

January 1994

YITP/K-1054  
UT-669  
SULDP-1994-1  
hep-th/9402001

**Determinant Formulae  
of Quasi-Finite Representation of  $W_{1+\infty}$  Algebra  
at Lower Levels**

H. Awata, M. Fukuma  
Yukawa Institute for Theoretical Physics  
Kyoto University, Kyoto 606, Japan

Y. Matsuo  
Department of Physics  
University of Tokyo, Tokyo 113, Japan

S. Odake  
Department of Physics, Faculty of Liberal Arts  
Shinshu University, Matsumoto 390, Japan

**Abstract**

We calculate the Kac determinant for the quasi-finite representation of  $W_{1+\infty}$  algebra up to level 8. It vanishes only when the central charge is integer. We give an algebraic construction of null states and propose the character formulae. The character of the Verma module is related to free fields in three dimensions which has rather exotic modular properties.

arXiv:hep-th/9402001v1 1 Feb 1994

# 1 Introduction

In the study of loop algebras, the Kac determinant [1] played an essential role for understanding the detailed structure of the Hilbert space. In fact, historically we could not have constructed the minimal models if we had no knowledge about it. On the other hand, the Kac determinant of the  $W_{1+\infty}$  algebra has not been discussed so far. It was mainly due to the fact that infinitely many states possibly appear at each energy level, reflecting the infinite number of generators. Although we have explicit representations by free fields [2, 3] and we can even construct its full character formula in a special case of  $C = 1$  [4, 5], it remained a longstanding problem to classify all the possible representations.

Recently, Kac and Radul [6] overcame this difficulty of infiniteness, explicitly showing the prescription for getting finite number of non-vanishing states at each energy level. The representation thus obtained is called quasi-finite.

Now that we have finite number of states at each energy level, the natural step which should follow is the computation of the Kac determinant and the character. In this letter, we report our preliminary results on this subject. Our computation was carried out by using *Mathematica* package. We give the Kac determinant up to level 8, where 160 relevant states exist. We find that additional null states appear only when the central charge is integer, the case in which the representation can be realized by free fermions or bosonic ghosts [7]. After explaining the structure of null states, we then propose the character formulae which are consistent with the Kac determinant. Furthermore, we find that the character of the Verma module is related to the character of *three*-dimensional free field first given by Cardy [8]. The modular property of  $W_{1+\infty}$  algebra thus becomes rather exotic due to this nature.

## 2 Definitions of $W_{1+\infty}$ algebra

Although there are some overlaps with our previous paper [7], we recapitulate some definitions of  $W_{1+\infty}$  algebra and give a brief review of [6] to make this paper self-contained to some extent.

The classical algebra is generated by the polynomials of  $z$  and  $D \equiv z \frac{\partial}{\partial z}$ . One of the typical generator may be written as  $z^r f(D)$  where the function  $f(w)$  is any regular function at  $w = 0$ . Their commutation relations are

$$[z^r f(D), z^s g(D)] = z^{r+s} (f(D+s)g(D) - f(D)g(D+r)). \quad (1)$$

We call the quantum version of this algebra the  $W_{1+\infty}$  algebra. For each classical generator  $z^r f(D)$ , we denote the corresponding quantum generator by  $W(z^r f(D))$ . The quantum

algebra is different from the classical one (1) only in the additional central term,

$$[W(z^r f(D)), W(z^s g(D))] = W(z^{r+s} f(D+s)g(D)) - W(z^{r+s} f(D)g(D+r)) + C\Psi(z^r f(D), z^s g(D)), \quad (2)$$

where two-cocycle  $\Psi$  is defined by

$$\begin{aligned} \Psi(z^r f(D), z^s g(D)) &= -\Psi(z^s g(D), z^r f(D)) \\ &= \begin{cases} \sum_{1 \leq j \leq r} f(-j)g(r-j) & \text{if } r = -s > 0 \\ 0 & \text{if } r + s \neq 0 \text{ or } r = s = 0. \end{cases} \end{aligned} \quad (3)$$

