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Representation Theory of The $W_{1+\infty}$ Algebra*

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Abstract

We review the recent development in the representation theory of the $W_{1+\infty}$ algebra. The topics that we concern are,

- Quasifinite representation
- Free field realizations
- (Super) Matrix Generalization
- Structure of subalgebras such as W_∞ algebra
- Determinant formula
- Character formula.

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1 Introduction

Symmetry is one of the most important concepts in modern physics, *e.g.* SU(3) symmetry in quark model, gauge symmetry in gauge theory, conformal symmetry in conformal field theory. To study physical system from symmetry point of view, we need the representation theory of the corresponding symmetry algebra; finite dimensional Lie algebra for quark model or gauge theory, infinite dimensional Lie algebra (the Virasoro algebra) for two-dimensional conformal field theory. Conformal symmetry restricts theories very severely due to its infinite dimensionality[13]. In fact, by combining the knowledge of the representation theory of the Virasoro algebra and the requirement of the modular invariance, the field contents of the minimal models were completely classified[16]. Another example of the powerfulness of the symmetry argument is that correlation functions of the XXZ model were determined by using the representation theory of affine quantum algebra $U_q\widehat{sl}_2$[18].

When conformal field theory has some extra symmetry, the Virasoro algebra must be extended, *i.e.* semi-direct products of the Virasoro algebra with Kac-Moody algebras, superconformal algebras, the W algebras and parafermions. The W_N algebra is generated by currents of conformal spin $2, 3, \cdots, N$, and their commutation relation has non-linear terms[45, 14]. The W infinity algebras are Lie algebras obtained by taking $N \to \infty$ limit of the W_N algebra.

The W infinity algebras naturally arise in various physical systems, such as two-dimensional quantum gravity[23, 21, 28, 31, 41, 25], the quantum Hall effects[17, 27], the membrane[15, 22], the large N QCD[26, 20], and also in the construction of gravitational instantons[43, 44, 36](see also [9]). To study these systems we first need to prepare the representation theory of W infinity algebras, especially the most fundamental one, the $W_{1+\infty}$ algebra.

To begin with, we present a short review of the history of the W infinity algebras before the appearance of ref.[30]. By taking an appropriate $N \to \infty$ limit of the W_N algebra, we can obtain a Lie algebra with infinite number of currents. Depending on how the background charge scales with N, there are many kinds of W infinity algebras. The first example is the w_∞ algebra[6]. Its generators w^k_n ($k, n \in \mathbb{Z}, k \geq 2$) have the commutation relation,

$$\Big[w^k_n, w^\ell_m\Big] = \Big((\ell-1)n - (k-1)m\Big)w^{k+\ell-2}_{n+m}. \tag{1.1}$$

w^2_n generates the Virasoro algebra without center and w^k_n has conformal spin k. This w_∞ algebra has a geometrical interpretation as the algebra of area-preserving diffeomorphisms of two-dimensional phase space. However, w_∞ admits a central extension only in the

Virasoro sector,

$$\left[w_n^k, w_m^\ell\right] = \Big((\ell-1)n-(k-1)m\Big)w_{n+m}^{k+\ell-2} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\delta^{k\ell}\delta^{k2}. \tag{1.2}$$

To introduce a central extension in all spin sectors, we must take another type of the limit $N \to \infty$ or the deformation of the w_∞ algebra. By deforming w_∞, Pope, Romans and Shen constructed such algebra, the W_∞ algebra, in algebraic way by requiring linearity, closure and the Jacobi identity[37]. The W_∞ algebra is generated by $\widetilde{W}_n^k$ $(k, n \in \mathbb{Z}, k \geq 2)$ and its commutation relation is given by

$$\begin{aligned}\left[\widetilde{W}_n^k, \widetilde{W}_m^\ell\right] &= \sum_{r=0}^{\infty} \tilde{g}_{2r}^{k\ell}(n,m)\widetilde{W}_{n+m}^{k+\ell-2-2r} \\ &\quad + \tilde{c}\delta^{k\ell}\delta_{n+m,0}\frac{1}{k-1}\binom{2(k-1)}{k-1}^{-1}\binom{2k}{k}^{-1}\prod_{j=-(k-1)}^{k-1}(n+j),\end{aligned} \tag{1.3}$$

where $\tilde{c}$ is the central charge of the Virasoro algebra generated by $\widetilde{W}_n^2$, and the structure constant $\tilde{g}_r^{k\ell}$ is given by

$$\tilde{g}_r^{k\ell}(n,m) = \frac{1}{2^{2r+1}(r+1)!}\phi_r^{k\ell}(0,0)N_r^{k,\ell}(n,m), \tag{1.4}$$

$$\begin{aligned}N_r^{x,y}(n,m) &= \sum_{s=0}^{r+1}(-1)^s\binom{r+1}{s}[x-1+n]_{r+1-s}[x-1-n]_s \\ &\qquad \times [y-1-m]_{r+1-s}[y-1+m]_s,\end{aligned} \tag{1.5}$$

$$\phi_r^{k\ell}(x,y) = {}_4F_3\left[\begin{matrix}-\frac{1}{2}-x-2y, \frac{3}{2}-x+2y, -\frac{r+1}{2}+x, -\frac{r}{2}+x \\ -k+\frac{3}{2}, -\ell+\frac{3}{2}, k+\ell-r-\frac{3}{2}\end{matrix}; 1\right], \tag{1.6}$$

$${}_4F_3\left[\begin{matrix}a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3\end{matrix}; z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n}\frac{z^n}{n!}, \tag{1.7}$$

$$[x]_n = \prod_{j=0}^{n-1}(x-j), \qquad (x)_n = \prod_{j=0}^{n-1}(x+j), \qquad \binom{x}{n} = \frac{[x]_n}{n!}. \tag{1.8}$$

The w_∞ algebra is obtained from W_∞ by contraction; we take the $q \to 0$ limit after rescaling $\widetilde{W}_n^k \to q^{2-k}\widetilde{W}_n^k$. Furthermore they constructed the $W_{1+\infty}$ algebra which contains a spin 1 current[38]. The $W_{1+\infty}$ algebra is generated by W_n^k $(k, n \in \mathbb{Z}, k \geq 1)$ and its commutation relation is given by

$$\begin{aligned}\left[W_n^k, W_m^\ell\right] &= \sum_{r=0}^{\infty} g_{2r}^{k\ell}(n,m)W_{n+m}^{k+\ell-2-2r} \\ &\quad + c\delta^{k\ell}\delta_{n+m,0}\frac{2}{k}\binom{2(k-1)}{k-1}^{-1}\binom{2k}{k}^{-1}\prod_{j=-(k-1)}^{k-1}(n+j),\end{aligned} \tag{1.9}$$

where c is the central charge of the Virasoro algebra generated by W_n^2, and the structure constant $g_r^{k\ell}$ is given by

$$g_r^{k\ell}(n,m) = \frac{1}{2^{2r+1}(r+1)!}\phi_r^{k\ell}(0,-\tfrac{1}{2})N_r^{k,\ell}(n,m). \tag{1.10}$$

Since $\tilde{g}_r^{k\ell}(n,m)=0$ for $k+\ell-r<4$ and $g_r^{k\ell}(n,m)=0$ for $k+\ell-r<3$, the summations over r are finite sum and the algebras close. These commutation relations are consistent with the hermitian conjugation $\widetilde{W}_n^{k\dagger}=\widetilde{W}_{-n}^k$, $W_n^{k\dagger}=W_{-n}^k$, and have diagonalized central terms. The $W_{1+\infty}$ algebra contains the W_∞ algebra as a subalgebra[39], but it is nontrivial in these basis. Moreover various extensions were constructed; super extension ($W_\infty^{1,1}$)[12, 7], u(M) matrix version of W_∞ (W_∞^M)[8], u(N) matrix version of $W_{1+\infty}$ ($W_{1+\infty}^N$)[35], and they were unified as $W_\infty^{M,N}$[34]. Based on the coset model $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$, a nonlinear deformation of W_∞, $\widehat{W}_\infty(k)$, was also constructed[10].

When we study the representation theory of W infinity algebras, we encounter the difficulty that infinitely many states possibly appear at each energy level, reflecting the infinite number of currents. For example, even at level 1, there are infinite number of states $W_{-1}^k|\mathrm{hws}\rangle$ ($k=1,2,3,\cdots$) for generic representation, so we could not treat these states, *e.g.* computation of the Kac determinant. Moreover they are not the simultaneous eigenstates of the Cartan generators W_0^k ($k=1,2,\cdots$). Only restricted class of the representation were studied by using $\mathbb{Z}_\infty$ parafermion and coset model[8] or free field realizations[34]. In the free field realization, there are only finite number of states at each energy level because the number of oscillators is finite at each level.

Last year Kac and Radul overcame this difficulty of infiniteness[30]. They proposed the quasifinite representation, which has only finite number of states at each energy level, and studied this class of representations in detail. From physicist point of view, this notion is the abstraction of the property that the free field realizations have.

In this article, we would like to review the recently developed representation theory of the W infinity algebras, mainly the $W_{1+\infty}$ algebra[30, 32, 5, 1, 2, 3, 4, 24]. In section 2 we give the definition of the $W_{1+\infty}$ algebra and its (super)matrix generalizations. Various subalgebras of $W_{1+\infty}$ are also given. In section 3 free field realizations of $W_{1+\infty}$ and $W_{1+\infty}^{M,N}$ are given. Using these we derive the full character formulae for those representations. In section 4 the quasifinite representation is introduced, and its general properties are presented. In section 5, after describing the Verma module, we compute the Kac determinant at lower levels for some representations (its results are given in appendix A). On the basis of this computation we derive the analytic form of the Kac determinant and the full character formulae. Appendix B is devoted to the description of the Schur function.

2 W infinity algebras

In this section we define the $W_{1+\infty}$ algebra and its (super)matrix generalization $W_{1+\infty}^{M,N}$. We also give a systematic method to construct a family of subalgebras of $W_{1+\infty}$.

2.1 The $W_{1+\infty}$ algebra

Since the W algebras were originally introduced as extensions of the Virasoro algebra, we first recall the Virasoro algebra. Let us consider the Lie algebra of the diffeomorphism group on the circle whose coordinate is z. The generator of this Lie algebra is $l_n = -z^{n+1}\frac{d}{dz}$ and its commutation relation is

$$[l_n, l_m] = (n-m)l_{n+m}.$$

The Virasoro algebra, whose generators are denoted as L_n, is the central extension of this algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}.$$

Besides l_n, we may consider the higher order differential operators on the circle, $z^n(\frac{d}{dz})^m$ $(n,m\in\mathbb{Z}, m\geq 0)$. Instead of $z^n(\frac{d}{dz})^m$, we take a basis z^nD^k $(n,k\in\mathbb{Z}, k\geq 0)$ with $D = z\frac{d}{dz}$. Since $f(D)z^n = z^nf(D+n)$, the commutation relation of the differential operators is

$$\Big[z^nf(D), z^mg(D)\Big] = z^{n+m}f(D+m)g(D) - z^{n+m}f(D)g(D+n), \tag{2.1}$$

where f and g are polynomials. The $W_{1+\infty}$ algebra is the central extension of this Lie algebra of differential operators on the circle[39, 11, 30]. We denote the corresponding generators by $W(z^nD^k)$ and the central charge by C. The commutation relation is[30]

$$\begin{aligned}&\Big[W(z^nf(D)), W(z^mg(D))\Big]\\ =\ & W(z^{n+m}f(D+m)g(D)) - W(z^{n+m}f(D)g(D+n)) + C\Psi(z^nf(D), z^mg(D)).\end{aligned} \tag{2.2}$$

Here the 2-cocycle Ψ is defined by

$$\begin{aligned}&\Psi(z^nf(D), z^mg(D))\\ =\ & \delta_{n+m,0}\Big(\theta(n\geq 1)\sum_{j=1}^{n}f(-j)g(n-j) - \theta(m\geq 1)\sum_{j=1}^{m}f(m-j)g(-j)\Big),\end{aligned} \tag{2.3}$$

where $\theta(P)=1$ (or 0) when the proposition P is true (or false). The 2-cocycle is unique up to coboundary[29]. By introducing z^ne^{xD} as a generating series for z^nD^k, the above 2-cocycle and commutation relation can be rewritten in a simpler form:

$$\Psi(z^ne^{xD}, z^me^{yD}) = -\frac{e^{mx}-e^{ny}}{e^{x+y}-1}\delta_{n+m,0}, \tag{2.4}$$

$$\Big[W(z^ne^{xD}), W(z^me^{yD})\Big] = (e^{mx}-e^{ny})\,W(z^{n+m}e^{(x+y)D}) - C\frac{e^{mx}-e^{ny}}{e^{x+y}-1}\delta_{n+m,0}. \tag{2.5}$$

Since $W_{1+\infty}$ is a Lie algebra, we can take any invertible linear combination of $W(z^n D^k)$ as a basis. The basis W_n^k in section 1, eq. (1.9), is expressed as

$$\begin{aligned} W_n^{k+1} &= W(z^n f_n^k(D)) \quad (k \geq 0), \quad c = C, \\ f_n^k(D) &= \binom{2k}{k}^{-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j}^2 [-D-n-1]_{k-j}[D]_j = (-1)^k D^k + \cdots. \end{aligned} \tag{2.6}$$

$W_{1+\infty}$ contains the $\hat{\mathrm{u}}(1)$ subalgebra generated by $J_n = W(z^n)$ and the Virasoro subalgebra generated by $L_n = -W(z^n D)$ with the central charge $c_{Vir} = -2C$. L_0 counts the energy level; $[L_0, W(z^n f(D))] = -nW(z^n f(D))$. We will regard $W(z^n f(D))$ with $n > 0$ ($n < 0$) as annihilation (creation) operators, respectively. The Cartan subalgebra of $W_{1+\infty}$ is generated by $W(D^k)$ ($k \geq 0$), so it is infinite dimensional. $W_n^2 = L_n - \frac{n+1}{2} J_n$ also generates the Virasoro algebra with $c_{Vir} = C$. Moreover there are two one-parameter families of the Virasoro subalgebras generated by[24]

$$L_n - (\alpha n + \beta) J_n, \quad (\alpha = \beta, 1-\beta; \beta \in \mathbb{C}), \tag{2.7}$$

whose central charge is

$$c_{Vir} = 2(-1 + 6\beta - 6\beta^2)C. \tag{2.8}$$

The $\hat{\mathrm{u}}(1)$ current J_n is anomalous except for $\beta = \frac{1}{2}$.

