November 1994

RIMS-997
YITP/U-94-29
SULDP-1994-8
UT-693
hep-th/9411053

Collective Field Theory, Calogero-Sutherland Model and Generalized Matrix Models

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Abstract

On the basis of the collective field method, we analyze the Calogero–Sutherland model (CSM) and the Selberg–Aomoto integral, which defines, in particular case, the partition function of the matrix models. Vertex operator realizations for some of the eigenstates (the Jack polynomials) of the CSM Hamiltonian are obtained. We derive Virasoro constraint for the generalized matrix models and indicate relations with the CSM operators. Similar results are presented for the q–deformed case (the Macdonald operator and polynomials), which gives the generating functional of infinitely many conserved charges in the CSM.

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1 Introduction

The purpose of this letter is to discuss some common properties which are shared by the Calogero–Sutherland model (CSM) [1] described by the Hamiltonian and momentum,

$$\widetilde{\mathcal{H}} = \frac{1}{2}\sum_{j=1}^{N}\Big(\frac{1}{i}\frac{\partial}{\partial q_j}\Big)^2 + \frac{1}{2}\Big(\frac{\pi}{L}\Big)^2\sum_{i\neq j}\frac{\beta(\beta-1)}{\sin^2\frac{\pi}{L}(q_i-q_j)}, \quad \widetilde{\mathcal{P}} = \sum_{j=1}^{N}\frac{1}{i}\frac{\partial}{\partial q_j}, \tag{1}$$

and the "generalized matrix model" whose partition function is defined by the following integral,

$$Z_\beta([g]) \equiv \int \prod_{i=1}^{M} dt_i\, |\Delta(t)|^{2\beta}\, e^{\sum_{n=0}^{\infty} g_n \sum_{i=1}^{M} t_i^n}, \quad \Delta(t) \equiv \prod_{i<j}(t_i - t_j). \tag{2}$$

For some specific values of β, the latter integral is related to the usual matrix models[2]; the hermitian matrix model ($\beta = 1$)[3], the orthogonal matrix model ($\beta = \frac{1}{2}$)[4] and the symplectic matrix model ($\beta = 2$)[5]. The integral of this type, called the Selberg–Aomoto integral, was studied in [6] as a multivariate generalization of the hypergeometric integral and has been used recently for calculating the correlation functions of the CSM [7][8].

In the present letter, we apply the collective field methods [9][10][11][12] to these two models. Firstly, in the CSM, by using a collective field Hamiltonian, we derive some of the eigenstates as the vertex operators. Mathematically, they are known as Jack symmetric polynomials [13] and are classified by the Young diagrams. Furthermore, the mutually commutative conserved charges of the CSM are known to be realized as the Cartan generators of the $W_{1+\infty}$ algebra [14]. These properties strongly indicates that the system has the $W_{1+\infty}$ symmetry [15][16][17].

Secondly, we study the integral (2) and show the appearance of the Virasoro constraint [18]. Although they are defined by the Virasoro generators with the mode $n \geq -1$, they have a unique relativistic extension with the following central charge,

$$c = 1 - \frac{6(1-\beta)^2}{\beta}. \tag{3}$$

This formula satisfies the duality symmetry, $\beta \leftrightarrow \frac{1}{\beta}$, which is the known property of the Jack polynomials [13].

The CSM and the Selberg–Aomoto integral might be related by the integral representation of the Jack polynomials. We show, through this relation, the vertex operators used in the CSM are the primary fields with dimension 1, *i.e.* the screening currents.

Finally, we refer to a q–deformation of the foregoing discussions. Mathematically, the corresponding Hamiltonian operator and the eigenstates are known as the Macdonald operator and the Macdonald polynomials. They are useful in obtaining the higher conservative charges of the CSM.

