A Note on Calogero-Sutherland Model, W_n Singular Vectors and Generalized Matrix Models[†]

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Abstract

We review some recent results on the Calogero-Sutherland model with emphasis upon its algebraic aspects. We give integral formulae for excited states (Jack polynomials) of this model and their relations with W_n singular vectors and generalized matrix models.

1. Introduction

The Calogero-Sutherland models¹ is a quantum mechanical system with a long-range interaction. It has been actively studied as a solvable system with anyonic statistics in 1+1 dimensions. Recently, it is greatly developed with the calculations of dynamical correlation functions^{2,3,4}. To evaluate more general ones, we may need to express the wave-functions of the excited states explicitly. In this note, we will present their algebraic construction by integral transformations^{5,6}.

For a special value of the coupling constant, the system reduces to that of free fermion. In this case, it has deep relations with the W algebras, matrix models, 2D quantum gravities and also 2D QCD. Such connections should remain even for the general cases. Indeed, we will demonstrate that the wavefunctions of excited states are identified with the W_n singular vectors^{5,7}. Furthermore, the W_n structure thus obtained causes the W constraints in the generalized matrix models.

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2. Calogero-Sutherland model and Jack polynomial.

We start with recapitulating some important properties of the Calogero-Sutherland Hamiltonian and momentum.

2.1. We consider a N-body problem on a unit circle. Denote their coordinates q_1, q_2, \dots, q_N . Hamiltonian and momentum are given by

$$H_{CS} = \sum_{j=1}^{N} \left(\frac{1}{i} \frac{\partial}{\partial q_j} \right)^2 + \frac{1}{2} \sum_{i < j} \frac{\beta(\beta - 1)}{\sin^2(q_i - q_j)/2}, \qquad P = \sum_{j=1}^{N} \frac{1}{i} \frac{\partial}{\partial q_j}. \quad (2.1)$$

Here β is a coupling constant with a reflection symmetry $\beta \leftrightarrow 1 - \beta$. It is known that when β is a real number, i.e., $\beta(\beta - 1) \ge -1/4$, the system becomes stable and has no bound states.

Define an inner-product $(f,g) \equiv \int_0^{2\pi} \prod_j dq_j f(q)^* g(q)$. Notice that H_{CS} and P are self-adjoint $H_{CS}^{\dagger} = H_{CS}$ and $P^{\dagger} = P$ under \dagger defined by $(f, \mathcal{O} g) \equiv (\mathcal{O}^{\dagger} f, g)$.

Since the Hamiltonian is rewritten as

$$H_{CS} = \sum_{j} h_{j}(\beta)^{\dagger} h_{j}(\beta) + \varepsilon_{0},$$

$$h_{j}(\beta) = \frac{1}{i} \frac{\partial}{\partial q_{j}} + \beta \sum_{i(\neq j)} \cot \left(\frac{q_{j} - q_{i}}{2}\right),$$

with a vacuum energy ε_0 , the energy is bounded from below and there are two minimal-energy states characterized as the states annihilated by $h_j(\beta)$ or $h_j(1-\beta)$. We restrict ourselves to the former vacuum, which is

$$\Delta^{\beta} \equiv \prod_{i < j} \sin^{\beta} \left(\frac{q_i - q_j}{2} \right).$$

The statistic of the particle is governed by the coupling β : if β is even (odd) then the particles become bosonic (fermionic).

2.2. We will write the excited states in the factorized form $J(q)\Delta(q)^{\beta}$ and change the variables to $x_j \equiv \exp(iq_j)$ on a complex plane. In these new variables, J(x) has to be a symmetric function, to possess the same statistic

as the vacuum. Define new Hamiltonian and momentum acted directly on J(x) as follows: $H \equiv \Delta^{-\beta} H_{CS} \Delta^{\beta} - \varepsilon_0$ and $P \equiv \Delta^{-\beta} P \Delta^{\beta}$, then

$$H = \sum_{i=1}^{N} D_i^2 + \beta \sum_{i < j} \frac{x_i + x_j}{x_i - x_j} (D_i - D_j), \qquad P = \sum_{i=1}^{N} D_i,$$
 (2.2)

with $D_i = x_i \frac{\partial}{\partial x_i}$. The vacuum wave-function becomes $\Delta^{\beta} = \prod_{i \neq j} (1 - x_i/x_j)^{\beta/2}$.

