YITP/K-1055
UT-670
SULDP-1994-2
March 1994

Quasifinite Highest Weight Modules over the Super $\mathcal{W}_{1+\infty}$ Algebra

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Abstract

We study quasifinite highest weight modules over the supersymmetric extension of the $\mathcal{W}_{1+\infty}$ algebra on the basis of the analysis by Kac and Radul. We find that the quasifiniteness of the modules is again characterized by polynomials, and obtain the differential equations for highest weights. The spectral flow, free field realization over the (B,C)–system, and the embedding into $\widehat{\mathrm{gl}}(\infty|\infty)$ are also presented.

hep-th/9404041

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1. Introduction

Conformal field theory has attracted much interest for the last ten years, since it describes classical vacua of string theory and two–dimensional statistical system at fixed points of the renormalization group. The representation theory of the Virasoro algebra plays a central role [BPZ]. However, when the systems have larger symmetries, the Virasoro algebra must be extended. For example, when supersymmetry exists one will be led to the super Virasoro algebra [NS, R, GS], while for $\mathbb{Z}_N$ symmetry what is called the $\mathcal{W}_N$ algebra will be relevant [Z, BS].

In the $\mathcal{W}_N$ algebra (or its supersymmetric extension [IMY]), there are $(N-1)$ generating currents with spins $s = 2, 3, ..., N$ (and their superpartners if supersymmetry exists). Here $s = 2$ corresponds to the energy momentum tensor. The peculiar nature of the algebra is in its nonlinearity, *i.e.*, the singular part of the operator product of generating currents is not expanded as a linear combination of the generating currents, and one has to introduce composite fields made of the currents. The occurrence of such operators implies that the corresponding algebra is not a Lie algebra in ordinary sense. Indeed, the check of the Jacobi identity gives severe restrictions on the structure constants.

The situation changes drastically if we take a suitable limit $N \to \infty$ [B]. The resulting algebra, called the $\mathcal{W}_\infty$ algebra, becomes a Lie algebra [PRS1]. This limiting procedure is essentially equal to regarding the composite fields needed in the operator product expansion of lower spin currents, as a new generating current with higher spin.

Great simplification further occurs if we add spin–1 current ($u(1)$ current) to the $\mathcal{W}_\infty$ algebra [PRS2]. The obtained algebra is named the $\mathcal{W}_{1+\infty}$ algebra for this historical reason. Although there are several types of $\mathcal{W}$ infinity–like algebras [BK, FFZ], we may say, at cost of rigor, that this is the most fundamental one. All other algebras such as $\mathcal{W}_\infty$ and $\mathcal{W}_N$ are obtained by imposing some suitable constraints on it.

The $\mathcal{W}_{1+\infty}$ algebra naturally arises in various physical systems. Firstly, in two–dimensional quantum gravity (the square root of) the generating function of scaling operators is identified with a τ–function of KP hierarchy and obeys the vacuum condition of the $\mathcal{W}_{1+\infty}$ algebra [FKN, DVV, IM, KS, S, G]. Secondly, in the quantum Hall effect, edge states satisfy the highest weight condition of the $\mathcal{W}_{1+\infty}$ algebra, reflecting the incompressibility of quantum fluid [IKS, CTZ]. Some interesting applications are also known in higher dimensional physics, such as the construction of gravitational instantons [T, YC, P]. Furthermore, the $\mathcal{W}_{1+\infty}$ algebra is known to be closely related to the central extension of gl(∞) algebra [KR]. The application to the large N two–dimensional QCD [GT, DLS] seems also intriguing in this context.

The major reason of such generality of the $\mathcal{W}_{1+\infty}$ algebra is that it is a central extension of the Lie algebra of differential operators on the circle [PRS1]. Recently, Kac and Radul gave a general framework on such Lie algebra and classified all the quasifinite representations [KR]. Since the purpose of this paper is to extend their work to the system with supersymmetry, it may be instructive to review their main results.

Let $\mathcal{G}$ be the Lie algebra of differential operators on the circle; $\mathcal{G}=\{\, z^n f(D) \mid n\in \mathbb{Z}\}$, where $f(w) \in \mathbb{C}[w]$ (polynomial ring with w indeterminate) and $D \equiv z\frac{d}{dz}$. Let then $\mathcal{W}_{1+\infty}$ be the central extension of $\mathcal{G}$, and for $z^n f(D) \in \mathcal{G}$ we denote the corresponding operator in $\mathcal{W}_{1+\infty}$ by $W(z^n f(D))$. The central extension is defined by the following commutation relations:

$$[W(z^n f(D)), W(z^m g(D))] = W\left([z^n f(D),\, z^m g(D)] \right) + C\,\Psi(z^n f(D), z^m g(D)),$$

where the two–cocycle Ψ is given by

$$\begin{aligned}\Psi(z^n f(D), z^m g(D)) &= -\Psi(z^m g(D), z^n f(D)) \\ &= \begin{cases} \sum_{j=1}^{n} f(-j) g(n-j) & \text{if } n=-m>0 \\ 0 & \text{if } n+m\neq 0 \quad \text{or} \quad n=m=0. \end{cases}\end{aligned}$$

More symmetrically, it is written as

$$\left[W(z^n e^{xD}),\, W(z^m e^{yD})\right] = \left(e^{xm} - e^{yn} \right) W(z^{n+m} e^{(x+y)D}) - C\,\delta_{n+m,0}\,\frac{e^{xm}-e^{yn}}{e^{x+y}-1}\,.$$

The two–cocycle is shown to be unique up to coboundaries [Li, F].

The $\mathcal{W}_{1+\infty}$ algebra has the following principal gradation:

$$\mathcal{W}_{1+\infty} = \bigoplus_{n\in\mathbb{Z}} (\mathcal{W}_{1+\infty})_n ,$$
$$(\mathcal{W}_{1+\infty})_n = \{ z^n f(D) \mid f(w) \in \mathbb{C}[w] \} .$$

Note that the Cartan subalgebra is given by $(\mathcal{W}_{1+\infty})_0 = \bigoplus_{s=0}^{\infty} \mathbb{C}W(D^s)$. A highest weight state $|\lambda\rangle$ is thus characterized by the condition

$$(\mathcal{W}_{1+\infty})_n |\lambda\rangle = 0 \quad (n \geq 1),$$
$$(\mathcal{W}_{1+\infty})_0 |\lambda\rangle \subset \mathbb{C}|\lambda\rangle.$$

We introduce the energy operator $L_0 \equiv -W(D)$ and call its eigenvalue of state the energy level. Note that $[L_0, W(z^{-k}f(D))] = +k\,W(z^{-k}f(D))$.

At each energy level k, there might be infinitely many states, reflecting the infinitely many degrees of freedom in the polynomial ring. The *quasifinite* representation is obtained if we require that all but finitely many states at each energy level vanish. More precisely, it is equivalent to saying that the set

$$I_{-k} \equiv \left\{ f(w) \in \mathbb{C}[w] \mid W(z^{-k}f(D))|\lambda\rangle = 0 \right\}$$

is different from $\{0\}$ for any $k \geq 1$. Since I_{-k} is an ideal in $\mathbb{C}[w]$, we can introduce the monic (with unit leading coefficient) generating polynomial $b_k(w)$; $I_{-k} = (b_k(w))$. These polynomials $\{b_k(w)\}_{k=1,2,3,\ldots}$ are called characteristic polynomials.

A surprising result obtained in ref. [KR] is that they are almost uniquely determined by the first characteristic polynomial $b(w) \equiv b_1(w)$. To show this, one has to observe that (i) $b_k(w)$ is divided by $\ell.c.m.(b(w), b(w-1), ..., b(w-k+1))$, (ii) $b(w)b(w-1)...b(w-k+1)$ is divided by $b_k(w)$. These statements are proved by using the null state conditions of the $\mathcal{W}_{1+\infty}$ algebra. Thus, if difference of any two distinct roots of $b(w)$ is not an integer, then $b_k(w)$ is uniquely expressed as $b_k(w) = b(w)b(w-1)...b(w-k+1)$.

It is further shown that the generating function $\Delta(x)$ of highest weights:

$$\Delta(x) \equiv -\sum_{s=0}^{\infty} \frac{x^s}{s!} \Delta^{(s)} \quad \text{for} \quad W(D^s)|\lambda\rangle = \Delta^{(s)}|\lambda\rangle$$

satisfies a simple differential equation:

$$b\left(\frac{d}{dx}\right)[\,(e^x-1)\Delta(x)+C\,]\;=\;0.$$

To cover all $\mathcal{W}$–like algebras with supersymmetry, the $\mathcal{W}_{1+\infty}$ algebra must be extended such as to contain supersymmetry. Such extension was first considered in ref. [MR, UY] in the context of supersymmetric Kadmtsev-Petviashvili hierarchy, and also in ref. [BdWV, BPRSS], where explicit form of (anti–) commutation relations are given. In this paper, we reformulate their work on the super $\mathcal{W}_{1+\infty}$ algebra (later denoted by $\mathcal{SW}_{1+\infty}$) and develop the representation theory, on the basis of the analysis by Kac and Radul for the $\mathcal{W}_{1+\infty}$ algebra. We find that quasifiniteness is again characterized by polynomials, and that the highest weights are expressed in terms of combined differential equations.

The present paper is organized as follows. In sect. 2 and sect. 3, we discuss the general theory of the super $\mathcal{W}_{1+\infty}$ algebra, viewing it as a central extension of the Lie superalgebra of superdifferential operators on the circle. In sect. 4, we classify the quasifinite highest weight representations of the super $\mathcal{W}_{1+\infty}$ algebra, and then, in sect. 5, derive the differential equation which determines the highest weights. In sect. 6, we discuss the spectral flow (two–parameter family of automorphisms) in $\mathcal{SW}_{1+\infty}$. In sect. 7, we consider the (B,C)–system as an example. Sect. 8 is devoted to conclusion and discussion. The embedding of $\mathcal{SW}_{1+\infty}$ into $\widehat{\mathrm{gl}}(\infty|\infty)$ and null vector condition are given in Appendices.

2. General Theory of Lie Superalgebra of Superdifferential Operators on the Circle

2.1. Let $\mathcal{A}=\mathcal{A}^{(0)}\oplus\mathcal{A}^{(1)}$ be a $\mathbb{Z}_2$–graded associative algebra and let $\sigma:\mathcal{A}\to\mathcal{A}$ be a $\mathbb{Z}_2$–preserving automorphism of $\mathcal{A}$; $\sigma\left(\mathcal{A}^{(0,1)}\right)=\mathcal{A}^{(0,1)}$. We specify the $\mathbb{Z}_2$–gradation of an element $a \in \mathcal{A}^{(0)}$ (*resp.* $a \in \mathcal{A}^{(1)}$) as $|a| = 0$ (*resp.* $|a| = 1$). We then introduce the *twisted Laurent polynomial algebra* $\mathcal{A}[z, z^{-1}]$ over $\mathcal{A}$:

$$\begin{aligned} \mathcal{A}[z, z^{-1}] &\equiv \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathcal{A} \\ &= \left\{ \sum_{n \in \mathbb{Z}} z^n \otimes a_n \,\middle|\, a_n \in \mathcal{A},\ \text{all but finite number of } a_n\text{'s vanish} \right\} \end{aligned}$$

with the following $*$–multiplication:

$$(z^n \otimes a) * (z^m \otimes b) \equiv z^{n+m} \otimes \sigma^m(a) \cdot b.$$

Note that the $\mathbb{Z}_2$–gradation of $\mathcal{A}$ naturally induces that of $\mathcal{A}[z, z^{-1}]$; $|z^n \otimes a| \equiv |a|$ if $a \in \mathcal{A}^{(0)}$ or $\mathcal{A}^{(1)}$. In what follows, we will denote $z^n \otimes a$ by $z^n a$ for simplicity.

2.2. Let $\tilde{\mathcal{A}}_\sigma$ denote the algebra $\mathcal{A}[z, z^{-1}]$ regarded as a Lie superalgebra with respect to the usual (anti–) bracket:

$$\begin{aligned} [\, z^n a, z^m b \,\} &\equiv (z^n a) * (z^m b) - (-1)^{|a||b|} (z^m b) * (z^n a) \\ &= z^{n+m} \sigma^m(a) \cdot b - (-1)^{|a||b|} z^{n+m} \sigma^n(b) \cdot a. \end{aligned}$$

2.3. Fix a linear map $\mathrm{str} : \mathcal{A} \to V$ such that $\mathrm{str}\, ab = (-1)^{|a||b|}\, \mathrm{str}\, ba$, where V is a vector space over $\mathbb{C}$. Then we can define a *central extension* $\hat{\mathcal{A}}_{\sigma,\,\mathrm{str}}$ of $\tilde{\mathcal{A}}_\sigma$ by V, $0 \to V \to \hat{\mathcal{A}}_{\sigma,\,\mathrm{str}} \to \tilde{\mathcal{A}}_\sigma \to 0$, as follows. First, we notice that the map $\Psi_{\sigma,\,\mathrm{str}} : \tilde{\mathcal{A}}_\sigma \times \tilde{\mathcal{A}}_\sigma \to V$ defined by

$$\begin{aligned} &\Psi_{\sigma,\,\mathrm{str}}(z^n a, z^m b) \\ &\quad \equiv -(-1)^{|a||b|} \Psi_{\sigma,\,\mathrm{str}}(z^m b, z^n a) \\ &\quad \equiv \begin{cases} \mathrm{str}\left((1 + \sigma + \cdots + \sigma^{n-1})(\sigma^{-n}(a) \cdot b) \right) & \text{if} \quad n = -m > 0, \\ 0 & \text{if} \quad n + m \neq 0 \text{ or } n = m = 0, \end{cases} \end{aligned}$$

satisfies the 2–*supercocycle condition*:

(1) $\Psi_{\sigma,\,\mathrm{str}}(A, B) = -(-1)^{|A||B|} \Psi_{\sigma,\,\mathrm{str}}(B, A)$,

(2) $(-1)^{|A||C|} \Psi_{\sigma,\,\mathrm{str}}\left([A, B\}, C \right) + \text{cyclic permutation} = 0$.