2 Hamiltonian and Collective coordinates

To fix some notations, we start from summarizing the collective field description of the CSM. Let us make the coordinate transformation, $x_j \equiv e^{2\pi i q_j/L}$. The eigenfunctions $\psi_\lambda(x)$ of the CSM are then factorized as,

$$\begin{aligned} \psi_\lambda(x) &= J_\lambda(x)\widetilde{\Delta}(x)^\beta, \\ \widetilde{\Delta}(x) &= \prod_{i<j} \sin\frac{\pi}{L}(q_i - q_j) \propto \prod_i x_i^{-(N-1)/2} \prod_{i<j}(x_i - x_j), \end{aligned} \tag{4}$$

where $J_\lambda(x)$ is the symmetric polynomial of the coordinates x_i $(i = 1, 2, \cdots, N)$. The Hamiltonian itself is modified when it is acted on $J_\lambda(x)$,

$$\begin{aligned} &\widetilde{\Delta}(x)^{-\beta}\widetilde{\mathcal{H}}\widetilde{\Delta}(x)^\beta = 2\Big(\frac{\pi}{L}\Big)^2\mathcal{H} + E_0, \\ &\mathcal{H} \equiv \sum_{i=1}^N D_i^2 + \beta\sum_{i<j}\frac{x_i + x_j}{x_i - x_j}(D_i - D_j), \end{aligned} \tag{5}$$

where $D_i \equiv x_i\frac{\partial}{\partial x_i}$ and $E_0 = \frac{1}{6}(\frac{\pi}{L})^2\beta^2(N^3 - N)$ is the eigenvalue of the ground-state $\widetilde{\Delta}(x)^\beta$. Similarly, the momentum is modified as,

$$\widetilde{\Delta}(x)^{-\beta}\widetilde{\mathcal{P}}\widetilde{\Delta}(x)^\beta = 2\frac{\pi}{L}\mathcal{P}, \qquad \mathcal{P} \equiv \sum_{i=1}^N D_i. \tag{6}$$

Eigenfunctions $J_\lambda(x)$ of $\mathcal{H}$ are known as the Jack symmetric polynomials [13]. They are parametrized by the Young diagrams and the eigenvalue associated with the diagram $\lambda = (\lambda_1, \cdots, \lambda_M)$ is given by,

$$\epsilon_\lambda = \sum_{i=1}^M\Big(\lambda_i^2 + \beta(N + 1 - 2i)\lambda_i\Big) = \sum_{i=1}^{M'}\Big(-\beta\lambda_i'^2 + (\beta N + 2i - 1)\lambda_i'\Big), \tag{7}$$

where ${}^t\lambda = (\lambda'_1, \cdots, \lambda'_{M'})$ is the conjugate of λ.

Because $J_\lambda(x)$ is symmetric with respect to x_i's, it can be written out by using the power sum polynomials $p_n \equiv \sum_i x_i^n$. Let us denote $a_n^\dagger$ as the "creation operator" which gives rise to this p_n. The "annihilation operator" associated with it is defined by the commutation relation,

$$\left[a_n, a_m^\dagger\right] = \frac{1}{\beta} n \delta_{n,m}, \quad (n, m > 0). \tag{8}$$

We introduce the vacuum states by $a_n|0\rangle = \langle 0|a_n^\dagger = 0$ ($n > 0$). One may translate the collective field Hilbert space and the space of symmetric polynomials by the formula,

$$\langle 0|e^{\beta \sum_i A(x_i)} a_{n_1}^\dagger \cdots a_{n_m}^\dagger|0\rangle = p_{n_1} \cdots p_{n_m}, \qquad A(x) \equiv \sum_{n=1}^\infty \frac{1}{n} a_n x^n. \tag{9}$$

These operators are related with the conventional collective field operators, $\rho(x) = \sum_{i=1}^N \delta(x - x_i)$, by $\int dx\, x^n \rho(x) = p_n$. Use of these combinations has the benefit to illuminate the relation with the matrix–type integral (2) more directly.

The convention of the definition of (8) is to reproduce the standard inner product between the symmetric polynomials [13][8], *i.e.*,

$$\begin{aligned} \langle p_1^{r_1} \cdots p_n^{r_n}, p_m^{s_m} \cdots p_1^{s_1}\rangle &\equiv \langle 0|a_1^{r_1} \cdots a_n^{r_n} a_m^{\dagger\, s_m} \cdots a_1^{\dagger\, s_1}|0\rangle, \\ &= \delta_{\{r\},\{s\}} \beta^{-\sum_{i=1}^n ri} \prod_{i\geq 1} i^{r_i} r_i!. \end{aligned} \tag{10}$$