These commutation relations can be written in a compact form:

$$[W(z^r e^{xD}), W(z^s e^{yD})] = (e^{xs} - e^{yr})W(z^{r+s} e^{(x+y)D}) + C \frac{e^{xs} - e^{yr}}{1 - e^{x+y}} \delta_{r+s,0}. \quad (4)$$

The basis given in [2],  $V_r^i = W_r^{i+2}$ , are expressed as  $W_r^{k+1} = W(z^r f_r^k(D))$  ( $k \geq 0$ ),

$$f_r^k(D) = \binom{2k}{k}^{-1} \sum_{j=0}^k (-1)^j \binom{k}{j}^2 [-D - r - 1]_{k-j} [D]_j, \quad (5)$$

where  $[x]_m = \prod_{j=0}^{m-1} (x-j)$  and  $\binom{x}{m} = [x]_m / m!$ . The generators of the  $\widehat{u(1)}$  and Virasoro<sup>1</sup> subalgebras are written as  $J_r = W(z^r)$ ,  $L_r = -W(z^r D)$ . In particular,  $L_0$  is used to count the energy level,  $[L_0, W(z^r f(D))] = -rW(z^r f(D))$ . One may immediately recognize that there are infinitely many generators at each energy level.

A representation of  $W_{1+\infty}$  algebra is specified by its highest weight conditions,

$$\begin{aligned} W(z^r D^k)|\lambda\rangle &= 0, & r \geq 1, k \geq 0, \\ W(D^k)|\lambda\rangle &= \Delta_k |\lambda\rangle, & k \geq 0. \end{aligned} \quad (6)$$

Unlike other two dimensional algebras, there are infinitely many parameters  $\Delta_k$  to specify the representation. In the quasi-finite representation, however, the number of relevant parameters is reduced to finite. It will be convenient to introduce,

$$\Delta(x) \equiv - \sum_{k=0}^{\infty} \frac{x^k}{k!} \Delta_k. \quad (7)$$

This is the eigenvalue of the operator,  $W(-e^{xD})$ .

### 3 Quasi-Finite Representation and Character of Verma Module

Kac and Radul studied the structure of the conditions to get finite number of non-vanishing states at each energy level. Their result may be summarized as follows.

---

<sup>1</sup>  $W_r^2 = L_r - \frac{1}{2}(r+1)J_r$ .

1. For each level  $r$ , generators which annihilate the highest weight state should take the form,  $W(z^{-r}b_r(D)g(D))$  where  $b_r(D)$  is a monic, finite degree polynomial of operator  $D$ .
2. The polynomial  $b_r(D)$  with  $r > 1$  is related to level 1 polynomial  $b(D) \equiv b_1(D)$  as
  - $b_r(D)$  can be divided by  $\text{lcm}(b(D), b(D-1), \dots, b(D-r+1))$ .
  - $b(D)b(D-1)\dots b(D-r+1)$  can be divided by  $b_r(D)$ .

$b_r$  can be uniquely determined as  $b_r(D) = \prod_{s=0}^{r-1} b(D-s)$  only when the differences of any two distinct roots of  $b(w) = 0$  are not integer.

3. The function  $\Delta(x)$  satisfies a differential equation,

$$b\left(\frac{d}{dx}\right) \left( (e^x - 1)\Delta(x) + C \right) = 0. \quad (8)$$

When  $b(w) = (w - \lambda_1)^{K_1} \dots (w - \lambda_\ell)^{K_\ell}$ , the solutions are

$$\Delta(x) = \frac{\sum_{i=1}^{\ell} p_{K_i}(x) e^{\lambda_i x} - C}{e^x - 1}, \quad \deg p_{K_i} \leq K_i - 1. \quad (9)$$

For the proof of these statements, see reference [6].