Thus, denoting by $W(A)$ the element in $\hat{\mathcal{A}}_{\sigma,\,\mathrm{str}}$ which corresponds to $A \in \tilde{\mathcal{A}}_\sigma$, we define the (anti–) bracket of two elements $W(A)$, $W(B) \in \hat{\mathcal{A}}_{\sigma,\,\mathrm{str}}$ by the following formula:

$$[\, W(A), W(B) \,\} \equiv W([\, A, B \,\}) + \Psi_{\sigma,\,\mathrm{str}}(A, B).$$

Hereafter, we will restrict ourselves to one–dimensional central extensions; $V = \mathbb{C}$.

3. The Super $\mathcal{W}_{1+\infty}$ Algebra $\mathcal{SW}_{1+\infty}$

3.1. In the rest of the present paper, we will exclusively consider the case where $\mathcal{A}$ is the polynomial algebra over (2×2) supermatrices:

$$\begin{aligned}\mathcal{A} &\equiv \left\{ \begin{bmatrix} f^0(w) & f^+(w) \\ f^-(w) & f^1(w)\end{bmatrix} \;\middle|\; f^A(w)\in\mathbb{C}[w]\,;\, A=0,1,\pm \right\} \\ &= \mathcal{A}^{(0)}\oplus\mathcal{A}^{(1)}.\end{aligned} \tag{3.1}$$

Here we assign the $\mathbb{Z}_2$–gradation as follows:

$$\begin{aligned}\mathcal{A}^{(0)} &= \left\{ \begin{bmatrix} f^0(w) & 0 \\ 0 & f^1(w)\end{bmatrix} \right\} \qquad : \quad \mathbb{Z}_2\text{−even}, \\ \mathcal{A}^{(1)} &= \left\{ \begin{bmatrix} 0 & f^+(w) \\ f^-(w) & 0\end{bmatrix} \right\} \qquad : \quad \mathbb{Z}_2\text{−odd}.\end{aligned}$$

Introducing a basis P_A $(A=0,1,\pm)$ in $\mathcal{A}$ as

$$\begin{aligned} P_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} \;\in \mathcal{A}^{(0)}, \\ P_+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}, \quad P_- = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} \;\in \mathcal{A}^{(1)},\end{aligned}$$

we may denote $F\in\mathcal{A}$ as $F(w)=f^A(w)P_A$. Note that the multiplication as matrices respects the $\mathbb{Z}_2$–gradation.

3.2. Following the general prescription given in the previous section, we fix a $\mathbb{Z}_2$–preserving automorphism $\sigma : \mathcal{A}\to\mathcal{A}$, and define a new $\mathbb{Z}_2$–graded associative algebra

$$\begin{aligned}\mathcal{A}[z,z^{-1}] &\equiv \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{A} \\ &= \left\{ \sum_{n\in\mathbb{Z}} z^n F_n(w) \;\middle|\; F_n(w)\in\mathcal{A}\,,\ \text{all but finite number of } F_n(w)\text{'s vanish} \right\}\end{aligned}$$

with the following $*$–multiplication:

$$(z^n F(w)) * (z^m G(w)) \equiv z^{n+m}\sigma^m\left(\,F(w)\,\right)\cdot G(w).$$

We will set σ as $\sigma(F(w))=\sigma(f^A(w)P_A)\equiv f^A(w+1)P_A$, so that we may replace $f^A(w)$ by $f^A(D)$ with $D=z\partial/\partial z$, and $*$–multiplication by the usual multiplication as matrices.† Note here that $f(D)\,z^m = z^m f(D+m)$ for any holomorphic function $f(w)$.

† This choice of σ is not unique. In ref. [KR], for example, they also consider the case, $\sigma'(F(w))=F(qw)$.

3.3. Let $\tilde{\mathcal{A}}_\sigma \equiv sw_{1+\infty}$ denote the algebra $\mathcal{A}[z, z^{-1}]$ regarded as a Lie superalgebra with the following (anti–) bracket:

$$\begin{aligned}
[\, z^n F(D), z^m G(D)\, \} &= \left[\, z^n f^A(D) P_A, z^m g^B(D) P_B \,\right\} \\
&\equiv (z^n f^A(D) P_A) \cdot (z^m g^B(D) P_B) - (-1)^{|P_A||P_B|} (z^m g^B(D) P_B) \cdot (z^n f^A(D) P_A).
\end{aligned}$$

3.4. We now introduce a linear map $\mathrm{str}_0 : \mathcal{A} \to \mathbb{C}$ as $\mathrm{str}_0 F(D) \equiv \mathrm{str} F(0)$, *i.e.*,

$$\mathrm{str}_0 \begin{bmatrix} f^0(D) & f^+(D) \\ f^-(D) & f^1(D) \end{bmatrix} \equiv f^0(0) - f^1(0).$$

We should notice that str_0 has the following property:

$$\mathrm{str}_0 F(D) G(D) = (-1)^{|F(D)||G(D)|} \mathrm{str}_0 G(D) F(D).$$

Thus, we can define a one–dimensional central extension $\hat{\mathcal{A}}_{\sigma,\,\mathrm{str}_0} \equiv \mathcal{SW}_{1+\infty}$ of $\tilde{\mathcal{A}}_\sigma = sw_{1+\infty}$ through the following (anti–) commutation relation:

$$[\, W(z^n F(D)), W(z^m G(D))\, \} \equiv W([\, z^n F(D), z^m G(D)\, \}) - C \Psi_{\sigma,\,\mathrm{str}_0}(z^n F(D), z^m G(D)), \tag{3.2}$$

where C is the central charge, and for $n = -m > 0$, the 2–supercocycle $\Psi_{\sigma,\,\mathrm{str}_0}$ is given by

$$\begin{aligned}
\Psi_{\sigma,\,\mathrm{str}_0}(z^n F(D), z^m G(D)) &= \mathrm{str}_0 \left((1 + \sigma + \cdots + \sigma^{n-1})(\sigma^{-n}(F(D)) \cdot G(D)) \right) \\
&= \sum_{j=1}^{n} f^A(-j) g^B(n-j)\, \mathrm{str}\, P_A P_B \\
&= \sum_{j=1}^{n} \{ f^0(-j) g^0(n-j) + f^+(-j) g^-(n-j) \\
&\qquad - f^-(-j) g^+(n-j) - f^1(-j) g^1(n-j) \}.
\end{aligned}$$

Note here that

$$\mathrm{str}\, P_A P_B = \begin{cases} 1 & \text{if} \quad (A, B) = (0,0) \quad \text{or} \quad (+,-) \\ -1 & \text{if} \quad (A, B) = (-,+) \quad \text{or} \quad (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

The above (anti–) commutation relations can be rewritten in a simpler form if we introduce $z^n e^{xD}$ as a generating series for $z^n D^k$:

$$
\begin{aligned}
&[W(z^n e^{xD} P_A), W(z^m e^{yD} P_B)\} \\
&\qquad = e^{xm}\, W(z^{n+m} e^{(x+y)D} P_A P_B) - (-1)^{|P_A||P_B|}\, e^{yn}\, W(z^{n+m} e^{(x+y)D} P_B P_A) \\
&\qquad\qquad + C\, \frac{e^{xm} - e^{yn}}{e^{x+y} - 1}\, \delta_{n+m,0}\, \mathrm{str} P_A P_B .
\end{aligned}
\tag{3.3}
$$

We remark that the indices n and m need not be integers in this expression.

The bosonic part of this algebra is the direct sum of two $\mathcal{W}_{1+\infty}$ algebras with central charges C and $-C$.

3.5. The *principal gradation* in $\mathcal{SW}_{1+\infty}$ may be introduced with half–integer labels $\alpha \in \mathbb{Z}/2$ as $\mathcal{SW}_{1+\infty} = \bigoplus_{\alpha \in \mathbb{Z}/2} (\,\mathcal{SW}_{1+\infty}\,)_\alpha$, where

$$
\begin{aligned}
(\,\mathcal{SW}_{1+\infty}\,)_{\alpha=n} &\equiv \left\{\; W\big(\, z^n \left(f^0(D) P_0 + f^1(D) P_1 \right) \big) \;\middle|\; f^{0,1}(w) \in \mathbb{C}[w] \,\right\}, \\
(\,\mathcal{SW}_{1+\infty}\,)_{\alpha=n+1/2} &\equiv \left\{\; W\big(\, z^n f^+(D) P_+ + z^{n+1} f^-(D) P_- \,\big) \;\middle|\; f^{\pm}(w) \in \mathbb{C}[w] \,\right\}.
\end{aligned}
$$

In fact, one can easily show that $\Big[\, (\,\mathcal{SW}_{1+\infty}\,)_\alpha \,, (\,\mathcal{SW}_{1+\infty}\,)_\beta \Big\} \subset (\,\mathcal{SW}_{1+\infty}\,)_{\alpha+\beta}$ with $\alpha, \beta \in \mathbb{Z}/2$. We notice that the Cartan subalgebra of $\mathcal{SW}_{1+\infty}$ is given by $(\,\mathcal{SW}_{1+\infty}\,)_0$.

3.6. We let θ be a Grassmann number, and identify

$$
\begin{bmatrix} P_0 & P_+ \\ P_- & P_1 \end{bmatrix} = \begin{bmatrix} \theta\partial_\theta & \theta \\ \partial_\theta & \partial_\theta\theta \end{bmatrix}.
$$

Then the multiplication $(f^A(D)P_A) \cdot (g^B(D)P_B)$ as superderivatives corresponds to the multiplication

$$
\begin{bmatrix} f^0(D) & f^+(D) \\ f^-(D) & f^1(D) \end{bmatrix} \cdot \begin{bmatrix} g^0(D) & g^+(D) \\ g^-(D) & g^1(D) \end{bmatrix}
$$

as matrices.

The (anti–) commutation relations of superderivatives (with central terms) are now easily obtained. For example, setting $n > 0$, we obtain[†]

$$
\begin{aligned}
\left[\, W\left(z^n f(D) P_a \right),\, W\left(z^m g(D) P_a \right) \right] &= W\left(\left[\, z^n f(D) P_a,\, z^m g(D) P_a \,\right] \right) \\
&\qquad -(-1)^a\, C \sum_{j=1}^{n} f(-j) g(n-j) \delta_{n+m,0}, \qquad (a=0,1), \\
\left\{ W\left(z^n f(D) P_\pm \right),\, W\left(z^m g(D) P_\mp \right) \right\} &= W\left(\left\{ z^n f(D) P_\pm,\, z^m g(D) P_\mp \right\} \right) \\
&\qquad \mp C \sum_{j=1}^{n} f(-j) g(n-j) \delta_{n+m,0}.
\end{aligned}
$$

Other (anti–) commutation relations have no central terms.

4. Quasifinite Representations

4.1. Let $V(\lambda)$ be a highest weight module over $\mathcal{SW}_{1+\infty}$ with the highest weight λ. The highest weight vector $|\lambda\rangle \in V(\lambda)$ is characterized via the principal gradation as $(\mathcal{SW}_{1+\infty})_\alpha |\lambda\rangle = 0$ for $\alpha \geq 1/2$ and $(\mathcal{SW}_{1+\infty})_0 |\lambda\rangle \subset \mathbb{C}|\lambda\rangle$. Explicitly, these conditions are written as

$$
\begin{aligned}
W\left(z^n f^A(D) P_A \right) |\lambda\rangle &= 0 \qquad \left(n \geq 1\,;\ {}^\forall f^A(w) \in \mathbb{C}[w] \right), \\
W\left(f(D) P_+ \right) |\lambda\rangle &= 0 \qquad \left({}^\forall f(w) \in \mathbb{C}[w] \right), \\
W\left(D^s P_a \right) |\lambda\rangle &= \Delta_a^{(s)} |\lambda\rangle \qquad (s \geq 0\,;\ a = 0,1)
\end{aligned}
\tag{4.1}
$$

for some functions $\Delta_a^{(s)}$ of λ.

It is convenient to introduce the generating functions $\Delta_a(x)$ of highest weights $\Delta_a^{(s)}$ $(a=0,1)$: $\Delta_a(x) \equiv -\sum_{s=0}^{\infty} \Delta_a^{(s)} x^s / s!$. Note that they are formally given as the eigenvalues of the operators $-W\left(e^{xD} P_a \right)$:

$$
W\left(e^{xD} P_a \right) |\lambda\rangle = -\Delta_a(x) |\lambda\rangle \qquad (a = 0,1).
$$

4.2. Let $U(\lambda)$ be a subspace of $V(\lambda)$ which is obtained from the highest weight state $|\lambda\rangle$ by acting on it $\mathcal{SW}_{1+\infty}$ once: $U(\lambda) = \mathcal{SW}_{1+\infty}|\lambda\rangle$. The principal gradation of $\mathcal{SW}_{1+\infty}$ naturally induces the labeling of $U(\lambda)$: $U(\lambda) = \bigoplus_{\alpha \geq 0} U_{-\alpha}(\lambda)$.

† $[X,Y] \equiv XY - YX$, $\{X,Y\} \equiv XY + YX$.