The eigenstate is labeled by a decreasing set of non-negative integers, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$, which is identified with a Young diagram with $\lambda_i(\geq 1)$ squares in *i*th row. The corresponding eigenvalue of H_{CS} and P can be written in terms of the momenta k_i of pseudo-particles as

$$\varepsilon_{\lambda} = \sum_{i=1}^{N} k_i^2, \qquad p_{\lambda} = \sum_{i=1}^{N} k_i, \qquad k_j \equiv \lambda_j + \frac{\beta}{2} (N+1-2j),$$

respectively. The wave-function $J_{\lambda}(x)$ is called the Jack symmetric polynomial in mathematical literatures^{1,8,9}. The neighboring pseudo-momenta should satisfy, $k_i - k_{i+1} \geq \beta$, which exhibits the nature of fractional statistics of this system. When $\beta = 1$, the Jack polynomial reduces to the Schur polynomial.

It may be instructive to illustrate the spectrum of the system for a positive integer β . We fill the momentum occupied by the pseudo-particle with "1" and the vacant state with "0" in the integer-valued momentum space. We denote the origin (p=0) by ":". For example, the 4-particles and $\beta=3$ case, (1) the vacuum; and (2) the excited state with $\lambda=(3,1,1,0)$ are respectively

- $(1), \cdots 0 1 0 0 1 0 : 0 1 0 0 1 0 0 0 \cdots,$
- $(2), \cdots 0 1 0 0 0 1 : 0 0 1 0 0 0 1 0 \cdots$
- **2.3.** Since H and P are symmetric in x_i 's, they can be expressed by the power-sums $p_n \equiv \sum_{i=1}^N x_i^n$ and their derivatives $\partial_n \equiv \frac{n}{\beta} \frac{\partial}{\partial p_n}$ as follows⁹:

$$H = \beta^2 \sum_{n,m>0} (p_{n+m}\partial_n \partial_m + p_n p_m \partial_{n+m}) + \beta \sum_{n>0} (n - n\beta + N\beta) p_n \partial_n,$$

$$P = \beta \sum_{n>0} p_n \partial_n. \tag{2.3}$$

Here we must treat p_n 's as formally independent variables, i.e., $\partial_n p_m = \frac{n}{\beta} \delta_{n,m}$ for all n, m > 0. Since this H and P do not depend on the number N of particles up to the last term of H, they are useful in analizing N-independent properties.

Define another inner-product $\langle f, g \rangle \equiv \oint \prod_j \frac{dp_j}{p_j} \overline{f(p)} g(p)$, with $\overline{p_n} \equiv \partial_n$. This is nothing but that of free bosons. Notice that H and P have a duality $p_n \leftrightarrow \partial_n$, i.e., they are self-adjoint $H^{\ddagger} = H$ and $P^{\ddagger} = P$ under \ddagger defined by $\langle f, \mathcal{O} g \rangle \equiv \langle \mathcal{O}^{\ddagger} f, g \rangle$.

There exists a following relation between H and the non-relativistic Virasoro generators \mathcal{L}_n :

$$H = \beta \sum_{n>0} p_n \mathcal{L}_n + (\beta - 1 + \beta N) P,$$

$$\mathcal{L}_n = \beta \sum_{m=1}^{n-1} \partial_m \partial_{n-m} + \beta \sum_{m>0} p_m \partial_{n+m} - (n+1) (\beta - 1) \partial_n.$$
(2.4)

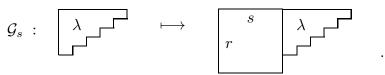
These relations with free bosons or Virasoro generators suggest algebraic aspects of the model.

3. Integral formula for the wave-functions of excited states

We next try to derive the explicit expression of all excited states. Our strategy is as follows: we introduce two types of (integral) transformations which maps the eigenstate into another while changing its energy and the number of particles. We can construct arbitrary state by applying them successively to the vacuum.