The independent components of the non-vanishing  $W_{1+\infty}$  generators may be taken as  $W(z^{-r}D^k)$  with  $k = 0, 1, \dots, \deg b_r - 1$ . If the condition in the second item is satisfied, the number of level  $r$  non-vanishing generators is  $rK$  where  $K \equiv \deg b(w)$ . This observation immediately yields the character of the Verma module,

$$\text{ch}(q, K) \equiv \text{tr } q^{L_0} = \chi(q)^K, \quad \chi(q) = \prod_{j=1}^{\infty} (1 - q^j)^{-j}. \quad (10)$$

It already reveals one of essential features of the  $W_{1+\infty}$  algebra. A few years ago, Cardy wrote an interesting paper on the modular invariance in higher dimensions [8]. By using  $L_0 = r \frac{\partial}{\partial r}$  as the dilatation generator in the radial direction, he shows that the partition function for the massless free fields in  $d$  dimensions is written as,

$$Z_d \equiv \text{tr } q^{L_0} = \prod_{j=1}^{\infty} (1 - q^j)^{-D_d(j)}, \quad (11)$$

where  $D_d(j)$  is the number of spherical harmonics with spin  $j$  on  $d-1$  sphere. For large  $j$ , it behaves as  $D_d(j) \sim j^{d-2}$ . From this viewpoint, the character of the Verma module (10) has *three-dimensional* nature. For this reason, its modular property is quite different from those of Virasoro or Kac-Moody minimal representations. In particular, the character itself is not modular covariant. However, if we write

$$\log \chi(q = e^{-2\pi\delta})^K = \frac{K}{2\pi i} \int_C \delta^{-s} F(s) ds, \quad F(s) = (2\pi)^{-s} \Gamma(s) \zeta(s+1) \zeta(s-1), \quad (12)$$

and define a modified character,

$$I(\delta) \equiv \frac{K}{2\pi i} \int_C \delta^{-s} \frac{\Gamma(\frac{1}{2}s - \frac{1}{4})}{\Gamma(\frac{1}{2}s + \frac{1}{4})} F(s + \frac{1}{2}) ds, \quad (13)$$

then we can realize modular covariance even in three dimensions:

$$I(\delta) = I(1/\delta) + RK(\delta^{-3/2} - \delta^{3/2}). \quad (14)$$

The second term on the right hand side may be regarded as Casimir energy term. We remark that it is not proportional to the central charge  $C$  but is related to the degree  $K$  of level one polynomial  $b(w)$ .

Later, we discuss the character formula for the degenerate representation ( $C = \text{integer}$ ). In those cases, they are written by free fields in two dimensions. It illustrates the hybrid nature of  $W_{1+\infty}$  algebra.

## 4 Determinant Formula

After this preparation, we would like to present our result on the Kac determinant for quasi-finite representations. We examine the simplest situations, *i.e.*,  $K = 1, 2, 3$ .

When  $K = 1$ , we write  $b(w) = w - \lambda$ , and thus (9) gives  $\Delta(x) = C(e^{\lambda x} - 1)/(e^x - 1)$ . In this case, the highest weight is parametrized by two constants  $C, \lambda$ . For the first three levels, the relevant ket states are,

$$\begin{aligned} \text{Level 1} & \quad W(z^{-1})|\lambda\rangle \\ \text{Level 2} & \quad W(z^{-2})|\lambda\rangle, W(z^{-1})^2|\lambda\rangle, W(z^{-2}D)|\lambda\rangle \\ \text{Level 3} & \quad W(z^{-3})|\lambda\rangle, W(z^{-1})W(z^{-2})|\lambda\rangle, W(z^{-1})^3|\lambda\rangle, \\ & \quad W(z^{-3}D)|\lambda\rangle, W(z^{-1})W(z^{-2}D)|\lambda\rangle, W(z^{-3}D^2)|\lambda\rangle. \end{aligned} \quad (15)$$

Corresponding bra states may be given by changing  $z^{-r}$  into  $z^r$ .

In general, the number of relevant states grows as,

$$\begin{aligned} \chi(q) = & \quad 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \\ & \quad 48q^6 + 86q^7 + 160q^8 + 282q^9 + 500q^{10} + \dots \end{aligned} \quad (16)$$

where the number of level  $m$  non-vanishing states is given by the coefficient of  $q^m$ .

