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Eigensystem and Full Character Formula of the $\mathcal{W}_{1+\infty}$ Algebra with c=1

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Abstract

By using the free field realizations, we analyze the representation theory of the $\mathcal{W}_{1+\infty}$ algebra with $c = 1$. The eigenvectors for the Cartan subalgebra of $\mathcal{W}_{1+\infty}$ are parametrized by the Young diagrams, and explicitly written down by $\mathcal{W}_{1+\infty}$ generators. Moreover, their eigenvalues and full character formula are also obtained.

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1. Introduction

The $\mathcal{W}_{1+\infty}$ algebra [1] appears in many two-dimensional physical systems, for example, quantum gravity [2, 3, 4, 5] and quantum Hall effect [6, 7]. However, we are far from applying the $\mathcal{W}_{1+\infty}$ algebra to these physical systems, since the representation theory of the $\mathcal{W}_{1+\infty}$ algebra has not been understood enough. On the other hand, we know that the $\mathcal{W}_{1+\infty}$ algebra is realized by free fields [8], as is the case for the $\mathcal{W}_{\infty}$ algebra [9] and its generalizations [10, 11]. The aim of the present letter is to demonstrate that the representation theory of the $\mathcal{W}_{1+\infty}$ algebra is easily investigated by using this free field realization. Although we here exclusively consider the case when the central charge of the $\mathcal{W}_{1+\infty}$ algebra is unity, the generalization of our analysis to other central charges is straightforward, and will be reported elsewhere with the relation of our work with the one by Kac and Radul [12].

This letter is arranged as follows. In sect. 2, we introduce free fermions and free bosons which play the fundamental role in the analysis of the $\mathcal{W}_{1+\infty}$ algebra. We then in sect. 3 study the representation theory of the $\mathcal{W}_{1+\infty}$ algebra on the basis of the fermionization. This section contains the main results of the letter. In sect. 4, we discuss the bosonization of the $\mathcal{W}_{1+\infty}$ algebra. Sect. 5 is devoted to conclusions.

2. Free fermions, bosons and their correspondence

2.1. We first fix some notations. The free fermion fields

$$\bar{\psi}(z) = \sum_{r \in \mathbf{Z}-\frac{1}{2}} \bar{\psi}_r z^{-r-\frac{1}{2}}, \qquad \psi(z) = \sum_{r \in \mathbf{Z}-\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}},$$

are defined with the anti-commutation relations $\{\bar{\psi}_r, \psi_s\} = \delta_{r+s,0}$, $\{\psi_r, \psi_s\} = \{\bar{\psi}_r, \bar{\psi}_s\} = 0$, and thus satisfy the following OPE relations:

$$\begin{aligned}
\bar{\psi}(z)\psi(w) &= \frac{1}{z-w} + {}^{\circ}_{\circ}\, \bar{\psi}(z)\psi(w)\, {}^{\circ}_{\circ}, \qquad & \psi(z)\psi(w) &= {}^{\circ}_{\circ}\, \psi(z)\psi(w)\, {}^{\circ}_{\circ}, \\
\psi(z)\bar{\psi}(w) &= \frac{1}{z-w} + {}^{\circ}_{\circ}\, \psi(z)\bar{\psi}(w)\, {}^{\circ}_{\circ}, \qquad & \bar{\psi}(z)\bar{\psi}(w) &= {}^{\circ}_{\circ}\, \bar{\psi}(z)\bar{\psi}(w)\, {}^{\circ}_{\circ}.
\end{aligned}$$

Here $\{A,B\}=AB+BA$, and $(1-x)^{-1}=\sum_{n\in\mathbf{Z}\geq 0}x^n$. We denote by ${}^\circ_\circ\,\mathcal{O}\,{}^\circ_\circ$ the fermionic normal ordering defined as ${}^\circ_\circ\, A_rB_s\, {}^\circ_\circ = A_rB_s\,\theta(r<1/2)-B_sA_r\,\theta(r\geq 1/2)$, where A_r and B_r are ψ_r or $\bar{\psi}_r$, and $\theta(P)$ is a step function such that $\theta(P)=1$ if the statement P is true, otherwise zero.

