear form $M_\lambda^*\otimes M_\lambda\to\mathbf{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle=1$ and $\big(\langle u|X\big)|v\rangle=\langle u|\big(X|v\rangle\big)$ for any $\langle u|\in M_\lambda^*$, $|v\rangle\in M_\lambda$ and $X\in U_q(sl_n)$.

3. Heisenberg realization for $U_q(sl_n)$

§ **3.1.** The Heisenberg algebra $\mathcal{H}_n$ is generated by the coordinate x_{ij}, $x_{ij}^{-1}\in\mathbf{C}$ and the differential operator $\vartheta_{ij}=x_{ij}\frac{\partial}{\partial x_{ij}}$ $(1\le i<j\le n)$ with relation $[\vartheta_{ij},x_{kl}]=\delta_k^i\delta_l^j x_{kl}$ or equivalently

$$q^{\vartheta_{ij}}x_{kl}q^{-\vartheta_{ij}}=q^{\delta_k^i\delta_l^j}x_{kl}.$$

The quantum algebra $U_q(sl_n)$ is realized by the Heisenberg algebra $\mathcal{H}_n$. We have

<u>Theorem I</u>. *There exists the algebra homomorphism $\pi_\lambda: U_q(sl_n)\to\mathcal{H}_n$ define as*

$$\begin{aligned}
\pi_\lambda(k_i)&=q^{\sum_{j=1}^{i-1}(\vartheta_{ji}-\vartheta_{j\,i+1})+(\lambda_i-2\vartheta_{i\,i+1})+\sum_{j=i+2}^{n}(\vartheta_{i+1\,j}-\vartheta_{ij})},\\
\pi_\lambda(e_i)&=\sum_{k=1}^{i}q^{\sum_{j=1}^{k-1}(\vartheta_{ji}-\vartheta_{j\,i+1})}\frac{x_{ki}}{x_{k\,i+1}}[\vartheta_{k\,i+1}],\\
\pi_\lambda(f_i)&=\sum_{k=1}^{i-1}\frac{x_{k\,i+1}}{x_{ki}}[\vartheta_{ki}]q^{-\sum_{j=k+1}^{i-1}(\vartheta_{ji}-\vartheta_{j\,i+1})-(\lambda_i-2\vartheta_{i\,i+1})-\sum_{j=i+2}^{n}(\vartheta_{i+1\,j}-\vartheta_{ij})}\\
&\quad+x_{i\,i+1}[(\lambda_i-\vartheta_{i\,i+1})+\sum_{j=i+2}^{n}(\vartheta_{i+1\,j}-\vartheta_{ij})]\\
&\quad-\sum_{k=i+2}^{n}\frac{x_{ik}}{x_{i+1\,k}}[\vartheta_{i+1\,k}]q^{\lambda_i+\sum_{j=k}^{n}(\vartheta_{i+1\,j}-\vartheta_{ij})},
\end{aligned}$$

with $x_{ii}=1$.

Here $[n]$ denotes the q integer, so $\pi_\lambda(g)$'s are the q-difference operators. The proof will be given in the next section.

We also have the following dual generators†

† These dual generators relate to the screening currents of the free field realization for $U_q(\widehat{sl_n})$ [AOS] which must important to the analysis of the q-KZ equation.

Theorem II. *There exists the algebra anti-homomorhpism* $\tilde{\pi}_\lambda : U_q(sl_n) \to \mathcal{H}_n$, $\tilde{\pi}_\lambda = \tilde{\sigma} \circ \pi_\lambda \circ \sigma$, *with* σ *such that* $\sigma(k_i) = k_{n-i}$, $\sigma(e_i) = e_{n-i}$, $\sigma(f_i) = f_{n-i}$ *and* $\tilde{\sigma}$ *such that* $\tilde{\sigma}(x_{ij}) = x_{n+1-j\, n+1-i}$, $\tilde{\sigma}(\vartheta_{ij}) = -\vartheta_{n+1-j\, n+1-i}$, $\tilde{\sigma}(\lambda_i) = -\lambda_{n+1-i}$.