We compute the Kac determinant up to level 8:

$$\begin{aligned} \det[1] & \quad \propto C \\ \det[2] & \quad \propto C^3(C - 1) \end{aligned}$$

$$\begin{aligned}
\det[3] &\propto C^6(C-1)^3(C-2) \\
\det[4] &\propto (C+1)C^{13}(C-1)^8(C-2)^3(C-3) \\
\det[5] &\propto (C+1)^3C^{24}(C-1)^{17}(C-2)^8(C-3)^3(C-4) \\
\det[6] &\propto (C+1)^{10}C^{48}(C-1)^{37}(C-2)^{19}(C-3)^8(C-4)^3(C-5) \\
\det[7] &\propto (C+1)^{23}C^{86}(C-1)^{71}(C-2)^{41}(C-3)^{19}(C-4)^8(C-5)^3(C-6) \\
\det[8] &\propto (C+1)^{54}C^{161}(C-1)^{138}(C-2)^{85}(C-3)^{43} \\
&\quad \times (C-4)^{19}(C-5)^8(C-6)^3(C-7). \tag{17}
\end{aligned}$$

We remark that  $\lambda$ -dependent terms disappear in the final expression due to nontrivial cancellations. In the next section, we will explain why it happens resorting to the spectral flow argument. The maximal power of  $C$  in  $\det[m]$  is given by the coefficient of  $q^m$  in  $t \frac{d}{dt} \prod_{j=1}^{\infty} (1 - tq^j)^{-j} |_{t=1}$ .

When  $K = 2$ ,  $b(w) = (w - \lambda_1)(w - \lambda_2)$  and we may parametrize the solution of the differential equation as

$$\Delta(x) = C_1 \frac{e^{\lambda_1 x} - 1}{e^x - 1} + C_2 \frac{e^{\lambda_2 x} - 1}{e^x - 1}, \quad C = C_1 + C_2. \tag{18}$$

The number of relevant states are generated by  $\chi(q)^2$ . We calculate the determinant up to level 4. The final result is,

$$\begin{aligned}
\det[1] &\propto (\lambda_1 - \lambda_2)^2 \prod_{k=1}^2 C_k \\
\det[2] &\propto (\lambda_1 - \lambda_2)^{10} (\lambda_1 - \lambda_2 - 1)^2 (\lambda_1 - \lambda_2 + 1)^2 \prod_{k=1}^2 C_k^4 (C_k - 1) \\
\det[3] &\propto (\lambda_1 - \lambda_2)^{34} \prod_{\epsilon=\pm 1} (\lambda_1 - \lambda_2 + \epsilon)^8 (\lambda_1 - \lambda_2 + 2\epsilon)^2 \prod_{k=1}^2 C_k^{12} (C_k - 1)^4 (C_k - 2) \\
\det[4] &\propto (\lambda_1 - \lambda_2)^{108} \prod_{\epsilon=\pm 1} (\lambda_1 - \lambda_2 + \epsilon)^{30} (\lambda_1 - \lambda_2 + 2\epsilon)^8 (\lambda_1 - \lambda_2 + 3\epsilon)^2 \\
&\quad \times \prod_{k=1}^2 (C_k + 1) C_k^{34} (C_k - 1)^{14} (C_k - 2)^4 (C_k - 3). \tag{19}
\end{aligned}$$

In this case,  $\lambda_1$  and  $\lambda_2$  appear only through their difference,  $\lambda_1 - \lambda_2$ , which will also be explained by spectral flow. In the previous section, we noted that something special happens when the difference of the roots is integer. We can observe it explicitly by the appearance of zeros in the determinant.

When two spins are identical,  $\lambda_1 = \lambda_2$ ,  $\Delta(x)$  becomes  $(C(e^{\lambda x} - 1) + p_1 x e^{\lambda x}) / (e^x - 1)$ . We may obtain the determinant formula for this case by writing,  $C_1 = M + C$ ,  $C_2 = -M$ ,  $\lambda_1 = \lambda + \frac{p_1}{M}$ ,  $\lambda_2 = \lambda$ , and by taking a limit  $M \rightarrow \infty$  [7]. Although there are many other

ways to reproduce the formula for  $\Delta(x)$ , the Kac determinant itself does not depend on them,

$$\det[1] \propto p_1^2, \quad \det[2] \propto p_1^{10}, \quad \det[3] \propto p_1^{34}, \quad \det[4] \propto p_1^{108}. \quad (20)$$

We note that they have no dependence on  $C$  or  $\lambda$ .

