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Character and Determinant Formulae of Quasifinite Representation of the $W_{1+\infty}$ Algebra

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Abstract

We diagonalize the Hilbert space of some subclass of the quasifinite module of the $W_{1+\infty}$ algebra. States are classified according to their eigenvalues for infinitely many commuting charges and the Young diagrams. The parameter dependence of their norms is explicitly derived. The full character formulae of the degenerate representations are given as summation of the bilinear combinations of the Schur polynomials.

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1 Introduction

The detailed study of (infinite dimensional) Lie algebras has been sometimes very essential in theoretical physics. The representation theory of finite dimensional Lie algebra is indispensable to understand quantum mechanics or gauge theories. If we extend the dimension by one, the loop algebras such as Virasoro [1] or Kac-Moody algebras are essential tools to describe two-dimensional statistical systems or string theories.

Recently, in many places such as two-dimensional quantum gravity [2]–[5], the quantum Hall effects [6][7], the membrane [8][9], or the large N QCD [10][11], the $W_{1+\infty}$ algebra is regarded as the fundamental symmetry of system.

As a member of loop algebras, the $W_{1+\infty}$ algebra has a unique character in that the number of currents is infinite. In a sense, it may be regarded as the symmetry of three-dimensional system since it is closely connected with the area-preserving diffeomorphism [12][13]. Due to this fact, the detailed representation theory was not fully developed until now although some attempts were made [14]. The situation is also similar in the extensions of the $W_{1+\infty}$ algebra [15]–[20]. One of the confusing feature of the $W_{1+\infty}$ algebra is its hybrid nature in dimensions. We remark that it has also definite "two-dimensional" aspects since we already knew the explicit realization in terms of two-dimensional free fields [13][21]. Furthermore, this symmetry is found even in instanton physics in four dimensions [22]–[24].

Last year, Kac and Radul [25] discovered a way to avert from the difficulty and proved that the Hilbert space at each energy level can be finite dimensional if we choose the weight vector properly. In our previous letter [26], we give the computer calculation of the Kac formula of the $W_{1+\infty}$ algebra at lower degree. In this article, we would like to give its analytical formula. Actually, we can go further to give the explicit form of the diagonal basis of the Hilbert space with respect to the inner product and give their parameter dependence. As corollaries, we give the full character formulae [27] of any degenerate representations. This will be the basis for the application of the representation theory of the $W_{1+\infty}$ algebra to physical systems, such as quantum gravity, the quantum Hall effects, the two-dimensional QCD which we would like to report in our future issues.

The plan of this paper is as follows. In section 2, we give a brief review of the result of Kac and Radul, and also a summary of our computer calculation of the determinant formula. The parameters of the system can be roughly classified into two groups, the central charges and the spins. Our determinant formula is factorized into functions which depend only on either of them. In section 3, we give the detailed account of the relation with the gl(∞) algebra. This is an essential step to understand the determinant formula. As we see in the following sections, the trasformation from the basis of the $W_{1+\infty}$ algebra to the corresponding ones of the gl(∞) algebra gives the spin dependent part of the determinant formula. On the other hand, the determinant of the gl(∞) algebra explains the central charges dependence. In section 4, we first derive the spin dependence from this viewpoint. In section 5, the central charge dependence is derived. There, the knowledge of the permutation group is essential to classifying the Hilbert space. Indeed, we derive the explicit form of the diagonal basis with respect to the inner product by using the Young diagrams. In section 6, we give the character formula for the degenerate representation as a bilinear form of the Schur polynomials. In appendix A, we give tables of the determinant formula which we previously derived by computer analysis. In appendix B, we explain the free-fermion method which was essential to calculating the inner product formula. In the $W_{1+\infty}$ algebra, there are an infinite number of "modular parameters" because the number of Cartan elements is infinite. The fermion which appears here is the "fermionization" of those modular parameters.

2 Brief review of the $W_{1+\infty}$ algebra

The $W_{1+\infty}$ algebra is a central extension of the Lie algebra of the (higher order) differential operators on the circle, which is generated by $z^r D^k$ with $r \in \mathbf{Z}$, $k \in \mathbf{Z}_{\geq 0}$ and $D \equiv z\frac{\partial}{\partial z}$. We write the generator of the $W_{1+\infty}$ algebra which correspond to the differential operator $z^r D^k$ as $W(z^r D^k)$. The commutation relations are,

$$\begin{aligned}&[W(z^r f(D)), W(z^s g(D))] = \\ &W(z^{r+s} f(D+s) g(D)) - W(z^{r+s} f(D) g(D+r)) \\ &+C\Psi(z^r f(D), z^s g(D)),\end{aligned} \qquad (1)$$

where $f(D)$ and $g(D)$ are polynomials of D and we introduce the *two-cocycle* Ψ,

$$\Psi(z^r f(D), z^s g(D)) = -\Psi(z^s g(D), z^r f(D))$$

$$= \left\{ \begin{array}{ll} \sum_{1\le j\le r} f(-j)g(r-j) & \text{if } r=-s>0 \\ 0 & \text{if } r+s\neq 0 \text{ or } r=s=0. \end{array} \right. \tag{2}$$

The principal gradation of the $W_{1+\infty}$ algebra is

$$\begin{aligned} \mathcal{W}_{1+\infty} &= \bigoplus_{r\in\mathbf{Z}} (\mathcal{W}_{1+\infty})_r \\ (\mathcal{W}_{1+\infty})_r &= \{z^r f(D) | f(w) \in \mathbf{C}[w]\} \end{aligned} \tag{3}$$

It is defined in terms of the eigenvalue of the "energy operator" $L_0 \equiv -W(D)$. The highest weight state of the $W_{1+\infty}$ module is defined in terms of this gradation,

$$\begin{array}{ll} W(z^r D^k)|\lambda\rangle = 0, & r\ge 1, k\ge 0, \\ W(D^k)|\lambda\rangle = \Delta_k|\lambda\rangle, & k\ge 0. \end{array} \tag{4}$$

We introduce,

$$\Delta(x) \equiv -\sum_{k=0}^{\infty} \frac{x^k}{k!}\Delta_k, \tag{5}$$

to rewrite the (infinite dimensional) weight vector, which will be called the weight function. The Verma module is spanned by the vectors which are obtained by applying the generators of negative gradation to the highest weight state,

$$W(z^{-r_1} f_1(D)) \cdots W(z^{-r_N} f_N(D))|\lambda\rangle,$$

and we define the energy level of this state by the sum, $\sum_{i=1}^{N} r_i$.

A representation of $W_{1+\infty}$ is called *quasifinite* if and only if there are only finite number of states at each energy level. The quasifinite module has the following properties [25]:

1. For each level r , there are infinitely many null generators of the form $W(z^{-r} b_r(D) g(D))$, where $b_r(D)$ is a monic, finite degree polynomial of operator D.

2. The polynomial $b_r(D)$ with $r > 1$ is related to level-1 polynomial $b(D) \equiv b_1(D)$ as

    - $b_r(D)$ is divided by l.c.m.$(b(D), b(D-1), \cdots b(D-r+1))$.
    - $b(D)b(D-1)\cdots b(D-r+1)$ is divided by $b_r(D)$.

If the difference of any two distinct roots is not an integer, b_r can be uniquely determined as $b_r(D) = \prod_{s=0}^{r-1} b(D-s)$.

3. The function $\Delta(x)$ satisfies a differential equation,

$$b(\tfrac{d}{dx})\left((e^x - 1)\Delta(x) + C\right) = 0. \tag{6}$$

When $b(w) = (w-\lambda_1)^{K_1}\cdots(w-\lambda_\ell)^{K_\ell}$, the solutions are

$$\Delta(x) = \frac{\sum_{i=1}^{\ell} p_{K_i}(x)e^{\lambda_i x} - C}{e^x - 1}, \quad \deg p_{K_i} = K_i - 1, \tag{7}$$

with $\sum_{i=1}^{\ell} p_{K_i}(0) = C$.

In this article, we analyze the irreducibility of the quasifinite module. We introduce,

Definition 1 Generalized Verma Module and Kac Determinant
The second property of the quasifinite representation means that there are at most rK independent generators at level r ($W(r^{-r}D^s)$ with $s = 0, 1, \cdots, rK-1$) if the characteristic polynomial $b(w)$ has degree K. We call the module freely generated by those generators as the generalized Verma module. The number of states at each level is given by the generating function [26],

$$\prod_{r=1}^{\infty} \frac{1}{(1-q^r)^{rK}} \equiv \sum_{\ell=0}^{\infty} n_\ell q^\ell. \tag{8}$$

At level ℓ, we define the determinant of the $n_\ell \times n_\ell$ matrix which consists of the inner products of the basis of the module as the Kac Determinant.

The purpose of this paper is to calculate this determinant, and with this knowledge, to give the character formula. We restrict ourselves to consider the special cases,

$$b(w) = \prod_{i=1}^{K}(w-\lambda_i), \qquad \Delta(x) = \sum_{i=1}^{K} C_i \frac{e^{\lambda_i x} - 1}{e^x - 1}. \tag{9}$$

In other word, we postulate first that there are only simple zeros in $b(w) = 0$ and the difference of their roots are not integers. In our previous letter [26], we made the computer calculation of the determinant formulae at lower levels with this assumption. We summarize our results in appendix A [26]. The determinant formula for general cases can be obtained by taking a suitable limit of the parameters. It may be symbolically written in the following form,

$$\det[r] = \prod_i A_r(C_i) \prod_{i<j} B_r(\lambda_i - \lambda_j). \tag{10}$$

The functions A_r and B_r have zero only when $\lambda_i - \lambda_j$ or C_i is integer. In the following sections, we derive these functions analytically.