§ **3.2.** Let $\mathcal{F} = \mathbf{C}[x_{ij}]\,|0\rangle$ be the Fock module over Heisenberg algebra $\mathcal{H}_n$ generated by the highest weight vector $|0\rangle$ such that $x_{ij}^{-1}|0\rangle = \vartheta_{ij}|0\rangle = 0$. The dual module $\mathcal{F}^* = \langle 0|\,\mathbf{C}[x_{ij}^{-1}]$ is generated by $\langle 0|$ which satisfies $\langle 0|x_{ij} = \langle 0|\vartheta_{ij} = 0$. The bilinear form $\mathcal{F}^* \otimes \mathcal{F} \to \mathbf{C}$ is uniquely defined by $\langle 0|0\rangle = 1$ and $\big(\langle u|X\big)|v\rangle = \langle u|\big(X|v\rangle\big)$ for any $\langle u| \in \mathcal{F}^*$, $|v\rangle \in \mathcal{F}$ and $X \in \mathcal{H}_n$. For $\langle 0|f(x_{ij}^{-1}) \in \mathcal{F}^*$ and $g(x_{ij})|0\rangle \in \mathcal{F}$, $\langle 0|f(x_{ij}^{-1})\, g(x_{ij})|0\rangle$ is nothing but the constant part of $f(x_{ij}^{-1})\, g(x_{ij})$.

4. Construction of the Heisenberg realization for $U_q(sl_n)$

Next we prove above Theorems by a Borel-Weil like approach, which is based on the method in Ref. [N]. First we give some notations.

§ **4.1.** The q-symmetric algebra $\mathcal{A}_q(Mat_n)$ is generated by t_{ij} $(1 \le i, j \le n)$ with relations

$$(4.1)\qquad \begin{aligned} t_{ik}t_{jk} &= qt_{jk}t_{ik}, & t_{il}t_{jk} &= t_{jk}t_{il}, \\ t_{ik}t_{il} &= qt_{il}t_{ik}, & t_{ik}t_{jl} - qt_{il}t_{jk} &= t_{jl}t_{ik} - q^{-1}t_{jk}t_{il}, \end{aligned}$$

for $i < j$ and $k < l$. Note that this algebra has the algebra automorphism ρ such that $\rho(t_{ij}) = t_{n+1-j\, n+1-i}$, $\rho(q) = q^{-1}$ and the algebra anti-automorphism $\tilde{\rho}$ such that $\tilde{\rho}(t_{ij}) = t_{n+1-j\, n+1-i}$, $\tilde{\rho}(q) = q$.

The algebra $\mathcal{A}_q(Mat_n)$ has the structure of a $U_q(sl_n)$-module. The action of $U_q(sl_n)$ on $\mathcal{A}_q(Mat_n)$ is

$$k_m t_{ij} = t_{ij} q^{\delta_{mj} - \delta_{m+1\, j}}, \qquad e_m t_{ij} = t_{i\, j-1}\delta_{m+1\, j}, \qquad f_m t_{ij} = t_{i\, j+1}\delta_{mj},$$
$$g(uv) = \sum_a (g_a^{'} u)(g_a^{''} v), \qquad g.1 = \epsilon(g)1,$$

for all u, $v \in \mathcal{A}_q(Mat_n)$ and for all $g \in U_q(sl_n)$ with $\Delta(g) = \sum_a g_a^{'} \otimes g_a^{''}$. Note that this action of $g \in U_q(sl_n)$ can be written by the matrix $\varrho(g)_{ij}$ as $g\, t_{ij} = \sum_k t_{jk}\varrho(g)_{kj}$ with $\varrho(k_m) = q^{E_{mm} - E_{m+1\, m+1}}$, $\varrho(e_m) = E_{m\, m+1}$, $\varrho(f_m) = E_{m+1\, m}$ and $(E_{\alpha\beta})_{ij} = \delta_{\alpha i}\delta_{\beta j}$. These matrices are noting but the vector representation for the $U_q(sl_n)$. The action for the rows of matrix t_{ij} is given by the above automorphism ρ or $\tilde{\rho}$.