The $\mathcal{SW}_{1+\infty}$ module $V(\lambda)$ is called *quasifinite* if $U_{-\alpha}(\lambda)$ is finite dimensional for each $\alpha \geq 0$. This condition is equivalent to the statement that

$$U^A_{-k}(\lambda) \equiv \left\{ \, W(z^{-k} f(D) P_A) |\lambda\,\rangle \, \middle| \, f(w) \in \mathbb{C}[w] \, \right\}$$

is finite dimensional for each $k \geq 0$ and $A = 0, 1, \pm$. It is straightforward to see that the following subsets of $\mathbb{C}[w]$ are all ideals of $\mathbb{C}[w]$:

$$I^-_0 \equiv \{ \, f(w) \in \mathbb{C}[w] \, | \, W \left(\, f(D) P_- \, \right) |\lambda\,\rangle = 0 \, \} \, ,$$
$$I^A_{-k} \equiv \left\{ \, f(w) \in \mathbb{C}[w] \, | \, W \left(z^{-k} f(D) P_A \right) |\lambda\,\rangle = 0 \, \right\} \qquad (k \geq 1 \, ; \, A = 0, 1, \pm).$$

Thus, if $U^A_{-k}(\lambda)$ is finite dimensional, all of I^-_0 and I^A_{-k} are different from $\{0\}$, so that I^-_0 and I^A_{-k} are generated by some monic polynomials $a^-(w)$ and $b^A_k(w)$, respectively: $I^-_0 = (\, a^-(w)\,)$ and $I^A_{-k} = \left(b^A_k(w) \right)$. Conversely, if I^-_0 and I^A_{-k} are generated by monic polynomials, then $U^A_{-k}(\lambda)$ become finite dimensional since

$$\dim U^A_{-k}(\lambda) = \dim \mathbb{C}[w]/I^A_{-k} = \deg b^A_k(w) < \infty.$$

Thus, we have proved the following theorem:

Theorem. *The highest weight module $V(\lambda)$ of $\mathcal{SW}_{1+\infty}$ is quasifinite if and only if the subsets I^-_0 and I^A_{-k} of $\mathbb{C}[w]$ are generated by monic polynomials;*

$$I^-_0 = \left(a^-(w) \right) \qquad I^A_{-k} = \left(b^A_k(w) \right) \qquad (k \geq 1 \, ; \, A = 0, 1, \pm). \tag{4.2}$$

We will call $a^-(w)$, $b^A_k(w)$ the characteristic polynomials for the highest weight module $V(\lambda)$.

For later convenience, we introduce the symbol

$$B_k(w) \equiv b^A_k(w) P_A = \begin{bmatrix} b^0_k(w) & b^+_k(w) \\ b^-_k(w) & b^1_k(w) \end{bmatrix},$$

and further denote $b^A(w) \equiv b^A_1(w)$, $B(w) \equiv B_1(w)$. In the following discussions, we will see that $a^-(w)$ and $b^+(w)$ play the central role in the quasifinite representations of $\mathcal{SW}_{1+\infty}$.

4.3. Theorem. *Characteristic polynomials* $a^-(w)$, $b^A(w)$ $(A=0,1,\pm)$ *are related to each other in the following manner:*

$$\begin{aligned} &a^-(w) \;\mid\; b^0(w),\\ &a^-(w-1) \;\mid\; b^1(w),\\ &a^-(w),\;\; a^-(w-1) \;\mid\; b^-(w), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &b^+(w) \;\mid\; b^0(w),\;\; b^1(w),\\ &b^0(w) \;\mid\; a^-(w)b^+(w),\;\; b^-(w),\\ &b^1(w) \;\mid\; a^-(w-1)b^+(w),\;\; b^-(w),\\ &b^-(w) \;\mid\; a^-(w-1)b^0(w),\;\; a^-(w)b^1(w). \end{aligned} \tag{4.4}$$

Here $f_1(w),\cdots,f_r(w)\,|\,g_1(w),\cdots,g_s(w)$ *implies that any* $f_i(w)$ *divides all* $g_j(w)$*'s.*

Proof .[†] We start from the identity

$$\left[W\left(z\begin{bmatrix}\epsilon^0 & \epsilon^+\\ \epsilon^- & \epsilon^1\end{bmatrix}\right), W\left(z^{-1}\begin{bmatrix}\alpha b^0(D) & \beta b^+(D)\\ \gamma b^-(D) & \delta b^1(D)\end{bmatrix}\right)\right\}|\lambda\rangle = 0$$

which holds for arbitrary constants α, β, γ, δ and ϵ^A $(A=0,1,\pm)$. Suitably choosing these constants, we can derive the following equations:

$$\begin{aligned} &W\left(\begin{bmatrix}0 & 0\\ b^0(D) & 0\end{bmatrix}\right)|\lambda\rangle = 0,\\ &W\left(\begin{bmatrix}0 & 0\\ b^1(D+1) & 0\end{bmatrix}\right)|\lambda\rangle = 0,\\ &W\left(\begin{bmatrix}0 & 0\\ b^-(D) & 0\end{bmatrix}\right)|\lambda\rangle = W\left(\begin{bmatrix}0 & 0\\ b^-(D+1) & 0\end{bmatrix}\right)|\lambda\rangle = 0, \end{aligned}$$

which assert the first statement, eq. (4.3). The second statement, eq. (4.4), can be similarly proved, by using the identity

$$\left[W\left(\begin{bmatrix}0 & \epsilon^+\\ \epsilon^- a^-(D) & 0\end{bmatrix}\right), W\left(z^{-1}\begin{bmatrix}\alpha b^0(D) & \beta b^+(D)\\ \gamma b^-(D) & \delta b^1(D)\end{bmatrix}\right)\right\}|\lambda\rangle = 0,$$

and taking a suitable choice of the constants α, β, γ, δ and $\epsilon^\pm$. $\square$

[†] Another proof of Theorems 4.3 and 4.4 is given in Appendix A, resorting to the embedding of $\mathcal{SW}_{1+\infty}$ into $\widehat{\mathrm{gl}}(\infty|\infty)$.

4.4. Theorem. *Characteristic polynomials $b_k^A(w)$ for $k \geq 1$ are related to each other in the following manner:*

$$\begin{aligned} &b_k^+(w),\ \ b_k^0(w-1),\ \ b_k^1(w),\ \ b_k^+(w-1)\ \mid\ b_{k+1}^+(w),\\ &b_k^0(w),\ \ b_k^0(w-1),\ \ b_k^-(w),\ \ b_k^+(w-1)\ \mid\ b_{k+1}^0(w),\\ &b_k^1(w),\ \ b_k^1(w-1),\ \ b_k^+(w),\ \ b_k^-(w-1)\ \mid\ b_{k+1}^1(w),\\ &b_k^-(w),\ \ b_k^1(w-1),\ \ b_k^0(w),\ \ b_k^-(w-1)\ \mid\ b_{k+1}^-(w), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} b_{k+\ell}^+(w)\ &\mid\ b_k^0(w-\ell)b_\ell^+(w),\ \ b_k^+(w-\ell)b_\ell^1(w),\\ b_{k+\ell}^0(w)\ &\mid\ b_k^0(w-\ell)b_\ell^0(w),\ \ b_k^+(w-\ell)b_\ell^-(w),\\ b_{k+\ell}^1(w)\ &\mid\ b_k^-(w-\ell)b_\ell^+(w),\ \ b_k^1(w-\ell)b_\ell^1(w),\\ b_{k+\ell}^-(w)\ &\mid\ b_k^-(w-\ell)b_\ell^0(w),\ \ b_k^1(w-\ell)b_\ell^-(w). \end{aligned} \tag{4.6}$$

Proof . The first statement, eq. (4.5), is obtained by combining the following two identities:

$$\left[W\left(z\begin{bmatrix} \epsilon^0 & \epsilon^+ \\ \epsilon^- & \epsilon^1 \end{bmatrix} \right), W\left(z^{-k-1}\begin{bmatrix} \alpha b_{k+1}^0(D) & \beta b_{k+1}^+(D) \\ \gamma b_{k+1}^-(D) & \delta b_{k+1}^1(D) \end{bmatrix} \right) \right\} |\lambda\,\rangle = 0,$$
$$\left[W\left(z\begin{bmatrix} \epsilon^0 & \epsilon^+ \\ \epsilon^- & \epsilon^1 \end{bmatrix} \right), W\left(z^{-k-1}D\begin{bmatrix} \alpha b_{k+1}^0(D) & \beta b_{k+1}^+(D) \\ \gamma b_{k+1}^-(D) & \delta b_{k+1}^1(D) \end{bmatrix} \right) \right\} |\lambda\,\rangle = 0$$

which hold for arbitrary constants α, β, γ, δ and ϵ^A $(A = 0, 1, \pm)$. The second statement, eq. (4.6), is obtained by looking at the identities

$$\left[W\left(z^{-k}\begin{bmatrix} b_k^0(D) & b_k^+(D) \\ b_k^-(D) & b_k^1(D) \end{bmatrix} \right), W\left(z^{-\ell}\begin{bmatrix} \alpha b_\ell^0(D) & \beta b_\ell^+(D) \\ \gamma b_\ell^-(D) & \delta b_\ell^1(D) \end{bmatrix} \right) \right\} |\lambda\,\rangle = 0,$$
$$\left[W\left(z^{-k}\begin{bmatrix} b_k^0(D) & b_k^+(D) \\ b_k^-(D) & b_k^1(D) \end{bmatrix} \right), W\left(z^{-\ell}D\begin{bmatrix} \alpha b_\ell^0(D) & \beta b_\ell^+(D) \\ \gamma b_\ell^-(D) & \delta b_\ell^1(D) \end{bmatrix} \right) \right\} |\lambda\,\rangle = 0.$$

□

Note that if we set $b_0^-(w) \equiv a^-(w)$, $b_0^+(w) \equiv 1$ and $b_0^0(w) \equiv b_0^1(w) \equiv 0$, then the Theorem 4.4 reduces to the Theorem 4.3 with some suitable choices of k and ℓ.

4.5. Iteratively using Theorems 4.3 and 4.4, we obtain the following Corollary:

Corollary. *Characteristic polynomials* $b_k^A(w)$ *for* $k \geq 1$ *are related to the polynomials* $a^-(w)$ *and* $b^+(w)$ *as*

$$\ell.c.m.\left(b^+(w), a^-(w-1), b^+(w-1), a^-(w-2), \cdots, b^+(w-k+1) \right) \mid b_k^+(w), \qquad (4.7a)$$

$$\ell.c.m.\left(a^-(w), b^+(w), a^-(w-1), b^+(w-1), \cdots, b^+(w-k+1) \right) \mid b_k^0(w), \qquad (4.7b)$$

$$\ell.c.m.\left(b^+(w), a^-(w-1), b^+(w-1), a^-(w-2), \cdots, a^-(w-k) \right) \mid b_k^1(w), \qquad (4.7c)$$

$$\ell.c.m.\left(a^-(w), b^+(w), a^-(w-1), b^+(w-1), \cdots, a^-(w-k) \right) \mid b_k^-(w), \qquad (4.7d)$$

and

$$b_k^+(w) \mid b^+(w)\, a^-(w-1)\, b^+(w-1)\, a^-(w-2) \cdots b^+(w-k+1), \qquad (4.8a)$$

$$b_k^0(w) \mid a^-(w)\, b^+(w)\, a^-(w-1)\, b^+(w-1) \cdots b^+(w-k+1), \qquad (4.8b)$$

$$b_k^1(w) \mid b^+(w)\, a^-(w-1)\, b^+(w-1)\, a^-(w-2) \cdots a^-(w-k), \qquad (4.8c)$$

$$b_k^-(w) \mid a^-(w)\, b^+(w)\, a^-(w-1)\, b^+(w-1) \cdots a^-(w-k). \qquad (4.8d)$$

Let $a^-(w) = \prod_{i=1}^{N_-}(w-\lambda_i^-)$ and $b^+(w) = \prod_{j=1}^{N_+}(w-\lambda_j^+)$. If difference of any two distinct elements of the set $\{\lambda_i^-\} \cup \{\lambda_j^+\}$ is not an integer, then $a^-(w)$, $b^+(w)$, $a^-(w-1)$, $b^+(w-1)$, $\cdots$ are all mutually prime. In this case, the characteristic plynomials $b_k^A(w)$ ($k \geq 1$, $A = 0, 1, \pm$) are uniquely determined due to the above corollary as follows:†

$$\begin{aligned} B_k(w) &= \left(\prod_{\ell=0}^{k-1} b^+(w-\ell) \right) \begin{bmatrix} \prod_{\ell=0}^{k-1} a^-(w-\ell) & \prod_{\ell=1}^{k-1} a^-(w-\ell) \\ \prod_{\ell=0}^{k} a^-(w-\ell) & \prod_{\ell=1}^{k} a^-(w-\ell) \end{bmatrix} \\ &= \left(\prod_{\ell=1}^{k-1} a^-(w-\ell) b^+(w-\ell) \right) \begin{bmatrix} a^-(w)b^+(w) & b^+(w) \\ a^-(w)a^-(w-k)b^+(w) & a^-(w-k)b^+(w) \end{bmatrix} \\ &= \frac{1}{2^{k-1}} B(w-k+1) \cdots B(w-1) B(w). \end{aligned} \qquad (4.9)$$

5. Differential Equations for Highest Weights

5.1. The structure of characteristic polynomials automatically determines that of highest weights. In the following subsections, we derive the differential equations for $\Delta_a(x)$ $(a = 0, 1)$. Recall that $W\left(e^{xD} P_a \right) |\lambda\rangle = -\Delta_a(x) |\lambda\rangle$.

† This equation is derived in a simpler way in Appendix.