One may derive the collective coordinate representation of the Hamiltonian and the momentum in a usual way. The Hamiltonian in terms of the collective coordinates, $\widehat{\mathcal{H}}$, is defined by the transformation in (9) as,

$$\mathcal{H}\langle 0|e^{\beta \sum_i A(x_i)} = \langle 0|e^{\beta \sum_i A(x_i)} \widehat{\mathcal{H}}. \tag{11}$$

For example, the momentum operator is obtained as follows,

$$\sum_i D_i \langle 0|e^{\beta \sum_i A(x_i)} = \sum_i \langle 0|e^{\beta \sum_i A(x_i)} \beta \sum_{n=1}^\infty x_i^n a_n = \langle 0|e^{\beta \sum_i A(x_i)} \sum_n (\beta a_n^\dagger a_n),$$

which gives, $\widehat{\mathcal{P}} = \beta \sum_{n=1}^{\infty} a_n^\dagger a_n$. Similarly the Hamiltonian is given by,

$$\begin{aligned}\widehat{\mathcal{H}} &= \beta^2 \sum_{n,m=1}^{\infty} (a_{n+m}^\dagger a_n a_m + a_n^\dagger a_m^\dagger a_{n+m}) \\ &\quad + \beta(1-\beta) \sum_{n=1}^{\infty} n a_n^\dagger a_n + \beta^2 N \sum_{n=1}^{\infty} a_n^\dagger a_n. \end{aligned} \tag{12}$$

This particular form of the Hamiltonian appeared previously, for example, in [11] (see also the recent work [20]).

3 Vertex operators and Energy eigenstates

Some of the simpler eigenfunctions (the Jack polynomials) of the Hamiltonian can be explicitly written down by using the vertex operators. As is known, they are parametrized by the Young diagrams. In this section, we give their bosonized forms when the Young diagram has, (i) one or two rows (columns) or (ii) one row and one column (one "hook").

Define, $A^\dagger(t) \equiv \sum_{n=1}^{\infty} \frac{1}{n} a_n^\dagger t^n$. There are two types of the vertex operators which are diagonalized by the action of the Hamiltonian $\widehat{\mathcal{H}}$ as,

$$\widehat{\mathcal{H}} e^{\gamma A^\dagger(t)} |0\rangle = \Big(\frac{\beta}{\gamma} D^2 + (\beta N - \gamma) D \Big) e^{\gamma A^\dagger(t)} |0\rangle, \tag{13}$$

where $\gamma = \beta, -1$ and $D = t\frac{\partial}{\partial t}$. Therefore, by expanding these vertex operators in terms of t,

$$e^{\gamma A^\dagger(t)} = \sum_{n=0}^{\infty} \widehat{J}_n^{(\gamma)}(a^\dagger) t^n, \tag{14}$$

$\widehat{J}_n^{(\beta)}(a^\dagger)|0\rangle$ and $\widehat{J}_n^{(-1)}(a^\dagger)|0\rangle$ become the eigenstates of the Hamiltonian. Indeed, the symmetric polynomials associated with them are the Jack polynomials of the Young diagram with single row (n) and single column (1^n), respectively.

For the state $\prod_{i=1}^{M} e^{\gamma A^\dagger(t_i)} |0\rangle$, we have,

$$\begin{aligned}\widehat{\mathcal{H}} \prod_{i=1}^{M} e^{\gamma A^\dagger(t_i)} |0\rangle &= \Big(\sum_{i=1}^{M} \Big(\frac{\beta}{\gamma} D_i^2 + (\beta N - \gamma) D_i \Big) \\ &\quad + 2\gamma \sum_{i<j} \frac{1}{1 - \frac{t_j}{t_i}} \Big(\frac{t_j}{t_i} D_i - D_j \Big) \Big) \prod_{i=1}^{M} e^{\gamma A^\dagger(t_i)} |0\rangle, \end{aligned} \tag{15}$$