The fermion Fock space F is spanned by the vectors

$$\bar{\psi}_{-r_1}\cdots\bar{\psi}_{-r_k}\psi_{-s_1}\cdots\psi_{-s_l}|0\rangle, \qquad r_i>r_{i+1}>0, \qquad s_i>s_{i+1}>0, \tag{2.1.1}$$

where $|0\rangle$ is the highest weight vector such that $\bar{\psi}_r|0\rangle=\psi_r|0\rangle=0$ for $r>0$.

2.2. The free boson field

$$\phi(z)=q+\alpha_0\log z-\sum_{n\in\mathbf{Z}_{\neq 0}}\frac{\alpha_n}{n}z^{-n},$$

is defined with the commutation relations $[\,\alpha_n,\alpha_m\,]=n\delta_{n+m,0}$, $[\,\alpha_0,q\,]=1$, and thus satisfies the following OPE relation:

$$\phi(z)\phi(w)=\log(z-w)+:\phi(z)\phi(w):.$$

Here $[\,A,B\,]=AB-BA$, and $\log(1-x)=-\sum_{n\in\mathbf{Z}>0}x^n/n$. We denote by $:\mathcal{O}:$ the bosonic normal ordering defined as $:\alpha_n\mathcal{O}:=\alpha_n:\mathcal{O}:\,\theta(n<0)+:\mathcal{O}:\alpha_n\,\theta(n\geq 0)$ and $:q\,\mathcal{O}:=q:\mathcal{O}:$, where $\mathcal{O}\in\mathbf{C}[\alpha_n,q]$.

The boson Fock space $B(\Lambda)$ is spanned by the vectors $\alpha_{-m_1}\cdots\alpha_{-m_k}|\Lambda\rangle$, with $m_i\geq m_{i+1}>0$, where $|\Lambda\rangle$ with $\Lambda\in\mathbf{C}$ is the highest weight vector such that $\alpha_n|\Lambda\rangle=0$ for $n>0$ and $\alpha_0|\Lambda\rangle=\Lambda|\Lambda\rangle$. Note that $|\Lambda\rangle=e^{\Lambda q}|0\rangle$.

2.3. As is well known, there exists the fermion-boson correspondence. The free fermion fields $\bar{\psi}(z)$ and $\psi(z)$ are realized by the free boson field $\phi(z)$ as $\bar{\psi}(z)=:e^{\phi(z)}:$ and $\psi(z)=:e^{-\phi(z)}:$. On the other hand, the $U(1)$ current $\partial\phi(z)$ is realized by the free fermion fields as the fermion number current $\partial\phi(z)={}^\circ_\circ\,\bar{\psi}(z)\psi(z)\,{}^\circ_\circ$, and the zero-mode operator q plays the role of the fermion number sift operator. The highest weight vector $|N\rangle$ for $N\in\mathbf{Z}$ of boson Fock space $B(N)$ is then expressed as

$$|N\rangle = \begin{cases} \bar\psi_{-N+\frac{1}{2}}\cdots\bar\psi_{-\frac{3}{2}}\bar\psi_{-\frac{1}{2}}|0\rangle, & N>0,\\ |0\rangle, & N=0,\\ \psi_{N+\frac{1}{2}}\cdots\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle, & N<0. \end{cases} \qquad (2.3.1)$$

Therefore, we have the relation $F=\oplus_{N\in\mathbf{Z}}B(N)$.

3. The $\mathcal{W}_{1+\infty}$ algebra

3.1. The $\mathcal{W}_{1+\infty}$ algebra with central charge $c=1$ [13] is nothing but the algebra of "local" bilinears of fermions; the generating currents $W^k(z)$ $(k\in\mathbf{Z}_{>0})$ of the $\mathcal{W}_{1+\infty}$ algebra are defined as follows:

$$W^k(z) = \sum_{\substack{p,q,t=0\\ p+q+t=k-1}}^{k-1} c^k_{p,q,t} \,{}^{\circ}_{\circ}\, \partial^p\bar\psi(z)\partial^q\psi(z) \,{}^{\circ}_{\circ}\, z^{-t},$$

with arbitrary constants $c^k_{p,q,t}\in\mathbf{C}$. Or equivalently,

$$W^k(z)=\sum_{n\in\mathbf{Z}} W^k_n z^{-n-k}, \qquad W^k_n = \sum_{\substack{r,s\in\mathbf{Z}-\frac{1}{2}\\ r+s=n}} c^k_{r,s} \,{}^{\circ}_{\circ}\, \bar\psi_r\psi_s \,{}^{\circ}_{\circ}\,, \qquad (3.1.1)$$