5.2. We first note that for arbitrary functions $f(w) \in \mathbb{C}[w]$, the following equation holds:

$$\left\{ W\left(\begin{bmatrix} 0 & f(D) \\ 0 & 0 \end{bmatrix} \right), W\left(\begin{bmatrix} 0 & 0 \\ a^-(D) & 0 \end{bmatrix} \right) \right\} |\lambda\rangle = 0.$$

The left–hand side can be rewritten as $W\,(\,f(D)a^-(D)(P_0+P_1)\,)\,|\lambda\rangle$, and thus, by setting $f(D) = \exp(xD)$, we obtain

$$a^-\left(\frac{d}{dx}\right)[\,\Delta_0(x) + \Delta_1(x)\,] = 0. \tag{5.1}$$

5.3. We then use the identity $\left[\,W\,(\,zG(D+1)\,)\,, W\left(\,z^{-1}B(D)\,\right)\,\right]|\lambda\rangle = 0$ which holds for arbitrary element $G(w) = g^A(w)P_A \in \mathcal{A}$. If we set $g^A(D) = \alpha^A \exp(xD)$ and pick up the coefficient of α^A $(A = 0, 1, \pm)$, we obtain the following set of differential equations:

$$b^+\left(\frac{d}{dx}\right)[\,e^x\Delta_0(x) + \Delta_1(x) - C\,] = 0, \tag{5.2a}$$

$$b^0\left(\frac{d}{dx}\right)[\,(\,1-e^x\,)\,\Delta_0(x) + C\,] = 0, \tag{5.2b}$$

$$b^1\left(\frac{d}{dx}\right)[\,(\,1-e^x\,)\,\Delta_1(x) - C\,] = 0, \tag{5.2c}$$

$$b^-\left(\frac{d}{dx}\right)[\,\Delta_0(x) + e^x\Delta_1(x) + C\,] = 0. \tag{5.2d}$$

Surprisingly, all of these four equations reduce to the first one if we use eq. (5.1). To prove this, we first notice that eq. (5.1) can be rewritten as

$$a^-\left(\frac{d}{dx} - 1\right)[\,e^x\Delta_0(x) + e^x\Delta_1(x)\,] = 0. \tag{5.3}$$

Since eq. (4.4) implies that $b^0(w)$, $b^1(w)$ and $b^-(w)$ are all divided by $b^+(w)$, we can replace $b^+\,(\,d/dx\,)$ in eq. (5.2a) by $b^A\,(\,d/dx\,)$ $(A = 0, 1, \pm)$:

$$b^A\left(\frac{d}{dx}\right)[\,e^x\Delta_0(x) + \Delta_1(x) - C\,] = 0 \qquad (A = 0, 1, \pm).$$

As for $A = 0$, $\Delta_1(x)$ can be replaced by $-\Delta_0(x)$, since $b^0(w)$ is divided by $a^-(w)$. As for $A = 1$, $e^x\Delta_0(x)$ can be replaced by $-e^x\Delta_1(x)$, since $b^1(w)$ is divided by $a^-(w-1)$ and so we can use eq. (5.3). Finally as for $A = -$, $e^x\Delta_0(x)$ and $\Delta_1(x)$ can be replaced by $-e^x\Delta_1(x)$ and $-\Delta_0(x)$, respectively, since $b^-(w)$ is divided by both of $a^-(w-1)$ and $a^-(w)$.

We summarize the results obtained above in the following theorem:

Theorem. *The generating functions $\Delta_a(x)$ $(a = 0,1)$ of highest weights satisfy the following differential equations:*

$$\begin{aligned} a^-\left(\frac{d}{dx}\right)[\,\Delta_0(x) + \Delta_1(x)\,] &= 0, \\ b^+\left(\frac{d}{dx}\right)[\,e^x\Delta_0(x) + \Delta_1(x) - C\,] &= 0. \end{aligned} \tag{5.4}$$

5.4. We assume that polynomials $a^-(w)$, $b^+(w)$ have the following form:

$$a^-(w) = \prod_{i=1}^{M}(w-\mu_i)^{m_i}, \qquad b^+(w) = \prod_{j=1}^{N}(w-\nu_j)^{n_j}, \tag{5.5}$$

where $\mu_i \neq \mu_{i'}$ if $i \neq i'$, and $\nu_j \neq \nu_{j'}$ if $j \neq j'$. Then the differential equations (5.4) may be solved as

$$\begin{aligned} \Delta_0(x) + \Delta_1(x) &= \sum_{i=1}^{M} p_i(x)e^{\mu_i x}, \\ e^x\Delta_0(x) + \Delta_1(x) - C &= -\sum_{j=1}^{N} q_j(x)e^{\nu_j x}. \end{aligned}$$

Here $p_i(x)$ and $q_j(x)$ are, respectively, degree $m_i - 1$ and $n_j - 1$ polynomials of x. Since these equations can be rewritten as

$$\begin{aligned} \Delta_0(x) &= -\frac{\sum_{i=1}^{M} p_i(x)e^{\mu_i x} + \sum_{j=1}^{N} q_j(x)e^{\nu_j x} - C}{e^x - 1}, \\ \Delta_1(x) &= +\frac{\sum_{i=1}^{M} p_i(x)e^{(\mu_i+1)x} + \sum_{j=1}^{N} q_j(x)e^{\nu_j x} - C}{e^x - 1}, \end{aligned} \tag{5.6}$$

we obtain four typical representations,

$$\Delta_0(x) = -C\,\frac{e^{\lambda x}-1}{e^x-1}, \qquad \Delta_1(x) = +C\,\frac{e^{\lambda x}-1}{e^x-1}, \tag{5.7a}$$

$$\Delta_0(x) = -\frac{q_k x^k e^{\lambda x}}{e^x-1}, \qquad \Delta_1(x) = +\frac{q_k x^k e^{\lambda x}}{e^x-1}, \tag{5.7b}$$

$$\Delta_0(x) = -C\,\frac{e^{\lambda x}-1}{e^x-1}, \qquad \Delta_1(x) = +C\,\frac{e^{(\lambda+1)x}-1}{e^x-1}, \tag{5.7c}$$

$$\Delta_0(x) = -\frac{p_k x^k e^{\lambda x}}{e^x-1}, \qquad \Delta_1(x) = +\frac{p_k x^k e^{(\lambda+1)x}}{e^x-1}, \tag{5.7d}$$

with $k \in \mathbb{Z}_{>0}$. Here eq. (5.7a) corresponds to the case where $a^-(w) = 1$ and $b^+(w) = w - \lambda$, and eq. (5.7c) to the case where $a^-(w) = w - \lambda$ and $b^+(w) = 1$. Eq. (5.7b) corresponds to the special case where $a^-(w) = 1$, $b^+(w) = (w-\lambda)^{k+1}$ and $C = 0$, while eq. (5.7d) to the special case where $a^-(w) = (w-\lambda)^{k+1}$, $b^+(w) = 1$ and $C = 0$ [M]. First two solutions describe the system having no degeneracy in the vacuum ($a^-(w) = 1$). On the other hand, in the last two representations, we have several states at level 0.

6. Spectral Flow

6.1. Since $\mathcal{SW}_{1+\infty}$ contains two $u(1)$ Kac-Moody algebras as subalgebras, $\mathcal{SW}_{1+\infty}$ has a two–parameter family of automorphisms which we will call the *spectral flow.*

Theorem. *There exist the following automorphisms $W(\cdot) \mapsto W'(\cdot)$:*

$$\begin{aligned} W'(z^n e^{xD} P_a) &= W(z^n e^{x(D+\lambda^a)} P_a) \pm C \frac{e^{\lambda^a x} - 1}{e^x - 1} \delta_{n0}, \quad && a = \{^0_1, \\ W'(z^n e^{xD} P_\pm) &= W(z^{n \pm (\lambda^1 - \lambda^0)} e^{x(D+\lambda^a)} P_\pm), && a = \{^1_0, \end{aligned} \tag{6.1}$$

with arbitrary parameters λ^a $(a = 0, 1)$.

Proof. One can easily show that this new generators $W'(\cdot)$ satisfy the same (anti–) commutation relations as those for the original ones $W(\cdot)$, eq. (3.3). □

6.2. Under the spectral flow, the highest weight state may change although the representation space as a set is kept invariant. We illustrate this phenomena by taking the $N = 2$ superconformal algebra [SS] as an example (see figure 1).

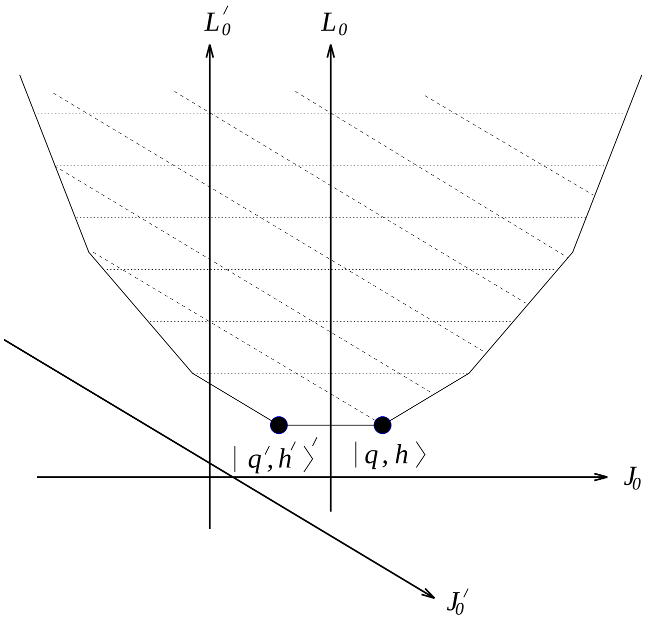

Figure 1. Spectral Flow for the $N = 2$ Superconformal Algebra

The generators of the $N = 2$ superconformal algebra consist of J_n ($U(1)$–current), L_n (energy–momentum tensor) and $G_r^{\pm}$ (supercurrents), and satisfy the following (anti–) commutation relation:

$$
\begin{aligned}
&[\,L_n, L_m\,] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0},\\
&\big[\,L_n, G_r^{\pm}\,\big] = \Big(\frac{n}{2}-r\Big)\,G_{n+r}^{\pm},\\
&\big\{\,G_r^{+}, G_s^{-}\,\big\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0},\\
&\big\{\,G_r^{\pm}, G_s^{\pm}\,\big\} = 0, \qquad\qquad [\,L_n, J_m\,] = -mJ_{n+m},\\
&[\,J_n, J_m\,] = \frac{c}{3}n\delta_{n+m,0}, \qquad \big[\,J_n, G_r^{\pm}\,\big] = \pm G_{n+r}^{\pm}.
\end{aligned}
$$

Here $n \in \mathbb{Z}$, and $r \in \mathbb{Z} + 1/2$ for Neveu–Schwarz (NS) sector or $r \in \mathbb{Z}$ for Ramond (R) sector. The highest weight state $|q, h\,\rangle$ is characterized by

$$
\begin{aligned}
J_n, L_n, G_r^{+}, G_s^{-}|q,h\,\rangle &= 0 \quad (n > 0, r \geq 0, s > 0),\\
J_0|q,h\,\rangle &= q|q,h\,\rangle,\\
L_0|q,h\,\rangle &= h|q,h\,\rangle.
\end{aligned}
$$

This algebra is invariant under the following transformation with arbitrary parameter λ:

$$\begin{aligned} J'_n &= J_n + \frac{c}{3}\lambda\delta_{n0}, \\ L'_n &= L_n + \lambda J_n + \frac{c}{6}\lambda^2\delta_{n0}, \\ G'^{\pm}_r &= G^{\pm}_{r\pm\lambda}. \end{aligned}$$

When λ is an integer (or half–odd integer), the spectral flow maps NS sector to NS sector and R sector to R sector (or NS to R, R to NS).

We first consider the case of R to R with $\lambda = 1$. For $h \neq c/24$, $|q, h\,\rangle$ is no longer the highest weight state with respect to the new generators, because $G'^{-}_1|h, q\,\rangle = G^{-}_0|q, h\,\rangle$ does not vanish. However, the new state $G^{-}_0|q, h\,\rangle$ satisfies the highest weight condition, and may be identified with the new highest weight state $|q', h'\,\rangle'$. Here, the new $u(1)$ charge q' and the new conformal weight h' are

$$q' = q - 1 + c/3 \qquad h' = h + q - 1 + c/6,$$

because $J'_0|q', h'\,\rangle' = (J_0 + c/3)G^{-}_0|q, h\,\rangle = (q - 1 + c/3)G^{-}_0|q, h\,\rangle$ and $L'_0|q', h'\,\rangle' = (L_0 + J_0 + c/6)G^{-}_0|q, h\,\rangle = (h + q - 1 + c/6)G^{-}_0|q, h\,\rangle$. For $h = c/24$, the new highest weight state is given by $|q', h'\,\rangle' = |q, h\,\rangle$ with $q' = q + c/3$ and $h' = h + q + c/6$.

Similarly, in the case of R to R with $\lambda = -1$, the new highest weight state for $h - q + c/8 \neq 0$ is given by $|q', h'\,\rangle' = G^{+}_{-1}|q, h\,\rangle$ with $q' = q + 1 - c/3$ and $h' = h - q + c/6$.