where $D_i = t_i \frac{\partial}{\partial t_i}$ and $|t_j/t_i| < 1$ for $i < j$. In the case of $\beta = 1$ when the Jack polynomial reduces to the Schur polynomial, the eigenstates of $\widehat{\mathcal{H}}$ are given by $\oint \prod_{j=1}^{M} \frac{dt_j}{2\pi i} t_j^{-\lambda_j - 1} \prod_{i<j}(1 - \frac{t_j}{t_i}) \prod_{i=1}^{M} e^{\pm A^\dagger(t_i)}|0\rangle$, which can be rewritten in a determinant form, $\det(\widehat{J}^{(\gamma)}_{\lambda_i - i + j})_{1 \le i,j \le M}|0\rangle$. For $\beta \neq 1$, however, this is no longer the case. Expanding (15), we obtain,

$$\begin{aligned} & \widehat{\mathcal{H}} \widehat{J}^{(\gamma)}_{n_1} \cdots \widehat{J}^{(\gamma)}_{n_M}|0\rangle \\ = & \sum_{i=1}^{M} \Big(\frac{\beta}{\gamma} n_i^2 + \Big(\beta N - \gamma(2i-1)\Big) n_i \Big) \widehat{J}^{(\gamma)}_{n_1} \cdots \widehat{J}^{(\gamma)}_{n_M}|0\rangle \\ & + 2\gamma \sum_{i<j} \sum_{r=1}^{n_j} (n_i - n_j + 2r) \widehat{J}^{(\gamma)}_{n_1} \cdots \widehat{J}^{(\gamma)}_{n_i + r} \cdots \widehat{J}^{(\gamma)}_{n_j - r} \cdots \widehat{J}^{(\gamma)}_{n_M}|0\rangle. \end{aligned} \tag{16}$$

At this moment, we are successful in diagonalizing this equation only for the cases $M = 1, 2$. For $M = 2$ case, the eigenstates correspond to the Young diagram with two rows ($\gamma = \beta$) or two columns ($\gamma = -1$). The explicit form of the diagonalized basis are given by,

$$\begin{aligned} \widehat{J}^{(\gamma)}_{(\lambda_1, \lambda_2)}(a^\dagger)|0\rangle &= \sum_{\ell=0}^{\lambda_2} c^{(\gamma)}(\lambda_1 - \lambda_2, \ell) \widehat{J}^{(\gamma)}_{\lambda_1 + \ell}(a^\dagger) \widehat{J}^{(\gamma)}_{\lambda_2 - \ell}(a^\dagger)|0\rangle, \\ c^{(\gamma)}(\lambda, \ell) &= \frac{\lambda + 2\ell}{\lambda + \ell} \prod_{j=1}^{\ell} \frac{\lambda + j}{j} \cdot \prod_{i=1}^{\ell} \frac{-\gamma + \frac{\beta}{\gamma}(i-1)}{\gamma + \frac{\beta}{\gamma}(\lambda + i)}. \end{aligned} \tag{17}$$

One can easily show that the Jack polynomials of single hook $(n, 1^m)$ are,

$$\widehat{J}_{(n,1^m)}(a^\dagger)|0\rangle = \sum_{\ell=0}^{m} (n + \ell + \beta(m - \ell\,))\ \widehat{J}^{(\beta)}_{n+\ell}(a^\dagger) \widehat{J}^{(-1)}_{m-\ell}(a^\dagger)|0\rangle. \tag{18}$$

4 Virasoro constraint

Let us go back to the Selberg–Aomoto integral (2). The collective field method can be also applied here. Namely, the insertion of the operator, $\sum_{i=1}^{M} t_i^n$, can be realized by taking a partial derivative with respect to the coupling g_n. These operators can be combined to give a single free collective field,

$$\partial\phi(z) = \sqrt{2\beta} \sum_{n=0}^{\infty} \frac{\partial}{\partial g_n} z^{-n-1} + \frac{1}{\sqrt{2\beta}} \sum_{n=1}^{\infty} n g_n z^{n-1}. \tag{19}$$

The coefficients for the bosonic field is chosen for the later convenience.