First, we introduce the Galilean boost \mathcal{G}_s , which uniformly shifts the pseudo-momentum of the pseudo-particles from $\lambda = (\lambda_1, \dots, \lambda_r)$ to $\lambda + s^r = (\lambda_1 + s, \dots, \lambda_r + s)$. It can be realized by multiplying the wave-function by $\prod_j e^{iq_js} = \prod_j x_j^s$. When it is operated to the eigenstate, the Young diagram is changed by adding a rectangle s^r from the left:

$$\mathcal{G}_s \cdot J_{\lambda}(x_1, \dots, x_r) = J_{\lambda + s^r}(x_1, \dots, x_r) = \prod_{i=1}^r x_i^s \cdot J_{\lambda}(x_1, \dots, x_r).$$
 (3.1)



The second integral transformation \mathcal{N}_{NM} changes the number of particles from M to N:

$$\mathcal{N}_{NM} \cdot J_{\lambda}(t_{1}, \dots, t_{M}) = J_{\lambda}(x_{1}, \dots, x_{N})$$

$$= \oint \prod_{j=1}^{M} \frac{dt_{j}}{t_{j}} \prod_{i,j} (1 - x_{i}/t_{j})^{-\beta} \prod_{i \neq j}^{M} (1 - t_{i}/t_{j})^{\beta} J_{\lambda}(t_{1}, \dots, t_{M}),$$
(3.2)

where the integration path is along the unit circle in the complex plane. *Proof.* This is proved by using two self-dualities of Hamiltonians and momentum. Replace the variables t_j of integration with t_j^{-1} . Let

$$\widetilde{H}(x) \equiv (H(x) - \beta N P(x)),$$

$$V \equiv \prod_{i,j} (1 - x_i t_j)^{-\beta} = e^{\beta \sum_{n>0} \frac{1}{n} \sum_{i,j} x_i^n t_j^n}.$$

Then \widetilde{H} commutes with \mathcal{N}_{NM} as

$$\widetilde{H}(x_1,\dots,x_N)\,\mathcal{N}_{NM}=\mathcal{N}_{NM}\,\widetilde{H}(t_1,\dots,t_M),$$

which is deduced as follows: first we change the action with $\widetilde{H}(x)$ on V to that with $\widetilde{H}(t)$ as $\widetilde{H}(x)V = \widetilde{H}(t)^{\ddagger}V = \widetilde{H}(t)V$; next we perform the integration by parts and pass the Hamiltonian through $\Delta^{2\beta}$ as $\Delta^{-2\beta}H^{\dagger}\Delta^{2\beta} = \Delta^{-\beta}H^{\dagger}_{CS}\Delta^{\beta} - \varepsilon_0 = H$. Momentum P also commutes with it*. \square

Therefore, starting from the vacuum with s_1 particles and combining these two transformations \mathcal{G}_s and \mathcal{N}_{nm} , we obtain all excited states of N particles^{5,6}

$$J_{\lambda}(x) = \mathcal{N}_{r_{n}, r_{n-1}} \mathcal{G}_{s_{n-1}} \mathcal{N}_{r_{n-1}, r_{n-2}} \cdots \mathcal{G}_{s_{2}} \mathcal{N}_{r_{2}, r_{1}} \mathcal{G}_{s_{1}} \cdot 1$$

$$= \oint \prod_{a=1}^{n-1} \prod_{j}^{r_{a}} \frac{dt_{j}^{(a)}}{t_{j}^{(a)}} \prod_{i,j} \left(1 - t_{i}^{(a+1)} / t_{j}^{(a)}\right)^{-\beta} \prod_{i \neq j}^{r_{a}} \left(1 - t_{i}^{(a)} / t_{j}^{(a)}\right)^{\beta} \prod_{j=1}^{r_{a}} \left(t_{j}^{(a)}\right)^{s_{a}},$$

$$(3.3)$$

^{*} Since the energy degenerates, we need one more condition to define the Jack polynomial uniquely. However, (3.2) is also compatible with it⁵.

with $x_j \equiv t_j^{(n)}$, $N = r_n$ and $\lambda = \sum_{a=1}^{n-1} (s_a^{r_a})$ such that

$$\lambda = \begin{bmatrix} s_{n-1} & s_{n-2} \\ r_{n-1} & r_{n-2} \end{bmatrix} \cdots \begin{bmatrix} s_2 & s_1 \\ r_2 & r_1 \end{bmatrix}$$

This formula reveals new algebraic aspects of the model as the following two sections.