3 Relation with the gl(∞) algebra

Some of the essential features of the $W_{1+\infty}$ algebra can be more clearly elucidated if we use the connection with its simpler cousin, the gl(∞) algebra. As explained in [25], we may construct a quasifinite representation of the gl(∞) algebra which is deeply connected with the corresponding one of the $W_{1+\infty}$ algebra. We here would like to explain the full detail of this correspondence since it illuminates the λ dependence of the determinant formula and also is essential to calculating the C dependence.

3.1 The gl(∞) algebra and its representation

The gl(∞) algebra is generated by the operators, $\bar{E}^{(\mu)}(i,j)$ $(\mu = 0, 1, \cdots, m)$, which act on the infinite dimensional space spanned by the basis, $\mathbf{v}_k^{(\mu)}$ with $k \in \mathbf{Z}$,

$$\bar{E}^{(\mu)}(i,j)\mathbf{v}_k^{(\nu)} = \theta(m-\mu-\nu)\delta_{j+k,0}\mathbf{v}_i^{(\mu+\nu)}. \tag{11}$$

Here $\theta(i) = 1$ for $i \geq 0$ and $\theta(i) = 0$ for $i < 0$. The commutation relation is,

$$\begin{aligned} &\left[\bar{E}^{(\mu)}(i,j), \bar{E}^{(\nu)}(k,\ell)\right] \\ &= \theta(m-\mu-\nu)\left(\delta_{j+k,0}\bar{E}^{(\mu+\nu)}(i,\ell) - \delta_{\ell+i,0}\bar{E}^{(\mu+\nu)}(k,j)\right). \end{aligned} \tag{12}$$

As usual, the highest weight state is defined by using the gradation, $\deg \bar{E}^{(\mu)}(i,j) = i + j$,

$$\begin{aligned} \bar{E}^{(\mu)}(i,j)|\lambda\rangle &= 0, \qquad i+j>0, \\ \bar{E}^{(\mu)}(i,-i)|\lambda\rangle &= \bar{q}_i^{(\mu)}|\lambda\rangle. \end{aligned} \tag{13}$$

For the finite dimensional case, the parameters $\bar{q}_i^{(\mu)}$ are arbitrary. However, as the $W_{1+\infty}$ algebra, there should be severe constraint on them once we require the quasifiniteness.

Define,

$$h_k^{(\mu)} \equiv \bar{q}_k^{(\mu)} - \bar{q}_{k-1}^{(\mu)}. \tag{14}$$

We introduce the set,

$$S^{(\mu)} \equiv \left\{ k \,|\, h_k^{(\nu)} \neq 0 \text{ for some } \nu \geq \mu \right\},$$

which satisfies the inclusion relation,

$$S^{(0)} \supseteq S^{(1)} \supseteq \cdots \supseteq S^{(m)}. \tag{15}$$

A quasifinite representation is then obtained [25] if and only if $S^{(\mu)}$ is a finite set for each μ.

Let us count the non-vanishing elements in the Hilbert space. At level 1, the Hilbert space consists of the vectors of the form, $\bar{E}^{(\mu)}(k-1,-k)|\lambda\rangle$. To see if they are null, we compute,

$$\bar{E}^{(\nu)}(k,-k+1)\bar{E}^{(\mu)}(k-1,-k)|\lambda\rangle = \theta(m-\mu-\nu)h_k^{(\mu+\nu)}|\lambda\rangle.$$

It shows that it becomes non-vanishing only if $k \in S^{(\mu)}$. Similar computation shows that more general state $\bar{E}^{(\mu)}(\ell,-k)|\lambda\rangle$ becomes non-vanishing only if there exist an integer $s \in S^{(\mu)}$ such that $k \geq s > \ell$. In the figure below, we show the elements which become non-vanishing for this case.

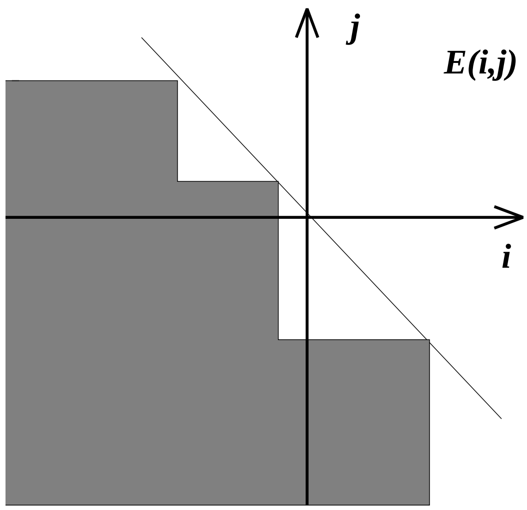


Figure: Surviving Generators

3.2 Definition of the $\hat{\mathrm{gl}}(\infty)$ algebra

In order to prevent the appearance of infinity once we try to relate it with $W_{1+\infty}$, we need to modify the generators of gl(∞) as follows:

$$E^{(\mu)}(i,j) = \bar{E}^{(\mu)}(i,j) - c^{(\mu)}\delta_{i+j,0}\theta(i). \tag{16}$$

Here the "central charges" are defined by,

$$c^{(\mu)} = \sum_{k\in S^{(\mu)}} h_k^{(\mu)}. \tag{17}$$

The algebra (12) and the highest weight condition (13) are also modified,

$$\begin{aligned}\left[E^{(\mu)}(i,j), E^{(\nu)}(k,\ell)\right] &= \theta(m-\mu-\nu)\times \\ &\quad \Big(\delta_{j+k,0}E^{(\mu+\nu)}(i,\ell) - \delta_{i+\ell,0}E^{(\mu+\nu)}(k,j) \\ &\quad +c^{(\mu+\nu)}\delta_{j+k,0}\delta_{\ell+i,0}(\theta(i)-\theta(k))\Big), \end{aligned} \tag{18}$$

and, $E^{(\mu)}(i,-i)|\lambda\rangle = q_i^{(\mu)}|\lambda\rangle$ with $q_i^{(\mu)} = \bar{q}_i^{(\mu)} - c^{(\mu)}\theta(i)$. We can also easily prove,

$$h_k^{(\mu)} = q_k^{(\mu)} - q_{k-1}^{(\mu)} + c^{(\mu)}\delta_{k,0}$$

We call the modified algebra (18) as the $\hat{\mathrm{gl}}(\infty)$ algebra. We remark that the quasifinite representations of gl(∞) and $\hat{\mathrm{gl}}(\infty)$ are identical since there appear no infinite sum in the definition.

3.3 Relation with the $W_{1+\infty}$ algebra

To find a relation between $\hat{\mathrm{gl}}(\infty)$ and $W_{1+\infty}$, we take the Hilbert space spanned by $\mathbf{v}_k$ as the space of functions on the circle spanned by $z^{\lambda+k+t}$ with $\lambda \in \mathbf{C}$, $k \in \mathbf{Z}$. Here the formal parameter t is defined by nilpotency condition, $t^{m+1} \equiv 0$. The action of differential operators on this basis is then given by,

$$\begin{aligned} z^r f(D) z^{\lambda+k+t} &= f(\lambda+k+t) z^{\lambda+k+r+t} \\ &= \left(\sum_{\mu=0}^{m} \frac{t^\mu}{\mu!} f^{(\mu)}(\lambda+k)\right) z^{\lambda+k+r+t}. \end{aligned} \tag{19}$$

By the identification, $\mathbf{v}_k^{(\mu)} \leftrightarrow t^\mu z^{\lambda+k+t}$, we define the correspondence between the generators,

$$W(z^r f(D)) = \sum_{k\in\mathbf{Z}} \sum_{\mu=0}^{m} \frac{f^{(\mu)}(\lambda+k)}{\mu!} E^{(\mu)}(r+k,-k), \tag{20}$$

for $r \neq 0$. Special care is needed to find the relation between the zero modes,

$$\begin{aligned} W(e^{xD}) &= \sum_{k\in\mathbf{Z}} \sum_{\mu=0}^{m} \frac{x^\mu e^{xk}}{\mu!} \left(e^{\lambda x} \bar{E}^{(\mu)}(k,-k) - \delta_{\mu,0} c^{(0)} \theta(k) \right) \\ &= \sum_{k\in\mathbf{Z}} \sum_{\mu=0}^{m} \frac{x^\mu e^{xk}}{\mu!} e^{\lambda x} E^{(\mu)}(k,-k) \\ &\quad -c^{(0)} \frac{e^{\lambda x}-1}{e^x-1} - \sum_{\mu=1}^{m} \frac{x^\mu c^{(\mu)} e^{\lambda x}}{\mu!(e^x-1)}. \end{aligned} \tag{21}$$

The central charges of the both algebras (C for $W_{1+\infty}$ and $c^{(0)}$ for $\hat{\mathrm{gl}}(\infty)$) are related by,

$$C = c^{(0)}. \tag{22}$$

The other central charges of $\hat{\mathrm{gl}}(\infty)$, $c^{(\mu)}$, can be related to the coefficients of x^μ in the polynomial $p_{K_i}(x)$ in (7) as we see in the next subsection.

3.4 Relation between the representations

In the correspondence (20), λ is a free parameter. This arbitrariness is removed once we consider the relation between the (quasifinite) representations of $W_{1+\infty}$ and $\hat{\mathrm{gl}}(\infty)$.