Since $W_{1+\infty}$ contains the $\hat{\mathrm{u}}(1)$ subalgebra, $W_{1+\infty}$ has a one-parameter family of automorphisms which we call the spectral flow[42, 12]. The transformation rule is given by[1]

$$W'(z^n e^{xD}) = W(z^n e^{x(D+\lambda)}) - C \frac{e^{\lambda x} - 1}{e^x - 1} \delta_{n0}, \tag{2.9}$$

where $\lambda \in \mathbb{C}$ is an arbitrary parameter. For lower components, for example, it is expressed as

$$\begin{aligned} J'_n &= J_n - \lambda C \delta_{n0}, \\ L'_n &= L_n - \lambda J_n + \tfrac{1}{2}\lambda(\lambda - 1) C \delta_{n0}. \end{aligned} \tag{2.10}$$

One can easily check that new generator $W'(\cdot)$ satisfies the same commutation relation as the original one $W(\cdot)$, eq. (2.5).

The Hermitian conjugation $\dagger$ is defined by

$$W(z^n D^k)^\dagger = W(z^{-n}(D-n)^k), \tag{2.11}$$

and $(aA + bB)^\dagger = \bar{a}A^\dagger + \bar{b}B^\dagger$, $(AB)^\dagger = B^\dagger A^\dagger$. The commutation relation eq. (2.5) is invariant under $\dagger$. $f_n^k(D)$, eq. (2.6), satisfies $f_n^k(D-n) = f_{-n}^k(D)$, which implies $W_n^{k\dagger} = W_{-n}^k$.

Finally we remark that $W_{1+\infty}$ is generated by $W(z^{\pm 1})$ and $W(D^2)$, namely $W(z^n D^k)$ is expressed as a commutator of $W(z^{\pm 1})$ and $W(D^2)$.

2.2 (Super)Matrix generalization of $W_{1+\infty}$

We can construct a (super)matrix generalization of $W_{1+\infty}$. Let us consider the $(M+N)\times(M+N)$ supermatrices $\mathrm{M}(M|N;\mathbb{C})$. An element of $\mathrm{M}(M|N;\mathbb{C})$ has the following form:

$$A=\begin{pmatrix} A^{(0)} & A^{(+)} \\ A^{(-)} & A^{(1)} \end{pmatrix}, \tag{2.12}$$

where $A^{(0)}, A^{(1)}, A^{(+)}, A^{(-)}$ are $M\times M$, $N\times N$, $M\times N$, $N\times M$ matrices, respectively, with complex entries. $\mathbb{Z}_2$-gradation is denoted by $|A|$; $|A|=0$ for $\mathbb{Z}_2$-even and $|A|=1$ for $\mathbb{Z}_2$-odd. $A^{(0)}$ and $A^{(1)}$ are $\mathbb{Z}_2$-even and $A^{(+)}$ and $A^{(-)}$ are $\mathbb{Z}_2$-odd. $\mathbb{Z}_2$-graded commutator is

$$[A,B\}=AB-(-1)^{|A||B|}BA. \tag{2.13}$$

The supertrace is

$$\mathrm{str}\,A=\mathrm{tr}\,A^{(0)}-\mathrm{tr}\,A^{(1)}, \tag{2.14}$$

and satisfies $\mathrm{str}\,(AB)=(-1)^{|A||B|}\mathrm{str}\,(BA)$.

$\mathrm{M}(M|N;\mathbb{C})$ generalization of $W_{1+\infty}$, whose generators are $W(z^nD^kA)$ $(n,k\in\mathbb{Z}, k\geq 0, A\in\mathrm{M}(M|N;\mathbb{C}))$ and the center C, is defined by the following (anti-)commutation relation:

$$\begin{aligned}&\Big[W(z^nf(D)A),W(z^mg(D)B)\Big\}\\ =\;& W(z^{n+m}f(D+m)g(D)AB)-(-1)^{|A||B|}W(z^{n+m}f(D)g(D+n)BA)\\ &-C\Psi(z^nf(D),z^mg(D))\mathrm{str}(AB).\end{aligned} \tag{2.15}$$

We call this ($\mathbb{Z}_2$-graded) Lie algebra the $W_{1+\infty}^{M,N}$ algebra, which satisfies the Jacobi identity

$$\begin{aligned}(-1)^{|A_1||A_3|}\Big[W(z^{n_1}f_1(D)A_1),\Big[W(z^{n_2}f_2(D)A_2),W(z^{n_3}f_3(D)A_3)\Big\}\Big\}&\\ +\text{ cyclic permutation}=0.&\end{aligned} \tag{2.16}$$

The original $W_{1+\infty}$ algebra corresponds to $M=0$, $N=1$. $M=0$ case was constructed in [35], and $M=N=1$ case in [2].

The $W_{1+\infty}^{M,N}$ algebra contains $\mathrm{M}(M|N;\mathbb{C})$ current algebra generated by $W(z^nA)$. For $M=0$, it is the $\widehat{\mathrm{gl}}(N)$ (or $\hat{\mathrm{u}}(N)$) algebra with level C. Since $W_{1+\infty}^{M,N}$ contains $M+N$ $\hat{\mathrm{u}}(1)$ subalgebras, $W_{1+\infty}^{M,N}$ has $(M+N)$-parameter family of automorphisms (spectral flow). Its transformation rule is

$$\begin{aligned}W'(z^ne^{xD}E^{(0)}_{ab}) &= W(z^{n-\mu^a+\mu^b}e^{x(D+\mu^b)}E^{(0)}_{ab})+C\frac{e^{\mu^ax}-1}{e^x-1}\delta_{ab}\delta_{n0},\\ W'(z^ne^{xD}E^{(1)}_{ij}) &= W(z^{n-\lambda^i+\lambda^j}e^{x(D+\lambda^j)}E^{(1)}_{ij})-C\frac{e^{\lambda^ix}-1}{e^x-1}\delta_{ij}\delta_{n0},\\ W'(z^ne^{xD}E^{(+)}_{aj}) &= W(z^{n-\mu^a+\lambda^j}e^{x(D+\lambda^j)}E^{(+)}_{aj}),\\ W'(z^ne^{xD}E^{(-)}_{ib}) &= W(z^{n-\lambda^i+\mu^b}e^{x(D+\mu^b)}E^{(-)}_{ib}),\end{aligned} \tag{2.17}$$

where μ^a $(a = 1, \cdots, M)$ and λ^i $(i = 1, \cdots, N)$ are arbitrary parameters, and $E^{(\alpha)}_{pq}$ is a matrix unit, $(E^{(\alpha)}_{pq})_{p'q'} = \delta_{pp'}\delta_{qq'}$.

$L_n = -W(z^n D \cdot 1)$ generates the Virasoro algebra with the central charge $c_{Vir} = 2(M - N)C$. L_0 counts the energy level. The Cartan subalgebra of $W^{M,N}_{1+\infty}$ is generated by $W(D^k E^{(0)}_{aa})$ $(k \geq 0, a = 1, \cdots, M)$ and $W(D^k E^{(1)}_{ii})$ $(k \geq 0, i = 1, \cdots, N)$.

2.3 Subalgebras of $W_{1+\infty}$

Although $W_{1+\infty}$ was constructed from W_∞ by adding a spin-1 current historically, it is natural to regard that W_∞ is obtained from $W_{1+\infty}$ by truncating a spin-1 current[39]. The higher spin truncation of $W_{1+\infty}$ was also constructed[11]. We will give a systematic method to construct a family of subalgebras of the $W_{1+\infty}$ algebra[4].

Let us choose a polynomial $p(D)$ and set

$$\widetilde{W}(z^n D^k) = W(z^n D^k p(D)), \quad (n, k \in \mathbb{Z}, k \geq 0). \tag{2.18}$$

Then commutator of $\widetilde{W}(z^n D^k)$ closes:

$$\begin{aligned} &\left[\widetilde{W}(z^n f(D)), \widetilde{W}(z^m g(D))\right] \\ &= \widetilde{W}(z^{n+m} f(D+m) g(D) p(D+m)) - \widetilde{W}(z^{n+m} f(D) g(D+n) p(D+n)) \\ &\quad + C\Psi(z^n f(D) p(D), z^m g(D) p(D)), \end{aligned} \tag{2.19}$$

or equivalently

$$\begin{aligned} \left[\widetilde{W}(z^n e^{xD}), \widetilde{W}(z^m e^{yD})\right] &= \left(p(\tfrac{d}{dx}) e^{mx} - p(\tfrac{d}{dy}) e^{ny}\right) \widetilde{W}(z^{n+m} e^{(x+y)D}) \\ &\quad - C p(\tfrac{d}{dx}) p(\tfrac{d}{dy}) \frac{e^{mx} - e^{ny}}{e^{x+y} - 1} \delta_{n+m,0}. \end{aligned} \tag{2.20}$$

We call this subalgebra $W_{1+\infty}[p(D)]$.

In this subalgebra there are no currents with spin $\leq \deg p(D)$. The W_∞ algebra corresponds to the choice $p(D) = D$. The basis $\widetilde{W}^k_n$ eq. (1.3) is expressed as

$$\begin{aligned} \widetilde{W}^{k+2}_n &= \widetilde{W}(z^n \tilde{f}^k_n(D)) \quad (k \geq 0), \\ \tilde{f}^k_n(D) &= -\binom{2(k+1)}{k+1}^{-1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k+2}{j+1} [-D - n - 1]_{k-j} [D - 1]_j \\ &= (-1)^{k-1} D^k + \cdots. \end{aligned} \tag{2.21}$$

We remark that the Virasoro generators exist only if $\deg p(w) \leq 1$. In the case of W_∞, the Virasoro generator L_n is given by $L_n = -\widetilde{W}(z^n)$ whose central charge, $\tilde{c}_{Vir}$, is related to C as $\tilde{c}_{Vir} = -2C$ [39]. For $\deg p(w) \geq 2$, we extend the algebra introducing the L_0 operator such as to count the energy level, $\left[L_0, \widetilde{W}(z^n f(D))\right] = -n\widetilde{W}(z^n f(D))$.

Next we give another type of subalgebra of $W_{1+\infty}$. For any positive integer p, $W_{1+\infty}$ with the central charge C contains $W_{1+\infty}$ with the central charge pC[23]. We denote its generator by $\bar{W}(z^nD^k)$ $(n,k\in\mathbb{Z},k\geq 0)$. $\bar{W}(\cdot)$ is given by

$$\begin{aligned}\bar{W}(z^ne^{xD}) &= W(z^{pn}e^{x\frac{1}{p}D})-C\Big(\frac{1}{e^{\frac{1}{p}x}-1}-\frac{p}{e^x-1}\Big)\delta_{n0}\\ &= W(z^{pn}e^{x\frac{1}{p}D})-C\sum_{j=0}^{p-1}\frac{e^{\frac{j}{p}x}-1}{e^x-1}\delta_{n0}.\end{aligned}\tag{2.22}$$

Essentially this is interpreted as the change of variable, $\zeta=z^p$, $\zeta\frac{d}{d\zeta}=\frac{1}{p}D$.

For $W_{1+\infty}^{M,N}$, these type of subalgebras such as W_∞^M[8] and $W_\infty^{M,N}$[34, 2] can be treated similarly.

3 Free field realizations

In this section we give the free field realizations of $W_{1+\infty}$ and $W_{1+\infty}^{M,N}$. Using these realizations, we give their full character formulae.