4. Relation with W_n singular vectors and WZNW correlation functions

The integrand of eq. (3.3) reminds us of a sl(n) type chain from $t^{(1)}$ to $t^{(n-1)}$. Indeed, they are realized by the sl(n) type boson $\vec{\phi}(z)$ such that

$$\vec{\alpha}^a \cdot \vec{\phi}(z) \ \vec{\alpha}^b \cdot \vec{\phi}(w) \sim A^{ab} \log(z - w),$$
$$\vec{\phi}(z) \equiv -\sum_{m \neq 0} \frac{1}{m} \vec{a}_m z^{-m} + \vec{a}_0 \log z + \vec{Q},$$

with simple roots $\vec{\alpha}^1, \dots, \vec{\alpha}^{n-1}$ and the sl(n) type Cartan matrix A^{ab} . Through this correspondence, we show that the Jack polynomials are identified with W_n singular vectors after a projection defined below.

Let us consider the bosonic Fock space generated by the highest weight state $|\vec{h}\rangle$ such that $\vec{\alpha}^a \cdot \vec{a}_0 |\vec{h}\rangle = h^a |\vec{h}\rangle$ and $\vec{a}_m |\vec{h}\rangle = 0$ (m > 0). For the W_n algebra with a Virasoro central charge

$$c = (n-1) \left\{ 1 - n(n+1) \left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}} \right)^2 \right\},\,$$

there exists a singular vector on the Fock space of the highest weight

$$h_{r,s}^{n-a} = (1 + r_a - r_{a+1})\sqrt{\beta} - (1 + s_a)\frac{1}{\sqrt{\beta}},$$

with positive-integers $r_a < r_{a+1}$ and s_a . Its Virasoro grade is $\sum_{a=1}^{n-1} r_a s_a$. It is constructed from the screening currents : $\exp\left\{\sqrt{\beta}\vec{\alpha}^a \cdot \vec{\phi}(t)\right\}$: as follows¹⁰

$$|\chi_{r,s}\rangle = \oint \prod_{a,j} dt_j^{(a)} : e^{\sqrt{\beta}\vec{\alpha}^a \cdot \vec{\phi}(t_j^{(a)})} : |\vec{h}'_{r,s}\rangle. \tag{4.1}$$

If we perform a OPE of this singular vector, then the OPE factor is almost the same as the integrand of eq. (3.3). In fact, the excited state and the singular vector relate as follows⁵ (ref. [7] for the Virasoro case):

$$J_{\lambda}(x) = \langle \vec{h}_{r,s} | V_1 | \chi_{r,s} \rangle. \tag{4.2}$$

Here

$$\langle \vec{h}|V_1 \equiv \langle \vec{h}|e^{\sqrt{\beta}\sum_{m>0}\frac{1}{m}\vec{\Lambda}_1 \cdot \vec{a}_m p_m},$$

with fundamental weights $\vec{\Lambda}_a$ such that $\vec{\alpha}^a \cdot \vec{\Lambda}_b = \delta_b^a$. It gives a projection from n-1 bosons to power-sums as

$$\langle \vec{h}|V_1 \vec{\alpha}^a \cdot \vec{a}_{-m} = \delta_1^a \sqrt{\beta} p_m \langle \vec{h}|V_1, \qquad \langle \vec{h}|V_1 \vec{\Lambda}_1 \cdot \vec{a}_m = \frac{m}{\sqrt{\beta}} \frac{\partial}{\partial p_m} \langle \vec{h}|V_1,$$

for a positive integer m.

By using eq. (2.3) and the above projection, one can consider a bosonic Hamiltonian \widehat{H} acted directly on the bosonic Fock space. Although it is not uniquely determined, the nontrivial part of \widehat{H} is a cubic form¹¹ similar to Ishibashi-Kawai Hamiltonian of string fields¹². Furthermore, \widehat{H} is expressed by using Virasoro generators L_m as (2.4):

$$\widehat{H} \sim \sum_{m>0} \vec{\alpha}^1 \cdot \vec{a}_{-m} L_m + \cdots,$$

which has a similarity with the BRST-operator of the two dimensional quantum gravity. Here \cdots are the Cartan parts and the trivial ones that are annihilated by the projection. Therefore, we obtain another view point for the integral formula (3.3): the W_n singular vectors in terms of bosons always become the eigenstates of the Calogero-Sutherland model because they are annihilated by the cubic part of the Hamiltonian \hat{H} .