6.3. Let us go back to $\mathcal{SW}_{1+\infty}$. We would like to derive the modification of the weights and the characteristic polynomials under the spectral flow. We restrict ourselves to the case $\lambda^1 - \lambda^0 \in \mathbb{Z}$. Thus, it is sufficient to consider three cases $\lambda^1 - \lambda^0 = 0, \pm 1$, because, for example, the flow with $\lambda^1 - \lambda^0 = 2$ is obtained by taking twice the flow with $\lambda^1 - \lambda^0 = 1$. We have the following Theorem:

Theorem. *Under the spectral flow, the new weights, $\Delta'_a(x)$, and the new characteristic polynomials, $a'^{-}(w)$ and $b'^{+}(w)$, are given as follows, for generic values of C and $\Delta_a(x)$:*

(i) *If $\lambda^1 - \lambda^0 = 0$, then*

$$\Delta'_a(x) = e^{\lambda^a x}\Delta_a(x) \mp C\frac{e^{\lambda^a x} - 1}{e^x - 1}, \quad a = \{^0_1, \tag{6.2}$$

and $a'^{-}(w) = a^{-}(w - \lambda^0)$, $b'^{+}(w) = b^{+}(w - \lambda^1)$.

(ii) *If* $\lambda^1 - \lambda^0 = 1$, *then*

$$\Delta'_a(x) = e^{\lambda^a x}\Delta_a(x) \mp C\frac{e^{\lambda^a x}-1}{e^x - 1} \pm \sum_{i=1}^{N_-} e^{(\lambda_i^- + \lambda^a)x}, \quad a = \{{}^0_1, \tag{6.3}$$

and $a'^-(w) = b^-(w - \lambda^0)$, $b'^+(w) = a^-(w - \lambda^1)$.

(iii) *If* $\lambda^1 - \lambda^0 = -1$, *then*

$$\Delta'_a(x) = e^{\lambda^a x}\Delta_a(x) \mp C\frac{e^{\lambda^a x}-1}{e^x - 1} \mp \sum_{j=1}^{N_+} e^{(\lambda_j^+ + \lambda^1)x}, \quad a = \{{}^0_1, \tag{6.4}$$

and $a'^-(w) = b^+(w - \lambda^0 + 1)$, $b'^+(w) = b_2^+(w - \lambda^1)$.

Eq. (6.2) is identical with the formula in the bosonic case [AFMO1].

Proof .

(i) $\lambda^1 - \lambda^0 = 0$.

The highest weight state $|\lambda\rangle$ with respect to the original generators W is also the highest weight state $|\lambda'\rangle'$ with respect to the new ones W', $|\lambda'\rangle' = |\lambda\rangle$. Hence, the new weights, $\Delta'_a(x)$, are given by eq. (6.2). Since $|\lambda'\rangle' = |\lambda\rangle$, we also have the following equation:

$$W'(a^-(D-\lambda^0)P_-)|\lambda'\rangle' = W(a^-(D)P_-)|\lambda'\rangle' = 0,$$
$$W'(z^{-1}b^+(D-\lambda^1)P_+)|\lambda'\rangle' = W(z^{-1}b^+(D)P_+)|\lambda'\rangle' = 0.$$

Therefore, the new characteristic polynomials are given by $a'^-(w) = a^-(w - \lambda^0)$ and $b'^+(w) = b^+(w - \lambda^1)$.

(ii) $\lambda^1 - \lambda^0 = 1$.

In this case $|\lambda\rangle$ is not the highest weight state with respect to the new generators W'. In generic situations[†], the new highest weight state is given by

$$|\lambda'\rangle' = \prod_{k=0}^{N_- -1} W(D^k P_-)|\lambda\rangle, \tag{6.5}$$

if $a^-(w) = \prod_{i=1}^{N_-}(w - \lambda_i^-)$. To prove it, we first remark that $W'(z^{n+1}f(D)P_0)$, $W'(z^{n+1}f(D)P_1)$, $W'(z^n f(D)P_+)$ and $W'(z^{n+2}f(D)P_-)$ with $n \geq 0$ annihilate $|\lambda'\rangle'$

[†] When eq. (6.5) is a null state, we must replace the upper bound of the product by a smaller number.

in a trivial way. On the other hand, $W'(zf(D)P_-)$ annihilates $|\lambda'\rangle'$ since the state $W'(zf(D)P_-)|\lambda'\rangle'$ can be rewritten in the following form:

$$\begin{aligned} W'(zf(D)P_-)|\lambda'\rangle' &= (-1)^{N_-}\prod_{k=0}^{N_--1} W(D^kP_-)\cdot W(f(D+\lambda^0)P_-)|\lambda\rangle \\ &= (-1)^{N_-}\prod_{k=0}^{N_--1} W(D^kP_-)\cdot \sum_{\ell=0}^{N_--1} c_\ell W(D^\ell P_-)|\lambda\rangle, \end{aligned}$$

where c_ℓ are some constants. In this expression, we have replaced $f(D+\lambda^0)$ by a polynomial with degree less than N_-, making use of the quasifinite condition $W(a^-(D)P_-)|\lambda\rangle = 0$. This state vanishes because $W(D^kP_-)^2 = 0$.

The weights of this new highest weight state $|\lambda'\rangle'$ are calculated as follows. First, we note the following equation:

$$\begin{aligned} &-W'(e^{xD}P_a)|\lambda'\rangle' \\ &= \prod_{k=0}^{N_--1} W(D^kP_-)\cdot\left(-W(e^{(D+\lambda^a)x}P_a)\mp C\frac{e^{\lambda^a x}-1}{e^x-1}\right)|\lambda\rangle \\ &\quad \pm\sum_{k=0}^{N_--1}\prod_{k_2=k+1}^{N_--1} W(D^{k_2}P_-)\cdot W(e^{x(D+\lambda^a)}D^kP_-)\cdot\prod_{k_1=0}^{k-1} W(D^{k_1}P_-)|\lambda\rangle \\ &= \left(e^{\lambda^a x}\Delta_a(x)\mp C\frac{e^{\lambda^a x}-1}{e^x-1}\right)|\lambda'\rangle' \\ &\quad \pm\sum_{k=0}^{N_--1}\prod_{k_2=k+1}^{N_--1} W(D^{k_2}P_-)\cdot e^{\lambda^a x}W(r_k^-(D,x)P_-)\cdot\prod_{k_1=0}^{k-1} W(D^{k_1}P_-)|\lambda\rangle. \end{aligned}$$

Here we first moved $W(e^{x(D+\lambda^A)}D^kP_-)$ to the right, and then, after reducing the degree in D using the quasifinite condition $W(a^-(D)P_-)|\lambda\rangle = 0$, we substituted it into the original position. The function $r_k^-(D,x) = \sum_{\ell=0}^{N_--1} r_{k,\ell}^-(x)D^\ell$ is defined as a remainder of $e^{xD}D^k$ by $a^-(D)$: $e^{xD}D^k = a^-(D)q_k^-(D,x) + r_k^-(D,x)$. Note that only D^k term in $r_k^-(D,x)$ contributes because $W(D^\ell P_-)^2 = 0$. Furthermore, we can also show that $r_{k,k}^-(x) = (\frac{d}{dx})^k r_{0,k}^-(x)$ and $\sum_{k=0}^{N_--1}(\frac{d}{dx})^k r_{0,k}^-(x) = \sum_{i=1}^{N_-} e^{\lambda_i^- x}$. We thus obtain eq. (6.3).

Moreover, one may show that $W(z^{-1}b^-(D)P_-)|\lambda'\rangle' = 0$ and $W(a^-(D)P_+)|\lambda'\rangle' = 0$ if and only if $W(z^{-1}b^-(D)P_-)|\lambda\rangle = 0$ and $a^-\left(\frac{d}{dx}\right)[\Delta_0(x)+\Delta_1(x)] = 0$, respectivly.

This can be proved as follows: First, since $\{\, W(z^m f(D)P_-), W(z^n g(D)P_-)\,\} = 0$,

$$W(z^{-1}b^-(D)P_-)|\lambda'\rangle' = (-1)^{N_-}\prod_{k=0}^{N_--1} W(D^k P_-)W(z^{-1}b^-(D)P_-)|\lambda\rangle.$$

Second, since $[\, W\,(\, f(D)(P_0+P_1)\,)\,, W(g(D)P_-)\,] = 0$,

$$\begin{aligned} W(a^-(D)P_+)|\lambda'\rangle' &= \sum_{\ell=0}^{N_--1} (-1)^{N_--1-\ell} \prod_{\substack{k=0\\k\neq\ell}}^{N_--1} W(D^k P_-)\cdot W\left(\, a^-(D)D^\ell(P_0+P_1)\,\right)|\lambda\rangle \\ &= \sum_{\ell=0}^{N_--1} (-1)^{N_--\ell} \prod_{\substack{k=0\\k\neq\ell}}^{N_--1} W(D^k P_-)|\lambda\rangle\, a^-\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)^\ell [\,\Delta_0(x)+\Delta_1(x)\,]\Big|_{x=0}. \end{aligned}$$

Hence,

$$W'(b^-(D-\lambda^0)P_-)|\lambda'\rangle' = W(z^{-1}b^-(D)P_-)|\lambda'\rangle' = 0,$$

$$W'(z^{-1}a^-(D-\lambda^1)P_+)|\lambda'\rangle' = W(a^-(D)P_+)|\lambda'\rangle' = 0.$$

Therefore, the new characteristic polynomials are given by $a'^-(w) = b^-(w-\lambda^0)$ and $b'^+(w) = a^-(w-\lambda^1)$.

(iii) $\lambda^1 - \lambda^0 = -1$.

Similarly to the case (ii), in generic situations the new highest weight state is given by

$$|\lambda'\rangle' = \prod_{k=0}^{N_+-1} W(z^{-1}D^k P_+)|\lambda\rangle, \tag{6.6}$$

if $b^+(w) = \prod_{j=1}^{N_+}(w-\lambda_j^+)$. The weights of this new state are also similarly calculated.

Moreover, one may show that $W(zb^+(D+1)P_-)|\lambda'\rangle' = 0$ and $W(z^{-2}b_2^+(D)P_+)|\lambda'\rangle' = 0$ if and only if $b^+\left(\frac{d}{dx}\right)[\,e^x\Delta_0(x)+\Delta_1(x)-C\,] = 0$ and $W(z^{-2}b_2^+(D)P_+)|\lambda\rangle = 0$, respectivly. This can be proved by using the facts that $\{\, W(z^m f(D)P_+), W(z^n g(D)P_+)\,\} = 0$ and $\left[\, W\,(\, f(D)P_1 + f(D+1)P_0\,)\,, W(z^{-1}g(D)P_+)\,\right] = 0$. Hence,

$$W'(b^+(D-\lambda^1)P_-)|\lambda'\rangle' = W(zb^+(D+1)P_-)|\lambda'\rangle' = 0,$$

$$W'(z^{-1}b_2^+(D-\lambda^1)P_+)|\lambda'\rangle' = W(z^{-2}b_2^+(D)P_+)|\lambda'\rangle' = 0.$$

Therefore, the new characteristic polynomials are given by $a'^-(w) = b^+(w-\lambda^1)$ and $b'^+(w) = b_2^+(w-\lambda^1)$.

This completes the proof of Theorem 6.3. □

6.4. The figure 2 illustrates the $W(P_\pm)$ part of the $\mathcal{SW}_{1+\infty}$ module: (i), (ii) and (iii) correspond to the cases $\lambda^1 - \lambda^0 = 0$, 1 and -1, respectively.

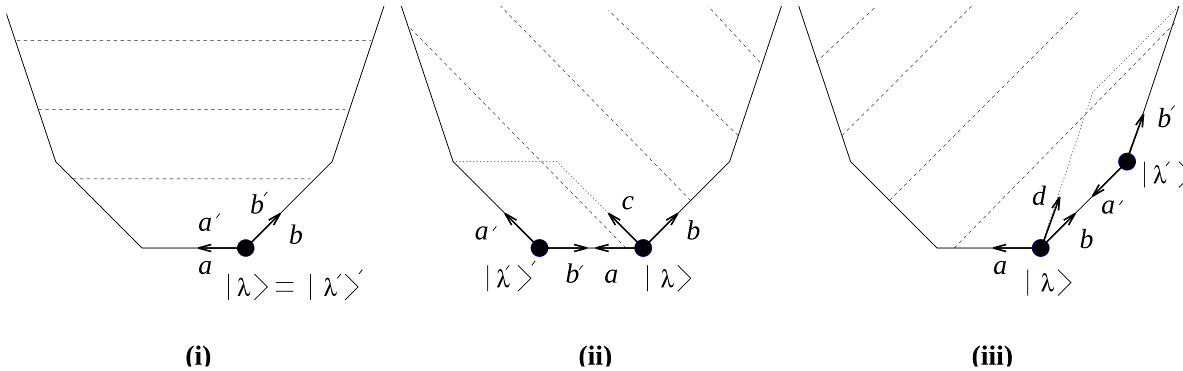

Figure 2. Spectral Flow for the $\mathcal{SW}_{1+\infty}$ Algebra

The arrow a corresponds to the generator $W(D^kP_-)$, b to $W(z^{-1}D^kP_+)$, c to $W(z^{-1}D^kP_-)$, d to $W(z^{-2}D^kP_+)$, a' to $W'(D^kP_-)$, and b' to $W'(z^{-1}D^kP_+)$. We can easily understand that if $\lambda^1 - \lambda^0 = 0$, then

$$W'(a'^-(w)P_-) = W(a^-(w)P_-), \qquad W'(z^{-1}b'^+(w)P_+) = W(z^{-1}b^+(w)P_+);$$

if $\lambda^1 - \lambda^0 = 1$, then

$$W'(a'^-(w)P_-) = W(z^{-1}b^-(w)P_-), \qquad W'(z^{-1}b'^+(w)P_+) = W(a^-(w)P_+);$$

if $\lambda^1 - \lambda^0 = -1$, then

$$W'(a'^-(w)P_-) = W(zb^+(w+1)P_-), \qquad W'(z^{-1}b'^+(w)P_+) = W(z^{-2}b_2^+(w)P_+).$$

7. Example: the (B, C)–system

7.1. In this section, we give the free-field realization of $\mathcal{SW}_{1+\infty}$ by using the (B, C)–system. Here the superfields $B(z, \theta) = \beta(z) + \theta b(w)$ and $C(z, \theta) = c(z) + \theta\gamma(w)$ are defined by the following OPE:

$$\gamma(z)\beta(0) \sim -\beta(z)\gamma(0) \sim \frac{1}{z}, \qquad c(z)b(0) \sim b(z)c(0) \sim \frac{1}{z}, \tag{7.1}$$

and the conformal weights of $(\beta,\ \gamma,\ b,\ c)$ are assigned as $(\lambda{+}1,\ -\lambda,\ \mu+1,\ -\mu)$ with $\lambda,\mu\in\mathbb{C}$. Conformal dimension of θ is thus $\lambda-\mu-1/2$.