3.1 $W_{1+\infty}$

The $W_{1+\infty}$ algebra is known to be realized by free fermion[12] or **bc** ghost[32]

$$\begin{aligned}&\mathbf{b}(z)=\sum_{r\in\mathbb{Z}}\mathbf{b}_rz^{-r-\lambda-1},\quad \mathbf{c}(z)=\sum_{s\in\mathbb{Z}}\mathbf{c}_sz^{-s+\lambda},\quad \mathbf{b}(z)\mathbf{c}(w)\sim\frac{\epsilon}{z-w},\\ &\mathbf{b}_r|\lambda\rangle=\mathbf{c}_s|\lambda\rangle=0\ (r\geq 0,s\geq 1),\quad \mathbf{c}_s^\dagger=\mathbf{b}_{-s},\end{aligned}\tag{3.1}$$

where $\epsilon=1$ for fermionic ghost bc or $\epsilon=-1$ for bosonic ghost $\beta\gamma$. The $W_{1+\infty}$ algebra with $C=\epsilon$ is realized by sandwiching a differential operator between **bc**:

$$\begin{aligned}W(z^ne^{xD}) &= \oint\frac{dz}{2\pi i}\,{}^\circ_\circ\mathbf{b}(z)z^ne^{xD}\mathbf{c}(z){}^\circ_\circ\\ &= \oint\frac{dz}{2\pi i}:\mathbf{b}(z)z^ne^{xD}\mathbf{c}(z):-\epsilon\frac{e^{\lambda x}-1}{e^x-1}\delta_{n0}\\ &= \sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n}}e^{x(\lambda-s)}E(r,s)-\epsilon\frac{e^{\lambda x}-1}{e^x-1}\delta_{n0}.\end{aligned}\tag{3.2}$$

Here the normal ordering ${}^\circ_\circ\ {}^\circ_\circ$ means subtracting the singular part and another normal ordering $:\mathbf{b}_r\mathbf{c}_s:$ means $\mathbf{b}_r\mathbf{c}_s$ if $r\leq -1$ and $\epsilon\mathbf{c}_s\mathbf{b}_r$ if $r\geq 0$. $E(r,s)$ is defined by

$$E(r,s)=\,:\mathbf{b}_r\mathbf{c}_s:,\tag{3.3}$$

and generates the $\widehat{\rm gl}(\infty)$ algebra:

$$\begin{aligned}\Big[E(r,s),E(r',s')\Big] &= \delta_{r'+s,0}E(r,s')-\delta_{r+s',0}E(r',s)\\ &\quad +C\delta_{r+s',0}\delta_{r'+s,0}\Big(\theta(r\ge 0)-\theta(r'\ge 0)\Big),\end{aligned} \tag{3.4}$$

where $C=\epsilon$ in this case. We remark that the spectral flow transformation eq. (2.9) with parameter λ' is obtained by replacing $\mathbf{b},\mathbf{c}$ in eq. (3.2) with $\mathbf{b}'(z)=z^{-\lambda'}\mathbf{b}(z)$, $\mathbf{c}'(z)=z^{\lambda'}\mathbf{c}(z)$.

From eq. (3.2) we obtain

$$\begin{aligned}W(z^nD^k)|\lambda\rangle &= 0 \quad (n\ge 1,k\ge 0),\\ -W(e^{xD})|\lambda\rangle &= \epsilon\frac{e^{\lambda x}-1}{e^x-1}|\lambda\rangle.\end{aligned} \tag{3.5}$$

This means that $|\lambda\rangle$ is the highest weight state of $W_{1+\infty}$ and its weight is

$$W(D^k)|\lambda\rangle=\epsilon\Delta_k^\lambda|\lambda\rangle, \tag{3.6}$$

where Δ_k^λ is the Bernoulli polynomial defined by

$$-\frac{e^{\lambda x}-1}{e^x-1}=\sum_{k=0}^\infty\Delta_k^\lambda\frac{x^k}{k!}. \tag{3.7}$$

To express how many states exist in the simultaneous eigenspace of the Cartan generators $W(D^k)$, the full character formula is introduced as

$$\mathrm{ch}=\mathrm{tr}\,\exp\Big(\sum_{k=0}^\infty g_kW(D^k)\Big), \tag{3.8}$$

where the trace is taken over the irreducible representation space and g_k are parameters. The states in the representation space are linear combinations of the following states:

$$W(z^{-n_1}D^{k_1})\cdots W(z^{-n_m}D^{k_m})|\lambda\rangle.$$

This state, however, is not the simultaneous eigenstate of $W(D^k)$, because

$$\Big[W(D^k),W(z^{-n}f(D))\Big]=W(z^{-n}((D-n)^k-D^k)f(D)). \tag{3.9}$$

On the other hand, the states in the Fock space of $\mathbf{bc}$ ghosts are linear combinations of the following states:

$$\mathbf{b}_{-r_1}\cdots\mathbf{b}_{-r_k}\mathbf{c}_{-s_1}\cdots\mathbf{c}_{-s_\ell}|\lambda\rangle,$$

which are simultaneous eigenstates of $W(D^k)$, because

$$\Big[W(D^k),\mathbf{b}_{-r}\Big]=(\lambda-r)^k\mathbf{b}_{-r},\quad \Big[W(D^k),\mathbf{c}_{-s}\Big]=-(\lambda+s)^k\mathbf{c}_{-s}. \tag{3.10}$$

Using this property, we derive the full character formula[5, 1]. For the fermionic case ($\epsilon = 1$), it is well known that the fermion Fock space can be decomposed into the irreducible representation spaces of $\hat{\mathrm{u}}(1)$ current algebra (cf. eq. (B.3)). Since $W_{1+\infty}$-generator does not change the U(1)–charge and $W_{1+\infty}$ contains $\hat{\mathrm{u}}(1)$ as a subalgebra, each $\hat{\mathrm{u}}(1)$ representation space is also the representation space of $W_{1+\infty}$ and irreducible[34, 5]. For the bosonic case ($\epsilon = -1$), the sector of vanishing U(1)–charge in the Fock space is the irreducible representation space of $W_{1+\infty}$ [1]. By taking a trace over whole Fock space, we define $S_m^{\lambda;\epsilon}$ as follows:

$$\sum_{m\in\mathbb{Z}} S_m^{\lambda;\epsilon} t^{-m} = e^{\epsilon\sum_{k=0}^{\infty} g_k \Delta_k^\lambda} \prod_{r=1}^{\infty}\Big(1+\epsilon t u_r(\lambda)\Big)^{\epsilon} \prod_{s=0}^{\infty}\Big(1+\epsilon t^{-1} v_s(\lambda)\Big)^{\epsilon}, \tag{3.11}$$

where t counts the U(1)–charge and $u_r(\lambda)$, $v_s(\lambda)$ are

$$u_r(\lambda) = e^{\sum_{k=0}^{\infty} g_k(\lambda-r)^k}, \quad v_s(\lambda) = e^{-\sum_{k=0}^{\infty} g_k(\lambda+s)^k}. \tag{3.12}$$

Then the full character for $|\lambda\rangle$ is given by

$$\mathrm{ch} \ = S_0^{\lambda;\epsilon}. \tag{3.13}$$

We remark that $S_m^{\lambda;1} = S_0^{\lambda+m;1}$ by the above statement. So we abbreviate $S_\lambda = S_0^{\lambda;1}$.

Products in eq. (3.11) can be written as

$$\prod_{r=1}^{\infty}\Big(1+\epsilon t u_r(\lambda)\Big)^{\epsilon} = e^{-\epsilon\sum_{\ell=1}^{\infty} x_\ell(\lambda)(-\epsilon t)^\ell}, \quad \prod_{s=0}^{\infty}\Big(1+\epsilon t^{-1} v_s(\lambda)\Big)^{\epsilon} = e^{-\epsilon\sum_{\ell=1}^{\infty} y_\ell(\lambda)(-\epsilon t)^{-\ell}}, \tag{3.14}$$

where

$$x_\ell(\lambda) = \frac{1}{\ell}\sum_{r=1}^{\infty} u_r(\lambda)^\ell, \quad y_\ell(\lambda) = \frac{1}{\ell}\sum_{s=0}^{\infty} v_s(\lambda)^\ell. \tag{3.15}$$

By introducing the elementary Schur polynomials P_n (see appendix B, eq. (B.18)), $S_m^{\lambda;\epsilon}$ is expressed as

$$S_m^{\lambda;\epsilon} = (-\epsilon)^m e^{\epsilon\sum_{k=0}^{\infty} g_k\Delta_k^\lambda} \sum_{a\in\mathbb{Z}} P_a(-\epsilon x(\lambda)) P_{a+m}(-\epsilon y(\lambda)). \tag{3.16}$$

To understand the full character formula, we specialize the parameters g_k as

$$g_k = -2\pi i\tau\delta_{k1}, \quad (q = e^{2\pi i\tau}), \tag{3.17}$$

which correspond to $\mathrm{tr}\, q^{L_0}$. For this choice, eq. (3.15) becomes

$$x_\ell(\lambda) = \frac{1}{\ell}\frac{q^{(1-\lambda)\ell}}{1-q^\ell}, \quad y_\ell(\lambda) = \frac{1}{\ell}\frac{q^{\lambda\ell}}{1-q^\ell}. \tag{3.18}$$

Then the specialized character is given by

$$S_\lambda = q^{\frac{1}{2}\lambda(\lambda-1)} \sum_{m=0}^{\infty} q^{m^2} \prod_{j=1}^{m} \frac{1}{(1-q^j)^2} \tag{3.19}$$

$$= q^{\frac{1}{2}\lambda(\lambda-1)} \prod_{j=1}^{\infty} \frac{1}{1-q^j}, \tag{3.20}$$

$$S_0^{\lambda;-1} = q^{-\frac{1}{2}\lambda(\lambda-1)} \sum_{m=0}^{\infty} q^{m} \prod_{j=1}^{m} \frac{1}{(1-q^j)^2} \tag{3.21}$$

$$= q^{-\frac{1}{2}\lambda(\lambda-1)} \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} \cdot \sum_{m=0}^{\infty} (-1)^m q^{\frac{1}{2}m(m+1)}. \tag{3.22}$$

Eqs. (3.19,3.21) are derived by eq. (3.16) and eqs. (B.32,B.33) in Appendix B. Eqs. (3.20,3.22) are derived by eq. (3.11) and Jacobi's triple product identity or characters of W_∞ with $c = 2$ [34, 1].

By tensoring the above free field realizations, we obtain free field realizations of $W_{1+\infty}$ with integer C;

$$C = \sum_i \epsilon_i, \quad -W(e^{xD})|\lambda\rangle = \sum_i \epsilon_i \frac{e^{\lambda_i x}-1}{e^x-1}|\lambda\rangle. \tag{3.23}$$

However, the character for this representation can not be obtained by the method given in this section. We will give another method in section 5.

Free field realization of $W_{1+\infty}[p(D)]$ is obtained from that of $W_{1+\infty}$ [4].

3.2 $W_{1+\infty}^{M,N}$

Results in the previous section are generalized easily[2]. Let us introduce M pairs of $\beta\gamma$ ghosts and N pairs of bc ghosts:

$$\begin{aligned}
&\beta^a(z) = \sum_{r\in\mathbb{Z}} \beta^a_r z^{-r-\mu_a-1}, \quad \gamma^a(z) = \sum_{s\in\mathbb{Z}} \gamma^a_s z^{-s+\mu_a}, \quad (a = 1,\cdots,M),\\
&b^i(z) = \sum_{r\in\mathbb{Z}} b^i_r z^{-r-\lambda_i-1}, \quad c^i(z) = \sum_{s\in\mathbb{Z}} c^i_s z^{-s+\lambda_i}, \quad (i = 1,\cdots,N),\\
&\beta^a_r, \gamma^a_s, b^i_r, c^i_s|\mu,\lambda\rangle = 0 \ (r \geq 0, s \geq 1),
\end{aligned} \tag{3.24}$$

where some conditions will be imposed on μ_a and λ_i later. Then the $W_{1+\infty}^{M,N}$ algebra with $C = 1$ is realized as follows:

$$\begin{aligned}
W(z^n e^{xD} A) &= \oint \frac{dz}{2\pi i} {}^\circ_\circ (\beta(z), b(z)) z^n e^{xD} \begin{pmatrix} A^{(0)} & A^{(+)} \\ A^{(-)} & A^{(1)} \end{pmatrix} \begin{pmatrix} \gamma(z) \\ c(z) \end{pmatrix} {}^\circ_\circ \\
&= \oint \frac{dz}{2\pi i} : (\beta(z), b(z)) z^n e^{xD} \begin{pmatrix} A^{(0)} & A^{(+)} \\ A^{(-)} & A^{(1)} \end{pmatrix} \begin{pmatrix} \gamma(z) \\ c(z) \end{pmatrix} :
\end{aligned}$$

$$
\begin{aligned}
&+1\cdot\Big(\sum_{a=1}^{M}\frac{e^{\mu_a x}-1}{e^x-1}A^{(0)}_{aa}-\sum_{i=1}^{N}\frac{e^{\lambda_i x}-1}{e^x-1}A^{(1)}_{ii}\Big)\delta_{n0}\\
=&\sum_{a=1}^{M}\sum_{b=1}^{M}\sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n-\mu_a+\mu_b}}A^{(0)}_{ab}e^{x(\mu_b-s)}E^{ab}_0(r,s)+1\cdot\sum_{a=1}^{M}\frac{e^{\mu_a x}-1}{e^x-1}A^{(0)}_{aa}\delta_{n0}\\
&+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n-\lambda_i+\lambda_j}}A^{(1)}_{ij}e^{x(\lambda_j-s)}E^{ij}_1(r,s)-1\cdot\sum_{i=1}^{N}\frac{e^{\lambda_i x}-1}{e^x-1}A^{(1)}_{ii}\delta_{n0}\\
&+\sum_{a=1}^{M}\sum_{j=1}^{N}\sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n-\mu_a+\lambda_j}}A^{(+)}_{aj}e^{x(\lambda_j-s)}E^{aj}_+(r,s)\\
&+\sum_{i=1}^{N}\sum_{b=1}^{M}\sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n-\lambda_i+\mu_b}}A^{(-)}_{ib}e^{x(\mu_b-s)}E^{ib}_-(r,s). \qquad (3.25)
\end{aligned}
$$

Here E's are defined by

$$
\begin{aligned}
E^{ab}_0(r,s)&=:\beta^a_r\gamma^b_s:,\quad & E^{aj}_+(r,s)&=\beta^a_r c^j_s,\\
E^{ib}_-(r,s)&=b^i_r\gamma^b_s, & E^{ij}_1(r,s)&=:b^i_r c^j_s:, \qquad (3.26)
\end{aligned}
$$

and they generate (super)matrix generalization of $\widehat{\mathrm{gl}}(\infty)$:

$$
\begin{aligned}
\Big[E^{ab}_0(r,s),E^{a'b'}_0(r',s')\Big]&=\delta^{a'b}\delta_{r'+s,0}E^{ab'}_0(r,s')-\delta^{ab'}\delta_{r+s',0}E^{a'b}_0(r',s)\\
&\quad-C\delta^{ab'}\delta^{a'b}\delta_{r+s',0}\delta_{r'+s,0}\Big(\theta(r\geq 0)-\theta(r'\geq 0)\Big),\\
\Big[E^{ij}_1(r,s),E^{i'j'}_1(r',s')\Big]&=\delta^{i'j}\delta_{r'+s,0}E^{ij'}_1(r,s')-\delta^{ij'}\delta_{r+s',0}E^{i'j}_1(r',s)\\
&\quad+C\delta^{ij'}\delta^{i'j}\delta_{r+s',0}\delta_{r'+s,0}\Big(\theta(r\geq 0)-\theta(r'\geq 0)\Big),\\
\Big\{E^{aj}_+(r,s),E^{ib}_-(r',s')\Big\}&=\delta^{ij}\delta_{r'+s,0}E^{ab}_0(r,s')+\delta^{ab}\delta_{r+s',0}E^{ij}_1(r',s)\\
&\quad-C\delta^{ab}\delta^{ij}\delta_{r+s',0}\delta_{r'+s,0}\Big(\theta(r\geq 0)-\theta(r'\geq 0)\Big),\\
\Big[E^{ab}_0(r,s),E^{a'j}_+(r',s')\Big]&=\delta^{a'b}\delta_{r'+s,0}E^{aj}_+(r,s'),\\
\Big[E^{ab}_0(r,s),E^{ib'}_-(r',s')\Big]&=-\delta^{ab'}\delta_{r+s',0}E^{ib}_-(r',s),\\
\Big[E^{ij}_1(r,s),E^{aj'}_+(r',s')\Big]&=-\delta^{ij'}\delta_{r+s',0}E^{aj}_+(r',s),\\
\Big[E^{ij}_1(r,s),E^{i'b}_-(r',s')\Big]&=\delta^{i'j}\delta_{r'+s,0}E^{ib}_-(r,s'), \qquad (3.27)
\end{aligned}
$$

where $C=1$ in this case and the other (anti-)commutation relations vanish.