Let us examine the null state conditions, $W(z^{-r} b_r(D))|\lambda\rangle = 0$ in the language of $\hat{\mathrm{gl}}(\infty)$. We first consider the case,

$$b(w) = \prod_{i=1}^{\kappa} (w - \lambda' - k_i)^{\mu_i}, \tag{23}$$

i.e. the differences of all the root of characteristic polynomial are integers. We put $m' \equiv \max(\mu_i)$. We introduce the set of integers associated with b as,

$$T^{(\mu)} \equiv \left\{ k \in \mathbf{Z} | b^{(\mu)}(\lambda'+k) = 0 \right\}.$$

It is obvious that they are determined uniquely from $b(w)$ and satisfy the inclusion relation,

$$T^{(0)} \supseteq T^{(1)} \supseteq \cdots \supseteq T^{(m')},$$

which is the same as (15).

Let λ in (20) be equal to λ' and $m' = m$, the null state condition,

$$0 = W(z^{-r} b_r(D))|\lambda\rangle = \sum_{k\in\mathbf{Z}} \sum_{\mu=0}^{m} \frac{b_r^{(\mu)}(\lambda+k)}{\mu!} E^{(\mu)}(-r+k,-k)|\lambda\rangle \tag{24}$$

implies that only the states of the form,

$$E^{(\mu)}(k-1,-k)|\lambda\rangle$$

with $k \in T^{(\mu)}$ may have non-vanishing norm since the coefficient in (20) vanishes. This condition is identical with the quasifinite representation of gl(∞) with,

$$S^{(\mu)} = T^{(\mu)}. \tag{25}$$

Another check to see the direct relation between the representations of $W_{1+\infty}$ and $\hat{\mathrm{gl}}(\infty)$ is to calculate the function $\Delta(x)$ from the highest weight of $\hat{\mathrm{gl}}(\infty)$. We observe that the weight $q_k^{(\mu)}$ is given by,

$$q_k^{(\mu)} = \sum_{s\in S^{(\mu)}} h_s^{(\mu)} \theta(k-s) - c^{(\mu)}\theta(k).$$

Combine it with (21) and $\Delta(x)|\lambda\rangle = -W(e^{xD})|\lambda\rangle$, we derive,

$$\Delta(x) = \sum_{k\in S^{(0)}} \frac{h_k^{(0)}(e^{x(\lambda+k)}-1)}{e^x-1} + \sum_{\mu=1}^{m} \sum_{k\in S^{(\mu)}} \frac{x^\mu h_k^{(\mu)} e^{x(\lambda+k)}}{\mu!(e^x-1)}. \tag{26}$$

This is exactly the solution of the differential equation (6) where $T^{(\mu)}$ of $b(w)$ is given by $S^{(\mu)}$.

In this way, the quasifinite representation of gl(∞) we have seen in section 3.1 can be identified with the representation of $W_{1+\infty}$ with the characteristic polynomial (23).

3.5 Characteristic polynomial for quasi-finite representation

By the correspondence with the gl(∞) module, the characteristic polynomials at higher levels for the quasifinite representation should be uniquely determined[5]. We claim,

Proposition 1 Irreducible qusifinite module
For the generic values of C_i, if the weight function $\Delta(x)$ satisfies (6)*, then there exist a unique irreducible quasifinite module such that the characteristic polynomial is*

$$b_r(D) = l.c.m.(b(D), b(D-1), \cdots, b(D-r+1)), \tag{27}$$

where $b(w)$ is a minimal degree monic polynomial satisfing (6)*.*

Proof: This follows from the following lemma and the relation with the $gl(\infty)$ given in this section. **Q.E.D.**

Lemma 1 Null state
If the weight function $\Delta(x)$ satisfies (6)*, then the state $W(z^{-r}e^{yD}f(D))|\lambda\rangle$ with*

$$f(D) = l.c.m.(b(D), b(D-1), \cdots, b(D-r+1)) \tag{28}$$

is a null state.

Proof: By combining relations,

$$W(z^{-r}e^{yD}f(D)) = f\left(\tfrac{d}{dy}\right) W(z^{-r}e^{yD})$$

and

$$\begin{aligned}&\left[W(z^s e^{x(D+s)}), W(z^{-r}e^{yD})\right]\\ &\qquad = (e^{x(s-r)} - e^{(x+y)s})\left\{W(z^{s-r}e^{(x+y)D}) + \frac{C\delta_{s,r}}{1-e^{x+y}}\right\},\end{aligned} \tag{29}$$

[5]We have to remark that the definition of the characteristic polynomial $b_r(D)$ in this paper is slightly different from [25]. In [25], it is introduced associated with the parabolic subalgebras. In this context, there are some arbitrariness from its consistency condition alone. On the other hand, we define $b_r(D)$ as the minimal monic polynomial that satisfies $W(z^{-r}b_r(D))|\lambda\rangle =$ null for the generic Cs.

we can derive the following equation for all s_i, $r \in \mathbf{Z}_{>0}$ with $\sum_{i=1}^{n} s_i = r$:

$$\langle\lambda|W(z^{s_n}e^{x_n(D+s_n)})\cdots W(z^{s_1}e^{x_1(D+s_1)})W(z^{-r}e^{yD}f(D))|\lambda\rangle$$
$$= f\left(\tfrac{d}{dy}\right)\prod_{j=1}^{n}\left(e^{x_j(s_{1,j}-r)} - e^{(x_{1,j}+y)s_j}\right)\left\{-\Delta(x_{1,n}+y) + \frac{C}{1-e^{(x_{1,n}+y)}}\right\}, \quad (30)$$

where $s_{1,j} \equiv \sum_{i=1}^{j} s_i$ and $x_{1,j} \equiv \sum_{i=1}^{j} x_i$. If we set $X = x_{1,n} + y$, then the right hand side of eq. (30) reduces to

$$\sum_{m=0}^{r-1} a_m(x,s)\ f\left(\tfrac{d}{dX}\right)e^{mX}(1-e^X)\left\{-\Delta(X) + \frac{C}{1-e^X}\right\}, \quad (31)$$

with some functions $a_m(x,s) = a_m(x_1,\cdots,x_n,s_1,\cdots,s_n)$. On the other hand, the differential equation (6) may be rewritten as

$$b(\tfrac{d}{dx} - s)e^{sx}\,(1-e^x)\left(-\Delta(x) + \frac{C}{1-e^x}\right) = 0. \quad (32)$$

Hence, the right hand side of eq. (30) vanishes. Since any inner product with $W(z^{-r}e^{yD}f(D))|\lambda\rangle$ is zero, it is a null state. **Q.E.D.**

4 λ dependence

We divide the derivation of the determinant formula (10) into two parts. As we have reviewed in the previous section, at each level, the same is the dimensions of the generalized Verma module of $W_{1+\infty}$ and gl(∞) algebras. Let us consider the Hilbert space at a spcific energy level. We denote $\{u_1,\cdots,u_N\}$ as the basis in terms of $W_{1+\infty}$ generators and $\{v_1,\cdots v_N\}$ as the basis in terms of gl(∞) generators. The relation between the two basis may be written as, $u_i = \sum_j \mathcal{A}_{ij}v_j$, with $N\times N$ matrix $\mathcal{A}$. The matrix $\mathcal{A}$ can be directly derived from the relation (20) and it depends only on λ_is. The determinant for the $W_{1+\infty}$ basis is rewritten as the determinant for the gl(∞) generators.

$$Det(\langle u_i|u_j\rangle) = Det(\mathcal{A})^2 Det(\langle v_i|v_j\rangle) \quad (33)$$

In the representation of the gl(∞) algebra, the only parameters which appear in the theory are C_is. This observation shows that (33) gives a natural decomposition of the determinant into a part which depends only on λ_is

$(Det(\mathcal{A})^2)$, and a part which depends only on C_is $(Det(\langle v_i|v_j\rangle))$. In this section, we derive the first factor.

The main theorem in this section is

Theorem 1 $\boldsymbol{\lambda}$ **dependence**
The factor B_r in (10) is given by,

$$B_r(\lambda) = \left(\lambda^{\mu_0^{(r)}} \prod_{s=1}^{\infty} (\lambda + s)^{\mu_s^{(r)}} (\lambda - s)^{\mu_s^{(r)}}\right)^2 . \tag{34}$$

Here the non-negative integers $\mu_s^{(r)}$ can be derived from the generating function,

$$\phi_s(q) \equiv \sum_{r=0}^{\infty} \mu_s^{(r)} q^r = \frac{\partial}{\partial \zeta} \left(\left(\prod_{r=1}^{\infty} \frac{1}{(1-q^r)^r} \right)^K \prod_{t=s+1}^{\infty} \frac{(1-q^t)^{t-s}}{(1-\zeta q^t)^{t-s}} \right) \Bigg|_{\zeta=1} . \tag{35}$$

For the simplest cases, $K = 2$ with $\lambda_1 - \lambda_2 = 0, \pm 1$, (35) gives respectively,

$$\begin{aligned} 2\phi_0(q) &= 2q + 10q^2 + 34q^3 + 108q^4 + 298q^5 \cdots, \\ 2\phi_1(q) &= 2q^2 + 8q^3 + 30q^4 + 88q^5 + \cdots, \end{aligned}$$

which correctly reproduce the table in appendix A.

Before we start the detailed proof, it may be useful in the future study to give the intuitive proof of this theorem.