Finally when  $K = 3$ ,  $b(w) = (w - \lambda_1)(w - \lambda_2)(w - \lambda_3)$ , the eigenvalues are given by  $\Delta(x) = \sum_{k=1}^3 C_k (e^{\lambda_k x} - 1)/(e^x - 1)$ , with  $C = C_1 + C_2 + C_3$ . The number of relevant states is given by generating function  $\chi(q)^3$ . The Kac determinant for the first two levels are,

$$\begin{aligned} \det[1] &\propto \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{k=1}^3 C_k \\ \det[2] &\propto \prod_{i < j} (\lambda_i - \lambda_j)^{12} (\lambda_i - \lambda_j - 1)^2 (\lambda_i - \lambda_j + 1)^2 \prod_{k=1}^3 C_k^5 (C_k - 1). \end{aligned} \quad (21)$$

When the  $\lambda_i$ 's are identical, the generating function becomes,

$$\Delta(x) = \frac{(C + p_1 x + p_2 x^2) e^{\lambda x} - C}{e^x - 1}. \quad (22)$$

This solution can be derived by putting  $C_1 = M^2$ ,  $C_2 = -2M^2 + C$ ,  $C_3 = M^2$ ,  $\lambda_2 = \lambda$ ,  $\lambda_1 = \lambda + \frac{\sqrt{p_2}}{M} + \frac{1}{2M^2} p_1$ ,  $\lambda_3 = \lambda - \frac{\sqrt{p_2}}{M} + \frac{1}{2M^2} p_1$ , and taking a limit  $M \rightarrow \infty$ . Again, there are many other ways to get the same limit, while the Kac determinant itself does not depend on them,

$$\det[1] \propto p_2^3, \quad \det[2] \propto p_2^{18}. \quad (23)$$

Not that they are free of other parameters  $(\lambda, C, p_1)$ .

By inspecting these formulae, one may summarize the properties of the Kac determinant for the quasi-finite representation of  $W_{1+\infty}$  algebra as follows:

1. The determinant is always factorized into two parts. The first depends only on  $C$  and the second on  $\lambda_i - \lambda_j$ .
2. The additional null states appear only when  $C$  is integer or  $\lambda_i - \lambda_j$  is integer.
3. When some  $\lambda_i$ 's are identical, only the top component of  $p_K(x)$  in (9) is relevant to the Kac determinant. In particular, when it does not vanish, there are no additional null states.

## 5 Spectral Flow

In the previous section, we see that the Kac determinants depend only on the difference between the parameters  $\lambda_i$ . This fact is naturally understood if we notice the existence of one parameter family of automorphisms (spectral flow) in  $W_{1+\infty}$  algebra [2].

The transformation rule is given by

$$W^\lambda(z^r e^{xD}) = W(z^r e^{x(D+\lambda)}) + C \frac{e^{\lambda x} - 1}{1 - e^x} \delta_{n0}, \quad (24)$$

where  $\lambda$  is an arbitrary parameter. For lower components, for example, it is expressed as

$$\begin{aligned} J'_n &= J_n - \lambda C \delta_{n0}, \\ L'_n &= L_n - \lambda J_n + \frac{1}{2} \lambda (\lambda - 1) C \delta_{n0}. \end{aligned} \quad (25)$$

One can easily show that  $W^\lambda(\cdot)$  satisfies the same commutation relation with (4). Furthermore, a highest weight state  $|\Lambda\rangle$  with respect to the original generators  $W(z^r D^k)$  is also a highest weight state with respect to the new generators  $W^\lambda(z^r D^k)$ , while the minimal polynomial  $b(w)$  and the weight  $\Delta(x)$  are replaced, respectively, by  $b(w - \lambda)$  and

$$e^{\lambda x} \Delta(x) + C \frac{e^{\lambda x} - 1}{e^x - 1}. \quad (26)$$

This implies that the spectral flow transforms  $\lambda_i$  into  $\lambda_i + \lambda$ . On the other hand, any automorphism does not change determinants. Thus, we conclude that the Kac determinant depends on the  $\lambda'_i$ s only through their differences.