The essential feature of the matrix–type integral can be extracted by considering a set of differential equations. For the hermitian matrix case, it is so–called the Virasoro constraint [18]. We may derive similar equations for our generalized integral (2). The method we employ here is essentially the same as the hermitian case, namely we start from the integral which is trivially zero and rewrite it as the differential operator of the source term acting on the original integral,

$$\begin{aligned} 0 &= \int \prod_{i=1}^{M} dt_i \sum_{i=1}^{M} \frac{\partial}{\partial t_i}\Big(t_i^{n+1} \, |\Delta(t)|^{2\beta} \, e^{\sum_{\ell=0}^{\infty} g_\ell \sum_{i=1}^{M} t_i^\ell} \Big) \\ &= L_n Z([g]), \qquad (n=-1,0,1,2,\ldots), \end{aligned} \tag{20}$$

$$L_n = \sum_{m=1}^{\infty} m g_m \frac{\partial}{\partial g_{n+m}} + \beta \sum_{m=0}^{n} \frac{\partial^2}{\partial g_m \partial g_{n-m}} + (1-\beta)(n+1)\frac{\partial}{\partial g_n}. \tag{21}$$

The Virasoro generators appearing here have the mode n greater than -1. Hence there is no central extension in the commutation relations between these operators. However, we may uniquely extend them as the components of the relativistic energy–momentum tensor,

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} = \frac{1}{2} : (\partial\phi(z))^2 : - \frac{1-\beta}{\sqrt{2\beta}} \partial^2 \phi(z). \tag{22}$$

This energy–momentum tensor satisfies the Virasoro algebra with central charge (3).

At this moment, the physical meaning of this central charge is obscure. For the hermitian case, the double scaling limit is described by the KP–hierarchy. The partition function is identified as the τ–function. Since KP–hierarchy is essentially the free fermion system with $c=1$, the central charge (3) looks plausible. The nontrivial values for other matrix models, (for orthogonal or symplectic case, $c=-2$), may indicate that the double scaling limit for those models is described by interacting system.

5 Integral representation and Virasoro symmetry

We now present the relations between two models defined by (1) and (2). The original Selberg–Aomoto integral studied in [6] is as follows,

$$\begin{aligned}\widetilde{S}_{M,N}(\lambda_1,\lambda_2,\lambda,\mu;[x]) &= \int_{[0,1]^M}\prod_{i=1}^{M} dt_i \cdot \prod_{i=1}^{M}\prod_{k=1}^{N}(1-t_i x_k)^{\mu} D_{\lambda_1,\lambda_2,\lambda}([t]),\\ D_{\lambda_1,\lambda_2,\lambda}([t]) &= \prod_{i=1}^{M} t_i^{\lambda_1}(1-t_i)^{\lambda_2}\prod_{i<j}|t_i-t_j|^{\lambda}.\end{aligned} \tag{23}$$

This integral satisfies the multivariate generalization of the hypergeometric differential equation when $\mu=-\lambda/2$ or 1. Furthermore, it can be expanded by Jack polynomials with $\beta=2\mu^2/\lambda$ in the similar way as the Taylor expansion of the hypergeometric function. This fact has been used [7] for discussing the correlation functions of the CSM.

The correspondence between (23) and our integral (2) is given when we make the transformation of variables, *i.e.* from x_k to $p_n=\sum_{k=1}^{N} x_k^n$,

$$\prod_{i=1}^{M}\prod_{k=1}^{N}(1-t_i x_k)^{\mu} = \prod_{i=1}^{M} e^{-\mu\sum_{n=1}^{\infty}\frac{1}{n}p_n t_i^n}. \tag{24}$$

The "Vertex operators" which appear on the right hand side are exactly the same as those which appeared in (13).

The Virasoro symmetry considered in the previous section is related to the CSM in this context. Indeed,

$$\widehat{\mathcal{H}} = \beta\sum_{n=1}^{\infty} a_n^{\dagger} L_n + (\beta(N+1-2a_0)-1)\widehat{\mathcal{P}}, \tag{25}$$

where $a_n=\frac{\partial}{\partial g_n}$ for $n\geq 0$ and $a_n^{\dagger}=\frac{n}{\beta}g_n$ for $n>0$. Furthermore, the vertex operators (13) are nothing but the screening currents in terms of the Virasoro generators,

$$L_n e^{\gamma A^{\dagger}(t)}|0\rangle = \partial_t\Big(t^{n+1}e^{\gamma A^{\dagger}(t)}\Big)|0\rangle. \tag{26}$$

for $n\geq -1$.