When μ_a and λ_i satisfy the following condition,

$$
\begin{aligned}
\mu_a-\mu_b&=0,\pm 1,\\
\lambda_i-\lambda_j&=0,\pm 1,\\
\mu_a-\lambda_i&=0,-1, \qquad (3.28)
\end{aligned}
$$

eq. (3.25) implies that $|\mu,\lambda\rangle$ is the highest weight state of $W_{1+\infty}^{M,N}$,

$$\begin{aligned}
W(z^nD^kA)|\mu,\lambda\rangle &= 0 \quad (n\geq 1, k\geq 0),\\
W(D^kA^{(+)})|\mu,\lambda\rangle &= 0 \quad (k\geq 0),\\
-W(e^{xD}E^{(0)}_{aa})|\mu,\lambda\rangle &= -\frac{e^{\mu_a x}-1}{e^x-1}|\mu,\lambda\rangle,\ \textit{i.e.},\ W(D^kE^{(0)}_{aa})|\mu,\lambda\rangle = -\Delta_k^{\mu_a}|\mu,\lambda\rangle,\\
-W(e^{xD}E^{(1)}_{ii})|\mu,\lambda\rangle &= \frac{e^{\lambda_i x}-1}{e^x-1}|\mu,\lambda\rangle,\ \textit{i.e.},\ W(D^kE^{(1)}_{ii})|\mu,\lambda\rangle = \Delta_k^{\lambda_i}|\mu,\lambda\rangle.
\end{aligned} \tag{3.29}$$

The full character is defined by

$$\mathrm{ch} = \mathrm{tr}\ \exp\sum_{k=0}^{\infty}\Big(\sum_{a=1}^{M} {g'}^a_k W(D^kE^{(0)}_{aa}) + \sum_{i=1}^{N} g^i_k W(D^kE^{(1)}_{ii})\Big), \tag{3.30}$$

where the trace is taken over the irreducible representation space. By taking a trace over whole Fock space, we define $S^{\mu_1,\cdots,\mu_M,\lambda_1,\cdots,\lambda_N}_{m'_1,\cdots,m'_M,m_1,\cdots,m_N}$ as follows:

$$\begin{aligned}
&\sum_{\substack{m'_1,\cdots,m'_M\\ m_1,\cdots,m_N}\in\mathbb{Z}} S^{\mu_1,\cdots,\mu_M,\lambda_1,\cdots,\lambda_N}_{m'_1,\cdots,m'_M,m_1,\cdots,m_N} {t'_1}^{-m'_1}\cdots {t'_N}^{-m'_N} t_1^{-m_1}\cdots t_N^{-m_N}\\
=\ & e^{\sum_{k=0}^{\infty}\left(-\sum_{a=1}^{M}{g'}^i_k\Delta_k^{\mu_a}+\sum_{i=1}^{N}g^i_k\Delta_k^{\lambda_i}\right)}\\
&\times\prod_{a=1}^{M}\prod_{r=1}^{\infty}\Big(1-t'_a u_r(\mu_a)\Big)^{-1}\prod_{s=0}^{\infty}\Big(1-{t'}^{-1}_a v_s(\mu_a)\Big)^{-1}\Big|_{g_k={g'}^a_k}\\
&\times\prod_{i=1}^{N}\prod_{r=1}^{\infty}\Big(1+t_i u_r(\lambda_i)\Big)\prod_{s=0}^{\infty}\Big(1+t_i^{-1}v_s(\lambda_i)\Big)\Big|_{g_k=g^i_k},
\end{aligned} \tag{3.31}$$

where we have used

$$\begin{aligned}
\Big[W(D^kE^{(0)}_{aa}),\beta^b_{-r}\Big] &= \delta_{ab}(\mu_a-r)^k\beta^b_{-r}, &\Big[W(D^kE^{(0)}_{aa}),\gamma^b_{-s}\Big] &= -\delta_{ab}(\mu_a+s)^k\gamma^b_{-s},\\
\Big[W(D^kE^{(1)}_{ii}),b^j_{-r}\Big] &= \delta_{ij}(\lambda_i-r)^k b^j_{-r}, &\Big[W(D^kE^{(1)}_{ii}),c^j_{-s}\Big] &= -\delta_{ij}(\lambda_i+s)^k c^j_{-s}.
\end{aligned} \tag{3.32}$$

$S^{\mu_1,\cdots,\mu_M,\lambda_1,\cdots,\lambda_N}_{m'_1,\cdots,m'_M,m_1,\cdots,m_N}$ can be expressed in terms of $S^{\lambda;\epsilon}_m$,

$$S^{\mu_1,\cdots,\mu_M,\lambda_1,\cdots,\lambda_N}_{m'_1,\cdots,m'_M,m_1,\cdots,m_N} = \prod_{a=1}^{M} S^{\mu_a;-1}_{m'_a}\Big|_{g_k={g'}^a_k}\cdot\prod_{i=1}^{N} S^{\lambda_i;1}_{m_i}\Big|_{g_k=g^i_k}. \tag{3.33}$$

Since the sector of vanishing U(1)–charge in the Fock space is again the irreducible representation space of $W_{1+\infty}^{M,N}$, the full character for the representation $|\mu,\lambda\rangle$ is given by

$$\mathrm{ch} \ = \sum_{\substack{m'_a,m_i\in\mathbb{Z}\\ \sum_a m'_a+\sum_i m_i=0}} S^{\mu_1,\cdots,\mu_M,\lambda_1,\cdots,\lambda_N}_{m'_1,\cdots,m'_M,m_1,\cdots,m_N}. \tag{3.34}$$

Setting all ${g'}^a_k$ and g^i_k to eq. (3.17), we obtain the specialized character. For example, in the case of $M=0$, the specialized character is essentially $\hat{\mathrm{u}}(1)$ character times level 1

$\widehat{su}(N)$ character[34]. For $N = M = 1$ and $\mu = \lambda$, the specialized character is given by[1]

$$\mathrm{ch} \;=\frac{1}{1+q^{\frac{1}{2}}}\prod_{j=1}^{\infty}\Big(\frac{1+q^{j-\frac{1}{2}}}{1-q^{j}}\Big)^{2}. \tag{3.35}$$

By interchanging $\beta\gamma$ with bc, we obtain the realization with $C = -1$. Although realizations with integer C can be obtained by tensoring, the character can not be derived by the method in this section.

4 Quasifinite representation of $W_{1+\infty}$

We study the highest weight representation of $W_{1+\infty}$. The highest weight state $|\lambda\rangle$ is characterized by

$$\begin{aligned} W(z^nD^k)|\lambda\rangle \;&=\; 0 \quad (n\geq 1, k\geq 0),\\ W(D^k)|\lambda\rangle \;&=\; \Delta_k|\lambda\rangle \quad (k\geq 0), \end{aligned} \tag{4.1}$$

where the weight Δ_k is some complex number. It is convenient to introduce the generating function $\Delta(x)$ for the weights Δ_k:

$$\Delta(x) = -\sum_{k=0}^{\infty}\Delta_k\frac{x^k}{k!}, \tag{4.2}$$

which we call the weight function. It is formally given as the eigenvalue of the operator $-W(e^{xD})$:

$$-W(e^{xD})|\lambda\rangle = \Delta(x)|\lambda\rangle. \tag{4.3}$$

The Verma module is spanned by the state

$$W(z^{-n_1}D^{k_1})\cdots W(z^{-n_m}D^{k_m})|\lambda\rangle. \tag{4.4}$$

The energy level, which is the relative L_0 eigenvalue, of this state is $\sum_{i=1}^{m} n_i$. Reflecting the infinitely many generators, the Verma module has infinitely many states at each level. The irreducible representation space is obtained by subtracting null states from the Verma module. A null state is the state which can not be brought back to $|\lambda\rangle$ by any successive actions of $W_{1+\infty}$ generators. Of course, in other words, a null state is the state which has vanishing inner products with any states.

In the rest of this article, we will study the quasifinite representations[30]. A representation is called *quasifinite* if there are only a finite number of non-vanishing states at each energy level. The representations obtained by free field realizations in the previous

[1] Eq.(79) in [34] can be expressed as $ch_n^{W_\infty^{1,1}}(\theta,\tau) = \frac{1-q}{(1+zq^{n+\frac{1}{2}})(1+z^{-1}q^{-n+\frac{1}{2}})}\prod_{j=1}^{\infty}\frac{(1+zq^{j-\frac{1}{2}})(1+z^{-1}q^{j-\frac{1}{2}})}{(1-q^j)^2}$.

section have this property, because there are only finite number of oscillators at each energy level. To achieve this, the weight function must be severely constrained. We will show that if there are a finite number of states at level 1, then it is so at any level.

Let us assume that there are only a finite number of non-vanishing states at level n. This means that the following linear relation exists:

$$W(z^{-n}f(D))|\lambda\rangle = \text{null}, \tag{4.5}$$

where f is some polynomial. Acting $W(e^{x(D+n)})$ to this state, we have

$$\begin{aligned}\text{null} &= W(e^{x(D+n)})W(z^{-n}f(D))|\lambda\rangle \\ &= \Big[W(e^{x(D+n)}), W(z^{-n}f(D))\Big]|\lambda\rangle + \text{ null} \\ &= (1-e^{xn})W(z^{-n}e^{xD}f(D))|\lambda\rangle + \text{ null},\end{aligned}$$

and thus the state $W(z^{-n}D^k f(D))|\lambda\rangle$ is also null for all $k \geq 0$. In other words, the set

$$I_{-n} = \Big\{f(w) \in \mathbb{C}[w] \;\Big|\; W(z^{-n}f(D))|\lambda\rangle = \text{null}\Big\} \tag{4.6}$$

is an ideal in the polynomial ring $\mathbb{C}[w]$. Since $\mathbb{C}[w]$ is a principal ideal domain, I_{-n} is generated by a monic polynomial $b_n(w)$, *i.e.* $I_{-n} = \{f(w)b_n(w)|f(w) \in \mathbb{C}[w]\}$. These polynomials $b_n(w)$ $(n = 1, 2, 3, \cdots)$ are called characteristic polynomials for the quasifinite representation.

Let $f_n(w)$ be the minimal-degree monic polynomial satisfying the following differential equation:

$$f_n\left(\tfrac{d}{dx}\right)\sum_{j=0}^{n-1} e^{jx}\Big((e^x-1)\Delta(x)+C\Big) = 0. \tag{4.7}$$

Then the characteristic polynomials $b_n(w)$ are related to each other as follows:[2]

(i) $b_n(w)$ divides both of $b_{n+m}(w+m)$ and $b_{n+m}(w)$ $(m \geq 1)$,

(ii) $f_n(w)$ divides $b_n(w)$.

The $b_n(w)$'s are determined as the minimal-degree monic polynomials satisfying both (i) and (ii). The property (i) is derived from the null state condition,

$$\begin{aligned}\text{null} &= W(z^m e^{x(D+n+m)})W(z^{-n-m}b_{n+m}(D))|\lambda\rangle \\ &= \Big[W(z^m e^{x(D+n+m)}), W(z^{-n-m}b_{n+m}(D))\Big]|\lambda\rangle \\ &= W(z^{-n}(e^{xD}b_{n+m}(D) - e^{x(D+n+m)}b_{n+m}(D+m)))|\lambda\rangle.\end{aligned}$$

[2] We can also show that $b_{n+m}(w)$ divides $b_n(w-m)b_m(w)$.