## 6 Characters for Degenerate Representations

From the Kac determinant, we can extract the information on null states in the Verma module with integer central charges. From the preceding discussions, it is plausible to assume the following form of Kac determinant for  $K = 1$ ,  $b(w) = w - \lambda$ :

$$\det[N] = \prod_{n=-\infty}^{\infty} (C - n)^{\tilde{\phi}(n,N)}, \quad \tilde{\phi}(n, N) \geq \phi(n, N), \quad (27)$$

where  $\phi(n, N)$  is the number of null states at level  $N$  in the  $C = n$  highest weight representation. Comparing the character of the Verma module, one may derive,<sup>2</sup>

$$\text{ch}_n(q) \equiv \text{tr } q^{L_0} = q^{\frac{1}{2}n\lambda(\lambda-1)} \left( \chi(q) - \sum_{N=0}^{\infty} \phi(n, N) q^N \right). \quad (28)$$

We now would like to propose the character formulae for  $C = n \geq -1$  degenerate representation with  $b(w) = w - \lambda$ , which is consistent with the Kac determinant above:

$$\text{ch}_n(q) = q^{\frac{1}{2}n\lambda(\lambda-1)} \prod_{j=1}^{\infty} \frac{1}{(1 - q^j)(1 - q^{j+1}) \cdots (1 - q^{j+n-1})} \quad (C = n > 0), \quad (29)$$

$$\text{ch}_{-1}(q) = q^{-\frac{1}{2}\lambda(\lambda-1)} \prod_{j=1}^{\infty} \frac{1}{(1 - q^j)^2} \cdot \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)} \quad (C = -1). \quad (30)$$

---

<sup>2</sup> For  $C = 0$  case,  $\text{ch}_0(q) = 1$ .



Recall that the former describes the  $n$  free-fermion system, while the latter the bosonic ghost. Actually, in these cases we can construct the full character formula as will be shown later, and (30) is obtained from the full character by restricting it such as counting the conformal weight only.

We here mention a remarkable similarity between the  $C = 1$  and  $C = -1$  characters. In fact, a short calculation shows the following equations:

$$\text{ch}_1(q) = q^{\frac{1}{2}\lambda(\lambda-1)} \sum_{m=0}^{\infty} q^m \left( q^{\frac{1}{2}m(m-1)} \prod_{j=1}^m \frac{1}{1-q^j} \right)^2, \quad (31)$$

$$\text{ch}_{-1}(q) = q^{-\frac{1}{2}\lambda(\lambda-1)} \sum_{m=0}^{\infty} q^m \left( \prod_{j=1}^m \frac{1}{1-q^j} \right)^2. \quad (32)$$

They may suggest that there exists a general correspondence between bosonic and fermionic characters.

The character formula (29) is a simple consequence of the following observation: *The representation space of  $W_{1+\infty}$  algebra with  $C = n > 0$ , is generated by  $W(z^{-r}D^k)$ , with  $0 \leq k \leq n-1$ ,  $r \geq k+1$ . We confirmed this statement for all generators in the case of  $C = 1$ , and for lower generators in the case of  $C = 2$ . Although we have no rigorous proof at this stage, (29) is consistent with the Kac determinant obtained above. In the appendix, we describe the general structure of null states for the free field theories.*

The character formula (30) for  $C = -1$  is rigorously proved as follows. Since we have already demonstrated the spectral flow, we can restrict ourselves to the simplest case,  $b(w) = w$ . We follow the argument of [5]: The basis of the representation space are  $\beta_{-r_1} \cdots \beta_{-r_k} \gamma_{-s_1} \cdots \gamma_{-s_k} |0\rangle$  with  $r_1 \geq \cdots \geq r_k \geq 1$  and  $s_1 \geq \cdots \geq s_k \geq 0$ . These states are simultaneous eigenstates of  $W(D^k)$ , since