with†

$$c^k_{r,s} = \sum_{\substack{p,q,t=0\\ p+q+t=k-1}}^{k-1} c^k_{p,q,t} \left[-r-\frac{1}{2}\right]_p \left[-s-\frac{1}{2}\right]_q,$$

where $[\,n\,]_m \equiv \prod_{j=0}^{m-1}(n-j)$. If we set $c^1_{0,0,0}=1$, then $c^1_{r,s}=1$, and thus $W^1(z)$ is just the fermion number current which realized by the free boson as $\partial\phi(z)$. Since

$$[\,W^k_n,\bar\psi_r\,] = c^k_{r+n,-r}\bar\psi_{r+n}, \qquad [\,W^k_n,\psi_r\,] = -c^k_{-r,r+n}\psi_{r+n},$$

the generators W^k_n satisfy the $\mathcal{W}_{1+\infty}$ algebra

$$[\,W^k_n, W^s_m\,] = \sum_{\ell\ge 2} g^{k,s,\ell}_{n,m} W^{k+s-\ell}_{n+m} + C^{k,s}_n \delta_{n+m,0}, \qquad (3.1.2)$$

† The $\mathcal{W}_{1+\infty}$ algebra spanned by the generators W^k_n in eq. (3.1.1) is a subalgebra of $gl(\infty)$. In particular, the $W^1(z)$ is the time evolution generator of KP hierarchy [14].

with some constants $g^{k,s,\ell}_{n,m}$, $C^{k,s}_n \in \mathbf{C}$.

For example, the standard basis for the $\mathcal{W}_{1+\infty}$ generators [13] is

$$c^k_{p,q,t} = (-1)^q \begin{pmatrix} p+q \\ q \end{pmatrix}^2 \begin{pmatrix} 2(k-1) \\ k-1 \end{pmatrix}^{-1} \delta_{t,0},$$

with $\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right) = [n]_m/m!$. In this basis, the anomaly terms in eq. (3.1.2) are diagonarized, that is to say $C^{k,s}_n \propto \delta_{k,s}$. Moreover, they preserve the hermiticity; namely, if we set $\psi^\dagger_r = \bar{\psi}_{-r}$, then $W^{k\dagger}_n = W^k_{-n}$. Since the $\mathcal{W}_{1+\infty}$ algebra is a Lie algebra, any basis obtained by the invertible linear transformation from this standerd basis as $\tilde{W}^k_n = \sum_{s\geq 1} T^k_s W^s_n$ with $T^k_s \in \mathbf{C}$ generate the same $\mathcal{W}_{1+\infty}$ algebra.

3.2. It is easy to see that the vector $|N\rangle$ in eq. (2.3.1) is the highest weight vector of the $\mathcal{W}_{1+\infty}$ algebra which satisfies $W^k_n|N\rangle = 0$ for $n > 0$ and $W^1_0|N\rangle = N|N\rangle$. Let then $M(N)$ and $L(N)$ be, respectively, the Verma module and the irreducible module over the $\mathcal{W}_{1+\infty}$ algebra with respect to the highest weight vector $|N\rangle$. Since any generators of the $\mathcal{W}_{1+\infty}$ algebra and the highest weight vector $|N\rangle$ are realized by fermions, we have the relation $F \supset \sum_{N\in\mathbf{Z}} M(N)$. Note here that $M(N) \cap M(N') = \emptyset$ for $N \neq N'$, since $\left[W^1_0, W^k_n \right] = 0$. Thus, we obtain the relation $F \supset \oplus_{N\in\mathbf{Z}} M(N)$. On the other hand, since the oscillator modes α_n of the free boson field belong to the $\mathcal{W}_{1+\infty}$ algebra, we also have the relation $\oplus_{N\in\mathbf{Z}} B(N) \subset \oplus_{N\in\mathbf{Z}} M(N)$. Thus, we conclude that $F = \oplus_{N\in\mathbf{Z}} M(N)$. Futhermore, since the Kac determinant for the fermion Fock space F does not vanish, the Verma module $M(N)$ is irreducible, *i.e.* $M(N) = L(N)$. We thus have proved the following theorem [11].