The property (ii) is derived from the following null state condition,

$$\begin{aligned} 0 &= W(z^n e^{x(D+n)}) W(z^{-n} b_n(D)) |\lambda\rangle \\ &= \Big[W(z^n e^{x(D+n)}), W(z^{-n} b_n(D)) \Big] |\lambda\rangle \\ &= \Big(W(e^{xD} b_n(D)) - W(e^{x(D+n)} b_n(D+n)) + C \sum_{j=1}^{n} e^{x(n-j)} b_n(n-j) \Big) |\lambda\rangle \\ &= b_n\Big(\tfrac{d}{dx}\Big) \sum_{j=0}^{n-1} e^{jx} \Big((e^x - 1)\Delta(x) + C \Big) |\lambda\rangle. \end{aligned}$$

The solution of these conditions are given by[30, 3]

$$b_n(w) = \mathrm{lcm}(b(w), b(w-1), \cdots, b(w-n+1)), \tag{4.8}$$

where $b(w) = b_1(w) = f_1(w)$ is the minimal-degree monic polynomial satisfying the differential equation,

$$b\Big(\tfrac{d}{dx}\Big)\Big((e^x - 1)\Delta(x) + C\Big) = 0. \tag{4.9}$$

Therefore, the necessary and sufficient condition for quasifiniteness is that the weight function satisfies this type of differential equation. Moreover it has been shown that the finiteness at level 1 (*i.e.* existence of $b(w)$) implies the finiteness at higher levels (*i.e.* existence of $b_n(w)$).

If we factorize the characteristic polynomial $b(w)$ as

$$b(w) = \prod_{i=1}^{K} (w - \lambda_i)^{m_i}, \quad (\lambda_i \neq \lambda_j), \tag{4.10}$$

then the solution of eq. (4.9) is given by

$$\Delta(x) = \frac{\sum_{i=1}^{K} p_i(x) e^{\lambda_i x} - C}{e^x - 1}, \tag{4.11}$$

where $p_i(x)$ is a polynomial of degree $m_i - 1$[3]. Since $\Delta(x)$ is regular at $x = 0$ by definition, p_i's satisfy $\sum_{i=1}^{K} p_i(0) = C$. Therefore $\Delta(x)$ has $\sum_{i=1}^{K}(m_i + 1)$ parameters; C, λ_i and coefficients in $p_i(x)$'s. In contrast to the weight function for general (non-quasifinite) representation, the weight function for quasifinite representation has thus only finite parameters. The representation realized by free field studied in section 3.1 has the weight function $\Delta(x) = \epsilon \frac{e^{\lambda x} - 1}{e^x - 1}$, which corresponds to the characteristic polynomial $b(w) = w - \lambda$. We can explicitly check that $b_n(w)$ is given by eq. (4.8)[32].

Under the spectral flow eq. (2.9), the representation space as a set is kept invariant. Furthermore the highest weight state with respect to the original generators $W(\cdot)$ is also

[3] Since $b(w)$ is minimal-degree, $\deg p_i(x)$ is exactly $m_i - 1$.

the highest weight state with respect to the new generators $W'(\cdot)$. On the other hand, the weight function $\Delta(x)$ and the characteristic polynomial $b(w)$ are replaced by the new ones[1]:

$$\begin{aligned} \Delta'(x) &= e^{\lambda x}\Delta(x) + C\frac{e^{\lambda x}-1}{e^x-1}, & (4.12)\\ b'(w) &= b(w-\lambda). & (4.13)\end{aligned}$$

This implies that the spectral flow transforms λ_i in eq. (4.10) into $\lambda_i + \lambda$.

To study the structure of null states, let us introduce the inner product as

$$\begin{aligned} &\langle\lambda|\lambda\rangle = 1, \\ &\big(\langle\lambda|W\big)|\lambda\rangle = \langle\lambda|\big(W|\lambda\rangle\big) = \langle\lambda|W|\lambda\rangle, \end{aligned} \tag{4.14}$$

and the corresponding bra state $\langle\lambda|$ as

$$\begin{aligned} \langle\lambda|W(z^nD^k) &= 0 \quad (n\leq -1, k\geq 0), \\ \langle\lambda|W(D^k) &= \Delta_k\langle\lambda| \quad (k\geq 0). \end{aligned} \tag{4.15}$$

Then the quasifinite condition for bra states is

$$\langle\lambda|W(z^nb_n(D+n)D^k) = \text{null} \quad (n\geq 1, k\geq 0). \tag{4.16}$$

These are consistent with the hermitian conjugation eq. (2.11) when $\Delta_k \in \mathbb{R}$ (or $\Delta_k \in \mathbb{C}$ if $\dagger$ is modified, $(aA+bB)^\dagger = aA^\dagger + bB^\dagger$).

Unitary representation was studied in [30]. The necessary and sufficient condition for unitary representation is that C is a non-negative integer and the weight function is

$$\Delta(x) = \sum_{i=1}^{C}\frac{e^{\lambda_i x}-1}{e^x-1}, \quad \lambda_i\in\mathbb{R}. \tag{4.17}$$

This weight function corresponds to the characteristic polynomial $b(w) = \prod_i'(w-\lambda_i)$ where the product is taken over different λ_i. We remark that all the unitary representations can be realized by tensoring C pairs of bc ghosts, eq. (3.23).

Quasifinite representations of $W_{1+\infty}^{M,N}$ and subalgebras can be treated similarly (For $M=N=1$, see [2], and for $W_{1+\infty}[p(D)]$ see [4]).

5 Kac determinant and full character formulae of $W_{1+\infty}$

In this section we compute the Kac determinant for some representations, on the basis of which we derive the analytic form of the Kac determinant and full character formulae.

5.1 The Verma module

Let us study the quasifinite representation of $W_{1+\infty}$ with central charge C and the weight function $\Delta(x)$. Characteristic polynomials $b_n(w)$ are determined from $\Delta(x)$. Since there are linear relations $W(z^{-n}D^k b_n(D))|\lambda\rangle$ = null, only $\deg b_n(w)$ states are independent in the states $\{W(z^{-n}D^k)|\lambda\rangle\}$. We may take independent states as follows:

$$W(z^{-n}D^k)|\lambda\rangle \quad (k = 0, 1, \cdots, \deg b_n(w) - 1). \tag{5.1}$$

The Verma module for the quasifinite representation is defined as the space spanned by these generators. Therefore the specialized character formula for the Verma module is

$$\operatorname{tr} q^{L_0} = q^{-\Delta_1} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\deg b_n(w)}}. \tag{5.2}$$

For example, for $b(w) = w - \lambda$, we have

$$\operatorname{tr} q^{L_0 + \Delta_1^\lambda} = \chi(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \tag{5.3}$$

This $\chi(q)$ has a close relationship with the partition function of three–dimensional free field theory (see [1]).

5.2 Determinant formulae at lower levels

In this subsection we will present explicit computation of the Kac determinant for quasifinite representations[1]. First let us consider the representation with $\Delta(x) = C\frac{e^{\lambda x}-1}{e^x-1}$. The characteristic polynomial is $b(w) = w - \lambda$ ($b(w) = 1$ for $C = \lambda = 0$). For the first three levels, the relevant ket states are,

$$\begin{array}{ll}
\text{Level 1} & W(z^{-1})|\lambda\rangle, \\
\text{Level 2} & W(z^{-2})|\lambda\rangle, W(z^{-1})^2|\lambda\rangle, W(z^{-2}D)|\lambda\rangle, \\
\text{Level 3} & W(z^{-3})|\lambda\rangle, W(z^{-1})W(z^{-2})|\lambda\rangle, W(z^{-1})^3|\lambda\rangle, \\
 & W(z^{-3}D)|\lambda\rangle, W(z^{-1})W(z^{-2}D)|\lambda\rangle, W(z^{-3}D^2)|\lambda\rangle.
\end{array} \tag{5.4}$$

Corresponding bra states may be given by changing z^{-n} into z^n. The number of relevant states grows as,

$$\chi(q) = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + 86q^7 + 160q^8 + 282q^9 + 500q^{10} + \cdots. \tag{5.5}$$

Inner product matrices are straightforwardly calculated; for example, at level 2,

$$\begin{pmatrix} 2C & 0 & (2\lambda+1)C \\ 0 & 2C^2 & -C \\ (2\lambda-3)C & -C & (2\lambda^2-2\lambda-1)C \end{pmatrix}.$$

The determinant for this matrix is $2C^3(C-1)$. We computed the Kac determinant up to level 8 by using computer[1]:

$$\begin{aligned}
\det[1] &\propto C, \\
\det[2] &\propto C^3(C-1), \\
\det[3] &\propto C^6(C-1)^3(C-2), \\
\det[4] &\propto (C+1)C^{13}(C-1)^8(C-2)^3(C-3), \\
\det[5] &\propto (C+1)^3C^{24}(C-1)^{17}(C-2)^8(C-3)^3(C-4), \\
\det[6] &\propto (C+1)^{10}C^{48}(C-1)^{37}(C-2)^{19}(C-3)^8(C-4)^3(C-5), \\
\det[7] &\propto (C+1)^{23}C^{86}(C-1)^{71}(C-2)^{41}(C-3)^{19}(C-4)^8(C-5)^3(C-6), \\
\det[8] &\propto (C+1)^{54}C^{161}(C-1)^{138}(C-2)^{85}(C-3)^{43}(C-4)^{19}(C-5)^8(C-6)^3(C-7).
\end{aligned}$$

We remark that λ-dependence disappears due to nontrivial cancellations. This is explained by the spectral flow [1]. We computed also the corank of the inner product matrix:

$$\begin{aligned}
\mathrm{cor}[n] &= \det[n], \quad (n=1,\cdots,7), \\
\mathrm{cor}[8] &= (C+1)^{54}C^{160}(C-1)^{138}(C-2)^{85}(C-3)^{43}(C-4)^{19}(C-5)^8(C-6)^3(C-7),
\end{aligned}$$

where the exponent stands for a corank, *i.e.* a number of null states. Subtracting this number from eq. (5.5), we get the specialized characters at lower levels. We will present the determinant and full character formulae in section 5.3.

Next we take the representation with

$$\Delta(x) = \sum_{i=1}^{K} C_i \frac{e^{\lambda_i x}-1}{e^x-1}, \quad C = \sum_{i=1}^{K} C_i, \tag{5.6}$$

where λ_i's are all different numbers. The characteristic polynomial is given by

$$b(w) = \prod_{i=1}^{K}(w-\lambda_i) \tag{5.7}$$

(if $C_i = \lambda_i = 0$, then the factor $w-\lambda_i$ in $b(w)$ should be omitted). Assuming that the difference of any two λ_i's is not an integer, we computed the Kac determinants at lower levels for $K = 1, \cdots, 5$, and they are given in appendix A[1]. The determinant formula may be written in the following form:

$$\det[n] \propto \prod_i A_n(C_i) \prod_{i<j} B_n(\lambda_i - \lambda_j). \tag{5.8}$$

λ_i-dependence appears only through their differences due to the spectral flow symmetry [1, 3]. The functions A_n and B_n have zero only when C_i or $\lambda_i - \lambda_j$ is integer.

We will give an analytic expression for $B_n(\lambda)$. Let us consider the case when one pair $\lambda_i - \lambda_j$ is an integer ℓ. In this case the characteristic polynomial $b_n(w)$ may have degree less than $n \deg b(w)$. The weight function becomes

$$\Delta'(x;\ell) = \Delta(x)\Big|_{\lambda_i-\lambda_j=\ell}, \tag{5.9}$$

and we denote the corresponding characteristic polynomial as $b'(w;\ell)$ and $b'_n(w;\ell)$, and the discrepancy of degree as

$$d(n;\ell) = n \deg b'(w;\ell) - \deg b'_n(w;\ell). \tag{5.10}$$

Then $B_n(\lambda)$ is

$$B_n(\lambda) = \prod_{\ell\in\mathbb{Z}} (\lambda - \ell)^{\beta_n(\ell)}, \tag{5.11}$$

where $\beta_n(\ell)$ is defined by

$$\sum_{n=0}^{\infty} \beta_n(\ell) q^n = 2t\frac{d}{dt}\prod_{n=1}^{\infty} \frac{(1-q^n)^{d(n;\ell)}}{(1-tq^n)^{d(n;\ell)}}\Big|_{t=1} \cdot \chi(q)^{\deg b'(w;\ell)}. \tag{5.12}$$

Its proof can be found in [3].

We remark that the determinant formula for the representation with $\Delta'(x;\ell)$ is not given by eq. (5.8), because eq. (5.8) is the determinant for the inner product matrix of size given by eq. (5.2) with $b_n(w)$ not $b'_n(w,\ell)$. Such a determinant has mixed C_i factors, *e.g.* $C_i + C_j + 1$. We here give some examples in the $\lambda_i - \lambda_{i+1} = 1$ case. When $K = 2$, the determinant for the first five levels are

$$\begin{aligned}
\det[1] &\propto C_1C_2,\\
\det[2] &\propto (C_1+C_2+1)\prod_{i=1,2} C_i^3(C_i-1),\\
\det[3] &\propto (C_1+C_2+1)^4\prod_{i=1,2} C_i^8(C_i-1)^3(C_i-2),\\
\det[4] &\propto (C_1+C_2+1)^{13}(C_1+C_2)\prod_{i=1,2} C_i^{20}(C_i-1)^9(C_i-2)^3(C_i-3),\\
\det[5] &\propto (C_1+C_2+1)^{34}(C_1+C_2)^4\prod_{i=1,2} C_i^{46}(C_i-1)^{22}(C_i-2)^9(C_i-3)^3(C_i-4),
\end{aligned}$$

and when $K = 3$, that of the first three levels are

$$\begin{aligned}
\det[1] &\propto \prod_{i=1,2,3} C_i,\\
\det[2] &\propto (C_1+C_2+1)(C_2+C_3+1)\prod_{i=1,2,3} C_i^4(C_i-1),\\
\det[3] &\propto (C_1+C_2+1)^4(C_2+C_3+1)^4(C_1+C_2+C_3+2)\prod_{i=1,2,3} C_i^{13}(C_i-1)^4(C_i-2).
\end{aligned}$$

5.3 $\Delta(x) = C\frac{e^{\lambda x}-1}{e^x-1}$ case

In this subsection we study the representation with $\Delta(x) = C\frac{e^{\lambda x}-1}{e^x-1}$[3]. The corresponding characteristic polynomial is $b(w) = w - \lambda$ ($b(w) = 1$ for $C = \lambda = 0$, but in this case non-vanishing states are $|\lambda\rangle$ only. So we do not care about this case). We have already known the full character formula for $C = \pm 1$. The numbers of states, eqs. (3.20,3.22), are less than that of the Verma module eq. (5.3). The determinant formula at lower levels given in the previous section suggests that additional null states appear only when C is an integer. This is indeed the case, and we will derive the analytic form of the determinant and full character formulae.