There is also a relation with the correlation function of the WZNW model. Let us consider the Kac-Moody algebra $\widehat{sl(n)}$ with a level $\kappa \equiv k+n$. The vertex operator V_1 in the projection operator corresponds to the product $\prod_{i=1}^N : \exp\left\{\sqrt{1/\kappa}\vec{\Lambda}_1\cdot\vec{\phi}(x_i)\right\}$: of N-vertex operators of $\widehat{sl(n)}$ with fundamental representations. Furthermore, the screening current of the W_n algebra is nothing but the ϕ -part (without $\beta\gamma$ -part): $\exp\left\{-\sqrt{1/\kappa}\vec{\alpha}^a\cdot\vec{\phi}(t)\right\}$: of that of $\widehat{sl(n)}$. If we decompose the Young diagram as $r_a=a$ and allow s_a 's to vanish, then N=n and the integrand of (3.3) is just the ϕ -part of the integral formula for a weight zero N-point function with fundamental representations of the WZNW model up to a non-symmetric part.

5. Generalized matrix model and Virasoro constraint

The excited state (3.3) has also some similarity to the partition function of the matrix model. Indeed, when n = 2 and $s_1 = (1 - r_1) - 1$, which is no longer a positive integers in general, then $Z_1(g) \equiv \bar{J}_{\lambda}(x)$ is

$$Z_1(g) = \int \prod_{i=1}^{r_1} dt_i \prod_{i < j} (t_i - t_j)^{2\beta} e^{\sum_{n>0} \sum_{i=1}^{r_1} g_n t_i^n},$$
 (5.1)

with $g_n \equiv \frac{\beta}{n} p_n$. The orthogonal, hermitian and symplectic matrix models correspond to when $\beta = 1/2$, 1 and 2, respectively.

Moreover, since the Jack polynomial is constructed from the screening current, this integral also satisfies Virasoro constraint $\mathcal{L}_m(g) Z_1(g) = 0$ for $m = -1, 0, \cdots$. This Virasoro generator is that of n = 2 in the last section with a central charge $c = 1 - 6 \left(\sqrt{\beta} - 1/\sqrt{\beta} \right)^2$. Hence, this integral is a generalization of one-matrix model for a general coupling constant β^{13} .

Although our correspondence between bosons and power-sums is a projection unless the Virasoro case, one can make it an invertible map by introducing many kinds of power-sums. In fact, the operator

$$\langle \vec{h}|V_{n-1} \equiv \langle \vec{h}| \prod_{b=1}^{n-1} e^{\sqrt{\beta} \sum_{m>0} \frac{1}{m} \vec{\Lambda}_b \cdot \vec{a}_m g_m^{(b)}},$$

gives that from bosons to n-1 kinds of power-sums $g_n^{(a)}$ by

$$\langle \vec{h}|V_{n-1} \vec{\alpha}^a \cdot \vec{a}_{-m} = \sqrt{\beta} g_m^{(a)} \langle \vec{h}|V_{n-1}, \quad \langle \vec{h}|V_{n-1} \vec{\Lambda}_a \cdot \vec{a}_m = \frac{m}{\sqrt{\beta}} \frac{\partial}{\partial g_m^{(a)}} \langle \vec{h}|V_{n-1},$$

for a positive integer m. When $s_a = \beta(1 - r_a + r_{a+1}) - 1$ with $r_n = 0$,

$$Z_{n-1}(g) \equiv \langle \vec{h}_{r,s} | V_{n-1} | \chi_{r,s} \rangle$$

$$= \int \prod_{a=1}^{n-1} \prod_{j=1}^{r_a} dt_j^{(a)} e^{\sum_{n>0} g_n^{(a)} (t_j^{(a)})^n} \prod_{i < j} (t_i^{(a)} - t_j^{(a)})^{2\beta} \prod_{a=1}^{n-2} \prod_{i,j} (t_i^{(a)} - t_j^{(a+1)})^{-\beta}.$$
(5.2)

Since $Z_{n-1}(g)$ now satisfies a W_n constraint, it is regarded as a generalization of the partition function of the conformal matrix model¹⁴ of $\beta = 1$.

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