7.2. For explicit calculation, it may be useful to "bosonize" the (B,C)–system as follows [FMS]. First we introduce free bosons $\phi(x)$, $\sigma(x)$ and free fermions $\xi(z)$, $\eta(z)$ with the following OPE:

$$\begin{aligned}&\phi(z)\phi(0)\sim+\log z, \qquad \sigma(z)\sigma(0)\sim-\log z,\\ &\eta(z)\xi(0)\sim\xi(z)\eta(0)\sim\frac{1}{z}.\end{aligned}$$

Then $\beta(z)$, $\gamma(z)$, $b(z)$ and $c(z)$ are expressed by $\phi(x)$, $\sigma(x)$, $\xi(z)$ and $\eta(z)$ as

$$\begin{aligned}\beta(z)&\equiv :e^{-\sigma(z)}:\partial\xi(z)=\sum_{n\in\mathbb{Z}}\beta_n z^{-n-\lambda-1},\\ \gamma(z)&\equiv :e^{\sigma(z)}:\eta(z)=\sum_{n\in\mathbb{Z}}\gamma_n z^{-n+\lambda},\\ b(z)&\equiv :e^{\phi(z)}:=\sum_{n\in\mathbb{Z}}b_n z^{-n-\mu-1},\\ c(z)&\equiv :e^{-\phi(z)}:=\sum_{n\in\mathbb{Z}}c_n z^{-n+\mu}.\end{aligned}$$

It is easy to show that the OPE (7.1) is actually reproduced in this representation.

Let the mode expansions of $\sigma(z)$ and $\phi(z)$ be as follows:

$$\begin{aligned}\sigma(z)&=-\sum_{n\in\mathbb{Z}_{\neq 0}}\frac{\alpha_n}{n}z^{-n}+\alpha_0\log z+\sigma_0,\\ \phi(z)&=-\sum_{n\in\mathbb{Z}_{\neq 0}}\frac{a_n}{n}z^{-n}+a_0\log z+\phi_0.\end{aligned}$$

Introducing the bosonic vacuum $|0\rangle$ satisfying $\alpha_n|0\rangle=a_n|0\rangle=0$ for $n\geq 0$, we define the (λ,μ)–vacuum $|\lambda,\mu\rangle$ by

$$|\lambda,\mu\rangle\equiv :e^{-\lambda\sigma(0)-\mu\phi(0)}:|0\rangle=e^{-\lambda\sigma_0-\mu\phi_0}|0\rangle. \tag{7.2}$$

Note that it satisfies the following equation:

$$\beta_n|\lambda,\mu\rangle=\gamma_{n+1}|\lambda,\mu\rangle=b_n|\lambda,\mu\rangle=c_{n+1}|\lambda,\mu\rangle=0 \qquad (n\geq 0). \tag{7.3}$$

7.3. Here we discuss the free–field realization of the fundamental representations of $\mathcal{SW}_{1+\infty}$, eqs. (5.7a) and (5.7c): Recall that eq. (5.7a) corresponds to the case where $a^-(w)=1$ and $b^+(w)=w-\lambda$, while eq. (5.7c) to the case where $a^-(w)=w-\lambda$ and $b^+(w)=1$.

Using the correspondence in sect. 3.6, we define the representation of $\mathcal{SW}_{1+\infty}$ over the (B,C)–system by

$$\begin{aligned} W(z^nF(D)) &\equiv \oint \frac{dzd\theta}{2\pi i} : B(z,\theta) z^n f^A(D) P_A C(z,\theta) : \\ &= \oint \frac{dz}{2\pi i} : \beta(z) z^n f^0(D)\gamma(z) : + \oint \frac{dz}{2\pi i}\beta(z) z^n f^+(D) c(z) \\ &\quad + \oint \frac{dz}{2\pi i} b(z) z^n f^-(D)\gamma(z) + \oint \frac{dz}{2\pi i} : b(z) z^n f^1(D) c(z) : . \end{aligned} \tag{7.4}$$

Explicit calculation shows that the central charge $C=1$, and $W(z^nF(D))$ has the following mode expansions:

$$\begin{aligned} W(z^nF(D)) =& \sum_{\ell\in\mathbb{Z}} f^0(\lambda+\ell) : \beta_{n+\ell}\gamma_{-\ell} : + \sum_{\ell\in\mathbb{Z}} f^+(\mu+\ell)\beta_{n+\ell-\lambda+\mu}c_{-\ell} \\ &+ \sum_{\ell\in\mathbb{Z}} f^-(\lambda+\ell) b_{n+\ell+\lambda-\mu}\gamma_{-\ell} + \sum_{\ell\in\mathbb{Z}} f^1(\mu+\ell) : b_{n+\ell}c_{-\ell} : . \end{aligned} \tag{7.5}$$

Thus, using eqs. (7.3) and (7.5), we can prove that $|\lambda,\mu\rangle$ is a highest weight vector if $\mu-\lambda=0$ or 1:

$$\begin{aligned} &W(z^nF(D))|\lambda,\mu\rangle = 0 \qquad (n\geq 1\,;\ {}^\forall F(w)\in\mathcal{A}), \\ &W(f(D)P_+)|\lambda,\mu\rangle = 0 \qquad ({}^\forall f(w)\in\mathbb{C}[w]), \\ &W(f^0(D)P_0+f^1(D)P_1)|\lambda,\mu\rangle \in \mathbb{C}|\lambda,\mu\rangle \qquad ({}^\forall f^{0,1}(w)\in\mathbb{C}[w]). \end{aligned}$$

7.4. When $\mu-\lambda=0$, eq. (7.3) implies that $W(P_-)|\lambda,\lambda\rangle = \sum_\ell b_{-\ell}\gamma_\ell|\lambda,\lambda\rangle = 0$, and thus we know that in this representation $a^-(w)=1$. For characteristic polynomials $b_k^A(w)$ ($k\geq 1\,;\ A=0,1,\pm$), one can prove the following equation as in the free field realization of the $C=\pm 1$ $\mathcal{W}_{1+\infty}$ algebra [M]:

$$b_k^A(\lambda+\ell) = 0 \qquad (\ell=0,1,\cdots,k-1\,;\ A=0,1,\pm). \tag{7.6}$$

As for $A=-$, for example, eq. (7.6) is obtained from the following equation by setting $\ell=0,1,\cdots,k-1$, and picking up the coefficient of $b_{-(k-\ell)}|\lambda,\lambda\rangle$:

$$\begin{aligned}0&=\left[W\left(z^{-k}b_k^-(D)P_-\right),\beta_\ell\right]|\lambda,\lambda\rangle\\&=b_k^-(\lambda+\ell)\,b_{-(k-\ell)}|\lambda,\lambda\rangle.\end{aligned}$$

Here we have used the fact that $b_{-n}|\lambda,\lambda\rangle$ $(n\geq 1)$ does not vanish. Thus, solving eq. (7.6), we obtain the explicit form of characteristic polynomials:

$$b_k^A(w)=(w-\lambda)(w-\lambda-1)\cdots(w-\lambda-k+1)\qquad (A=0,1,\pm). \tag{7.7}$$

In particular, noticing that $b^+(w)$ $(=b_1^+(w))$ $=w-\lambda$, we can rewrite eq. (7.7) into the form $b_k^A(w)=\prod_{\ell=0}^{k-1}b^+(w-\ell)$, which is consistent with eq. (4.9) for $a^-(w)=1$.

When $\mu-\lambda=1$, we can similarly show that $a^-(w)=w-\lambda$ and $b^+(w)=1$.

7.5. The eigenvalue $\Delta_0(x)$ of the operator $-W\left(e^{xD}P_0\right)$ is calculated as follows [M]:

$$\begin{aligned}-W&\left(e^{xD}P_0\right)|\lambda,\mu\rangle\\&=-\oint\frac{dz}{2\pi i}:\beta(z)e^{xD}\gamma(z):|\lambda,\mu\rangle\\&=-\oint\frac{dz}{2\pi i}:\beta(z)\gamma(e^xz):|\lambda,\mu\rangle\\&=-\oint\frac{dz}{2\pi i}\left[:e^{-\sigma(z)}:\partial\xi(z):e^{\sigma(e^xz)}:\eta(e^xz)+\frac{1}{z-e^xz}\right]:e^{-\lambda\sigma(0)-\mu\phi(0)}:|0\rangle\\&=\frac{-1}{e^x-1}\oint\frac{dz}{2\pi i}\frac{1}{z}\left[:e^{-\sigma(z)+\sigma(e^xz)}:-1\right]:e^{-\lambda\sigma(0)-\mu\phi(0)}:|0\rangle\\&=-\frac{e^{\lambda x}-1}{e^x-1}|\lambda,\mu\rangle.\end{aligned}$$

Namely, we obtain

$$\Delta_0(x)=-\frac{e^{\lambda x}-1}{e^x-1}. \tag{7.8}$$

Similarly we can calculate $\Delta_1(x)$ as

$$\begin{aligned}-W\left(e^{xD}P_1\right)|\lambda,\mu\rangle&=-\oint\frac{dz}{2\pi i}:b(z)c(e^xz):|\lambda,\mu\rangle\\&=+\frac{e^{\mu x}-1}{e^x-1}|\lambda,\mu\rangle,\end{aligned}$$

i.e.,

$$\Delta_1(x) = + \frac{e^{\mu x} - 1}{e^x - 1}. \tag{7.9}$$

7.6. If $\mu - \lambda = 0$, then the generating functions $\Delta_a(x)$ $(a = 0, 1)$ given in eqs. (7.8) and (7.9) actually satisfy the differential equations in the previous section with $a^-(w) = 1$, $b^+(w) = w - \lambda$ and $C = 1$:

$$\Delta_0(x) + \Delta_1(x) = 0,$$
$$\left(\frac{d}{dx} - \lambda \right) [\, e^x \Delta_0(x) + \Delta_1(x) - 1 \,] = 0.$$

If $\mu - \lambda = 1$, then they satisfy the differential equations with $a^-(w) = w - \lambda$, $b^+(w) = 1$ and $C = 1$.

8. Conclusion and Discussion

In this paper we have formulated the super $\mathcal{W}_{1+\infty}$ algebra, $\mathcal{SW}_{1+\infty}$, as a central extension of the Lie super algebra of superdifferential operators acting on the polynomial algebra over 2×2 supermatrices. We then have studied the quasifinite highest weight modules over $\mathcal{SW}_{1+\infty}$. Our discussion is parallel with Kac and Radul's one. The quasifiniteness of the modules is characterized by polynomials, and the generating functions of highest weights, $\Delta_a(x)$ $(a = 0, 1)$, satisfy a set of differential equations.

Mathematically, there are many things to be clarified. In the bosonic counterpart, we have already obtained the determinant formulae of the $\mathcal{W}_{1+\infty}$ module and the character formulae of the degenerate representations [AFOQ,AFMO1,AFMO2]. Furthermore, we study the structure of subalgebras of the $\mathcal{W}_{1+\infty}$ algebra [AFMO3], especially, the $\mathcal{W}_\infty$ algebra (algebra without spin one current). The supersymmetric extension of these analysis seems to be of some interest.

Since $\mathcal{SW}_{1+\infty}$ contains the $N = 2$ superconformal algebra as a subalgebra, $\mathcal{SW}_{1+\infty}$ has another interesting application, *geometry*. In fact, geometry of complex manifolds (especially the Calabi–Yau manifolds and their mirrors), and topological field theory have been studied by using the $N = 2$ superconformal algebra. For example, the

Calabi–Yau manifolds are described by the $N = 2$ supersymmetric non-linear σ model or by Landau–Ginzburg orbifolds, whose elliptic genera have been computed recently in refs. [EOTY, KYY]. $\mathcal{SW}_{1+\infty}$ naturally appears there through the free field realization, and thus, using $\mathcal{SW}_{1+\infty}$ we may obtain more information than using the $N = 2$ superconformal algebra only. We hope to report on these subjects in our future communication.

Finally we comment on the family of (super) $\mathcal{W}$ infinity algebras. The super $\mathcal{W}_\infty$ algebra given in ref. [BPRSS] is a subalgebra of $\mathcal{SW}_{1+\infty}$, which corresponds to the relation between $\mathcal{W}_{1+\infty}$ and $\mathcal{W}_\infty$ [AFMO3]. The super $\mathcal{W}_\infty$ ($\mathcal{W}_\infty^{1,1}$) was extended to $\mathcal{W}_\infty^{M,N}$ [O]. Similarly, by replacing 2×2 supermatrices by $(M+N) \times (M+N)$ supermatrices, one can easy extend $\mathcal{SW}_{1+\infty}$ ($\mathcal{W}_{1+\infty}^{1,1}$) to $\mathcal{W}_{1+\infty}^{M,N}$, which contains $\mathcal{W}_\infty^{M,N}$ as a subalgebra.