From (27), the factor $t - s$ in (35) can be regarded as the number of additional null generators at level t when a pair (λ_i, λ_j) satisfies the relation $\lambda_i - \lambda_j = \pm s$. More explicitly,

$$t - s = tK - \text{degree }(\text{l.c.m}(b(w), b(w-1), b(w-2), \ldots, b(w-t+1))). \tag{36}$$

If a state in the Verma module has the form, $N(\bullet)^m \cdot W(\bullet)^n|\lambda\rangle$, where $N(\bullet)$s are any null operators, the inner product of this state with any bra state will get a factor $(\lambda_i - \lambda_j - s)^m$. In order to collect the power factor m for the all state at energy level t, we attach a factor ζ with q in order to mark the null generators. A state of the form, $N(\bullet)^m \cdot W(\bullet)^n|\lambda\rangle$, will get a factor ζ^m. We take a derivative with respect to ζ to pick up the multiplicity factor m. Since the bra states should get the same factor, we multiply the coefficient of q^m by two. This argument shows that the determinant can be divisible by the factor in (34).

4.1 Relation betweem generators

The proof of the theorem is straightforward but a little lengthy. We will divide the argument into small steps.

The independent generators in the $W_{1+\infty}$ algebra at level r are $W(z^{-r}D^s)$ with $s = 0, 1, \cdots, rK - 1$. On the other hand, those in the gl(∞) algebra may be taken as $E_{\lambda_s}(-r+j, -j)$ with $s = 1, \cdots, K$ and $j = 0, \cdots, r-1$. We denote the gl(∞) generator associated with parameter λ as $E_\lambda(i, j)$. From (20), those generators are related by $rK \times rK$ matrix A_r as, $W(z^{-r}D^i) = \sum_{s=1}^{K} \sum_{j=0}^{r-1} (A_r)_{i,(s-1)r+j} E_{\lambda_s}(-r+j, -j)$ with the matrix element,

$$(A_r)_{i,(s-1)r+j} = (\lambda_s + j)^i. \tag{37}$$

The matrix A_r has the form of the Vandermonde matrix. It is hence quite easy to derive its determinant as,

Lemma 2 *Up to the multiplication of constant,*

$$\begin{aligned} det(A_r) &= \prod_{1 \le i < j \le K} \prod_{k,\ell=0}^{r-1} (\lambda_i - \lambda_j + k - \ell) \\ &= \prod_{1 \le i < j \le K} (\lambda_i - \lambda_j)^r \prod_{\epsilon=\pm 1} \prod_{s=1}^{r-1} (\lambda_i - \lambda_j + \epsilon s)^{r-s}. \end{aligned} \tag{38}$$

4.2 Relation between Hilbert spaces

Let $\mathcal{H}(n_1, n_2, n_3, \cdots)$ be the Hilbert space spanned by the product of n_1 elements of level 1 generators, n_2 elements of level 2 generators, n_3 elements of level 3 generators and so on, which are acting on the highest weight state. Here, n_is are the non-negative integers. In order to consider the determinant at finite level, only finite number of them can be non-vanishing. The energy level can be written out of them as,

$$N = \sum_{\ell=1}^{\infty} \ell n_\ell. \tag{39}$$

The basis of this Hilbert space may be written either in terms of the $W_{1+\infty}$ generators or in terms of the gl(∞) generators. The transformation matrix between those basis can be constructed out of the matrices A_r which is introduced in the previous subsection. For the Hilbert space $\mathcal{H}(n_1, n_2, \cdots)$, it is given by,

$$A_1^{(n_1)} \otimes A_2^{(n_2)} \otimes A_3^{(n_3)} \otimes \cdots = \bigotimes_{r=1}^{\infty} A_r^{(n_r)}, \qquad (40)$$

where $A_r^{(n_r)}$ is the transformation matrix between the n_r-th symmetrized product of the original rK basis. In this case, $A_r^{n_r}$ becomes $\begin{pmatrix} rK + n_r - 1 \\ n_r \end{pmatrix} \times \begin{pmatrix} rK + n_r - 1 \\ n_r \end{pmatrix}$ matrix.

To derive the determinant for the matrix (40), we need to remark some identities of the linear algebra, which can be proved easily.

Lemma 3 *(1) Let B_r be an arbitrary $N_r \times N_r$ matrix ($r = 1, 2, \cdots, M$). The determinant for the direct product matrix is given by,*

$$det(B_1 \otimes \cdots \otimes B_M) = \prod_{r=1}^{M} (detB_r)^{\nu_r}, \qquad \nu_r = (\prod_{s=1}^{M} n_s)/n_r. \qquad (41)$$

(2) Let B be an arbitrary $N \times N$ matrix. If we denote $B^{(M)}$ as the representation of B in terms of Mth symmetric basis. Then,

$$detB^{(M)} = (detB)^{\sigma_M}, \qquad \sigma_M = \begin{pmatrix} N + M - 1 \\ N \end{pmatrix}. \qquad (42)$$

The determinant formula for the space $\mathcal{H}(n_1, n_2, n_3, \cdots)$ is derived as,

Lemma 4

$$det(\bigotimes_{r=1}^{\infty} A_r^{(n_r)}) = \prod_{r=1}^{\infty} (detA_r)^{q_r([n])}, \qquad (43)$$

with

$$q_r([n]) = \begin{pmatrix} n_r + rK - 1 \\ rK \end{pmatrix} \prod_{\substack{s=1 \\ s \neq r}}^{\infty} \begin{pmatrix} n_s + sK - 1 \\ n_s \end{pmatrix}.$$

If we use (38), this formula becomes,

$$\prod_{i<j} \left((\lambda_i - \lambda_j)^{\alpha_0([n])} \prod_{s=1}^{\infty} ((\lambda_i - \lambda_j + s)(\lambda_i - \lambda_j + s))^{\alpha_s([n])} \right) \qquad (44)$$

with

$$\alpha_s([n]) = \sum_{t=s+1}^{\infty} (t - s) \begin{pmatrix} n_t + tK - 1 \\ tK \end{pmatrix} \prod_{\substack{u=1 \\ u \neq t}}^{\infty} \begin{pmatrix} n_u + uK - 1 \\ n_u \end{pmatrix}. \qquad (45)$$

4.3 Generating function

Finally, to derive the generating functional (35), we take the summation over infinite indices $(n_1, n_2, \cdots)$ with parameter q,

$$\phi_s(q) = \sum_{n_1,n_2,\cdots} \alpha_s([n]) q^{\sum_{j=1}^{\infty} j n_j}. \tag{46}$$

Combining it with (45), and by using the Taylor expansions,

$$\begin{aligned} \frac{1}{(1-q^u)^{Ku}} &= \sum_{n=0}^{\infty} \begin{pmatrix} n+uK-1 \\ n \end{pmatrix} q^{un} \\ \frac{q^t}{(1-q^t)^{Kt+1}} &= \sum_{n=1}^{\infty} \begin{pmatrix} n+tK-1 \\ tK \end{pmatrix} q^{tn}, \end{aligned} \tag{47}$$

we get the explicit form of the summation,

$$\begin{aligned} \phi_s(q) &= \sum_{t=s+1}^{\infty} \frac{(t-s)q^t}{(1-q^t)^{Kt+1}} \prod_{\substack{u=1 \\ u\neq t}}^{\infty} \frac{1}{(1-q^u)^{Ku}} \\ &= \frac{\partial}{\partial \zeta} \left(\left(\prod_{r=1}^{\infty} \frac{1}{(1-q^r)^r} \right)^K \prod_{t=s+1}^{\infty} \frac{(1-q^t)^{t-s}}{(1-\zeta q^t)^{t-s}} \right) \Bigg|_{\zeta=1}. \end{aligned} \tag{48}$$

It completes our derivation of theorem 1. **Q.E.D.**

5 C dependence

In the following section, we derive the C dependence of the determinant formula for the case, $K = 1$, $b(w) = w - \lambda$, $\Delta(x) = C\frac{e^{\lambda x}-1}{e^x-1}$. The computation is basically carried out by using the $\hat{\mathrm{gl}}(\infty)$ generators. The relation between the nonvanishing generators contain the dependence on λ but it will disappear if we take the determinant. Hence the determinant formula of $\hat{\mathrm{gl}}(\infty)$ is identical with that of $W_{1+\infty}$.

Computation for $K = 1$ is sufficient for understanding the result in our previous computation appendix A, since they are the direct product of $K = 1$ representations.

Unfortunately, the determinant formula for more non-trivial cases, where the characteristic polynomial has roots whose mutual difference is an integer, is still beyond the scope of the present paper.

5.1 Classification by the complete Cartan elements

There are an infinite number of commuting charges (forming the Cartan subalgebra) in the $W_{1+\infty}$ algebra, $W(D^k)$ with $k = 0, 1, 2, \ldots$. In our previous computation, we used only $L_0 \equiv -W(D)$ to classify states. However, much more detailed analysis should be possible if we diagonalize the Hilbert state with respect to the action of all the Cartan elements.

In the framework of the $W_{1+\infty}$ algebra, however, the construction of the Weyl basis is not so straightforward since a simple commutation shows that

$$\Big[W(D^k), W(z^r f(D))\Big] = W(z^r((D+r)^k - D^k)f(D)). \tag{49}$$

It is obvious that we need to diagonalize the operator Q which acts on the $W_{1+\infty}$ generator as

$$Q[W(z^r f(D))] = W(z^r D f(D)).$$

If we restrict $f(D)$ to be a polynomial, we can not find any solution to this equation.