Although linear combinations of the Jack polynomials can be obtained by the integral (23), it is interesting if one may derive the direct integral representation of the Jack polynomials. However, it is still difficult to find the general form of such integral representation, some of the simpler ones can be written as follows,

$$\oint \prod_{j=1}^{M} \frac{dt_j}{2\pi i} \prod_{i=1}^{M} \prod_{k=1}^{N} (1 - t_i x_k)^{-\gamma} \prod_{i<j} (t_i - t_j)^{2\frac{\gamma^2}{\beta}} \prod_{i=1}^{M} t_i^{-\lambda_i - 1 - \frac{\gamma^2}{\beta}(M-1)}, \tag{27}$$

where,

$$\lambda_i = \begin{cases} \lambda + 1 & (1 \leq i \leq m) \\ \lambda & (m+1 \leq i \leq M) \end{cases}, \quad 0 \leq m \leq M - 1. \tag{28}$$

These Jack polynomials correspond to the Young diagram $\lambda = (\lambda_1, \cdots, \lambda_M)$ and ${}^t\lambda = (\lambda_1, \cdots, \lambda_M)$ for $\gamma = \beta$ and -1, respectively.

6 q-Deformation and Macdonald polynomials

Finally, we briefly discuss the q–deformation and the Macdonald polynomials Q_λ [19] by using the method developed in the previous sections. The detail will appear elsewhere. The Macdonald operator,

$$D_{q,t} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,x_i}, \tag{29}$$

plays the same role as the CSM Hamiltonian $\mathcal{H}$, where T_{q,x_i} is the q-shift operator,

$$T_{q,x_i} f(x_1, \cdots, x_N) = f(x_1, \cdots, q x_i, \cdots, x_N). \tag{30}$$

Here, a new complex deformation parameter q is introduced and t is related to β by $t = q^\beta$.

The situation in previous sections can be obtained by taking the limit $\hbar \to 0$ with $q = e^\hbar$. In this limit, the Macdonald operator behaves as $D_{q,t} = \sum_{n \geq 0} D_{q,t}^{(n)} \hbar^n / n!$ with,

$$\begin{aligned} D_{q,t}^{(0)} &= N, \qquad D_{q,t}^{(1)} = \mathcal{P} + \frac{\beta}{2} N(N-1), \\ D_{q,t}^{(2)} &= \mathcal{H} + \beta(N-1)\mathcal{P} + \frac{\beta^2}{6} N(N-1)(2N-1). \end{aligned} \tag{31}$$

here $\mathcal{H}$ and $\mathcal{P}$ are in (5) and (6). In this sense, the Macdonald operator can be regarded as the generating functional of the infinitely many conserved charges in the CSM.

Amazingly, one may find a closed form of collective field representation for the Macdonald operator $D_{q,t}$ as follows,

$$\begin{aligned} D_{q,t}\langle 0|e^{\sum_{n=1}^\infty \frac{1-t^n}{1-q^n}\frac{a_n}{n}p_n} &= \langle 0|e^{\sum_{n=1}^\infty \frac{1-t^n}{1-q^n}\frac{a_n}{n}p_n}\widehat{D}_{q,t}, \\ \widehat{D}_{q,t} &= \frac{t^N}{t-1}\oint\frac{dz}{2\pi i}\frac{1}{z}e^{\sum_{n=1}^\infty \frac{1-t^{-n}}{n}a_n^\dagger z^n}e^{-\sum_{n=1}^\infty \frac{1-t^n}{n}a_n z^{-n}} - \frac{1}{t-1}, \end{aligned} \tag{32}$$

where the commutation relations for the bosonic oscillators are deformed as $[a_n, a_m^\dagger] = n\frac{1-q^n}{1-t^n}\delta_{n,m}$. Expanding this expression, we find the bosonized momentum and Hamiltonian as the coefficients of $\hbar$ and $\hbar^2$.