THEOREM 3.2. *The fermion Fock space F is the direct sum of the irreducible modules $L(N)$ $(N \in \mathbf{Z})$ over $\mathcal{W}_{1+\infty}$:*

$$F = \bigoplus_{N\in\mathbf{Z}} L(N).$$

3.3. The Cartan subalgebra of $\mathcal{W}_{1+\infty}$ is spanned by W^k_0, for which the following equation holds:

$$\left[\, W^k_0, \bar{\psi}_r \,\right] = \bar{a}^k_r \bar{\psi}_r, \qquad \left[\, W^k_0, \psi_r \,\right] = a^k_r \psi_r, \tag{3.3.1}$$

with $\bar{a}_r^k = c_{r,-r}^k$ and $a_r^k = -c_{-r,r}^k$. Therefore, any vector in the form of eq. (2.1.1) is an eigenvector of the Cartan subalgebra. Hence, the full character $\mathrm{ch}L(N)$ of the $\mathcal{W}_{1+\infty}$ algebra,

$$\mathrm{ch}L(N) = \mathrm{tr}_{L(N)} \prod_{k\geq 1} x_k^{W_0^k},$$

is now easily calculated by taking a trace over the fermion Fock space $F = \oplus_{N\in\mathbf{Z}} L(N)$ as follows:

Theorem 3.3. *The generating function of the full characters* $\mathrm{ch}L(N)$ *is*

$$\sum_{N\in\mathbf{Z}} z^N \mathrm{ch}L(N) = \prod_{r-\frac{1}{2}\in\mathbf{Z}_{\geq 0}} \left(1 + z\prod_{k\geq 1} x_k^{\bar{a}_{-r}^k}\right)\left(1 + z^{-1}\prod_{k\geq 1} x_k^{a_{-r}^k}\right). \tag{3.3.2}$$

The character formula with $x_k = 1$ for $k \geq 3$ was obtained in Ref. [11].

The right hand side of eq. (3.3.2) can be rewritten as

$$\exp\left\{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \bar{f}(x_k^n) z^n\right\} \exp\left\{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} f(x_k^n) z^{-n}\right\},$$

where

$$\bar{f}(x_k) = \sum_{r-\frac{1}{2}\in\mathbf{Z}_{\geq 0}} \prod_{k\geq 1} x_k^{\bar{a}_{-r}^k}, \qquad f(x_k) = \sum_{r-\frac{1}{2}\in\mathbf{Z}_{\geq 0}} \prod_{k\geq 1} x_k^{a_{-r}^k}.$$

Thus, by introducing the elementary Schur polynomials as

$$\sum_{n\in\mathbf{Z}} \bar{P}_n z^n = \exp\left\{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \bar{f}(x_k^n) z^n\right\},$$
$$\sum_{n\in\mathbf{Z}} P_n z^n = \exp\left\{\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} f(x_k^n) z^{-n}\right\},$$

the full character $\mathrm{ch}L(N)$ is now expressed as

$$\mathrm{ch}L(N) = \sum_{\substack{n,m\in\mathbf{Z}\\ n+m=N}} \bar{P}_n P_m.$$

3.4. We next discuss eigenvectors and eigenvalues for the Cartan subalgebra in $L(N)$. Due to eq. (3.3.1), we know that the eigenvectors in $L(N)$ have the following form:

$$\bar{\psi}_{-r_1}\cdots\bar{\psi}_{-r_t}\psi_{-s_1}\cdots\psi_{-s_t}|N\rangle, \qquad r_i > r_{i+1} > N, \quad s_i > s_{i+1} > -N.$$

Here the fermion number of the above state is N, since $\bar{\psi}$ and ψ appear the same times. The eigenvalues are easily calculated by using eq. (3.3.1), and we have the following theorem.

THEOREM 3.4. *Eigenvectors of the Cartan subalgebra of* $\mathcal{W}_{1+\infty}$ *in* $L(N)$ *are parametrized by ordered sets* $Y = (\, n_1 > \cdots > n_t \,|\, m_1 > \cdots > m_t \,)$ *with* $n_i,\, m_i \in \mathbf{Z}_{\geq 0}$:[†]

$$|N;Y\,\rangle = \bar{\psi}_{-N-n_1-\frac{1}{2}} \cdots \bar{\psi}_{-N-n_t-\frac{1}{2}} \psi_{N-m_1-\frac{1}{2}} \cdots \psi_{N-m_t-\frac{1}{2}} |N\,\rangle (-1)^{\sum_{i=1}^{t}(m_i+i-1)},$$