The construction of the diagonal basis becomes possible if we view the $W_{1+\infty}$ algebra from the equivalent $\hat{\mathrm{gl}}(\infty)$ algebra. In [25], they proved that the quasifinite representation of those algebras coincide.

In the language of $\hat{\mathrm{gl}}(\infty)$, the generators $E^{(0)}(i,j)$ are already diagonal with respect to the action of the Cartan elements[6],

$$\Big[W(D^k), E(i,j)\Big] = ((\lambda + i)^k - (\lambda - j)^k)E(i,j). \tag{50}$$

The state $E(-i_1,-j_1)\cdots E(-i_n,-j_n)|\lambda\rangle$ has the eigenvalue,

$$\sum_{a=1}^{n} \Big[(\lambda - i_a)^k - (\lambda + j_a)^k \Big] + \Delta_k, \tag{51}$$

with respect to the action of $W(D^k)$. To summarize, we may claim (for $K = 1$ case),

Proposition 2 Classification of states
Let $I \equiv \{i_1, \cdots, i_n\}$ *(resp.* $J \equiv \{j_1, \cdots, j_n\}$*) be a set of positive (resp. non-negative) integers and* σ *be a permutation of the set of integers* $1, \cdots, n$*. The*

[6] Here and in the following discussion, we omit the superscript (0) in $E(i,j)$ since we are only considering $K = 1$ cases.

eigenvectors with respect to all $W(D^k)$ are given as the linear combinations of the form,

$$\sum_\sigma c_\sigma \prod_{a=1}^n E(-i_a, -j_{\sigma(a)})|\lambda\rangle, \tag{52}$$

with $c_\sigma \in \mathbf{C}$.

5.2 Explicit calculation of inner product

Due to the above theorem, we understand that we need to consider only the class of states of the form (52) to diagonalize the Hilbert space. For that purpose, we would like to prove the explicit form of inner product between those states,

$$\langle\lambda| \prod_{a=1}^n (E(j_{\sigma(a)}, i_a) \prod_{b=1}^n E(-i_b, -j_{\sigma'(b)}))|\lambda\rangle = (-1)^n(-C)^{L(\sigma^{-1}\sigma')}. \tag{53}$$

This equation is valid if all the indices i (or j) are given by different integers. The function $L(\sigma)$ is the "depth" of the permutation σ. It is known that any element of the permutation group can be written as the product of cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix} = (134)(26)(5). \tag{54}$$

The function $L(\sigma)$ is then given by the number of the cycles (including trivial one cycle). In the above example, $L(\sigma) = 3$.

In order to prove (53), we observe that, due to the nature of the $\hat{\mathrm{gl}}(\infty)$ algebra, the indices which appear in (53) may be replaced by other integers,

$$\langle\lambda| \prod_{a=1}^n E(a,a) \prod_{b=1}^n E(-b, -\sigma^{-1}\sigma'(b))|\lambda\rangle.$$

In the following, we will write $\sigma^{-1}\sigma'$ as σ for simplicity. Let us first consider the case $L(\sigma) = 1$. We postulate that the inner product that consists of a cycle of length m is given by $(-1)^m(-C)$ up to $m = n-1$ and prove the statement by induction. This assumption is straightforwardly proved for $m = 1$ since the only non-vanishing contribution comes from the central charge of the algebra. The typical element which consists of one cycle with n element may be taken as,

$$\langle\lambda|E(n,n)\cdots E(1,1)E(-1,-2)E(-2,-3)\cdots E(-n,-1)|\lambda\rangle.$$

We move the element $E(1,1)$ to the right. Non-vanishing commutation relation happens only with $E(-1,-2)$ and $E(-n,-1)$, generating $E(1,-2)$ and $-E(-n,1)$, respectively. However, the latter one vanishes after it is operated on the vacuum. Next, we move thus obtained element $E(1,-2)$ to the right. This time, only nontrivial element is the commutation with $E(-n,-1)$. It gives the contribution $-E(-n,-2)$. In this way, one arrives at the expression,

$$-\langle\lambda|E(n,n)\cdots E(2,2)E(-2,-3)\cdots E(-n,-2)|\lambda\rangle.$$

However, this is the inner product which consists of one cycle with $n-1$ element. By induction assumption, it is equal to $(-1)^n(-C)$.

If there are several cycles, the argument similar to the above can be used to reduce the inner product to the product of cycles. Therefore, we have,

$$\prod_{cycles}(-1)^m(-C) = (-1)^n(-C)^{L(\sigma)}$$

5.3 Young Diagram Classification

Since the inner product formula is written in terms of the permutation group and its representation, we can easily believe that the diagonal basis is explicitly constructed by organizing them such as to give the irreducible representation of the permutation group. To accomplish this, we first prepare some notations.

Let $\mathcal{S}_n$ be the permutation group for n objects. Conjugacy classes of $\mathcal{S}_n$ are classified according to the type of cycle decomposition (as in eq. (54)). Denoting by k_j the number of length-j cycles, we represent a conjugacy class as $(k) = 1^{k_1}2^{k_2}\cdots n^{k_n}$. Note that $k_1 + 2k_2 + \cdots + nk_n = n$, and the number of elements in the class (k) is $N(k) \equiv n!/(1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!)$. Irreducible representations are classified by Young diagrams Y, and we denote the character and dimension of irreducible representation Y by χ_Y and d_Y, respectively.

We define the action of $\sigma \in \mathcal{S}_n$ on the state $\prod_{a=1}^n E(-i_a,-j_a)|\lambda\rangle$ by

$$\sigma \prod_{a=1}^n E(-i_a,-j_a)|\lambda\rangle \equiv \prod_{a=1}^n E(-i_a,-j_{\sigma(a)})|\lambda\rangle. \tag{55}$$

We then introduce the operator

$$B^Y_{\alpha\beta} \equiv \frac{d_Y}{n!} \sum_{\sigma \in \mathcal{S}_n} D^Y(\sigma)_{\alpha\beta} \sigma. \tag{56}$$

Here $D^Y(\sigma)_{\alpha\beta}$ (α, $\beta = 1, 2, ..., d_Y$) is the (real-valued) representation matrix of element σ. Since $\sigma^\dagger = \sigma^{-1}$, we obtain the following relations (see, for example, [31] for the proof):

$$B^{Y\,\dagger}_{\alpha\beta} = B^Y_{\beta\alpha}, \tag{57}$$

$$B^Y_{\alpha\beta} B^{Y'}_{\mu\nu} = \delta_{YY'} \delta_{\beta\mu} B^Y_{\alpha\nu}. \tag{58}$$

With this operator we define new vectors as follows:

$$|Y; \alpha\beta\rangle \equiv B^Y_{\alpha\beta} \prod_{a=1}^{n} E(-i_a, -j_a)|\lambda\rangle. \tag{59}$$

In the following, we will restrict our discussion to the case where no degeneracy exists in the set of indices, $\{j_a\}$ (and also in $\{i_a\}$).

We are now in a position to prove the following theorem:

Theorem 2 Young Diagram Classification
The vectors $|Y; \alpha\beta\rangle$ form an orthogonal basis in the subspace spanned by (52):

$$\langle Y; \alpha\beta | Y'; \mu\nu\rangle = \delta_{YY'} \delta_{\alpha\mu} \delta_{\beta\nu} a^Y_n, \quad a^Y_n = \frac{d_Y}{n!} \prod_{b \in Y} (C - C_b). \tag{60}$$

Here to each box b in the Young diagram, we assign a number C_b as,

0	1	2	3	$\cdots$
-1	0	1	2	$\cdots$
-2	-1	0	1	$\cdots$
-3	-2	-1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(61)

Proof: By using eqs. (57) and (58), the left-hand side of eq. (60) is rewritten as

$$\langle Y; \alpha\beta | Y'; \mu\nu\rangle$$
$$= \delta_{YY'} \delta_{\alpha\mu} \frac{d_Y}{n!} \sum_{\sigma \in \mathcal{S}_n} D^Y(\sigma)_{\beta\nu} \langle\lambda| \prod_a E(j_a, i_a) \sigma \prod_b E(-i_b, -j_b)|\lambda\rangle. \tag{62}$$

Due to eq. (53), $\langle\lambda|\prod_a E(j_a,i_a)\sigma\prod_b E(-i_b,-j_b)|\lambda\rangle = (-1)^n(-C)^{L(\sigma)}$. Since $L(\sigma)$ is a class function, we may denote it by $L(k)$ if $\sigma\in(k)$. We thus obtain

$$\langle Y;\alpha\beta|Y';\mu\nu\rangle = \delta_{YY'}\delta_{\alpha\mu}\frac{d_Y}{n!}(-1)^n\sum_{(k)}(-C)^{L(k)}\sum_{\sigma\in(k)}D^Y(\sigma)_{\beta\nu}. \tag{63}$$

Here we can show that the matrix $\sum_{\sigma\in(k)}D^Y(\sigma)$ always commutes with the actions of any elements in $\mathcal{S}_n$, and thus, due to Schur's lemma, we conclude that $\sum_{\sigma\in(k)}D^Y(\sigma)$ is proportional to the unit matrix. The coefficient is easily calculated by taking its trace, and we obtain

$$\sum_{\sigma\in(k)}D^Y(\sigma)_{\beta\nu} = \frac{N(k)}{d_Y}\chi_Y(k)\delta_{\beta\nu}. \tag{64}$$

Substituting this expression into eq. (63), we obtain

$$\langle Y;\alpha\beta|Y';\mu\nu\rangle = \delta_{YY'}\delta_{\alpha\mu}\delta_{\beta\nu}\frac{(-1)^n}{n!}\sum_{(k)}(-C)^{L(k)}N(k)\chi_Y(k). \tag{65}$$

Since we have the following identity as is proved in appendix B.3:

$$\frac{(-1)^n}{n!}\sum_{(k)}(-C)^{L(k)}N(k)\chi_Y(k) = \frac{d_Y}{n!}\prod_{b\in Y}(C-C_b), \tag{66}$$

we finally obtain eq. (60). **Q.E.D.**

As a simple corollary of the inner product formula, we may derive the condition for the unitarity. The positivity of the Hilbert space may be rephrased as the positivity of the factor a_n^Y for any Y and n. From the table (61), we can immediately prove that this condition is achieved only when C is positive integer.