§ **4.2.** For the ordered set $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$, let ξ^I_J be the quantum r-minor determinant with respect to rows I and columns J such that [TT, NYM]

$$\xi^I_J = \sum_{\sigma \in \mathbf{S}_r} (-q)^{l(\sigma)} t_{i_{\sigma(1)} j_1} \cdots t_{i_{\sigma(r)} j_r}.$$

Here $\mathbf{S}_r$ is the permutation group of the set $\{1, \cdots, r\}$ and $l(\sigma)$ stands for the number of inversions involved in σ ; $l(\sigma) = \sharp\{(i,j)\,;\, i < j,\, \sigma(i) > \sigma(j)\}$. From now on, $\xi^I_J = 0$ if I or J has same elements. Note that $\xi^I_J \xi^{I'}_{J'} = \xi^{I'}_{J'} \xi^I_J$ if $I' \subset I$, $J' \subset J$. We have

__**Proposition**__. *With the lower triangular matrix B, the Gauss decomposition of the matrix $T = (t_{ij})$ of the q-coordinates is given as*

$$t_{ij} = \sum_k B_{ik} X_{kj}, \qquad B_{ij} = (\xi^{1 \cdots j-1}_{1 \cdots j-1})^{-1} \xi^{1 \cdots j-1\, i}_{1 \cdots j}, \qquad X_{ij} = (\xi^{1 \cdots i}_{1 \cdots i})^{-1} \xi^{1 \cdots i}_{1 \cdots i-1\, j},$$

and $B_{ij} = 0$ for $i < j$ and $X_{ij} = 0$ for $i > j$. Here $\{1 \cdots 0\} = \{\ \}$.

Proof. follows from

$$t_{ij} = B_{i1} X_{1j} + (\xi^1_1)^{-1} \xi^{1i}_{1j}, \qquad (\xi^{1 \cdots r}_{1 \cdots r})^{-1} \xi^{1 \cdots r\, i}_{1 \cdots r\, j} = B_{i\, r+1} X_{r+1\, j} + (\xi^{1 \cdots r+1}_{1 \cdots r+1})^{-1} \xi^{1 \cdots r+1\, i}_{1 \cdots r+1\, j},$$

which are obtained from the q-deformed Jacobi identity

$$\xi^{1 \cdots r}_{1 \cdots r} \xi^{1 \cdots r\, r+1\, r+2}_{1 \cdots r\, r+1\, r+2} = \xi^{1 \cdots r\, r+1}_{1 \cdots r\, r+1} \xi^{1 \cdots r\, r+2}_{1 \cdots r\, r+2} - q \xi^{1 \cdots r\, r+1}_{1 \cdots r\, r+2} \xi^{1 \cdots r\, r+2}_{1 \cdots r\, r+1}. \qquad \text{Q.E.D.}$$

We regard X_{ij} $(i < j)$ as a q-analogue of local coordinates of the flag manifold $B \backslash GL_n$. For $i < i_1$ and $I = \{i_1 < \cdots < i_r\}$, we denote $\eta^i_I = \xi^{1 \cdots i\, i+1 \cdots i+r}_{1 \cdots i\, i_1 \cdots i_r}$, then $X_{ij} = (\eta^{i-1}_i)^{-1} \eta^{i-1}_j$. Since the principal minors $\xi^{1 \cdots i}_{1 \cdots i}$'s $1 \le i \le n$ commute with each other, one can consistently adjoin their inverse to the algebra $\mathbf{C}[\xi^I_J]$.

§ **4.3.** The quantum minor η^r_{ij}'s satisfy, for $r < i < j < k < l$, the same relations as t_{ij}'s in (4.1) and Plüker relation (Young symmetry) [TT, NYM, N]

$$\eta^r_i \eta^r_{jk} - q \eta^r_j \eta^r_{ik} + q^2 \eta^r_k \eta^r_{ij} = 0,$$

and the commutation relations

$$\eta^r_i\eta^r_{jk} = q\eta^r_{jk}\eta^r_i, \qquad \eta^r_{ij}\eta^r_k = q\eta^r_k\eta^r_{ij}, \qquad \eta^r_{ik}\eta^r_j = \eta^r_k\eta^r_{ij} + \eta^r_i\eta^r_{jk},$$

$$\eta^r_{ij}\eta^r_{jk} = q\eta^r_{jk}\eta^r_{ij}, \qquad \eta^r_{ij}\eta^r_{kl} = q^2\eta^r_{kl}\eta^r_{ij}.$$

The action of $U_q(sl_n)$ on the quantum minor η^i_j is

$$k_m\eta^{i-1}_{ij} = \eta^{i-1}_{ij}q^{\delta_{mj}-\delta_{m+1\,j}+\delta_{mi}},$$

$$e_m\eta^{i-1}_{ij} = \eta^{i-1}_{i\,j-1}\delta_{m+1\,j}, \qquad f_m\eta^{i-1}_{ij} = \eta^{i-1}_{i\,j+1}\delta_{mj} + \eta^{i-1}_{i+1\,j}\delta_{mi}.$$

Owning to the Plüker relation, $\eta^{i-1}_{i+1\,j} = \eta^{i-1}_{i+1}(\eta^{i-1}_i)^{-1}\eta^{i-1}_{ij} - q\eta^{i-1}_j(\eta^{i-1}_i)^{-1}\eta^{i-1}_{i\,i+1}$, the algebra $\mathcal{A} = \mathbf{C}[\eta^{i-1}_j, (\eta^{i-1}_i)^{-1}]_{1\le i\le n-1,\ i\le j\le n}$ has the structure of a $U_q(sl_n)$-module.