and the eigenvalue of $|N;Y\,\rangle$ *is*

$$W_0^k |N;Y\,\rangle = \big(w_N^k + Y_N^k(Y) \big) \, |N;Y\,\rangle, \tag{3.4.1}$$

with

$$\begin{aligned} w_N^k &= \sum_{r=\frac{1}{2}}^{N-\frac{1}{2}} \bar{a}_{-r}^k \, \theta(N \geq 0) + \sum_{s=\frac{1}{2}}^{-N-\frac{1}{2}} a_{-s}^k \, \theta(N < 0), \\ Y_N^k(Y) &= \sum_{i=1}^{t} \Big(\bar{a}_{-N-n_i-\frac{1}{2}}^k + a_{N-m_i-\frac{1}{2}}^k \Big), \end{aligned}$$

where w_N^k *is the weight of the highest weight vector* $|N\,\rangle$.

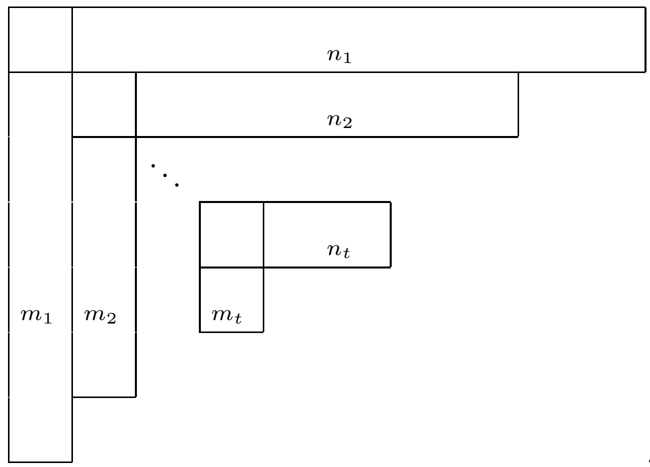
Besides the above parametrization Y, we have other expressions for the eigenvectors, for example,

$$\begin{aligned} \bar{\psi}_{-N-f_1+\frac{1}{2}} \cdots \bar{\psi}_{-N-f_p+p-\frac{1}{2}} |N-p\,\rangle, \qquad & f_i \geq f_{i+1} \in \mathbf{Z}_{>0}, \\ \psi_{N-g_1+\frac{1}{2}} \cdots \psi_{N-g_q+q-\frac{1}{2}} |N+q\,\rangle, \qquad & g_i \geq g_{i+1} \in \mathbf{Z}_{>0}. \end{aligned} \tag{3.4.2}$$

Such many different expressions for the same eigenvector are understood as the different parametrizations for the same Young diagram. The first expression for the eigenvectors in the theorem 3.4 corresponds to the following parametrization for the Young diagram

[†] We have inserted a phase factor for later convenience.

$Y = (\, n_1 > \cdots > n_t \,|\, m_1 > \cdots > m_t \,)$ with $n_i,\, m_i \in \mathbf{Z}_{\geq 0}$:



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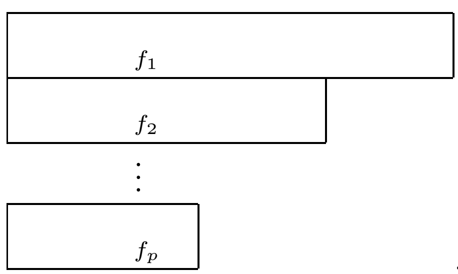
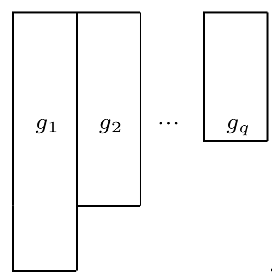
The other two expressions for the eigenvectors in eq. (3.4.2) correspond to the following parametrizations for the Young diagram $Y = (\, f_1 \geq \cdots \geq f_p \,)$ or $Y = (\, g_1 \geq \cdots \geq g_q \,)$ with $f_i,\, g_i \in \mathbf{Z}_{>0}$, respectively:



, or



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3.5. We now calculate the eigenvalues explicitly. To do so, we introduce a new basis of $\mathcal{W}_{1+\infty}$ for which the eigenvalues have a simple form;

$$c^{2k-1}_{p,q,t} = (-1)^{k-1}\delta^{k-1}_p \delta^{k-1}_q \delta^0_t, \qquad c^{2k}_{p,q,t} = \frac{(-1)^{k-1}}{2}\left(\delta^k_p\delta^{k-1}_q - \delta^{k-1}_p\delta^k_q\right)\delta^0_t,$$

or equivalently,

$$\begin{aligned} W^{2k-1}(z) &= (-1)^{k-1} \, {}^{\circ}_{\circ}\, \partial^{k-1}\bar{\psi}(z)\partial^{k-1}\psi(z) \,{}^{\circ}_{\circ}, \\ W^{2k}(z) &= \frac{(-1)^{k-1}}{2} \, {}^{\circ}_{\circ} \left(\partial^k\bar{\psi}(z)\partial^{k-1}\psi(z) - \partial^{k-1}\bar{\psi}(z)\partial^k\psi(z) \right) {}^{\circ}_{\circ}. \end{aligned}$$

For this basis, one can easily obtain

$$\bar{a}^{2k-1}_r = \prod_{s=\frac{3}{2}-k}^{k-\frac{3}{2}} (r+s), \qquad \bar{a}^{2k}_r = r \prod_{s=\frac{3}{2}-k}^{k-\frac{3}{2}} (r+s), \qquad a^k_r = (-1)^k \bar{a}^k_r.$$

Thus, the eigenvalue of the eigenvector $|N;Y\rangle$ is evaluated as eq. (3.4.1) with

$$w_N^{2k-1} = \frac{1}{2k-1}\prod_{\ell=1-k}^{k-1}(N+\ell), \qquad w_N^{2k} = \frac{N}{2k}\prod_{\ell=1-k}^{k-1}(N+\ell),$$

and

$$Y_N^{2k-1}(Y) = (2k-2)\sum_{(i,j)\in Y}\prod_{\ell=2-k}^{k-2}(N+j-i+\ell),$$
$$Y_N^{2k}(Y) = (2k-1)\sum_{(i,j)\in Y}(N+j-i)\prod_{\ell=2-k}^{k-2}(N+j-i+\ell), \qquad Y_N^{2}(Y) = \sum_{(i,j)\in Y} 1,$$

where $(i,j)\in Y$ means that the Young diagram Y has a box in the place of the i-th row and j-th column. Here we have used the identities

$$\sum_{m=a}^{b}\prod_{n=-c}^{c}(m+n) = \frac{1}{2c+2}\left\{\prod_{n=-c}^{c+1}(b+n) - \prod_{n=-c-1}^{c}(a+n)\right\},$$
$$\sum_{m=a}^{b} m\prod_{n=-c}^{c}(m+n) = \frac{1}{2c+3}\left\{(b+\frac{1}{2})\prod_{n=-c}^{c+1}(b+n) - (a-\frac{1}{2})\prod_{n=-c-1}^{c}(a+n)\right\}.$$

4. Bosonization of the $\mathcal{W}_{1+\infty}$ algebra

4.1. We now study the representation theory of the $\mathcal{W}_{1+\infty}$ algebra in terms of the free boson field. Since

$$\circ\!\!\!\circ\, \partial^p\bar{\psi}(z)\partial^q\psi(z) \,\circ\!\!\!\circ = \sum_{\substack{m,n\in\mathbf{Z}_{\ge 0}\\ m+n=p+q}} b^{p,q}_{m,n}\partial^m P^{(n+1)}(z),$$
$$P^{(n)}(z) =: e^{-\phi(z)}\partial^n e^{\phi(z)} : \, =: (\,\partial + \partial\phi(z)\,)^n \cdot 1 :, \qquad b^{p,q}_{m,n} = \frac{(-1)^{q-m}}{n+1}\binom{q}{m},$$

the generators of the $\mathcal{W}_{1+\infty}$ algebra are realized by the free boson field as follows [2]:

$$W^k(z) = \sum_{\substack{m,n,t=0\\ m+n+t=k-1}}^{k-1} \tilde{c}^k_{m,n,t}\partial^m P^{(n+1)}(z) z^{-t},$$

with $\tilde{c}^k_{m,n,t} = \sum_{p,q=0}^{k-1} c^k_{p,q,t} b^{p,q}_{m,n}\delta_{p+q+t,k-1}$. Obviously, $B(N) = L(N)$.