6 Character formulae for $K=1$ module

In the previous section, we get the explicit form of the norm of diagonal basis in terms of the Young diagrams. To understand the structure of the Hilbert space, we need to count the number of the states which belong to each diagram and have the same eigenvalues for all the Cartan elements.

The generating functional for such degeneracy is neatly expressed by introducing the full character,

$$\chi([g]) \equiv \mathrm{Tr}_{\mathcal{H}} \exp\left(\sum_{k=0}^{\infty} g_k W(D^k)\right). \tag{67}$$

For the $K=1$ module, if there are no null states aside from those coming from characteristic polynomial, the non-vanishing generators are given by $E(-r,-s)$ with $r \geq 1, s \geq 0$. If we combine it with (50), we get the following theorem

Theorem 3 Full character for the generalized Verma module
The full character for the generalized Verma module is

$$\begin{aligned} \chi([g]) &= e^{\sum_{k=0}^{\infty} g_k \Delta_k} \prod_{r=1}^{\infty}\prod_{s=0}^{\infty} \frac{1}{1-u_r v_s}, & (68)\\ &= e^{\sum_{k=0}^{\infty} g_k \Delta_k} \sum_{Y} \tau_Y(x)\tau_Y(y), & (69) \end{aligned}$$

where

$$u_r \equiv e^{\sum_{k=0}^{\infty} g_k(\lambda - r)^k}, \qquad v_s \equiv e^{-\sum_{k=0}^{\infty} g_k(\lambda+s)^k}, \tag{70}$$

τ_Y *is the character of irreducible representation* Y *of* $\mathrm{gl}(\infty)$, (*see appendix B.1*), *and the parameters* x *and* y *are the Miwa variables for* u *and* v, *respectively:*

$$x_\ell = \frac{1}{\ell}\sum_{r=1}^{\infty} u_r^\ell, \qquad y_\ell = \frac{1}{\ell}\sum_{s=0}^{\infty} v_s^\ell, \qquad \ell = 1,2,3,\cdots. \tag{71}$$

Δ_k is defined in (5) with $\Delta(x) = C\frac{e^{\lambda x}-1}{e^x-1}$. The proof of (69) is given in appendix B.2.

If we expand (68) in (infinitely many) parameters u_r and v_s as

$$\sum_{n=0}^{\infty}\sum_{I_n,J_n} N(I_n,J_n) \prod_{i\in I_n} u_i \prod_{j \in J_n} v_j,$$

then $N(I_n, J_n)$ gives the number of the states of the form (52). If we expand each factor in the summation of (69), we can get the degeneracy with respect to each Young diagram Y, and the eigenvalues.

For example, some of the simpler Schur polynomials are expanded as follows:

$$\begin{aligned}
\tau_2(x) &= \frac{x_1^2}{2} + x_2 = \sum_{i<j} u_i u_j + \sum_i u_i^2, \\
\tau_{11}(x) &= \frac{x_1^2}{2} - x_2 = \sum_{i<j} u_i u_j, \\
\tau_3(x) &= \frac{x_1^3}{6} + x_1 x_2 + x_3 = \sum_{i<j<k} u_i u_j u_k + \sum_{i\neq j} u_i^2 u_j + \sum_i u_i^3, \\
\tau_{21}(x) &= \frac{x_1^3}{3} - x_3 = 2\sum_{i<j<k} u_i u_j u_k + \sum_{i\neq j} u_i^2 u_j, \\
\tau_{111}(x) &= \frac{x_1^3}{6} - x_1 x_2 + x_3 = \sum_{i<j<k} u_i u_j u_k. \qquad (72)
\end{aligned}$$

The result in the previous section shows that the generalized Verma module for $K = 1$ becomes reducible when $C =$ integer. We call this represenatation as the *degenerate representation*. The full character formula for the irreducible module can be obtained by combining previous theorem with (60).

Theorem 4 Full Character of Degenerate Representations
Let V_n (resp. H_n) be the set of the Young diagrams the number of whose columns (resp. rows) does not exceed n, then the full characters of $C = \pm n$ are given by,

$$\begin{aligned}
\chi_{C=n} &= e^{\sum_{k=0}^{\infty} g_k \Delta_k} \sum_{Y\in V_n} \tau_Y(x)\tau_Y(y), \\
\chi_{C=-n} &= e^{\sum_{k=0}^{\infty} g_k \Delta_k} \sum_{Y\in H_n} \tau_Y(x)\tau_Y(y). \qquad (73)
\end{aligned}$$

In the character formula for non-integer C (69), the summation is over every Young diagram, or in other words, the two-dimensional sum. On the other hand, the character formula for degenerate representation (73), the summation is restricted to one-dimensional indices. The degeneracy of the Hilbert space reduces the dimensionality of the system from three to two, which naturally explains the hybrid nature of $W_{1+\infty}$ symmetry.

To make our formula (73) into more familiar form, we give the explicit form of the characters which depend only on the parameter q associated with the eigenvalue of $L_0 \equiv -W(D)$. For this restriction, we replace

$$u_r = q^r, \qquad v_r = q^r. \tag{74}$$

Namely, we put $g_k = -2\pi i\tau\delta_{k,1}$ with $q = e^{2\pi i\tau}$. The Miwa variables (71) are then rewritten as,

$$x_\ell = \frac{1}{\ell}\frac{q^\ell}{1-q^\ell}, \qquad y_\ell = \frac{1}{\ell}\frac{1}{1-q^\ell}. \tag{75}$$

After these replacements, a compact form of the Schur polynomial can be given. We introduce,

$$\begin{aligned}
&f_k(q; m_1, \cdots, m_k) \\
&\equiv \prod_{j=1}^{m_k} \frac{1}{(1-q^j)(1-q^{m_{k-1}+1+j})\cdots(1-q^{m_{k-1}+\cdots+m_1+k-1+j})} \\
&= \prod_{j=1}^{m_k}\prod_{s=0}^{k-1}\left(1-q^{\sum_{t=1}^{s} m_{k-t}+s+j}\right)^{-1}.
\end{aligned} \tag{76}$$

for non-negative integers m_i. When $m_k = 0$, we put $f_k(q; m_1, \cdots, 0) = 1$. The Schur polynomial is then rewritten as,

$$\begin{aligned}
\tau_Y(x) &= q^{\sum_{j=1}^n \frac{j(j+1)}{2} m_j} \prod_{k=1}^n f_k(q; m_1, \cdots, m_k) \\
\tau_Y(y) &= q^{\sum_{j=1}^n \frac{j(j-1)}{2} m_j} \prod_{k=1}^n f_k(q; m_1, \cdots, m_k) \\
\tau_{Y'}(x) &= q^{\frac{1}{2}\sum_{j=1}^n (\sum_{s=j}^n m_s)(\sum_{s=j}^n m_s+1)} \prod_{k=1}^n f_k(q; m_1, \cdots, m_k) \\
\tau_{Y'}(y) &= q^{\frac{1}{2}\sum_{j=1}^n (\sum_{s=j}^n m_s)(\sum_{s=j}^n m_s-1)} \prod_{k=1}^n f_k(q; m_1, \cdots, m_k).
\end{aligned} \tag{77}$$

where $Y = \{m_1 + \cdots + m_n, m_2 + \cdots + m_n, \cdots, m_n\}$, and Y' is the transpose of the Young diagram Y.

The full character formulae (73) then give,

$$\begin{aligned}
\chi_{C=n}(q) &= q^{n\lambda(\lambda-1)/2}\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} q^{\sum_{j=1}^{n}(\sum_{s=j}^{n} m_s)^2}\prod_{k=1}^{n} f_k(q;m_1,\cdots,m_k)^2,\\
\chi_{C=-n}(q) &= q^{-n\lambda(\lambda-1)/2}\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} q^{\sum_{j=1}^{n} j^2 m_j}\prod_{k=1}^{n} f_k(q;m_1,\cdots,m_k)^2, \quad (78)
\end{aligned}$$

with positive integer n.

In our previous letter [26], we gave the character formulae for $C=\pm 1$ and $C=n>0$. For $C=\pm 1$ cases, our general formula (78) gives,

$$\begin{aligned}
\chi_{C=1}(q) &= q^{\lambda(\lambda-1)/2}\sum_{m=0}^{\infty} q^{m^2}\prod_{j=1}^{m}\frac{1}{(1-q^j)^2},\\
\chi_{C=-1}(q) &= q^{-\lambda(\lambda-1)/2}\sum_{m=0}^{\infty} q^{m}\prod_{j=1}^{m}\frac{1}{(1-q^j)^2}, \qquad (79)
\end{aligned}$$

which are exactly same as our previous formulae. For $C=n>0$ cases, what we derived previously were,

$$\chi_{C=n}(q) = q^{n\lambda(\lambda-1)/2}\prod_{j=1}^{\infty}\prod_{k=1}^{n}(1-q^{j+k-1})^{-1}. \qquad (80)$$

We have confirmed that it is equivalent to (78) by Taylor expansion up to q^{30}.