§ **4.4.** To relate the non-commutative algebra $\mathbf{C}[X_{ij}]$ with the commutative one $\mathbf{C}[x_{ij}]$, we fix the ordering of η^i_j's. The algebra $\mathcal{A}$ has the basis

$$\{\ (\eta^0_n)^{a_{1n}}\cdots(\eta^0_1)^{a_{11}}(\eta^1_n)^{a_{2n}}\cdots(\eta^1_2)^{a_{22}}\cdots(\eta^{n-2}_{n-1})^{a_{n-1\,n-1}} \mid a_{ij}\in\mathbf{Z}_{\ge0},\ i<j,\ a_{ii}\in\mathbf{Z}\ \},$$

which ordering we call *normal ordering.* We introduce the projection $\circ\!\!\circ * \circ\!\!\circ : \mathcal{A}\to\mathcal{A}$ such that

$$\circ\!\!\circ\ \textit{any ordered}\ \prod_{i\le j}(\eta^{i-1}_j)^{a_{ij}}\ \circ\!\!\circ = \textit{normal ordered}\ \prod_{i\le j}(\eta^{i-1}_j)^{a_{ij}}.$$

Let $Z^a_\lambda = \circ\!\!\circ \prod_i(\eta^{i-1}_i)^{\lambda_i}\prod_{j<k}(X_{jk})^{a_{jk}} \circ\!\!\circ$ with $\lambda_i\in\mathbf{Z}$ and $a_{ij}\in\mathbf{Z}_{\ge0}$. If we denote $Y^i = (\eta^{i-1}_n)^{a_{in}}\cdots(\eta^{i-1}_i)^{a_{ii}}$ with $a_{ii} = \lambda_i - \sum_{j=i+1}^n a_{ij}$, then $Z^a_\lambda = Y^1\cdots Y^{n-1}$. The algebra $\mathcal{A}$ has the decomposition $\mathcal{A} = \oplus_{\lambda_i\in\mathbf{Z}}\mathcal{A}_\lambda$ such that $\mathcal{A}_\lambda$ is the vector space spanned by the vectors $\{Z^a_\lambda \mid a_{ij}\in\mathbf{Z}_{\ge0},\ i<j\ \}$. The algebra $\mathcal{A}_\lambda$ also has the structure of a $U_q(sl_n)$-module, and we have

<u>Lemma</u>. *The left action of $U_q(sl_n)$ on $\mathcal{A}_\lambda$ is as follows*

$$k_iZ^a_\lambda = Z^a_\lambda q^{\sum_{j=1}^{i-1}(a_{ji}-a_{j\,i+1})+(\lambda_i-2a_{i\,i+1})+\sum_{j=i+2}^n(a_{i+1\,j}-a_{ij})},$$

$$e_iZ^a_\lambda = \sum_{k=1}^{i} q^{\sum_{j=1}^{k-1}(a_{ji}-a_{j\,i+1})}[a_{k\,i+1}] \circ\!\!\circ Z^a_\lambda(X_{k\,i+1})^{-1}X_{ki} \circ\!\!\circ,$$

$$\begin{aligned} f_iZ^a_\lambda = &\sum_{k=1}^{i-1}[a_{ki}] \circ\!\!\circ X_{k\,i+1}(X_{ki})^{-1}Z^a_\lambda \circ\!\!\circ q^{-\sum_{j=k+1}^{i-1}(a_{ji}-a_{j\,i+1})-(\lambda_i-2a_{i\,i+1})-\sum_{j=i+2}^n(a_{i+1\,j}-a_{ij})} \\ &+ [(\lambda_i - a_{i\,i+1}) + \sum_{j=i+2}^n(a_{i+1\,j}-a_{ij})] \circ\!\!\circ X_{i\,i+1}Z^a_\lambda \circ\!\!\circ \\ &- \sum_{k=i+2}^n [a_{i+1\,k}] \circ\!\!\circ X_{ik}(X_{i+1\,k})^{-1}Z^a_\lambda \circ\!\!\circ q^{\lambda_i+\sum_{j=k}^n(a_{i+1\,j}-a_{ij})}. \end{aligned}$$