4.2. Let $\bar{S}_n$ and S_n with $n \in \mathbf{Z}$ be the elementary Schur polynomials defined by

$$\sum_{n\in\mathbf{Z}} \bar{S}_n z^n = \exp\left\{\sum_{n>0} \frac{W^1_{-n}}{n} z^n\right\}, \qquad \sum_{n\in\mathbf{Z}} S_n z^n = \exp\left\{-\sum_{n>0} \frac{W^1_{-n}}{n} z^n\right\},$$

and let $\chi_{m,n}$ with $m, n \in \mathbf{Z}$ be the following operators:

$$\chi_{m,n} \equiv (-1)^{m+1} \sum_{\ell\geq 0} \bar{S}_{n-\ell} S_{m+\ell+1} = (-1)^m \sum_{\ell\geq 0} \bar{S}_{n+\ell+1} S_{m-\ell}.$$

Then we have the following theorem.

THEOREM 4.2. *The eigenvectors* $|N;Y\rangle$ *associated with the Young diagram* $Y = (\, n_1 > \cdots > n_t \,|\, m_1 > \cdots > m_t \,)$ *with* n_i*,* $m_i \in \mathbf{Z}_{\geq 0}$ *is realized as*

$$|N;Y\rangle = \det\left(\, \chi_{m_i,n_j} \,\right)_{1\leq i,j\leq t} |N\rangle.$$

Proof . We consider the generating function of the eigenvectors,

$$\bar{\psi}(z_1)\cdots\bar{\psi}(z_t)\psi(w_1)\cdots\psi(w_t)|N\rangle = \sum_{\{r_i,s_i\}} \bar{\psi}_{r_1}\cdots\bar{\psi}_{r_t}\psi_{s_1}\cdots\psi_{s_t}|N\rangle \prod_{i=1}^{t} z_i^{-r_i-\frac{1}{2}} w_i^{-s_i-\frac{1}{2}}.$$

Note that the left hand side can be rewritten in terms of bosons as

$$\frac{\prod_{i<j}(z_i-z_j)(w_i-w_j)}{\prod_{i,j}(z_i-w_j)} : \prod_{i=1}^{t} e^{\phi(z_i)} \prod_{j=1}^{t} e^{-\phi(w_i)} : |N\rangle.$$

Thus, the theorem is obtained if we use the following identity:

$$\frac{\prod_{i<j}(z_i-z_j)(w_i-w_j)}{\prod_{i,j}(z_i-w_j)} = (-1)^{\frac{1}{2}t(t-1)} \det\left(\frac{1}{z_i-w_j}\right)_{1\leq i,j\leq t}. \qquad \square$$

Note that the polynomial $\det\left(\, \chi_{m_i,n_j} \,\right)_{1\leq i,j\leq t}$ is exactly the character polynomial appearing in Ref. [14] as the τ function of the KP hierarchy.

4.3. If we consider the other two parametrizations for the eigenvectors in eq. (3.4.2), then we obtain the following proposition.

PROPOSITION 4.3. *The eigenvectors* $|N;Y\rangle$ *associated with the Young diagram* $Y = (\, f_1 \geq \cdots \geq f_p \,)$ *or* $Y = (\, g_1 \geq \cdots \geq g_q \,)$ *with* f_i*,* $g_i \in \mathbf{Z}_{>0}$ *is realized by the Schur polynomials of bosons as follows:*

$$|N;Y\rangle = \det\left(\, \bar{S}_{f_i+j-i} \,\right)_{1\leq i,j\leq p} |N\rangle.$$
$$= \det\left(\, S_{g_i+j-i} \,\right)_{1\leq i,j\leq q} |N\rangle (-1)^{\sum_i g_i}.$$

Proof . We now bosonize the generating function of the eigenvectors as

$$\bar{\psi}(z_1)\cdots\bar{\psi}(z_p)|N-p\,\rangle = \prod_{i<j}(z_i - z_j) : \prod_{i=1}^{p} e^{\phi(z_i)} : |N-p\,\rangle.$$

By using the Vandermonde's determinant

$$\prod_{i<j}(z_i - z_j) = (-1)^{\frac{1}{2}p(p-1)} \det\left(z_i^{j-1} \right)_{1\le i,j\le p},$$

we obtain the first part of the proposition. The second part is proved similarly. □

5. Conclusion and Discussion

On the basis of the free fermion realization, we have identified the eigenvectors and eigenvalues for the Cartan subalgebra of $\mathcal{W}_{1+\infty}$ with $c=1$, which are parametrized by the Young diagrams. Furthermore, we have obtained the full character formula for the $\mathcal{W}_{1+\infty}$ algebra.