$$[W(D^k), \beta_r] = \bar{a}_r^k \beta_r, \quad [W(D^k), \gamma_s] = a_s^k \gamma_s, \quad (33)$$

where  $\bar{a}_r^k = r^k$  and  $a_s^k = -(-s)^k$ . Thus, the full character is calculated as

$$\text{ch}_{-1}^{\lambda=0} \equiv \text{tr} \prod_{k=0}^{\infty} x_k^{W(D^k)} = t^0 \text{ term of } \prod_{r=1}^{\infty} \left( 1 - t \prod_{k=0}^{\infty} x_k^{\bar{a}_r^k} \right)^{-1} \prod_{s=0}^{\infty} \left( 1 - t^{-1} \prod_{k=0}^{\infty} x_k^{a_s^k} \right)^{-1}, \quad (34)$$

from which the character (30) is obtained as

$$\begin{aligned} \text{ch}_{-1}^{\lambda=0}(q) &= t^0 \text{ term of } \prod_{r=1}^{\infty} (1 - tq^r)^{-1} \prod_{s=0}^{\infty} (1 - t^{-1}q^s)^{-1} \\ &= \sum_{\ell=0}^{\infty} \text{ch}_{\ell}^{W_{\infty}^{c=2}} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} \cdot \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)}. \end{aligned}$$

Here we have used the character of  $W_{\infty}$  algebra with  $c = 2$  [4].

## 7 Future Issues

In this letter, we examined the Kac determinant of  $W_{1+\infty}$  algebra for the first few levels. Although the information we obtained is still restricted, it helps us to understand the structure of degenerate representations and their characters. In particular, our result seems to suggest that non-trivial representation happens only when  $C$  is an integer.

There are many things which are worth thorough investigation. As mathematical problems, besides the completion of rigorous proofs for the above formulae on the Kac determinants and characters, it must be interesting to investigate the system where  $\lambda_i - \lambda_j$  is integer. As problems of physics, we should clarify the role of  $W_{1+\infty}$  algebra in three-dimensional systems, and the origin of the disappearance of the modular invariance in ordinary sense. We hope to report on the above subjects in our future issues.

**Acknowledgements:** Two of the authors (Y.M. and S.O.) would like to thank members of YITP for their hospitality where part of this work was carried out. Y.M. and S.O. are obliged to Prof. Kenzo Inoue and Prof. Takeo Inami for financial support. This work is also supported in part by Grant-in-Aid for Scientific Research from Ministry of Science and Culture, and by Soryushi-Shogakkai.

### Appendix: Null States for Free Fields

In this appendix, we show a general characterization of null states of free field theories. We treat bosonic and fermionic cases in parallel fashion, *i.e.*,  $\mathbf{b}(z) = \sum_{r \in \mathbf{Z}} \mathbf{b}_r z^{-r-1+\lambda}$  stands for  $\beta$  or  $b$ , and  $\mathbf{c}(z) = \sum_{r \in \mathbf{Z}} \mathbf{c}_r z^{-r-\lambda}$  stands for  $\gamma$  or  $c$ , depending on which we treat, free boson ( $\epsilon = -1$ ) or fermion ( $\epsilon = 1$ ).  $|\lambda\rangle$  is characterized by  $\mathbf{b}_r |\lambda\rangle = \mathbf{c}_s |\lambda\rangle = 0$  ( $r \geq 0, s \geq 1$ ). The  $W_{1+\infty}$  algebra with central charge  $C = \epsilon n$  is realized by  $W(z^r f(D)) = \sum_{\alpha=1}^n \oint \frac{dz}{2\pi i} : \mathbf{b}^{(\alpha)}(z) z^r f(D) \mathbf{c}^{(\alpha)}(z) :$ . In the representation space, the element of  $W_{1+\infty}$  module is generated by  $E(r, s) = \sum_{\alpha=1}^n \mathbf{b}_{-r}^{(\alpha)} \mathbf{c}_{-s}^{(\alpha)}$ , where  $r \geq 1$  and  $s \geq 0$ .