Proof. follows from

$$k_m Y^i = Y^i q^{(a_{im}-a_{i\,m+1})\sum_{j=i}^{n-1}\delta_{mj}+\delta_{m\,i-1}\sum_{j=i+1}^{n}a_{m+1\,j}},$$

$$e_m Y^i = [a_{i\,m+1}] \circ\atop\circ Y^i(\eta_{m+1}^{i-1})^{-1}\eta_m^{i-1} \circ\atop\circ \sum_{j=i+1}^{n}\delta_{m+1\,j},$$

$$\begin{aligned} f_m Y^i &= [a_{im}] {\circ\atop\circ} \eta_{m+1}^{i-1}(\eta_m^{i-1})^{-1}Y^i {\circ\atop\circ} \sum_{j=i}^{n-1}\delta_{mj} + \delta_{m\,i-1}\sum_{k=i+1}^{n}[a_{ik}]\eta_{ik}^{i-2} {\circ\atop\circ} (\eta_k^{i-1})^{-1}Y^i {\circ\atop\circ} q^{-\sum_{j=i+1}^{k-1}a_{ij}} \\ &= [a_{im}] {\circ\atop\circ} \eta_{m+1}^{i-1}(\eta_m^{i-1})^{-1}Y^i {\circ\atop\circ} \sum_{j=i}^{n-1}\delta_{mj} + \delta_{m\,i-1}[\sum_{k=i+1}^{n}a_{ik}] {\circ\atop\circ} \eta_i^{i-2}(\eta_{i-1}^{i-2})^{-1}Y^i {\circ\atop\circ} \\ &- \delta_{m\,i-1}\sum_{k=i+1}^{n}[a_{ik}] {\circ\atop\circ} \eta_k^{i-2}(\eta_{i-1}^{i-2})^{-1}\eta_k^{i-1}(\eta_i^{i-1})^{-1}Y^i {\circ\atop\circ} q^{-\sum_{j=k}^{n}a_{ij}}, \end{aligned}$$

here we use $k_m(\eta_i^r)^a = (k_m\eta_i^r)^a$, $e_m(\eta_i^r)^a = [a](\eta_i^r)^{a-1}(e_m\eta_i^r)$, $f_m(\eta_i^r)^a = [a](f_m\eta_i^r)(\eta_i^r)^{a-1}$ with $a \in \mathbf{Z}$ and the identity $\sum_k [a_k]q^{\left(\sum_{j<k}-\sum_{j>k}\right)a_j} = [\sum_k a_k]$. The polynomials of q in $e_i Z_\lambda^a$ and $f_i Z_\lambda^a$ come from the Cartan parts of the comultiplication of e_i and f_i respectively.

Q.E.D.

§ **4.5.** *Proof of Theorem I.*

We consider the commutative algebra $\mathbf{C}[x_{ij}]_{1\le i<j\le n}$ and define an isomorphism $\pi_\lambda : \mathcal{A}_\lambda \to \mathbf{C}[x_{ij}]$ by $\pi_\lambda(Z_\lambda^a) = z^a$, with $z^a = \prod_{r<j}(x_{rj})^{a_{rj}}$. Applying this isomorphism π_λ to above Lemma, we obtain the q-difference operators on $\mathbf{C}[x_{ij}]$ in Theorem I. Q.E.D.

§ **4.6.** *Proof of Theorem II.*

With the lower triangular matrix $\tilde{B}$, the Gauss decomposition of inverse direction $T = \tilde{X}\tilde{B}$ is obtained by the algebra anti-automorphism $\tilde{\rho}$ in §4.1 from the Gauss decomposition $T = BX$. By the algebra automorphism ρ with some sign changing, we get the action of $U_q(sl_n)$ on $\mathbf{C}[\tilde{X}_{ij}]$ and the dual generators of Theorem II. Q.E.D.

Conclusion and Discussion.

We constructed the Heisenberg realization for the $U_q(sl_n)$ by the flag coordinate, which is applicable to the construction of the free field realization for the $U_q(\widehat{sl_n})$ [AOS]. In the Ref. [DJMM], they also gave the similar realization for the $U_q(sl_n)$ but it seems that it can not be affinized.

