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Excited States of Calogero-Sutherland Model and Singular Vectors of the W_N Algebra

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Abstract

Using the collective field method, we find a relation between the Jack symmetric polynomials, which describe the excited states of the Calogero-Sutherland model, and the singular vectors of the W_N algebra. Based on this relation, we obtain their integral representations. We also give a direct algebraic method which leads to the same result, and integral representations of the skew-Jack polynomials.

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1 Introduction

Calogero-Sutherland model [1] describes a system of non-relativistic particles on a circle under the inverse square potential. Its Hamiltonian and momentum are

$$H_{CS}=\sum_{j=1}^{N_0}\frac{1}{2}\Big(\frac{1}{i}\frac{\partial}{\partial q_j}\Big)^2+\Big(\frac{\pi}{L}\Big)^2\sum_{\substack{i,j=1\\ i<j}}^{N_0}\frac{\beta(\beta-1)}{\sin^2\frac{\pi}{L}(q_i-q_j)},\quad P_{CS}=\sum_{j=1}^{N_0}\frac{1}{i}\frac{\partial}{\partial q_j},\tag{1}$$

where β is a coupling constant. This model was introduced by Sutherland several years ago and has been known to describe a system with the generalized exclusion principle in $1+1$ dimension [2][13]. Recently, this model and its various cousins (Haldane-Shastry models [3] and similar models with internal degree of freedom [4]) have been intensively studied. Among many beautiful results, we may mention Yangian symmetry [5][6], $W_{1+\infty}$ symmetry [7], and their relations with 2D Yang-Mills theory [8][9] and the matrix models [10][11].

Among others, the development which is particularly relevant to our study may be the evaluation of the dynamical correlation functions [11]–[15]. In these calculations, they essentially used the mathematical properties of the Jack symmetric polynomial, namely the eigenstates of the Calogero-Sutherland model, developed by Stanley and Macdonald [16]. To go further to get higher correlation functions, it is desirable to obtain the explicit expression of the Jack polynomial. In this paper, we derive such formula as the multiple-integrals which typically appeared in the conformal field theory in the Coulomb-gas representation.

The Jack symmetric polynomial is a deformation of the Schur symmetric polynomial ($\beta=1$ case) which can be expressed in terms of a free fermion [17]. Natural questions arise ; does the Calogero-Sutherland system have some field theoretical reformulation in terms of free bosons? In [18], we studied this problem and obtained the collective field description [19] of the Calogero-Sutherland system. In particular, the Hamiltonian becomes cubic in free bosons and takes the following form,

$$\hat{H}_\beta=\sqrt{2\beta}\sum_{n>0}a_{-n}L_n+\sum_{n>0}a_{-n}a_n(N_0\beta+\beta-1-\sqrt{2\beta}a_0).\tag{2}$$

Here L_n is the Coulomb-gas representation of the Virasoro generator whose central charge is given by $1-\frac{6(1-\beta)^2}{\beta}$. By using this Hamiltonian, we derived the explicit form of some of the Jack symmetric polynomials. We observed that the eigenstates for a single pseudo-particle (hole) excitation have an interpretation as the screening charges of the Virasoro algebra. We obtained also the integral representation of the Jack symmetric polynomial with the rectangular Young diagram. These observations shows that there are some relations between Calogero-Sutherland model and the representation theory of the Virasoro algebra.

In fact, Mimachi and Yamada showed that the Virasoro singular vectors are expressed in terms of the Jack symmetric polynomial with the rectangular Young diagram [20]. Their derivation is based on the direct computation. However, we may give more instructive proof by our bosonized Hamiltonian (2). Indeed, when the Hamiltonian is acted on the singular vectors, the cubic term vanishes because of the highest weight condition and the action of the bilinear term is trivially diagonal. By this observation, the Virasoro singular vector naturally becomes the eigenstate of the Hamiltonian.

In this paper we generalize this result. *When the Young diagram consists of $N-1$ rectangles, the Jack symmetric polynomials with such Young diagram are related to the singular vectors of the W_N algebra.*

After projecting out some of the redundant degrees of freedoms, we obtain the integral representations of the Jack polynomials with arbitrary Young diagram [25].

The nature of this formula is revealed by decomposing the integral into several alternating actions of two types of operators, (a) G_s: the Galilean boost which amounts to adding (r,s) rectangle to the original Young diagram from the left. Here r is the number of pseudo-particles. (b) $N_{n,m}$ which increases the number of pseudo-particle from m to n without changing the Young diagram associated with the state. This operator can be realized as the integral transformation for the wave function. By those operators, the Jack polynomial for the Young diagram $\lambda' = ((r^1)^{s^1}, (r^2)^{s^2}, \cdots, (r^{N-1})^{s^{N-1}})$ with N_0 variables is written as

$$J_\lambda \propto N_{N_0,r^1} G_{s^1} N_{r^1,r^2} G_{s^2} \cdots N_{r^{N-2},r^{N-1}} G_{s^{N-1}} |\phi\rangle_{r^{N-1}}. \qquad (3)$$

Here $|\phi\rangle_{r^{N-1}}$ is the vacuum for r^{N-1} pseudo-particles. The following figure illustrates our construction for $\beta = 3$ and $r^{N-1} = 3$ case in the momentum space of the pseudo-particles. Here 1 indicates the momentum occupied by the pseudo-particles.