Acknowledgements

Two of the authors (Y.M. and S.O.) would like to thank members of YITP for their hospitality. This work is supported in part by Grant-in-Aid for Scientific Research from Ministry of Science and Culture, and by Soryushi-Shogakkai.

Appendix A. Embedding into $\widehat{\mathrm{gl}}(\infty|\infty)$

A.1. Let E_{mn} $(m, n \in \mathbb{Z})$ denote the matrix unit of infinite size: $E_{mn} = (\delta_{im}\delta_{jn})_{i,j\in\mathbb{Z}}$. An infinite dimensional Lie algebra $\mathrm{gl}(\infty)$ is then defined as

$$\mathrm{gl}(\infty) \equiv \left\{ \sum_{m,n\in\mathbb{Z}} a_{mn}E_{mn} \,\middle|\, a_{mn} = 0 \quad \text{for} \quad |m-n| \gg 0 \right\}.$$

We further define $\mathrm{gl}(\infty|\infty) \equiv \mathrm{gl}(\infty) \otimes_{\mathbb{C}} \mathcal{A}$ with $\mathcal{A}$ the algebra over (2×2) supermatrices

$$\mathcal{A} \equiv \left\{ \begin{bmatrix} f^1(m) & f^-(m) \\ f^+(m) & f^0(m) \end{bmatrix} \,\middle|\, f^A(m) \in \mathbb{C}[m]\,;\, A = 0, 1, \pm \right\},$$

with the same $\mathbb{Z}_2$–gradation as eq. (3.1). We have changed the arrangement of matrix elements from eq. (3.1) for later convenience.

A.2. Let θ be a Grassmann number and $z^n F(D) = \sum_A z^n f^A(D) P_A \in sw_{1+\infty}$. The embedding map $\varphi : sw_{1+\infty} \hookrightarrow \mathrm{gl}(\infty|\infty)$ is defined through the action of $sw_{1+\infty}$ on $\mathbb{C}[z, z^{-1}] \oplus \mathbb{C}[z, z^{-1}]\theta$ as follows:

$$z^n F(D) \cdot z^m (1, \theta) \equiv \sum_{\ell \in \mathbb{Z}} z^\ell (1, \theta) \cdot \varphi\,(\, z^n F(D)\,)_{\ell, m}\,. \tag{A.1}$$

Since the action of P_A's on $(1, \theta)$ is given as

$$\begin{bmatrix} P_1 & P_- \\ P_+ & P_0 \end{bmatrix} \cdot 1 = \begin{bmatrix} 1 & 0 \\ \theta & 0 \end{bmatrix}, \qquad \begin{bmatrix} P_1 & P_- \\ P_+ & P_0 \end{bmatrix} \cdot \theta = \begin{bmatrix} 0 & 1 \\ 0 & \theta \end{bmatrix},$$

the left hand side of eq. (A.1) reduces to

$$z^{m+n}(1, \theta) \cdot \begin{bmatrix} f^1(m) & f^-(m) \\ f^+(m) & f^0(m) \end{bmatrix} = \sum_\ell z^\ell (1, \theta) \cdot \delta_{\ell, m+n} \begin{bmatrix} f^1(m) & f^-(m) \\ f^+(m) & f^0(m) \end{bmatrix}.$$

Thus, we obtain

$$\varphi\,(\, z^n F(D)\,)_{\ell, m} = \delta_{\ell, m+n} \begin{bmatrix} f^1(m) & f^-(m) \\ f^+(m) & f^0(m) \end{bmatrix} \equiv (\, \Lambda^n [F(d)]\,)_{\ell, m}\,.$$

Here Λ and $[F(d)]$ stand for the following infinite matrices:

$$\Lambda = \sum_{m \in \mathbb{Z}} E_{m, m-1}, \qquad [F(d)] = \sum_{m \in \mathbb{Z}} F(m) E_{mm}, \qquad F(m) = \begin{bmatrix} f^1(m) & f^-(m) \\ f^+(m) & f^0(m) \end{bmatrix}.$$

By definition, the map $\varphi : sw_{1+\infty} \to \mathrm{gl}(\infty|\infty)$ is homomorphic, *i.e.*,

$$\varphi(AB) = \varphi(A)\varphi(B).$$

A.3. Let us introduce new variables $\mu = \theta + z\partial_\theta$ and $\mu^{-1} = z^{-1}\theta + \partial_\theta$. Note that they satisfy $\mu^2 = z$, $\mu\mu^{-1} = \mu^{-1}\mu = 1$, and also that

$$\begin{bmatrix} P_1 & P_- \\ P_+ & P_0 \end{bmatrix} = \begin{bmatrix} P_1 & \mu^{-1} P_0 \\ \mu P_1 & P_0 \end{bmatrix}.$$

Thus, we can think of the diagonal elements P_0 and P_1 as the fundamental elements. Hence, elements of $\mathrm{gl}(\infty|\infty)$ can be represented by matrices with half–integer indices as follows:

$$\mathrm{gl}(\infty|\infty) = \left\{ \sum_{\alpha, \beta \in \mathbb{Z}/2} a_{\alpha\beta} E_{\alpha\beta} \;\middle|\; a_{\alpha\beta} = 0 \quad \text{for} \quad |\alpha - \beta| \gg 0 \right\},$$

where we denote $E_{\alpha\beta} = (\delta_{\mu\alpha}\delta_{\nu\beta})_{\mu,\nu\in\mathbb{Z}/2}$. The $\mathbb{Z}_2$–gradation is assigned as $\mathrm{gl}(\infty|\infty) = \mathrm{gl}(\infty|\infty)^{(0)} \oplus \mathrm{gl}(\infty|\infty)^{(1)}$ and $E_{\alpha\beta} \in \mathrm{gl}(\infty|\infty)^{(0)}$ if and only if $\alpha - \beta \in \mathbb{Z}$, otherwise $E_{\alpha\beta} \in \mathrm{gl}(\infty|\infty)^{(1)}$.

A.4. Denoting by $\widetilde{W}(A)$ the element in $\widehat{\mathrm{gl}}(\infty|\infty)$ which corresponds to an element A in $\mathrm{gl}(\infty|\infty)$, we introduce $\widehat{\mathrm{gl}}(\infty|\infty)$ as the central extension of $\mathrm{gl}(\infty|\infty)$ with the following (anti–) commutation relation:

$$\left[\widetilde{W}(A), \widetilde{W}(B)\right\} \equiv \widetilde{W}([\,A, B\,\}) - C\widetilde{\Psi}(A, B), \qquad \widetilde{\Psi}(A, B) = \mathrm{str}J\,[\,A, B\,\}\,,$$

where $J = \sum_{\alpha\geq 0} E_{\alpha\alpha}$ and $\mathrm{str}A = -\sum_{\alpha\in\mathbb{Z}/2}(-1)^{2\alpha}(A)_{\alpha\alpha}$. The fundamental (anti–) commutation relation for $\widehat{\mathrm{gl}}(\infty|\infty)$ is

$$\begin{aligned}\left[\widetilde{W}(E_{\alpha\beta}), \widetilde{W}(E_{\gamma\delta})\right\} = \delta_{\beta\gamma}\widetilde{W}(E_{\alpha\delta}) - (-1)^{2(\alpha-\beta)2(\gamma-\delta)}\delta_{\delta\alpha}\widetilde{W}(E_{\gamma\beta}) \\ + C\delta_{\beta\gamma}\delta_{\delta\alpha}(-1)^{2\alpha}\,(\,\theta(\alpha) - \theta(\gamma)\,)\,,\end{aligned}$$

where $\theta(\alpha) = 1$ if $\alpha \geq 0$ and otherwise $\theta(\alpha) = 0$. The (anti–) commutation relation for $\mathcal{SW}_{1+\infty}$ embedded in $\widehat{\mathrm{gl}}(\infty|\infty)$ is thus given by

$$\begin{aligned}\left[\widetilde{W}(\varphi(\,z^n F(D)\,)), \widetilde{W}(\varphi(\,z^m G(D)\,))\right\} \equiv \widetilde{W}\,(\,[\,\varphi(\,z^n F(D)\,)\,, \varphi(\,z^m G(D)\,)\,\}\,) \\ + C\delta_{n+m,0}\left(\textstyle\sum_{j\geq 0} - \sum_{j\geq n}\right)\left\{h^{11}(j) + h^{-+}(j) - h^{+-}(j) - h^{00}(j)\right\},\end{aligned}$$

with $h^{A,B}(j) = f^A(j+m)g^B(j)$, which is the same as that in eq. (3.2).

A.5. We can easily understand automorphisms of $\mathcal{SW}_{1+\infty}$ in eq. (6.1) as the ones of $\widehat{\mathrm{gl}}(\infty|\infty)$ as follows.

Since any automorphisms $\pi_w : sw_{1+\infty} \to sw_{1+\infty}$ and $\pi_g : \mathrm{gl}(\infty|\infty) \to \mathrm{gl}(\infty|\infty)$ are realized by the basis transformation $\pi_z : \mathbb{C}[z, z^{-1}, \theta] \to \mathbb{C}[z, z^{-1}, \theta]$ as

$$\begin{aligned} z^n F(D) \cdot \pi_z\,(\,z^m(1,\theta)\,) &= \sum_\ell \pi_z\left(z^\ell(1,\theta)\right) \cdot \pi_g\,(\,\varphi\,(\,z^n F(D)\,)\,)_{\ell,m}\,, \\ \pi_w\,(\,z^n F(D)\,) \cdot \pi_z^{-1}\,(\,z^m(1,\theta)\,) &= \sum_\ell \pi_z^{-1}\left(z^\ell(1,\theta)\right) \cdot \varphi\,(\,z^n F(D)\,)_{\ell,m}\,,\end{aligned}$$

we have

$$\pi_w\,(\,z^n F(D)\,) \cdot z^m(1,\theta) = \sum_\ell z^\ell(1,\theta) \cdot \pi_g\,(\,\varphi\,(\,z^n F(D)\,)\,)_{\ell,m}\,.$$

Furthermore, one can also easily obtain the induced automorphisms $\pi_w : \mathcal{SW}_{1+\infty} \to \mathcal{SW}_{1+\infty}$ and $\pi_g : \widehat{\mathrm{gl}}(\infty|\infty) \to \widehat{\mathrm{gl}}(\infty|\infty)$ with some modifications coming from central terms. For example, for the transformation $\pi_z\left(z^m(1,\theta)\right) = z^m(z^{\lambda^1}, z^{\lambda^0}\theta)$, the automorphisms are given by

$$\begin{aligned}
&\pi_g\left(\widetilde{W}\left(\varphi\left(z^n F(D)\right)\right)\right) \\
&\quad = \widetilde{W}\left(\begin{bmatrix} \delta_{\ell,m+n} f^1(m+\lambda^1) & \delta_{\ell,m+n+\lambda^0-\lambda^1} f^-(m+\lambda^0) \\ \delta_{\ell,m+n+\lambda^1-\lambda^0} f^+(m+\lambda^1) & \delta_{\ell,m+n} f^0(m+\lambda^0) \end{bmatrix}_{\ell,m\in\mathbb{Z}}\right) \\
&\qquad\qquad + C\delta_{n,0}\left\{\left(\sum_{j\geq\lambda^1} - \sum_{j\geq 0}\right) f^1(j) - \left(\sum_{j\geq\lambda^0} - \sum_{j\geq 0}\right) f^0(j)\right\},
\end{aligned}$$

$$\begin{aligned}
\pi_w\left(W\left(z^n F(D)\right)\right) = W\Big(& z^n\{f^1(D+\lambda^1)P_1 + z^{\lambda^0-\lambda^1} f^-(D+\lambda^0)P_- \\
& + z^{\lambda^1-\lambda^0} f^+(D+\lambda^1)P_+ + f^0(D+\lambda^0)P_0\}\Big) \\
& + C\delta_{n,0}\left\{\left(\sum_{j\geq\lambda^1} - \sum_{j\geq 0}\right) f^1(j) - \left(\sum_{j\geq\lambda^0} - \sum_{j\geq 0}\right) f^0(j)\right\},
\end{aligned}$$

which corresponds to the spectral flow in eq. (6.1).

A.6. We can reformulate the quasifinite highest weight representation of $\mathcal{SW}_{1+\infty}$ in terms of $\widehat{\mathrm{gl}}(\infty|\infty)$. We denote $\hat{A}_\alpha = M^{-2\alpha} A_\alpha$ with a diagonal matrix A_α and $M = \sum_{\alpha\in\mathbb{Z}/2} E_{\alpha,\alpha-1/2}$, $\Lambda = M^2$.