Acknowledgments.

The authors would like to thank E. Frenkel, K. Hasegawa, M. Jimbo, K. Kimura, G. Kuroki, F. Malikov, A. Matsuo, T. Miwa, J. Shiraishi, Y. Yamada and the members of KEK, RIMS and YITP for valuable discussions. H.A and S.O are supported by Soryushi-syougakkai.

Appendix. The Jordan-Schwinger type realization and q-oscillator

§ **A.1.** If we consider only $i=1$ of $t_{ij}\in\mathcal{A}_q(Mat_n)$, then we can obtain the n variables Jordan-Schwinger type realization for $U_q(sl_n)$ [H, Z]. Let us denote $t_j=t_{1j}$, the algebra $\mathcal{A}=\mathbf{C}[t_i]_{1\le i\le n}$ has the basis $\{\ t_n^{a_n}\cdots t_1^{a_1}\ |\ a_i\in\mathbf{Z}_{\ge 0}\ \}$, which ordering we call *normal ordering*, and has the structure of a $U_q(sl_n)$-module. By an isomorphism $\pi:\mathcal{A}\to\mathbf{C}[x_i]$, $\pi(t_n^{a_n}\cdots t_1^{a_1})=x_1^{a_1}\cdots x_n^{a_n}$ and by the action of $U_q(sl_n)$ on $\mathcal{A}$, we obtain

Proposition. *There exists the algebra homomorphism $\pi:U_q(sl_n)\to\mathcal{H}_n$ define as*

$$\pi(k_i)=q^{\vartheta_i-\vartheta_{i+1}},\quad \pi(e_i)=\frac{x_i}{x_{i+1}}[\vartheta_{i+1}],\quad \pi(f_i)=\frac{x_{i+1}}{x_i}[\vartheta_i].$$

We introduce the projection $\circ\!\!\!\circ * \circ\!\!\!\circ$ same as before. Denote $X_i=t_1^{-1}t_i$ $(2\le i\le n)$ and $Z_\lambda^a = {}^\circ_\circ t_1^\lambda\prod_{i=2}^n X_i^{a_i}{}^\circ_\circ = t_n^{a_n}\cdots t_1^{a_1}$ with $a_1=\lambda-\sum_{i=2}^n a_i$, then the algebra $\mathcal{A}[t_1^{-1}]$ has the decomposition $\mathcal{A}[t_1^{-1}]=\oplus_{\lambda\in\mathbf{Z}}\mathcal{A}_\lambda$ such that $\mathcal{A}_\lambda$ is the vector space spanned by the vectors $\{Z_\lambda^a\ |\ a_i\in\mathbf{Z}_{\ge 0},\ i>1\ \}$. By an isomorphism $\pi_\lambda:\mathcal{A}_\lambda\to\mathbf{C}[x_i]$, $\pi_\lambda(t_n^{a_n}\cdots t_1^{a_1})=x_2^{a_2}\cdots x_n^{a_n}$ and by the action of $U_q(sl_n)$ on $\mathcal{A}_\lambda$, we obtain the $n-1$ variables inhomogeneous realization for $U_q(sl_n)$, which is the same as above Proposition with additional conditions $x_1=1$ and $\vartheta_1=\lambda-\sum_{i=2}^n\vartheta_i$. This realization corresponds with that in Theorem-I on $\mathbf{C}[x_{1j}]$ with $\lambda_i=0$ for $i\ne 1$.

§ **A.2.** For the Heisenberg algebra $\langle x,\vartheta\rangle$ with $q^\vartheta x q^{-\vartheta}=qx$, if we denote

$$a=x,\quad a^\dagger=\frac{1}{x}[\vartheta],\quad N=\vartheta,$$

then $\langle a,a^\dagger,N\rangle$ satisfies the q-oscillator algebra such that

$$aa^\dagger=[N],\quad a^\dagger a=[N+1],$$

which is equivalent to $a^\dagger a-q^{\pm1}aa^\dagger=q^{\mp N}$. And they satisfy $[N,a]=a$, $[N,a^\dagger]=-a^\dagger$.

So we can rewrite our Theorem by the q-oscillator algebra.

References

[ABG] A. Abada, A. Bougourzi and M. El Gradechi, *Deformation of the Wakimoto construction,* (hep-th/9209009) preprint CRM-1829 (1992).