7 Discussion

Although we understand the representation of $W_{1+\infty}$ to some extent, there are still many things to be understood. In particular, the C dependence for $K>1$ is not still well-understood. We hope to report on the full detail in our future issue. In the mathematical side, we are currently working on the supersymmetric extension [20], the structure of the subalgebras [30]. Those works will be related to the topological field theory and/or the matrix models.

The relation of the $W_{1+\infty}$ algebra with extended objects seems also interesting from the geometrical viewpoint. In this work, we used the basis D^k to parametrize the generators. However, as Kac and Radul observed, there is another parametrization of generators which leads to different representation. One example is to use q^{kD} basis. One may regard it as the representation based on torus (instead of sphere). In general, one may imagine the possibility of the representation theories based on higher-genus Riemann surfaces. If we want to apply the $W_{1+\infty}$ algebra to membranes, for example, we need to consider the "degenerate" three manifolds where such Riemann surface may appear. The hybrid nature of the $W_{1+\infty}$ algebra that we have observed in this paper may be important to understand such phenomena.

After we submited this paper, the full character for the unitary representation has been given in [32]. One may check [33] that for the special case ($K = 1$), the formula obtained there coincides with ours (73) with $C = n > 0$.

Appendix A: Determinant formulae at lower degrees

In this appendix, we give the explicit form of the functions $A_r(C)$ and $B_r(\lambda)$ defined in (10) We can parametrize those functions in the form,

$$A_r(C) = \prod_{\ell \in \mathbf{Z}} (C - \ell)^{\alpha(\ell)}, \qquad B_r(\lambda) = \prod_{\ell \in \mathbf{Z}} (\lambda - \ell)^{\beta(\ell)}$$

We make tables for the index $\alpha(\ell)$ and $\beta(\ell)$. We note that $\beta(\ell) = \beta(-\ell)$. Hence we will write them only for $\ell \geq 0$.

<u>$K = 1$</u>: $B_r \equiv 1$ due to the spectral flow symmetry [26].

r	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\alpha(4)$	$\alpha(5)$	$\alpha(6)$	$\alpha(7)$
1	0	1	0	0	0	0	0	0	0
2	0	3	1	0	0	0	0	0	0
3	0	6	3	1	0	0	0	0	0
4	1	13	8	3	1	0	0	0	0
5	3	24	17	8	3	1	0	0	0
6	10	48	37	19	8	3	1	0	0
7	23	86	71	41	19	8	3	1	0
8	54	161	138	85	43	19	8	3	1

<u>$K = 2$</u>

r	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	4	1	0	0	10	2	0	0
3	0	12	4	1	0	34	8	2	0
4	1	34	14	4	1	108	30	8	2

$\underline{K=3}$

r	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	5	1	0	0	12	2	0	0
3	0	19	5	1	0	50	10	2	0

$\underline{K=4}$

r	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	6	1	0	0	14	2	0	0
3	0	27	6	1	0	68	12	2	0

$\underline{K=5}$

r	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	7	1	0	0	16	2	0	0

Appendix B: Free-fermion representation of characters for the permutation and the general linear groups

Characters of the permutation group and the general linear group can be expressed in terms of free fermions [28][29]. In this appendix we summarize the useful formulae.

B.1

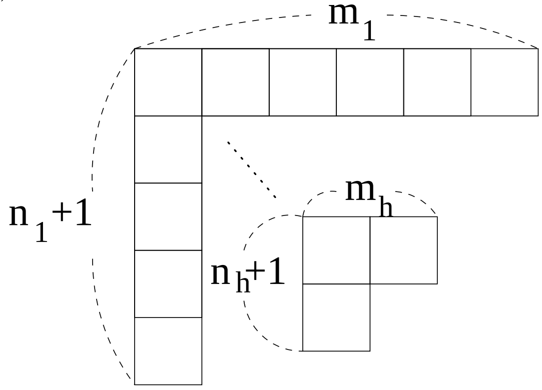
Free fermions[7] $\bar{\flat}(z) = \sum_{n\in\mathbf{Z}} \bar{\flat}_n z^{-n-1}$, $\flat(z) = \sum_{n\in\mathbf{Z}} \flat_n z^{-n}$ and the vacuum state $\|0\rangle\rangle$ are defined by

$$\begin{aligned} &\{\bar{\flat}_m, \flat_n\} = \delta_{m+n,0}, \quad \{\bar{\flat}_m, \bar{\flat}_n\} = \{\flat_m, \flat_n\} = 0, \\ &\bar{\flat}_m\|0\rangle\rangle = \flat_n\|0\rangle\rangle = 0, \qquad (m \geq 0, n \geq 1). \end{aligned} \tag{1}$$

The fermion Fock space is a linear span of $\prod_i \bar{\flat}_{-m_i} \prod_j \flat_{-n_j}\|0\rangle\rangle$. The $U(1)$ current $\mathcal{J}(z) = \sum_{n\in\mathbf{Z}} \mathcal{J}_n z^{-n-1}$ is defined by $\mathcal{J}(z) =: \bar{\flat}(z)\flat(z) :$, *i.e.*, $\mathcal{J}_n = \sum_{m\in\mathbf{Z}} : \bar{\flat}_m \flat_{n-m} :$, where the normal ordering $: \bar{\flat}_m \flat_n :$ means $\bar{\flat}_m \flat_n$ if $m \leq -1$ and $-\flat_n \bar{\flat}_m$ if $m \geq 0$. Their commutation relations are

$$[\mathcal{J}_n, \mathcal{J}_m] = n\delta_{n+m,0}, \quad [\mathcal{J}_n, \bar{\flat}_m] = \bar{\flat}_{n+m}, \quad [\mathcal{J}_n, \flat_m] = -\flat_{n+m}. \tag{2}$$

To the Young diagram Y of the following form ($m_1 > \cdots > m_h \geq 1$, $n_1 > \cdots > n_h \geq 0$):



we define the corresponding state $\|Y\rangle\rangle$ as

$$\|Y\rangle\rangle \equiv \prod_{i=1}^{h} \bar{\flat}_{-m_i} \flat_{-n_i} (-1)^{n_i} \|0\rangle\rangle. \tag{3}$$

Note that the number of fermion bilinears, h, corresponds to that of hooks in the Young diagram. Bra states are obtained from ket states by † operation ($\bar{\flat}_n{}^\dagger = \flat_{-n}$) with the normalization $\langle\langle 0\|0\rangle\rangle = 1$; for example, $\langle\langle Y\| =$

[7] We use this notation to avoid a confusion with the free fermions used in the free-field realization of $W_{1+\infty}$. Relation to usual free fermions $\bar{\psi}(z) = \sum_{r\in\mathbf{Z}+1/2} \bar{\psi}_r z^{-r-1/2}$, $\psi(z) = \sum_{r\in\mathbf{Z}+1/2} \psi_r z^{-r-1/2}$ is given by $\bar{\flat}_n = \bar{\psi}_{n+1/2}$, $\flat_n = \psi_{n-1/2}$.

$\|Y\rangle\!\rangle^\dagger = \langle\!\langle 0\| \prod_{i=1}^h \bar{\flat}_{n_i} \flat_{m_i} (-1)^{n_i}$ and $\langle\!\langle Y\|Y'\rangle\!\rangle = \delta_{YY'}$. Note that $\{\|Y\rangle\!\rangle\}$ is an orthonormal basis of the fermion Fock space with vanishing $U(1)$ charge.

Irreducible representations of the permutation group $\mathcal{S}_n$ and the general linear group $GL(N)$ are both characterized by the Young diagrams Y. We denote their characters by $\chi_Y(k)$ and $\tau_Y(x)$, respectively. Here $(k) = 1^{k_1} 2^{k_2} \cdots n^{k_n}$ stands for the conjugacy class of $\mathcal{S}_n$; $k_1 + 2k_2 + \cdots + nk_n = n =$the number of boxes in Y. $x = [x_\ell]$ $(\ell = 1, 2, 3, \cdots)$ stands for $x_\ell = \frac{1}{\ell}\mathrm{tr}\ g^\ell = \frac{1}{\ell}\sum_{i=1}^N \epsilon_i^\ell$ for an element g of $GL(N)$, and in this case the number of boxes in Y is a rank of tensor for $GL(N)$. In section 6, we consider the Lie algebra of $GL(N)$, gl(N), for sufficiently large N.

$\chi_Y(k)$ and $\tau_Y(x)$ are expressed as follows:

$$\chi_Y(k) \;=\; \langle\!\langle 0\| \mathcal{J}_1^{k_1} \mathcal{J}_2^{k_2} \cdots \mathcal{J}_n^{k_n} \|Y\rangle\!\rangle, \tag{4}$$

$$\tau_Y(x) \;=\; \langle\!\langle 0\| \exp\Big(\sum_{\ell=1}^\infty x_\ell \mathcal{J}_\ell\Big) \|Y\rangle\!\rangle. \tag{5}$$

We remark that they can also be written as $\chi_Y(k) = \langle\!\langle Y\| \mathcal{J}_{-1}^{k_1} \mathcal{J}_{-2}^{k_2} \cdots \mathcal{J}_{-n}^{k_n} \|0\rangle\!\rangle$ and $\tau_Y(x) = \langle\!\langle Y\| \exp\left(\sum_{\ell=1}^\infty x_\ell \mathcal{J}_{-\ell}\right) \|0\rangle\!\rangle$.