In addition, we wish to make several further comments.

First, in the case of the free boson realization, not only the vector $|N\,\rangle$ with $N \in \mathbf{Z}$ but also $|\Lambda\,\rangle$ with $\Lambda \in \mathbf{C}$ is the highest weight vector of the $\mathcal{W}_{1+\infty}$ algebra. Since N is treated as an indeterminate variable in deriving the formulas of characters $\mathrm{ch}L(N)$ and eigenvectors $|N;Y\,\rangle$, we can replace $N \in \mathbf{Z}$ by $\Lambda \in \mathbf{C}$ in the formulas.

Second, the $\mathcal{W}_{1+\infty}$ algebra can also be realized by the bc system with spins λ and $1-\lambda$,

$$b(z) = \sum_{r\in\mathbf{Z}-\lambda} b_r z^{-r-\lambda}, \qquad c(z) = \sum_{r\in\mathbf{Z}+\lambda} c_r z^{-r+\lambda-1}.$$

In fact, it is achieved simply by replacing $\bar{\psi}_r$ and ψ_s with $b_{r+\frac{1}{2}-\lambda}$ and $c_{s-\frac{1}{2}+\lambda}$, respectively. Eigenvectors and eigenvalues for the bc system are the same as those of the free fermion system. If we redefine the Virasoro generator $W^2(z)$ by adding the derivative of $U(1)$ current $W^1(z)$, then the central charge of the Virasoro generator varies. However, the $\mathcal{W}_{1+\infty}$ algebra itself does not change because this redefinition is nothing but a linear transformation of the $\mathcal{W}_{1+\infty}$ generators W^k_n.

Third, even for the other quasi-finite case $c \neq 1$ [12, 15], we expect that the eigenvectors are also parametrized by the Young diagrams.

Finally, for other W infinity algebras considered in Ref. [11], we can easily write down the full characters or the generating functions of them as the theorem 3.3.

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References

[1] See references in P. Bouwknegt and K. Schoutens, Phys. Rept. **223** (1993) 183-276.

[2] M. Fukuma, H. Kawai and R. Nakayama, Commun. Math. Phys. (1991) 371-403.

[3] H. Itoyama and Y. Matsuo, Phys. Lett. **B262** (1991) 233-239.

[4] V. Kac and A. Schwarz, Phys. Lett. **B257** (1991) 329-334;
A. Schwarz, Mod. Phys. Lett. **A6** (1991) 2713-2726.

[5] J. Goeree, Nucl. Phys. **B358** (1991) 737-757.

[6] S. Iso, D. Karabali and B. Sakita, Phys. Lett. **B296** (1992) 143-150.

[7] A. Cappelli, C. Trugenberger and G. Zemba, Nucl. Phys. **B396** (1993) 465-490; *Classification of quantum Hall universality classes by* $\mathcal{W}_{1+\infty}$ *symmetry,* Preprint MPI-PH-93-75, November 1993, (hepth/9310181).

[8] E. Bergshoeff, C. Pope, L. Romans, E. Sezgin and X. Shen, Phys. Lett. **B245** (1990) 447-452.

[9] I. Bakas and E. Kiritsis, Nucl. Phys. **B343** (1990) 185-204; Mod. Phys. Lett. **A5** (1990) 2039-2050.

[10] S. Odake and T. Sano, Phys. Lett. **B258** (1991) 369-374.

[11] S. Odake, Int. J. of Mod. Phys. **A7** (1992) 6339-6355.

[12] V. Kac and A. Radul, *Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle,* MIT Mathematics preprint, July 1993, (hepth/9308153).

[13] C. Pope, L. Romans and X. Shen, Nucl. Phys. **B339** (1990) 191-221.

[14] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Proc. of RIMS Symposium on Non-Linear Integrable Systems - Classical Theory and Quantum Theory, (ed. M. Jimbo and T. Miwa), 39-120, World Sci., 1983.

[15] Y. Matsuo, *Free fields and quasi-finite representation of* $\mathcal{W}_{1+\infty}$ *algebra,* Preprint UT-661, December 1993, (hepth/9312192).