Similar to our discussion in section 3, we obtain the bosonized realization for some of the eigenstates (the Macdonald polynomials) of the Macdonald operator. Let,

$$\exp\left(\sum_{n=1}^\infty \frac{1-q^{\gamma n}}{1-q^n}\frac{a_n^\dagger}{n}z^n\right) = \sum_{n=0}^\infty \widehat{Q}_n^{(\gamma)}z^n \tag{33}$$

the states $\widehat{Q}_n^{(\gamma)}|0\rangle$ with $\gamma=\beta$ or -1 are the Macdonald polynomials corresponding to the Young diagram with single row (n) or single column (1^n), respectively. That of two rows $\lambda=(\lambda_1,\lambda_2)$ or two columns ${}^t\lambda=(\lambda_1,\lambda_2)$ are given by,[5]

$$\begin{aligned} \widehat{Q}^{(\gamma)}_{(\lambda_1,\lambda_2)}|0\rangle &= \sum_{\ell=0}^{\lambda_2} c^{(\gamma)}(\lambda_1-\lambda_2,\ell)\widehat{Q}^{(\gamma)}_{\lambda_1+\ell}\widehat{Q}^{(\gamma)}_{\lambda_2-\ell}|0\rangle, \\ c^{(\gamma)}(\lambda,\ell) &= \frac{1-q^{\frac{\beta}{\gamma}(\lambda+2\ell)}}{1-q^{\frac{\beta}{\gamma}(\lambda+\ell)}}\prod_{j=1}^{\ell}\frac{1-q^{\frac{\beta}{\gamma}(\lambda+j)}}{1-q^{\frac{\beta}{\gamma}j}}\cdot\prod_{i=1}^{\ell}\frac{q^\gamma-q^{\frac{\beta}{\gamma}(i-1)}}{1-q^{\gamma+\frac{\beta}{\gamma}(\lambda+i)}}, \end{aligned} \tag{34}$$

with $\gamma=\beta$ or -1, respectively. The Macdonald polynomials of single hook $(n,1^m)$ are,

$$\widehat{Q}_{(n,1^m)}|0\rangle = \sum_{\ell=0}^{m}\frac{1-q^{n+\ell}t^{m-\ell}}{1-q}q^{m-\ell}\,\widehat{Q}^{(\beta)}_{n+\ell}\widehat{Q}^{(-1)}_{m-\ell}|0\rangle. \tag{35}$$

[5] A conjecture for the special case of this expression was derived during the discussion with H. Kubo. We understand that this result is independently obtained by A. N. Kirillov.

Acknowledgments:

We would like to thank M. Fukuma, T. Inami, S. Iso, N. Kawakami, A. N. Kirillov, H. Kubo, M. Ninomiya and P. Wiegmann for discussions and encouragements. Some parts of this work were done in Hotaka-Sanso. This work is supported in part by Grant–in–Aid for Scientific Research from Ministry of Science and Culture.

Note Added:

After we submitted this paper, we learned that Avan and Jevicki [21] discussed the connection between $c = 1$ matrix model and the Calogero Moser model. In this context, we have to mention the work by Simons et. al. [22] where the matrix model technique was used to derive two-point correlation functions for the CSM with $\beta = 1, 2, \frac{1}{2}$. We were also indicated that G. Harris [23] studied the Virasoro constraint of the matrix model for non-orientable surfaces. Although the purpose of these works is different from ours, i.e. to relate the CSM with the conformal field theory with $c < 1$, they give complementary viewpoints to the problem.

As for the approach which is the closest to ours, we would like to mention the recent announcement by Mimachi and Yamada [24] where they expressed the Virasoro singular vectors as the Jack polynomials of rectangular Young diagrams, i.e. our equation (27).

References

[1] F. Calogero, *Jour. Math. Phys.* **10** (1969) 2197-2200;
B. Sutherland, *Jour. Math. Phys.* **12** (1970) 246-250, 251-256.

[2] E. Brezin, C. Itzykson, G. Parisi, J. Zuber, *Comm. Math. Phys.*, **59** (1978) 35-51;
M. Mehta, “Random Matrix”, Academic Press 1967.

[3] E. Brezin, V. Kazakov, *Phys. Lett.*, **B236**, (1990), 144-150;
D. Gross, A. Migdal, *Phys. Rev. Lett.*, **64**, (1990) 127-130;
M. Douglas, S. Shenker, *Nucl. Phys.*, **B335**, (1990) 635-654.

[4] E. Brézin and H. Neuberger, *Phys. Rev. Lett.* **65** (1990) 2098-2101.

[5] R.C. Myers and V. Periwal, *Phys. Rev. Lett.* **64** (1990) 3111-3114.