B.2

For arbitrary parameters u_r and v_s such that the following (infinite) product converges, we can show the following identity,

$$\prod_r \prod_s \frac{1}{1 - u_r v_s} = \sum_Y \tau_Y(x) \tau_Y(y), \tag{6}$$

where the summation runs over all the Young diagrams and x, y are the Miwa variables for u, v,

$$x_\ell \equiv \frac{1}{\ell} \sum_r u_r^\ell, \quad y_\ell \equiv \frac{1}{\ell} \sum_s v_s^\ell, \quad (\ell = 1, 2, 3, \cdots). \tag{7}$$

Proof:

$$\begin{aligned} \prod_r \prod_s \frac{1}{1 - u_r v_s} &= \exp\Big(\sum_r \sum_s \log \frac{1}{1 - u_r v_s}\Big) = \exp\Big(\sum_r \sum_s \sum_{\ell=1}^\infty \frac{1}{\ell} (u_r v_s)^\ell\Big) \\ &= \exp\Big(\sum_{\ell=1}^\infty \ell x_\ell y_\ell\Big) \end{aligned}$$

$$
\begin{aligned}
&= \langle\langle 0\| \exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) \exp\Big(\sum_{\ell=1}^{\infty} y_\ell \mathcal{J}_{-\ell}\Big)\|0\rangle\rangle \\
&= \sum_Y \langle\langle 0\| \exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big)\|Y\rangle\rangle\langle\langle Y\| \exp\Big(\sum_{\ell=1}^{\infty} y_\ell \mathcal{J}_{-\ell}\Big)\|0\rangle\rangle \\
&= \sum_Y \tau_Y(x)\tau_Y(y).
\end{aligned}
$$

We have used the completeness of $\{\|Y\rangle\rangle\}$ in the fermion Fock space with vanishing $U(1)$ charge. **Q.E.D.**

B.3

In subsection 5.3 we need the following quantity for $\mathcal{S}_n$,

$$
a_n^Y \equiv \frac{(-1)^n}{n!} \sum_{(k)} (-C)^{L(k)} N(k) \chi_Y(k), \tag{8}
$$

where $(k) = 1^{k_1}2^{k_2}\cdots n^{k_n}$, $k_1 + 2k_2 + \cdots + nk_n = n$, $L(k) = k_1 + k_2 + \cdots + k_n$ and $N(k) = n!/(1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!)$. We remark that a_n^Y is a polynomial of C with degree n,

$$
a_n^Y = \frac{d_Y}{n!} C^n + \cdots .
$$

To calculate a_n^Y, we introduce its generating function $a^Y(t) = \sum_{n=0}^{\infty} a_n^Y t^n$. By eq. (4), $a^Y(t)$ becomes

$$
a^Y(t) = \langle\langle 0\| \exp\Big(C \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} \mathcal{J}_\ell t^\ell\Big)\|Y\rangle\rangle. \tag{9}
$$

By rewriting eq. (3) as

$$
\begin{aligned}
\|Y\rangle\rangle &= \prod_{i=1}^{h} (-1)^{n_i+i-1} \cdot \bar{\flat}_{-m_1} \cdots \bar{\flat}_{-m_h} \flat_{-n_1} \cdots \flat_{-n_h} \|0\rangle\rangle \\
&= \oint \prod_{i=1}^{h} \frac{dz_i}{2\pi i}\frac{dw_i}{2\pi i} \prod_{i=1}^{h} z_i^{-m_i} w_i^{-n_i-1} \\
&\quad \times \prod_{i=1}^{h} (-1)^{n_i+i-1} \cdot \bar{\flat}(z_1) \cdots \bar{\flat}(z_h) \flat(w_1) \cdots \flat(w_h) \|0\rangle\rangle,
\end{aligned}
$$

$a^Y(t)$ can be expressed as

$$
\begin{aligned}
a^Y(t) \;=\; & \oint \prod_{i=1}^{h} \frac{dz_i}{2\pi i}\frac{dw_i}{2\pi i} \prod_{i=1}^{h} z_i^{-m_i} w_i^{-n_i-1} \\
& \times \prod_{i=1}^{h} \frac{(1+tz_i)^C}{(1+tw_i)^C}(-1)^{n_i+i-1} \cdot \langle\!\langle 0 \| \bar{\flat}(z_1)\cdots\bar{\flat}(z_h)\flat(w_1)\cdots\flat(w_h)\|0\rangle\!\rangle .
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
\exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big)\bar{\flat}(z)\exp\Big(-\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) \;&=\; \exp\Big(\sum_{\ell=1}^{\infty} x_\ell z^\ell\Big)\bar{\flat}(z), \\
\exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big)\flat(z)\exp\Big(-\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) \;&=\; \exp\Big(-\sum_{\ell=1}^{\infty} x_\ell z^\ell\Big)\flat(z),
\end{aligned} \tag{10}
$$

in particular, for $U(C) = \exp\Big(C\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}}{\ell}\mathcal{J}_\ell t^\ell\Big)$

$$
\begin{aligned}
U(C)\bar{\flat}(z)U(-C) \;&=\; (1+tz)^C\bar{\flat}(z), \\
U(C)\flat(z)U(-C) \;&=\; (1+tz)^{-C}\flat(z).
\end{aligned}
$$

By expanding $a^Y(t)$, we obtain

$$
a_n^Y = \sum_{r_i,s_i}\prod_{i=1}^{h}\binom{C}{m_i+r_i}\binom{-C}{n_i-s_i}(-1)^{n_i+i-1}\cdot\langle\!\langle 0\|\bar{\flat}_{r_1}\cdots\bar{\flat}_{r_h}\flat_{-s_1}\cdots\flat_{-s_h}\|0\rangle\!\rangle,
$$

where the summation runs over $r_i \geq 0$, $0 \leq s_i \leq n_i$, $\sum_{i=1}^{h} r_i = \sum_{i=1}^{h} s_i$. $\binom{x}{n}$ is defined by $\binom{x}{n} = [x]_n/n!$ and $[x]_n = \prod_{i=0}^{n-1}(x-i)$. Thus, a_n^Y is divided by $\prod_{i=1}^{h}\binom{C}{m_i}$ as a polynomial of C. Similarly, starting from

$$
\begin{aligned}
\|Y\rangle\!\rangle \;=\; & \oint \prod_{i=1}^{h} \frac{dz_i}{2\pi i}\frac{dw_i}{2\pi i} \prod_{i=1}^{h} z_i^{-m_i} w_i^{-n_i-1} \\
& \times \prod_{i=1}^{h}(-1)^{n_i+i-1+h}\cdot \flat(w_1)\cdots\flat(w_h)\bar{\flat}(z_1)\cdots\bar{\flat}(z_h)\|0\rangle\!\rangle,
\end{aligned}
$$

we obtain

$$
a_n^Y = \sum_{r_i,s_i}\prod_{i=1}^{h}\binom{C}{m_i-r_i}\binom{-C}{n_i+s_i}(-1)^{n_i+i-1+h}\cdot\langle\!\langle 0\|\flat_{s_1}\cdots\flat_{s_h}\bar{\flat}_{-r_1}\cdots\bar{\flat}_{-r_h}\|0\rangle\!\rangle,
$$

where the summation runs over $1 \leq r_i \leq m_i$, $s_i \geq 1$, $\sum_{i=1}^h r_i = \sum_{i=1}^h s_i$. Therefore, a_n^Y is divided by $\prod_{i=1}^h \binom{-C}{n_i+1}$ as a polynomial of C. Combining these results, we can conclude that a_n^Y is $\prod_{i=1}^h \binom{C}{m_i}\binom{-C}{n_i+1}/C^h$ up to constant. We thus finally obtain

$$a_n^Y = \frac{d_Y}{n!} \prod_{i=1}^h [C]_{m_i}[-C-1]_{n_i}(-1)^{n_i}. \tag{11}$$

This result can be converted into a simpler form as given in subsection 5.3:

$$a_n^Y = \frac{d_Y}{n!} \prod_{b \in Y} (C - C_b). \tag{12}$$

We can give another easy proof of the above result. The relation between the transformed basis $U(C)\bar{\flat}_{-m}U(-C)$ with the original ones $\bar{\flat}_{-m}$ can be obtained if we expand the factor $(1+tz)^C$ in z around 0. For non-integer C, $U(C)\bar{\flat}_{-m}U(-C)$ is written by infinite sum with respect to $\bar{\flat}_{-m+\ell}$ with $\ell = 0, 1, 2, 3, \cdots$. However, if $C \in \mathbf{Z}$, truncation of the summation happens. We move each operator $\bar{\flat}_{-m}\flat_{-n}$ in (3) to the left of $U(C)$. When it is acted on the bra vacuum, it vanishes when $C = -n, -n+1, \cdots, m-2, m-1$. It gives the following assignment of the polynomial of C to each pair of the fermion operators:

$$\bar{\flat}_{-m}\flat_{-n} \Longleftrightarrow (C+n)(C+n-1)\ldots(C-m+2)(C-m+1). \tag{13}$$

Combining these factors for each hook, we get the assignment in (61).

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