We claim that the following operator for  $C = \epsilon n$  becomes null,

$$\det_{\epsilon} \begin{pmatrix} E(r_1, s_1) & \cdots & E(r_1, s_{n+1}) \\ \vdots & & \vdots \\ E(r_{n+1}, s_1) & \cdots & E(r_{n+1}, s_{n+1}) \end{pmatrix}, \quad (35)$$

where  $r_1 \leq \cdots \leq r_{n+1}$ ,  $s_1 \leq \cdots \leq s_{n+1}$  for fermionic cases, and  $r_1 < \cdots < r_{n+1}$ ,  $s_1 < \cdots < s_{n+1}$  for bosonic cases.  $\det_{\epsilon} A$  stands for a permanent<sup>3</sup> (determinant) of a matrix  $A$  for  $\epsilon = +1$  ( $\epsilon = -1$ ), respectively.

The proof of this statement is straightforward. By definition, the  $m \times m$  matrix

---

<sup>3</sup> Permanent of a  $m \times m$  matrix  $A$  is defined by  $\det_+(A) \equiv \sum_{\sigma} \prod_i A_{i\sigma(i)}$ , where  $\sigma$  is the permutations of the set  $\{1, 2, \dots, m\}$ .

$(E(r_i, s_j))$  is a product of the  $m \times n$  matrix  $(\mathbf{b}_{r_i}^{(\alpha)})$  and the  $n \times m$  matrix  $(\mathbf{c}_{s_j}^{(\alpha)})$ . If  $m > n$ ,  $\det_{\pm}$  vanishes since the matrix  $(E(r_i, s_j))$  becomes a projection operator.

From this observation, the appearance of first null state for the fermionic case should occur at level  $n + 1$  for  $C = n$  and at level  $(n + 1)^2$  for the bosonic case  $C = -n$ . We can easily confirm it in our determinant formula.

## References

- [1] V.G. Kac, *Lecture Notes in Physics* **94** (1979) 441-445;  
 D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **151B** (1985) 31-36;  
 M. Kato and S. Matsuda, *Phys. Lett.* **B184** (1987) 184-190; *Advanced Studies in Pure Mathematics* **16** (1988) 205-254;  
 V.G. Kac and D.A. Kazhdan, *Adv. Math.* **34** (1979) 97-108;  
 S. Mizoguchi, *Phys. Lett.* **B222** (1989) 226-230.
- [2] C.N. Pope, L.J. Romans and X. Shen, *Phys. Lett.* **B242** (1990) 401-406; *Nucl. Phys.* **339B** (1990) 191-221;  
 E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, *Phys. Lett.* **B245** (1990) 447-452.
- [3] I. Bakas and E. Kiritsis, *Nucl. Phys.* **B343** (1990) 185-204; *Mod. Phys. Lett.* **A5** (1990) 2039-2050;  
 S. Odake and T. Sano, *Phys. Lett.* **B258** (1991) 369-374;  
 M. Fukuma, H. Kawai and R. Nakayama, *Comm. Math. Phys.* **143** (1992) 371-403;  
 A. Dhar, G. Mandal and S. Wadia, *Mod. Phys. Lett.* **A7** (1992) 3129-3146;  
 J. de Boer, L. Fehér, A. Honecker, “A Class of W-algebra with Infinitely generated Classical Limit”, ITP-SB-93-84, BONN-HE-93-49, December 1993, hep-th/9312049.
- [4] S. Odake, *Int. J. of Mod. Phys.* **A7** (1992) 6339-6355.
- [5] H. Awata, M. Fukuma, S. Odake and Y. Quano, “Eigensystem and Full Character Formula of the  $W_{1+\infty}$  Algebra with  $c = 1$ ”, preprint YITP/K-1049, SULDP-1993-1, RIMS-959, hep-th/9312208 (December 1993).
- [6] V. Kac and A. Radul, *Comm. Math. Phys.* **157** (1993) 429-457.
- [7] Y. Matsuo, “Free Fields and Quasi-Finite Representation of  $W_{1+\infty}$  algebra”, preprint UT-661, hep-th/9312192 (December 1993).
- [8] J.L. Cardy, *Nucl. Phys.* **B366** (1991) 403-419.