[6] J. Kaneko, *SIAM. J. Math. Anal.* **24** (1993) 1086-1110;
See also A. Korányi, "International symposium in memory of Hua Loo Keng" Vol. II, Beijing 1988, (S.Gong et.al. eds.), 169-180, Springer-Verlag 1991;
K. Aomoto, *SIAM. J. Math. Anal.* **18** (1987) 545-549;
A. Selberg, *Norsk. Mat. Tidsskr.* **26** (1941) 71–78.

[7] P.J. Forrester, *Nucl. Phys.* **B388** (1992) 671-699; *Nucl. Phys.* **B416** (1994) 377-385; *J. Stat. Phys.* **72** (1993) 39-50.

[8] F. Lesage, V. Pasquier and D. Serban, *Dynamical Correlation Functions in the Calogero–Sutherland Model*, Sacley preprint, SPhT/94-051 (April 1994) hep-th/9405008.

[9] A. Jevicki and B. Sakita, *Nucl. Phys.* **B165** (1980) 511-527.

[10] I. Andrić, A. Jevicki and H. Levin, *Nucl. Phys.* **B215 [FS7]** (1983) 307-315.

[11] J. A. Minahan and A. P. Polychronakos, *Density Correlation Functions in Calogero–Sutherland Models*, preprint, CERN-TH.7243/94, USC-94/008, hep-th/9404192;

[12] S. Iso and S.-J. Rey, *Collective Field Theory of the Fractional Quantum Hall Edge State and the Calogero–Sutherland Model*, preprint, UT-682, SNUTP 94-55, hep-th/9406192.

[13] R. Stanley, *Adv. Math.* **77** (1989) 76-115;
I.G. Macdonald, Lect. Note in Math. **1271**, 189-200, Springer 1987;
For general informations for the symmetric polynomial, see I.G. Macdonald "Symmetric Functions and Hall Polynomials", Oxford University Press 1979.

[14] K. Hikami and M. Wadati, *J. Phys. Soc. Jpn.* **62** (1993) 4203-4217.

[15] S. Iso, D. Karabali and B. Sakita, *Phys. Lett.* **B296** (1992) 143-150;
A. Cappelli, C. Trugenberger and G. Zemba, *Nucl. Phys.***B396** (1993) 465-490; *Phys. Rev. Lett.***72** (1994) 1902-1905.

[16] V. Kac and A. Radul, *Comm. Math. Phys.* **157** (1993) 429-457;
E. Frenkel, V. Kac, A. Radul and W. Wang, preprint (May 1994) hep-th/9405121.

[17] H. Awata, M. Fukuma, S. Odake and Y. Quano, *Lett. Math. Phys.* **31** (1994) 289-298;
Y. Matsuo, *Phys. Lett.* **B326** (1994) 95-100;
H. Awata, M. Fukuma, Y. Matsuo and S. Odake, *Phys. Lett.***B332**(1994) 336-344; preprint, hep-th/9404041, to appear in *Commun. Math. Phys.*; preprint, hep-th/9405093 to appear in *Commun. Math. Phys.*; preprint, hep-th/9406111 to appear in *Journ. of Phys.***A**; preprint, hep-th/9408158 to appear in *Prog. Theor. Phys. Suppl.*

[18] M. Fukuma, H. Kawai and R. Nakayama, *Int. J. Mod. Phys.* **A6** (1991) 1385-1406;
H. Itoyama and Y. Matsuo, *Phys. Lett.* **B262** (1991) 233-239.

[19] I. Macdonald, *Publ. I.R.M.A. Strasbourg* **372/S-20** (1988) 131–171.

[20] S. Iso, *Anyon Basis of* $c = 1$ *Conformal Field Theory*, preprint UT-692, hep-th/9411051, November 1994.

[21] J. Avan and A. Jevicki, *Phys. Lett.* **B266** (1991) 35;
J. Avan and A. Jevicki, *Comm. Math. Phys.* **150** (1992) 149.

[22] B. D. Simons, P. A. Lee, B. L. Altshuler, *Nucl. Phys.* **B409** (1993) 487.

[23] G. Harris, *Nucl. Phys.* **B356** (1991) 685-702.

[24] K. Mimachi and Y. Yamada, private communication (to appear).