As we have shown, the basis eq. (4.4) is not a good basis because of eq. (3.9). The construction of the diagonal basis becomes possible if we view the $W_{1+\infty}$ algebra from the equivalent $\widehat{\mathrm{gl}}(\infty)$ algebra, which is implicitly used in section 3.1. It was proved that the quasifinite representations of those algebras coincide[30]. The $\widehat{\mathrm{gl}}(\infty)$ algebra is defined by eq. (3.4) and the relation with $W_{1+\infty}$ is[4]

$$W(z^n e^{xD}) = \sum_{\substack{r,s\in\mathbb{Z}\\ r+s=n}} e^{x(\lambda-s)} E(r,s) - C\frac{e^{\lambda x}-1}{e^x-1}\delta_{n0}. \tag{5.13}$$

The highest weight state of $\widehat{\mathrm{gl}}(\infty)$ is defined by

$$\begin{aligned} E(r,s)|\lambda\rangle &= 0 \quad (r+s>0), \\ E(r,-r)|\lambda\rangle &= q_r|\lambda\rangle \quad (r\in\mathbb{Z}). \end{aligned} \tag{5.14}$$

The quasifiniteness of the representation is achieved only when finite number of $h_r = q_r - q_{r-1} + C\delta_{r0}$ are non-vanishing[30]. In this case the following $E(r,s)$ annihilates the highest weight state[3]:

$$E(r,s)|\lambda\rangle = 0 \quad (r\geq 0, s\geq 1). \tag{5.15}$$

The generator $E(r,s)$ is already diagonal with respect to the action of the Cartan elements,

$$\Big[W(D^k), E(r,s)\Big] = \Big((\lambda+r)^k - (\lambda-s)^k\Big)E(r,s). \tag{5.16}$$

Therefore the state

$$E(-r_1,-s_1)\cdots E(-r_n,-s_n)|\lambda\rangle, \quad (r_a\geq 1, s_a\geq 0), \tag{5.17}$$

is the simultaneous eigenstate of $W(D^k)$ with the eigenvalues

$$\Delta_k^\lambda + \sum_{a=1}^n\Big((\lambda-r_a)^k - (\lambda+s_a)^k\Big). \tag{5.18}$$

[4] This relation should be modified for different $b(w)$. When $b(w)=0$ has multiple roots, for example $b(w) = (w-\lambda)^m$, $\widehat{\mathrm{gl}}(\infty)$ also need to be modified[30, 3].

The representation space is decomposed into the eigenspace with above eigenvalues. So we need to consider only this subspace, which is spanned by

$$\prod_{a=1}^{n} E(-r_a, -s_{\sigma(a)})|\lambda\rangle, \quad (r_a \geq 1, s_a \geq 0), \tag{5.19}$$

where σ is a permutation of n objects. The number of these states is equal to the number of onto-map from $I = \{r_1, \cdots, r_n\}$ to $J = \{s_1, \cdots, s_n\}$.

We calculated the inner product matrix of these state[3]. By symmetrizing the indices of eq. (5.19) according to the Young diagram with n boxes, this matrix can be block-diagonalized. Each block can be further diagonalized. In fact, when r_a's and s_a's are all different respectively, we obtained an explicit form $|Y; \alpha, \beta\rangle$ (see [3] for details). In general, by taking appropriate linear combination of eq. (5.19), an orthogonal basis $|Y; \alpha, \beta\rangle$ ($\alpha = 1, \cdots, d_Y^I$; $\beta = 1, \cdots, d_Y^J$) is obtained:

$$\langle Y; \alpha, \beta | Y'; \alpha', \beta'\rangle = \delta_{YY'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\frac{\sqrt{d_Y^I d_Y^J}}{n!}\prod_{b\in Y}(C - C_b). \tag{5.20}$$

Here, to each box b in the Young diagram Y, we assign a number C_b as

$$\begin{array}{|c|c|c|c|c} \hline 0 & 1 & 2 & 3 & \cdots \\ \hline -1 & 0 & 1 & 2 & \cdots \\ \hline -2 & -1 & 0 & 1 & \cdots \\ \hline -3 & -2 & -1 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array} . \tag{5.21}$$

d_Y^I is the number of assignment of r_a to each box in the Young diagram Y with n boxes such that r_a's are non-decreasing from left to right, and increasing from top to bottom. When r_a's are all different, d_Y^I is equal to d_Y, the dimension of irreducible representation Y of permutation group $\mathcal{S}_n$.

The determinant formula given in section 5.2 is reproduced from above results. For example, at level 4,

$$\begin{aligned} \det[4] &\propto (C+1)C^{13}(C-1)^8(C-2)^3(C-3) \\ &= C \times C \times C \times C \times C(C-1) \times C(C-1) \\ &\quad \times C(C-1) \cdot C(C+1) \times C(C-1) \times C(C-1) \times C(C-1)(C-2) \\ &\quad \times C(C-1)(C-2) \times C(C-1)(C-2)(C-3), \end{aligned}$$

where each factor comes from

$$\begin{aligned} (I, J) &= (\{4\}, \{0\}), (\{3\}, \{1\}), (\{2\}, \{2\}), (\{1\}, \{3\}), (\{3, 1\}, \{0, 0\}), (\{2, 2\}, \{0, 0\}), \\ &\quad (\{2, 1\}, \{1, 0\}), (\{1, 1\}, \{2, 0\}), (\{1, 1\}, \{1, 1\}), (\{2, 1, 1\}, \{0, 0, 0\}), \\ &\quad (\{1, 1, 1\}, \{1, 0, 0\}), (\{1, 1, 1, 1\}, \{0, 0, 0, 0\}). \end{aligned}$$

As a simple corollary of the inner product formula, we may derive the condition for the unitarity. The positivity of the representation space may be rephrased as the positivity of the right hand side of eq. (5.20) for any Y. From eq. (5.21), we can immediately prove that this condition is achieved only when C is non-negative integer.

The full character is defined by eq. (3.8). From above determinant formula, when C is not an integer, there are no null states aside from those coming from characteristic polynomials. Therefore combining eqs. (5.16,5.17) we get the full character formula for non-integer C,

$$\mathrm{ch} = e^{C\sum_{k=0}^{\infty} g_k \Delta_k^{\lambda}} \prod_{r=1}^{\infty} \prod_{s=0}^{\infty} \frac{1}{1 - u_r(\lambda) v_s(\lambda)}, \tag{5.22}$$

where Δ_k^{λ}, $u_r(\lambda)$ and $v_s(\lambda)$ are defined by eq. (3.7),(3.12) respectively.

If we expand this product as

$$\sum_{n=0}^{\infty} \sum_{\substack{I,J \\ |I|=|J|=n}} N(I,J) \prod_{r\in I} u_r(\lambda) \prod_{s\in J} v_s(\lambda),$$

then $N(I,J)$ gives the number of the states of the form eq. (5.19). We need to go further to classify those states after the Young diagram. The following result gives such classification (see appendix B),

$$\prod_{r=1}^{\infty} \prod_{s=0}^{\infty} \frac{1}{1 - u_r(\lambda) v_s(\lambda)} = \sum_{Y} \tau_Y(x(\lambda)) \tau_Y(y(\lambda)), \tag{5.23}$$

where the summation is taken over all Young diagrams, and τ_Y is the character of irreducible representation Y of $\widehat{\mathrm{gl}}(\infty)$, and the parameters x and y are the Miwa variables for u and v defined by eq. (3.15). If we expand each factor in the summation, we can get the degeneracy with respect to each Young diagram Y, and the eigenvalues. The coefficient of $\prod_{r\in I} u_r(\lambda)$ in $\tau_Y(x(\lambda))$ is d_Y^I, and $N(I,J) = \sum_Y d_Y^I d_Y^J$.

Combining these Young diagram classification eqs. (5.20,5.23), we get the full character formula with integer C [3],

$$\mathrm{ch}_{C=n} = e^{n\sum_{k=0}^{\infty} g_k \Delta_k^{\lambda}} \sum_{\substack{Y \\ wd(Y)\le n}} \tau_Y(x(\lambda)) \tau_Y(y(\lambda)), \tag{5.24}$$

$$\mathrm{ch}_{C=-n} = e^{-n\sum_{k=0}^{\infty} g_k \Delta_k^{\lambda}} \sum_{\substack{Y \\ ht(Y)\le n}} \tau_Y(x(\lambda)) \tau_Y(y(\lambda)), \tag{5.25}$$

where n is an non-negative integer, and $wd(Y)$ ($ht(Y)$) stands for the number of columns (rows) of Y. The full characters obtained in section 3.1 agree with this result.

By setting g_k to eq. (3.17), we obtain the specialized character. In this case the Schur polynomial is expressed as

$$\tau_Y(x(\lambda)) = q^{(1-\lambda)\sum_{j=1}^{n} j m_j + \sum_{j=1}^{n} \frac{1}{2} j(j-1) m_j} \prod_{k=1}^{n} F_k(q; m_1, \cdots, m_k),$$

$$\begin{aligned}
\tau_Y(y(\lambda)) &= q^{\lambda\sum_{j=1}^n jm_j+\sum_{j=1}^n \frac{1}{2}j(j-1)m_j}\prod_{k=1}^n F_k(q;m_1,\cdots,m_k),\\
\tau_{{}^tY}(x(\lambda)) &= q^{(1-\lambda)\sum_{j=1}^n jm_j+\frac{1}{2}\sum_{j=1}^n(\sum_{s=j}^n m_s)(\sum_{s=j}^n m_s-1)}\prod_{k=1}^n F_k(q;m_1,\cdots,m_k),\\
\tau_{{}^tY}(y(\lambda)) &= q^{\lambda\sum_{j=1}^n jm_j+\frac{1}{2}\sum_{j=1}^n(\sum_{s=j}^n m_s)(\sum_{s=j}^n m_s-1)}\prod_{k=1}^n F_k(q;m_1,\cdots,m_k), \qquad (5.26)
\end{aligned}$$

where $Y=(m_1+\cdots+m_n, m_2+\cdots+m_n,\cdots,m_n)$, and ${}^tY$ is the transpose of the Young diagram Y, and $F_k(q;m_1,\cdots,m_k)$ is

$$F_k(q;m_1,\cdots,m_k)=\prod_{j=1}^{m_k}\prod_{s=0}^{k-1}\Big(1-q^{\sum_{t=1}^s m_{k-t}+s+j}\Big)^{-1}. \qquad (5.27)$$

This expression is obtained from eq. (B.36) by setting $f_i-f_{i+1}=m_i$ (or eq. (B.37) by setting $g_i-g_{i+1}=m_i$). The full characters eqs. (5.24,5.25) reduce to the specialized characters,

$$\begin{aligned}
\mathrm{ch}_{C=n} &= q^{\frac{1}{2}\lambda(\lambda-1)n}\sum_{m_1=0}^\infty\cdots\sum_{m_n=0}^\infty q^{\sum_{j=1}^n(\sum_{s=j}^n m_s)^2}\prod_{k=1}^n F_k(q;m_1,\cdots,m_k)^2, \qquad (5.28)\\
\mathrm{ch}_{C=-n} &= q^{-\frac{1}{2}\lambda(\lambda-1)n}\sum_{m_1=0}^\infty\cdots\sum_{m_n=0}^\infty q^{\sum_{j=1}^n j^2m_j}\prod_{k=1}^n F_k(q;m_1,\cdots,m_k)^2. \qquad (5.29)
\end{aligned}$$

As we will show in the next subsection, eq. (5.28) can be rewritten in a product form:

$$\mathrm{ch}_{C=n}=q^{\frac{1}{2}\lambda(\lambda-1)n}\prod_{j=1}^\infty\prod_{k=1}^n\frac{1}{1-q^{j+k-1}}. \qquad (5.30)$$

This character is consistent with the conjecture that the representation space is spanned by $W(z^{-j}D^{k-1})$ with $1\le k\le n$ (of course with $j\ge1$, $1\le k\le j$)[1].