We define $I_\alpha \equiv \left\{ \hat{A}_\alpha \,\middle|\, \widetilde{W}(\hat{A}_\alpha)|\widetilde{\lambda}\rangle = 0 \right\}$, where $|\widetilde{\lambda}\rangle$ is the highest weight vector such that $\widetilde{W}(\hat{A}_\alpha)|\widetilde{\lambda}\rangle = 0$ for all $\hat{A}_\alpha$ with $\alpha > 0$. Then we can show that I_α is an ideal, *i.e.*, if $\hat{A}_\alpha \in I_\alpha$, then $H\hat{A}_\alpha \in I_\alpha$ for any diagonal matrix H. Hence, I_α is generated by a characteristic matrix $\hat{C}_\alpha$, *i.e.*, if $\hat{A}_\alpha \in I_\alpha$, then there exists a diagonal matrix H such that $\hat{A}_\alpha = H\hat{C}_\alpha$. The relation between C_α and the characteristic polynomials $a^-(w)$ and $b_k^A(w)$ in eq. (4.2) is

$$\begin{bmatrix} (C_n)_{kk} & (C_{n+\frac{1}{2}})_{k+\frac{1}{2},k+\frac{1}{2}} \\ (C_{n-\frac{1}{2}})_{kk} & (C_n)_{k+\frac{1}{2},k+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} b_n^1(k) & b_n^-(k) \\ b_n^+(k) & b_n^0(k) \end{bmatrix},$$

where $n \geq 0$, $k \in \mathbb{Z}$, and we set $b_0^-(k) = a^-(k)$, $b_0^0(k) = b_0^1(k) = b_0^+(k) = 0$. The matrix

$\left(\sum_{\alpha>0}\hat{C}_\alpha\right)_{\mu,\nu\in\mathbb{Z}/2}$ is arranged explicitly as follows:

$$
\begin{array}{c|cccccccc|}
\mu\backslash\nu & \cdots & \underbrace{-2} & \underbrace{-\tfrac{3}{2}} & \underbrace{-1} & \underbrace{-\tfrac{1}{2}} & \underbrace{0} & \underbrace{\tfrac{1}{2}} & \cdots \\
\vdots & \ddots & & & & & & & \\
-3) & & b^1(-2) & b^-(-2) & b_2^1(-1) & b_2^-(-1) & b_3^1(0) & b_3^-(0) & \\
-\tfrac{5}{2}) & & b^+(-2) & b^0(-2) & b_2^+(-1) & b_2^0(-1) & b_3^+(0) & b_3^0(0) & \\
-2) & & 0 & a^-(-2) & b^1(-1) & b^-(-1) & b_2^1(0) & b_2^-(0) & \\
-\tfrac{3}{2}) & & 0 & 0 & b^+(-1) & b^0(-1) & b_2^+(0) & b_2^0(0) & \\
-1) & & 0 & 0 & 0 & a^-(-1) & b^1(0) & b^-(0) & \\
-\tfrac{1}{2}) & & 0 & 0 & 0 & 0 & b^+(0) & b^0(0) & \\
0) & & 0 & 0 & 0 & 0 & 0 & a^-(0) & \\
\tfrac{1}{2}) & & 0 & 0 & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & \ddots
\end{array}.
$$

We can show that there exist diagonal matrices H_n $(n=1,2,3)$ such that

$$
\begin{aligned}
&\hat{C}_{\alpha+\frac{1}{2}} = (M^{-1}H_1)\hat{C}_\alpha = \hat{C}_\alpha(M^{-1}H_2),\\
&\hat{C}_\alpha\hat{C}_\beta = H_3\hat{C}_{\alpha+\beta}.
\end{aligned}
$$

This equation can be solved recursively, and we obtain

$$
\begin{aligned}
&\ell.c.m.\Big((\hat{C}_{\frac{1}{2}})_{\mu,\mu+\frac{1}{2}},(\hat{C}_{\frac{1}{2}})_{\mu+\frac{1}{2},\mu+1},\cdots,(\hat{C}_{\frac{1}{2}})_{\mu+\alpha-\frac{1}{2},\mu+\alpha}\Big) \quad | \quad (\hat{C}_\alpha)_{\mu,\mu+\alpha},\\
&(\hat{C}_\alpha)_{\mu,\mu+\alpha} \quad | \quad (\hat{C}_{\frac{1}{2}})_{\mu,\mu+\frac{1}{2}}(\hat{C}_{\frac{1}{2}})_{\mu+\frac{1}{2},\mu+1}\cdots(\hat{C}_{\frac{1}{2}})_{\mu+\alpha-\frac{1}{2},\mu+\alpha}.
\end{aligned}
$$

for all μ, $\alpha\in\mathbb{Z}/2$. This is equivalent to the relations in eqs. (4.3)–(4.6). Furthermore, if the elements of $\hat{C}_{\frac{1}{2}}$ are mutually prime, then we have the relation $\hat{C}_\alpha=(\hat{C}_{\frac{1}{2}})^{2\alpha}$, which is the same as eq. (4.9). Note that if we set

$$
(\hat{C}_\alpha^{\ell cm})_{\mu,\mu+\alpha} = \ell.c.m.\Big((\hat{C}_{\frac{1}{2}})_{\mu,\mu+\frac{1}{2}},(\hat{C}_{\frac{1}{2}})_{\mu+\frac{1}{2},\mu+1},\cdots,(\hat{C}_{\frac{1}{2}})_{\mu+\alpha-\frac{1}{2},\mu+\alpha}\Big),
$$

then one may show that $\widetilde{W}(\hat{C}_\alpha^{\ell cm})|\widetilde{\lambda}\rangle$ is a null state. We will prove it in Appendix B.

Appendix B. Null Vector Condition

B.1. We discussed the quasifinite highest weight module as the *generalized Verma module* [KR], which is annihilated by the parabolic subalgebra. However, as seen in Corollary 4.5, the characteristic polynomials $b_k^A(w)$ are not fixed uniquely. Here we will show that the characteristic polynomials are uniquely determined if we demand that the quasifinite highest weight module be *irreducible*.

We first introduce a bilinear form. Recall that $V(\lambda)$ is the Verma module over $\mathcal{SW}_{1+\infty}$, generated by the highest weight vector $|\lambda\rangle$, such that

$$W(D^k P_+)|\lambda\rangle = 0, \qquad W(z^{n+1}D^k P_A)|\lambda\rangle = 0, \qquad W(e^{xD}P_a)|\lambda\rangle = -\Delta_a(x)|\lambda\rangle$$

with $n, k \in \mathbb{Z}_{\geq 0}$, $A = 0, 1, \pm$ and $a = 0, 1$. The dual module $V(\lambda)^*$ is generated by $\langle\lambda|$ which satisfies

$$\langle\lambda|W(D^k P_-) = 0, \qquad \langle\lambda|W(z^{-n-1}D^k P_A) = 0, \qquad \langle\lambda|W(e^{xD}P_a) = -\Delta_a(x)\langle\lambda|$$

with $n, k \in \mathbb{Z}_{\geq 0}$, $A = 0, 1, \pm$ and $a = 0, 1$. The bilinear form $V(\lambda)^* \otimes V(\lambda) \to \mathbb{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$ and $(\langle u|W)|v\rangle = \langle u|(W|v\rangle)$ for any $\langle u| \in V(\lambda)^*$, $|v\rangle \in V(\lambda)$ and $W \in \mathcal{SW}_{1+\infty}$.

The null vector $|\chi\rangle$ is defined by the condition that $\langle u|\chi\rangle = 0$ for all $\langle u| \in V(\lambda)^*$.

B.2. We let $b_0^-(w) = a^-(w)$, $b_0^A(w) = 0$ with $A = 0, 1, +$ and

$$\begin{aligned}
b_k^+(w) &= \ell.c.m.\left(b^+(w), a^-(w-1), b^+(w-1), a^-(w-2), \cdots, b^+(w-k+1)\right), \\
b_k^0(w) &= \ell.c.m.\left(a^-(w), b^+(w), a^-(w-1), b^+(w-1), \cdots, b^+(w-k+1)\right), \\
b_k^1(w) &= \ell.c.m.\left(b^+(w), a^-(w-1), b^+(w-1), a^-(w-2), \cdots, a^-(w-k)\right), \\
b_k^-(w) &= \ell.c.m.\left(a^-(w), b^+(w), a^-(w-1), b^+(w-1), \cdots, a^-(w-k)\right),
\end{aligned} \tag{B.1}$$

for $k \in \mathbb{Z}_{>0}$. We will show the following Theorem:

Theorem. *If the weight functions $\Delta_0(x)$ and $\Delta_1(x)$ satisfy the differential equation (5.4), then $|\chi_k^A\rangle \equiv W(z^{-k}e^{yD}b_k^A(D)P_A)|\lambda\rangle$ is a null vector for all $y \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$ and $A \in \{0, 1, +, -\}$.*

To obtain the quasifinite *irreducible* highest weight module, we must factor out the null vectors which are characterized by the polynomials in eq. (B.1). Since there possibly exist additional null vectors for some special values of C and $\Delta_a(x)$, we here discuss the generic case.

B.3. *Proof of the Theorem.*

First we have the following Lemma:

Lemma. *The subalgebra*

$$\mathcal{SW}^{+}_{1+\infty} \equiv \{\, W(D^k P_+), W(z^{n+1} D^k P_A) \,|\, n,k \in \mathbb{Z}_{\geq 0},\ A = 0,1,\pm \,\}$$

of $\mathcal{SW}_{1+\infty}$ *is generated by* $W(D^k P_+)$ *and* $W(zD^k P_-)$ *with* $k \in \mathbb{Z}_{\geq 0}$.[†]

Proof . $W(zD^k P_a)$ with $a = 0,1$ are obtained as follows:

$$W(zD^k P_0) = \{W(DP_+), W(zD^k P_-)\} - \{W(P_+), W(zD^{k+1} P_-)\},$$
$$W(zD^k P_1) = \{W(P_+), W(zD^k (D+1) P_-)\} - \{W(DP_+), W(zD^k P_-)\}.$$

One can further obtain $W(z^n D^k P_A)$ from $W(z^{n-1} D^\ell P_A)$ by taking (anti–) commutators with $W(zP_0)$ or $W(zP_1)$. □

Hence, to prove that $|\chi\rangle$ is a null vector, it is sufficient to show that $W(e^{xD} P_+)|\chi\rangle$ and $W(ze^{x(D+1)} P_-)|\chi\rangle$ are null vectors or vanish for all $x \in \mathbb{C}$.[‡] The proof of the Theorem is given by induction as follows:

Step 1. $|\chi_0^-\rangle$ *and* $|\chi_1^+\rangle$ *are null vectors.*

Proof . From the differential equation for $\Delta_0(x)$ and $\Delta_1(x)$, we obtain

$$W(e^{xD} P_+)|\chi_0^-\rangle = a^- \left(\frac{d}{dX}\right) [\,\Delta_0(X) + \Delta_1(X)\,]\,|\lambda\rangle = 0,$$
$$W(ze^{x(D+1)} P_-)|\chi_1^+\rangle = b^+ \left(\frac{d}{dX}\right) \big[\,\Delta_1(X) + e^X \Delta_0(X) - C\,\big]\,|\lambda\rangle = 0,$$

with $X \equiv x + y$. Moreover, $W(ze^{x(D+1)} P_-)|\chi_0^-\rangle = 0$ and $W(e^{xD} P_+)|\chi_1^+\rangle = 0$. □

[†] Note that the whole algebra $\mathcal{SW}_{1+\infty}$ is generated by $W(P_\pm)$, $W(z^{\pm 1} P_\mp)$ and $W(DP_0)$.

[‡] In the bosonic case, the $\mathcal{W}_{1+\infty}$ is generated by $W(z^{\pm 1})$ and $W(D^2)$. To show the null vector condition, it is sufficient that $W(ze^{x(D+1)})|\chi\rangle$ is a null vector or vanishes.

Step 2. *If $|\chi^-_{k-1}\rangle$ and $|\chi^+_k\rangle$ are null vectors for a positive integer k, then $|\chi^0_k\rangle$ and $|\chi^1_k\rangle$ are also null vectors.*

Proof . Since $b^+_k(w)\,|\,b^0_k(w),\,b^1_k(w)$, and $b^-_{k-1}(w)\,|\,b^0_k(w),\,b^1_k(w+1)$, the following four vectors are null (here $X\equiv x+y$):

$$W(e^{xD}P_+)|\chi^0_k\rangle = -W(z^{-k}e^{XD}b^0_k(D)P_+)|\lambda\rangle,$$
$$W(e^{xD}P_+)|\chi^1_k\rangle = e^{-kx}W(z^{-k}e^{XD}b^1_k(D)P_+)|\lambda\rangle,$$
$$W(ze^{x(D+1)}P_-)|\chi^0_k\rangle = e^{(-k+1)x}W(z^{-k+1}e^{XD}b^0_k(D)P_-)|\lambda\rangle,$$
$$W(ze^{x(D+1)}P_-)|\chi^1_k\rangle = -W(z^{-k+1}e^{X(D+1)}b^1_k(D+1)P_-)|\lambda\rangle. \qquad \square$$

Step 3. *If $|\chi^0_k\rangle$ and $|\chi^1_k\rangle$ are null vectors for a positive integer k, then $|\chi^-_k\rangle$ and $|\chi^+_{k+1}\rangle$ are also null vectors.*

Proof . Since $b^0_k(w),\,b^1_k(w)\,|\,b^-_k(w)$ and $b^1_k(w)\,|\,b^+_{k+1}(w)$ and $b^0_k(w)\,|\,b^+_{k+1}(w+1)$, the following two vectors are null:

$$W(e^{xD}P_+)|\chi^-_k\rangle = W(z^{-k}(e^{-kx}e^{XD}b^-_k(D)P_0 + e^{XD}b^-_k(D)P_1))|\lambda\rangle,$$
$$W(ze^{x(D+1)}P_-)|\chi^+_{k+1}\rangle = W(z^{-k}(e^{-kx}e^{XD}b^+_{k+1}(D)P_1 + e^{X(D+1)}b^+_{k+1}(D+1)P_0))|\lambda\rangle.$$

Moreover, $W(e^{xD}P_+)|\chi^+_k\rangle = 0$ and $W(ze^{x(D+1)}P_-)|\chi^-_{k+1}\rangle = 0$. $\square$

Thus we have completed the proof of Theorem B.1.

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