[AOS] H. Awata, S. Odake and J. Shiraishi, *Free Boson representation of* $U_q(\widehat{sl_3})$, (hep-th/9305017) preprint RIMS-920,YITP/K-1017;*Free Boson realization of* $U_q(\widehat{sl_N})$, (hep-th/9305146) preprint RIMS-924,YITP/K-1018

[ATY] H. Awata, A. Tsuchiya and Y. Yamada, *Integral formulas for the WZNW correlation functions,* Nucl. Phys. **B365** (1991) 680-696.
H. Awata, *Screening current Ward identity and integral formulas for the WZNW correlation functions,* Prog. Theor. Phys. Supplement **110** (1992) 303-319.

[DFJMN] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators,* Commun. Math. Phys. **151** (1993) 89-154.

[DJMM] E. Date, M. Jimbo, K. Miki and T. Miwa, *Cyclic representations of* $U_q(sl(n+1,\mathbf{C}))$ *at* $q^N = 1$, Publ. RIMS. Kyoto Univ. **27** (1991) 347-366.

[DJO] E. Date, M. Jimbo and M. Okado, *Crystal base and q-vertex operators,* Osaka Univ. Math. Sci. preprint **1** (1991).

[FF] B. Feigin and E. Frenkel, *A family of representations of affine Lie algebras,* Russian Math. Surveys **43** (1988) 221-222; *Affine Kac-Moody algebras and semi-infinite Flag manifolds,* Commun. Math. Phys. **128** (1990) 161-189.

[FJ] I. Frenkel and N. Jing, *Vertex representations of quantum affine algebras,* Proc. Nat'l. Acad. Sci. USA **85** (1988) 9373-9377.

[FM] B. Feigin and F. Malikov, *Integral intertwining operators and complex powers of differential (q-difference) operators,* preprint RIMS-894 (1992).

[FR] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations,* Commun. Math. Phys. **146** (1992) 1-60.

[H] T. Hayashi, *Q-analogues of Clifford and Weyl algebras - Spinor and Oscillator representations of quantum enveloping algebras,* Commun. Math. Phys. **127** (1990) 129-144.

[JMMN] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Correlation functions of the XXZ model for* $\Delta < -1$, Phys. Lett. **A168** (1992) 256-263.

[KQS] A. Kato, Y. Quano and J. Shiraishi, *Free Boson Representation of q-Vertex Operators and their Correlation Functions,* Tokyo Univ. preprint UT-618 (1992).

[M1] A. Matsuo, *Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations,* Nagoya Univ. preprint (1992);*Quantum algebra structure of certain Jackson integrals,* Nagoya Univ. preprint (1992).

[M2] A. Matsuo, *Free field representation of quantum affine algebra* $U_q(\widehat{sl_2})$, (hep-th/9208079) Nagoya Univ. preprint (1992).

[M3] A. Matsuo, *Free field representation of q-deformed primary fields for $U_q(\widehat{sl_2})$,* (hep-th/9212040) Nagoya Univ. preprint (1992).

[N] M. Noumi, *Quantum Grassmannians and q-hypergeometric series,* CWI Quarterly **5** (1992) 293-307.

[NYM] M. Noumi, H. Yamada and K. Mimachi, *Finite dimensional representations of the quantum group $GL_q(n;\mathbf{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$,* to appear in Japanese J. Math.

[R] N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system,* Lett. Math. Phys. **26** (1992) 153-165.

[Sh] J. Shiraishi, *Free boson representation of $U_q(\widehat{sl_2})$,* Phys. Lett. **A171** (1992) 243-248.

[Sm] F. Smirnov, *Dynamical symmetries of massive integrable models,* Int. J. Mod. Phys. **A7** Supplement 1 (1992) 813-837;839-858.

[SV] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology,* Invent. Math. **106** (1991) 139-194.

[TT] E. Taft and J. Towber, *Quantum deformation of flag schemes and Grassmann schemes I,* J, Algebra **142** (1991) 1-36.

[W] M. Wakimoto, *Fock representations of the affine Lie algebra $A_1^{(1)}$,* Commun. Math. Phys. **104** (1986) 605-609.

[Z] see references in C. Zachos, *Paradigms of quantum algebras,* Symmetries in Science **V**, B. Gruber et al. (eds.), Plenum, 1991, p.593-609.