5.4 Other cases

In ref.[24], the quasifinite representation with the weight function,

$$\Delta(x)=\sum_{i=1}^N\frac{e^{\lambda'_ix}-1}{e^x-1},\quad C=N, \qquad (5.31)$$

was studied. This representation is realized by bc ghost, eq. (3.23). We review their results. Let us break the set $\{\lambda'_1,\cdots,\lambda'_N\}$ in the following way:

$$\begin{aligned}
&\{\lambda'_1,\cdots,\lambda'_N\}=S_1\cup\cdots\cup S_m,\\
&S_i=\{\lambda_i+k_1^{(i)},\cdots,\lambda_i+k_{n_i}^{(i)}\},\quad \lambda_i-\lambda_j\notin\mathbb{Z},\quad k_1^{(i)}\ge\cdots\ge k_{n_i}^{(i)}\in\mathbb{Z}. \qquad (5.32)
\end{aligned}$$

Then the representation for the weight function $\Delta(x)$ is a direct product of the representations for $\Delta_i(x)=\sum_{j=1}^{n_i}\frac{e^{(\lambda_i+k_j^{(i)})x}-1}{e^x-1}$ [30]. Therefore, the character is factorized as

$$\mathrm{ch}=\prod_{i=1}^{m}\mathrm{ch}_i, \tag{5.33}$$

where ch_i is the character of the representation for $\Delta_i(x)$, and we need to consider only the quasifinite representations with

$$\Delta(x)=\sum_{i=1}^{n}\frac{e^{(\lambda+k_i)x}-1}{e^x-1},\quad C=n,\quad k_1\geq\cdots\geq k_n\geq 0\in\mathbb{Z}. \tag{5.34}$$

The full characters for these representations are given by [24]

$$\mathrm{ch}=\det\Bigl(S_{\lambda+k_i-i+j}\Bigr)_{1\leq i,j\leq n}, \tag{5.35}$$

where S_λ is defined in section 3.1. By setting g_k to eq. (3.17) and using eq. (3.20), eq. (5.35) reduces to the specialized character,

$$\mathrm{ch}=q^{\sum_{i=1}^{n}\frac{1}{2}(\lambda+k_i)(\lambda+k_i-1)}\prod_{j=1}^{\infty}\frac{1}{(1-q^j)^n}\prod_{1\leq i<j\leq n}(1-q^{k_i-k_j-i+j}). \tag{5.36}$$

For $C=n>0$, the weight function $\Delta(x)=C\frac{e^{\lambda x}-1}{e^x-1}$ studied in the previous section is a special case of eq. (5.34); $k_1=\cdots=k_n=0$. So the full character eq. (5.24) must be obtained from eq. (5.35). In fact we can show that the full character eq. (5.35) is expressed as a summation over all Young diagrams:

$$\begin{aligned}\mathrm{ch} &= \det\Bigl(S_{\lambda+k_i-i+j}\Bigr)_{1\leq i,j\leq n}\\ &= \det\Bigl((-1)^{j-i+k_i}e^{\sum_{k=0}^{\infty}g_k\Delta_k^\lambda}\sum_{m\in\mathbb{Z}}P_{m-n-1+i-k_i}(-x(\lambda))P_{m-n-1+j}(-y(\lambda))\Bigr)_{1\leq i,j\leq n}\\ &= (-1)^{|Y|}e^{n\sum_{k=0}^{\infty}g_k\Delta_k^\lambda}\sum_{h_1\geq\cdots\geq h_n\geq 0}\det\Bigl(P_{h_j-k_i+i-j}(-x(\lambda))\Bigr)_{1\leq i,j\leq n}\det\Bigl(P_{g_i-i+j}(-y(\lambda))\Bigr)_{1\leq i,j\leq n}\\ &= e^{n\sum_{k=0}^{\infty}g_k\Delta_k^\lambda}\sum_{\substack{Y\\ wd(Y)\leq n}}\tau_{Y/Y_k}(x(\lambda))\tau_Y(y(\lambda)),\end{aligned} \tag{5.37}$$

where Y_k is a Young diagram with ${}^tY_k=(k_1,\cdots,k_n)$, and τ_{Y/Y_k} is a skew S–function (see appendix B). Here we have used the determinant formula for the product of non-square matrices,

$$\det\Bigl(\sum_{m=1}^{N}a_{im}b_{jm}\Bigr)_{1\leq i,j\leq n}=\sum_{1\leq m_1<\cdots<m_n\leq N}\det\Bigl(a_{im_j}\Bigr)_{1\leq i,j\leq n}\det\Bigl(b_{im_j}\Bigr)_{1\leq i,j\leq n}. \tag{5.38}$$

For $k_1=\cdots=k_n=0$, τ_{Y/Y_k} reduces to τ_Y. So we establish the equivalence of eq. (5.24) and eq. (5.35) in this case. Eq. (5.30) is obtained from eq. (5.36).

Similarly we can rewrite the full character eq. (5.25) as

$$\mathrm{ch}_{C=-n}=\det\Bigl(S_{j-i}^{\lambda;-1}\Bigr)_{1\leq i,j\leq n}. \tag{5.39}$$

5.5 Differential equation for full characters

Finally we comment on the differential equation of the full character. From eq. (3.11), $S_m^{\lambda;\epsilon}$ as a function of x and y satisfies the differential equation,

$$\frac{\partial}{\partial x_\ell} S_m^{\lambda;\epsilon} = (-\epsilon)^{\ell+1} S_{m+\ell}^{\lambda;\epsilon}, \quad \frac{\partial}{\partial y_\ell} S_m^{\lambda;\epsilon} = (-\epsilon)^{\ell+1} S_{m-\ell}^{\lambda;\epsilon}. \tag{5.40}$$

Thus the full character eq. (5.35) satisfies the following differential equation,

$$\begin{aligned} \frac{\partial}{\partial x_\ell} S^\lambda_{\{k_1,\cdots,k_n\}} &= (-1)^{\ell+1} \sum_{i=1}^n S^\lambda_{\{k_1,\cdots,k_i+\ell,\cdots,k_n\}}, \\ \frac{\partial}{\partial y_\ell} S^\lambda_{\{k_1,\cdots,k_n\}} &= (-1)^{\ell+1} \sum_{i=1}^n S^\lambda_{\{k_1,\cdots,k_i-\ell,\cdots,k_n\}}, \end{aligned} \tag{5.41}$$

where $S^\lambda_{\{k_1,\cdots,k_n\}} = \det(S_{\lambda+k_i-i+j})_{1\le i,j\le n}$.

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Appendix A: Determinant formulae at lower degrees

In this appendix, we give the explicit form of the functions $A_n(C)$ and $B_n(\lambda)$ defined in eq. (5.8)[1]. We can parametrize those functions in the form,

$$A_n(C) = \prod_{\ell\in\mathbb{Z}} (C-\ell)^{\alpha(\ell)}, \qquad B_n(\lambda) = \prod_{\ell\in\mathbb{Z}} (\lambda-\ell)^{\beta(\ell)}$$

We make tables for the index $\alpha(\ell)$ and $\beta(\ell)$. We note that $\beta(\ell)=\beta(-\ell)$. Hence we will write them only for $\ell \ge 0$.

$\underline{K=1}$: $B_n = 1$ due to the spectral flow symmetry [1].

n	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\alpha(4)$	$\alpha(5)$	$\alpha(6)$	$\alpha(7)$
1	0	1	0	0	0	0	0	0	0
2	0	3	1	0	0	0	0	0	0
3	0	6	3	1	0	0	0	0	0
4	1	13	8	3	1	0	0	0	0
5	3	24	17	8	3	1	0	0	0
6	10	48	37	19	8	3	1	0	0
7	23	86	71	41	19	8	3	1	0
8	54	161	138	85	43	19	8	3	1

$K=2$

n	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	4	1	0	0	10	2	0	0
3	0	12	4	1	0	34	8	2	0
4	1	34	14	4	1	108	30	8	2

$K=3$

n	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	5	1	0	0	12	2	0	0
3	0	19	5	1	0	50	10	2	0

$K=4$

n	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	6	1	0	0	14	2	0	0
3	0	27	6	1	0	68	12	2	0

$K=5$

n	$\alpha(-1)$	$\alpha(0)$	$\alpha(1)$	$\alpha(2)$	$\alpha(3)$	$\beta(0)$	$\beta(1)$	$\beta(2)$	$\beta(3)$
1	0	1	0	0	0	2	0	0	0
2	0	7	1	0	0	16	2	0	0

B Appendix B: The Schur function

The Schur function, which is the character of the general linear group, can be expressed in terms of free fermions [40, 19]. In this appendix we summarize the useful formulae([3], see also [33]).

B.1

Free fermions[5] $\bar{\flat}(z)$, $\flat(z)$ and the vacuum state $\|0\rangle\!\rangle$ are defined by

$$\begin{aligned} &\bar{\flat}(z)=\sum_{n\in\mathbb{Z}}\bar{\flat}_n z^{-n-1}, \quad \flat(z)=\sum_{n\in\mathbb{Z}}\flat_n z^{-n}, \\ &\{\bar{\flat}_m,\flat_n\}=\delta_{m+n,0}, \quad \{\bar{\flat}_m,\bar{\flat}_n\}=\{\flat_m,\flat_n\}=0, \\ &\bar{\flat}_m\|0\rangle\!\rangle=\flat_n\|0\rangle\!\rangle=0, \qquad (m\geq 0, n\geq 1). \end{aligned} \tag{B.1}$$

[5] We use this notation to avoid a confusion with the free fermions used in the free-field realization of $W_{1+\infty}$. Relation to usual free fermions $\bar{\psi}(z)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\bar{\psi}_r z^{-r-\frac{1}{2}}$, $\psi(z)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}\psi_r z^{-r-\frac{1}{2}}$ is given by $\bar{\flat}_n=\bar{\psi}_{n+\frac{1}{2}}$, $\flat_n=\psi_{n-\frac{1}{2}}$.

The fermion Fock space is a linear span of $\prod_i \bar{\flat}_{-m_i} \prod_j \flat_{-n_j} \| 0 \rangle\!\rangle$. The U(1) current $\mathcal{J}(z) = \sum_{n\in\mathbb{Z}} \mathcal{J}_n z^{-n-1}$ is defined by $\mathcal{J}(z) = :\bar{\flat}(z)\flat(z):$, *i.e.* $\mathcal{J}_n = \sum_{m\in\mathbb{Z}} :\bar{\flat}_m \flat_{n-m}:$, where the normal ordering $:\bar{\flat}_m\flat_n:$ means $\bar{\flat}_m\flat_n$ if $m \le -1$ and $-\flat_n\bar{\flat}_m$ if $m \ge 0$. Their commutation relations are

$$[\mathcal{J}_n, \mathcal{J}_m] = n\delta_{n+m,0}, \quad [\mathcal{J}_n, \bar{\flat}_m] = \bar{\flat}_{n+m}, \quad [\mathcal{J}_n, \flat_m] = -\flat_{n+m}. \tag{B.2}$$

The fermion Fock space is decomposed into the irreducible representations of $\hat{\mathrm{u}}(1)$ with the highest weight state $\|N\rangle\!\rangle$ ($N \in \mathbb{Z}$),

$$\|N\rangle\!\rangle = \begin{cases} \bar{\flat}_{-N}\cdots\bar{\flat}_{-2}\bar{\flat}_{-1}\|0\rangle\!\rangle & N \ge 1 \\ \|0\rangle\!\rangle & N = 0 \\ \flat_{N+1}\cdots\flat_{-1}\flat_0\|0\rangle\!\rangle & N \le -1. \end{cases} \tag{B.3}$$

A free boson $\phi(z)$ and the vacuum state $\|p\rangle\!\rangle_B$ are defined by

$$\begin{aligned} &\phi(z) = \hat{q} + \alpha_0 \log z - \sum_{n\neq 0} \frac{\alpha_n}{n} z^{-n}, \\ &[\alpha_n, \alpha_m] = n\delta_{n+m,0}, \quad [\alpha_0, \hat{q}] = 1, \\ &\alpha_n \|p\rangle\!\rangle_B = 0 \quad (n > 0), \quad \alpha_0\|p\rangle\!\rangle_B = p\|p\rangle\!\rangle_B. \end{aligned} \tag{B.4}$$

The boson Fock space is a linear span of $\prod_i \alpha_{-n_i}\|p\rangle\!\rangle_B$. The normal ordering $:\ :$ means that α_n ($n \ge 0$) is moved to the right of α_m ($m < 0$) and $\hat{q}$. $\|p\rangle\!\rangle_B$ is obtained from $\|0\rangle\!\rangle$ as $\|p\rangle\!\rangle_B = :e^{p\phi(0)}: \|p\rangle\!\rangle_B$. The vertex operator satisfies

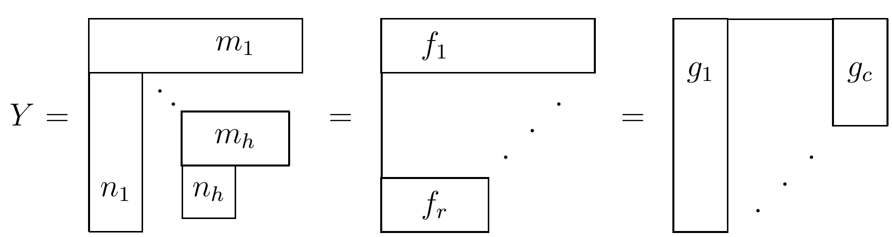
$$:e^{p\phi(z)}::e^{p'\phi(w)}: = (z-w)^{pp'} :e^{p\phi(z)+p'\phi(w)}:. \tag{B.5}$$

Boson-fermion correspondence is

$$\bar{\flat}(z) = :e^{\phi(z)}:, \quad \flat(z) = :e^{-\phi(z)}:, \quad \|N\rangle\!\rangle = \|N\rangle\!\rangle_B. \tag{B.6}$$

U(1) current is $\mathcal{J}(z) = \partial\phi(z)$.

A Young diagram has various parametrization:



(B.7)

where $m_1 > \cdots > m_h \ge 1$, $n_1 > \cdots > n_h \ge 0$, $f_1 \ge \cdots \ge f_r \ge 1$, $g_1 \ge \cdots \ge g_c \ge 1$. According to these parametrizations, we denote the Young diagram Y by

$Y = (m_1, \cdots, m_h; n_1, \cdots, n_h)$, $Y = (f_1, \cdots, f_r)$ or ${}^tY = (g_1, \cdots, g_c)$ respectively, and the number of boxes as $|Y| = \sum_{i=1}^h (m_i + n_i) = \sum_{i=1}^r f_i = \sum_{i=1}^c g_i$. Corresponding to the Young diagram eq. (B.7), we define a state $\|N; Y\rangle\!\rangle$ as follows:

$$\|N; Y\rangle\!\rangle = \prod_{i=1}^{h} \bar{\flat}_{-m_i-N} \flat_{-n_i+N} (-1)^{n_i} \|N\rangle\!\rangle \tag{B.8}$$

$$= \bar{\flat}_{-\bar{f}_1-N} \bar{\flat}_{-\bar{f}_2-N} \cdots \bar{\flat}_{-\bar{f}_r-N} \|N - r\rangle\!\rangle \tag{B.9}$$

$$= (-1)^{|Y|} \flat_{-\bar{g}_1+N} \flat_{-\bar{g}_2+N} \cdots \flat_{-\bar{g}_r+N} \|N + c\rangle\!\rangle, \tag{B.10}$$

where

$$\bar{f}_i = f_i - i + 1, \quad \bar{g}_i = g_i - i. \tag{B.11}$$

These states $\|N; Y\rangle\!\rangle$ with all Young diagrams are a basis of $\widehat{\mathrm{u}}(1)$ representation space of the highest weight $\|N\rangle\!\rangle$. We abbreviate $\|0; Y\rangle\!\rangle$ as $\|Y\rangle\!\rangle$.

Bra states are obtained from ket states by $\dagger$ operation ($\bar{\flat}_n{}^\dagger = \flat_{-n}$) with the normalization $\langle\!\langle 0\|0\rangle\!\rangle = 1$; for example, $\langle\!\langle N\| = \|N\rangle\!\rangle^\dagger$ and $\langle\!\langle N\|N'\rangle\!\rangle = \delta_{NN'}$, $\langle\!\langle Y\| = \|Y\rangle\!\rangle^\dagger = \langle\!\langle 0\| \prod_{i=1}^h \bar{\flat}_{n_i} \flat_{m_i} (-1)^{n_i}$ and $\langle\!\langle Y\|Y'\rangle\!\rangle = \delta_{YY'}$. Note that $\{\|N; Y\rangle\!\rangle\}$ is an orthonormal basis of the fermion Fock space with U(1)–charge N.

Irreducible representations of the permutation group $\mathcal{S}_n$ and the general linear group $GL(N)$ are both characterized by the Young diagrams Y. We denote their characters by $\chi_Y(k)$ and $\tau_Y(x)$, respectively. Here $(k) = 1^{k_1} 2^{k_2} \cdots n^{k_n}$ stands for the conjugacy class of $\mathcal{S}_n$; $k_1 + 2k_2 + \cdots + nk_n = n$ =the number of boxes in Y. $x = [x_\ell]$ ($\ell = 1, 2, 3, \cdots$) stands for $x_\ell = \frac{1}{\ell} \mathrm{tr}\, g^\ell = \frac{1}{\ell} \sum_{i=1}^N \epsilon_i^\ell$ for an element g of $GL(N)$ whose diagonalized form is $g = \mathrm{diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_N]$. In this case the number of boxes in Y is a rank of tensor for $GL(N)$. We take $N \to \infty$ limit formally. τ_Y is called the Schur function. The skew S-function $\tau_{Y/Y'}$ is defined by

$$\tau_{Y/Y'}(x) = \sum_{Y''} C^Y_{Y'Y''} \tau_{Y''}(x), \tag{B.12}$$

where the Clebsch-Gordan coefficients $C^Y_{Y'Y''}$ are

$$\tau_{Y'}(x) \tau_{Y''}(x) = \sum_{Y} C^Y_{Y'Y''} \tau_Y(x), \tag{B.13}$$

namely decomposition of the tensor product of representations Y' and Y''; $Y' \otimes Y'' = \bigoplus_Y C^Y_{Y'Y''} Y$. $\tau_{Y/Y'}(x)$ is non-vanishing only for $Y' \subseteq Y$.

$\chi_Y(k)$, $\tau_Y(x)$ and $\tau_{Y/Y'}(x)$ are expressed in terms of free fermion as follows:

$$\chi_Y(k) = \langle\!\langle 0\| \mathcal{J}_1^{k_1} \mathcal{J}_2^{k_2} \cdots \mathcal{J}_n^{k_n} \|Y\rangle\!\rangle, \tag{B.14}$$

$$\tau_Y(x) = \langle\!\langle 0\| \exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) \|Y\rangle\!\rangle, \tag{B.15}$$

$$\tau_{Y/Y'}(x) = \langle\!\langle Y'\| \exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) \|Y\rangle\!\rangle. \tag{B.16}$$

We remark that they can also be written as $\chi_Y(k) = \langle\!\langle Y \| \mathcal{J}_{-1}^{k_1}\mathcal{J}_{-2}^{k_2}\cdots\mathcal{J}_{-n}^{k_n}\|0\rangle\!\rangle$, $\tau_Y(x) = \langle\!\langle Y\| \exp(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_{-\ell})\|0\rangle\!\rangle$ and $\tau_{Y/Y'}(x) = \langle\!\langle Y\| \exp(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_{-\ell})\|Y'\rangle\!\rangle$.

Under the adjoint action of $\exp(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell)$, $\bar{\flat}(z)$ and $\flat(z)$ transform as

$$\begin{aligned}\exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big)\bar{\flat}(z)\exp\Big(-\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) &= \exp\Big(\sum_{\ell=1}^{\infty} x_\ell z^\ell\Big)\bar{\flat}(z),\\ \exp\Big(\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big)\flat(z)\exp\Big(-\sum_{\ell=1}^{\infty} x_\ell \mathcal{J}_\ell\Big) &= \exp\Big(-\sum_{\ell=1}^{\infty} x_\ell z^\ell\Big)\flat(z).\end{aligned} \tag{B.17}$$

B.2

Let us introduce the elementary Schur polynomial $P_n(x)$,

$$\exp\Big(\sum_{\ell=1}^{\infty} x_\ell z^\ell\Big) = \sum_{n\in\mathbb{Z}} P_n(x) z^n. \tag{B.18}$$

Note that $P_n(x) = 0$ for $n < 0$. Then the Schur functions with one row, one column and one hook are respectively given by

$$\tau_f(x) = P_f(x), \tag{B.19}$$

$$\tau_{1^g}(x) = (-1)^g P_g(-x), \tag{B.20}$$

$$\begin{aligned}\tau_{m;n}(x) &= (-1)^n \sum_{\ell=0}^{\infty} P_{m+\ell}(x) P_{n-\ell}(-x)\\ &= (-1)^{n-1}\sum_{\ell=0}^{\infty} P_{m-1-\ell}(x) P_{n+1+\ell}(-x).\end{aligned} \tag{B.21}$$

Using this, the Schur function with the Young diagram eq. (B.7) is given by

$$\tau_Y(x) = \det\Big(\tau_{m_i;n_j}(x)\Big)_{1\le i,j\le h} \tag{B.22}$$

$$= \det\Big(P_{f_i-i+j}(x)\Big)_{1\le i,j\le r} \tag{B.23}$$

$$= (-1)^{|Y|}\det\Big(P_{g_i-i+j}(-x)\Big)_{1\le i,j\le c}. \tag{B.24}$$

The Schur function with the transposed Young diagram ${}^tY$ is

$$\tau_{{}^tY}(x) = (-1)^{|Y|}\tau_Y(-x). \tag{B.25}$$

The skew S-function with Young diagrams parametrized in the second and third form of eq. (B.7), is given by

$$\tau_{Y/Y'}(x) = \det\Big(P_{f_i-f'_j-i+j}(x)\Big)_{1\le i,j\le r} \tag{B.26}$$

$$= (-1)^{|Y|-|Y'|}\det\Big(P_{g_i-g'_j-i+j}(-x)\Big)_{1\le i,j\le c}. \tag{B.27}$$

Since τ_Y's are a basis of the space of symmetric functions, we have

$$\prod_r \prod_s \frac{1}{1-u_r v_s} = \sum_Y \tau_Y(x)\tau_Y(y), \tag{B.28}$$

where the summation runs over all the Young diagrams and x, y are the Miwa variables for u, v,

$$x_\ell = \frac{1}{\ell}\sum_r u_r^\ell, \quad y_\ell = \frac{1}{\ell}\sum_s v_s^\ell. \tag{B.29}$$

Similarly we have

$$\sum_Y \tau_{Y/Y'}(x)\tau_Y(y) = \sum_Y \tau_Y(x)\tau_Y(y)\tau_{Y'}(y). \tag{B.30}$$

Those formulae are easily proved by using free field expression eqs. (B.15,B.16) and bosonization. For example, eq. (B.23) is obtained as follows. By rewriting $\|Y\rangle\!\rangle$ eq. (B.9) as

$$\|Y\rangle\!\rangle = \oint \prod_{i=1}^r \frac{dz_i}{2\pi i} z_i^{-\bar{f}_i} \cdot \bar{\mathfrak{b}}(z_1)\cdots\bar{\mathfrak{b}}(z_r)\| - r\rangle\!\rangle,$$

eq. (B.15) becomes

$$\tau_Y(x) = \oint \prod_{i=1}^r \frac{dz_i}{2\pi i} z_i^{-\bar{f}_i} e^{\sum_{\ell=1}^\infty x_\ell z_i^\ell} \cdot \langle\!\langle 0\|\bar{\mathfrak{b}}(z_1)\cdots\bar{\mathfrak{b}}(z_r)\| - r\rangle\!\rangle.$$

Bosonization tells us that

$$\langle\!\langle 0\|\bar{\mathfrak{b}}(z_1)\cdots\bar{\mathfrak{b}}(z_r)\| - r\rangle\!\rangle = \prod_{i<j}(z_i - z_j) \cdot \prod_{i=1}^r z_i^{-r}.$$

Since $\prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant $(-1)^{\frac{1}{2}r(r-1)} \det(z_i^{j-1})_{1\le i,j\le r}$, we obtain eq. (B.23) after picking up residues.

Eq. (B.28) is proved as follows:

$$\begin{aligned}
\prod_r \prod_s \frac{1}{1-u_r v_s} &= \exp\Big(\sum_r\sum_s \log\frac{1}{1-u_r v_s}\Big) = \exp\Big(\sum_r\sum_s\sum_{\ell=1}^\infty \frac{1}{\ell}(u_r v_s)^\ell\Big) \\
&= \exp\Big(\sum_{\ell=1}^\infty \ell x_\ell y_\ell\Big) = \langle\!\langle 0\|\exp\Big(\sum_{\ell=1}^\infty x_\ell \mathcal{J}_\ell\Big)\exp\Big(\sum_{\ell=1}^\infty y_\ell \mathcal{J}_{-\ell}\Big)\|0\rangle\!\rangle \\
&= \sum_Y \langle\!\langle 0\|\exp\Big(\sum_{\ell=1}^\infty x_\ell \mathcal{J}_\ell\Big)\|Y\rangle\!\rangle\langle\!\langle Y\|\exp\Big(\sum_{\ell=1}^\infty y_\ell \mathcal{J}_{-\ell}\Big)\|0\rangle\!\rangle = \sum_Y \tau_Y(x)\tau_Y(y).
\end{aligned}$$

Here we have used the completeness of $\{\|Y\rangle\!\rangle\}$ in the fermion Fock space with vanishing U(1)–charge.

B.3

In this subsection we set x_ℓ as follows:

$$x_\ell = \frac{1}{\ell}\frac{q^{a\ell}}{1-q^\ell}. \tag{B.31}$$

Then the Schur functions with one row and one column are given by

$$P_n(x) = \prod_{j=1}^{n}\frac{q^a}{1-q^j}, \tag{B.32}$$

$$(-1)^n P_n(-x) = \prod_{j=1}^{n}\frac{q^{j-1+a}}{1-q^j}. \tag{B.33}$$

From eq. (B.21), the Schur function with one hook becomes

$$\tau_{m;n}(x) = q^{a(m+n)+\frac{1}{2}n(n+1)}\prod_{j=1}^{m-1}\frac{1}{1-q^j}\prod_{j=1}^{n}\frac{1}{1-q^j}\cdot\frac{1}{1-q^{m+n}}. \tag{B.34}$$

By combining those and eqs. (B.22,B.23,B.24), the Schur function with the Young diagram eq. (B.7) is given by

$$\begin{aligned}\tau_Y(x) &= q^{a|Y|+\sum_{i=1}^{h}\left(\frac{1}{2}n_i(n_i+1)+(i-1)(m_i+n_i)\right)} \\ &\quad\times\prod_{i=1}^{h}\left(\prod_{j=1}^{m_i-1}\frac{1}{1-q^j}\prod_{j=1}^{n_i}\frac{1}{1-q^j}\right)\cdot\frac{\prod_{i<j}(1-q^{m_i-m_j})(1-q^{n_i-n_j})}{\prod_{i,j}(1-q^{m_i+n_j})} \qquad (B.35)\\ &= q^{a|Y|+\sum_{i=1}^{r}(i-1)f_i}\prod_{i=1}^{r}\prod_{j=1}^{f_i-i+r}\frac{1}{1-q^j}\cdot\prod_{i<j}(1-q^{f_i-f_j-i+j}) \qquad (B.36)\\ &= q^{a|Y|+\sum_{i=1}^{r}\frac{1}{2}g_i(g_i-1)}\prod_{i=1}^{c}\prod_{j=1}^{g_i-i+c}\frac{1}{1-q^j}\cdot\prod_{i<j}(1-q^{g_i-g_j-i+j}). \qquad (B.37)\end{aligned}$$

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