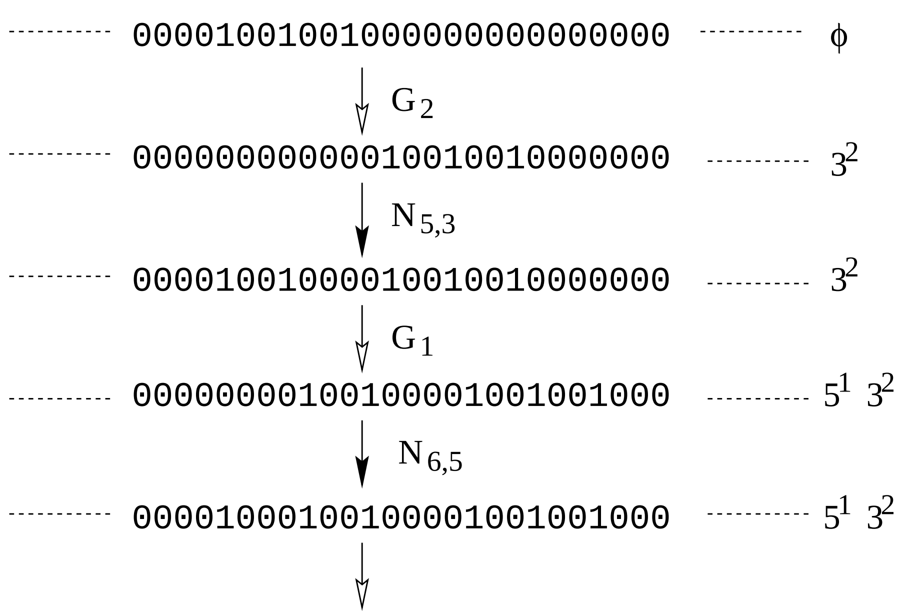


Figure 1: Construction of Jack Polynomials

This paper is organized as follows. In section 2 we give a short summary of the Calogero-Sutherland model and the Jack polynomial. In section 3 we apply the collective field method to the Calogero-Sutherland model. The Hamiltonian and momentum operators are realized by the bosonic operators. In section 4 we show the relation between the singular vectors of the Virasoro algebra and the Jack symmetric polynomials with the rectangular Young diagram. This result is generalized in section 5. The singular vectors of the W_N algebra are related to the Jack symmetric polynomials with the Young diagrams which consist of $N-1$ rectangles. Using this relations, we obtain the integral representations of the Jack polynomial with arbitrary Young diagram. In section 6, we define integral transformations which directly give the Jack polynomials as we explain the formula (3). In section 7 we give the integral representations of the skew-Jack polynomials. Section 8 is devoted to discussions on many relevant topics. In appendix A we discuss how the Jack symmetric polynomials are realized on the boson Fock space. In appendix B we discuss the analytic continuations of some integrals. We give explicit examples in appendix C.

2 Short summary of the Calogero-Sutherland model and Jack polynomials

The ground state of H_{CS} is given by [1]

$$\Delta_{CS}^{\beta} = \Big(\frac{L}{\pi}\prod_{\substack{i,j=1\\ i<j}}^{N_0} \sin\frac{\pi}{L}(q_i - q_j)\Big)^{\beta} \tag{4}$$

with the ground state energy $E_0 = \frac{1}{6}(\frac{\pi}{L})^2\beta^2(N_0^3 - N_0)$. For $\beta = 1$, this is nothing but the free fermion vacuum (Vandermonde determinant).

Let us make the coordinate transformation, $x_j = e^{2\pi i q_j/L}$. We are interested in the excited states of the form $J_\lambda(x)\Delta_{CS}^{\beta}$, where $J_\lambda(x)$ is the symmetric polynomial of the coordinates x_i. Hamiltonian and momentum acted on $J_\lambda(x)$ are given by

$$\begin{aligned} &\Delta_{CS}^{-\beta} H_{CS} \Delta_{CS}^{\beta} = \frac{1}{2}\Big(\frac{2\pi}{L}\Big)^2 H_\beta + E_0, \qquad \Delta_{CS}^{-\beta} P_{CS} \Delta_{CS}^{\beta} = \frac{2\pi}{L} P, \\ &H_\beta = \sum_{i=1}^{N_0} D_i^2 + \beta \sum_{\substack{i,j=1\\ i<j}}^{N_0} \frac{x_i + x_j}{x_i - x_j}(D_i - D_j), \quad P = \sum_{i=1}^{N_0} D_i, \end{aligned} \tag{5}$$

where $D_i = x_i \frac{\partial}{\partial x_i}$.

Eigenfunctions of H_β and P are called in mathematical literature as the Jack symmetric polynomials [16], $J_\lambda(x)$. They are indexed by the Young diagram λ, which may be physically interpreted as the distribution of the momentum of pseudo-particles (holes) of the system.

The Young diagram is parametrized by the numbers of boxes in each row, $\lambda = (\lambda_1, \cdots, \lambda_{N_0})$, $\lambda_1 \geq \cdots \geq \lambda_{N_0} \geq 0$. The length $\ell(\lambda)$ of λ is the number of the non-zero λ_i's. Then $\lambda = (\lambda_1, \cdots, \lambda_{N_0})$ and $(\lambda_1, \cdots, \lambda_{\ell(\lambda)})$ stand for the same Young diagram. The conjugate Young diagram is defined by interchanging rows with columns, denoted by $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ or $(\lambda'_1, \cdots, \lambda'_{\lambda_1})$. The total number of boxes is denoted by $|\lambda| = \sum_i \lambda_i$.

The energy eigenvalue was obtained as [1],

$$\epsilon_{\beta,\lambda} = \sum_{i=1}^{N_0} \Big(\lambda_i^2 + \beta(N_0 + 1 - 2i)\lambda_i \Big) = \sum_{i=1}^{\infty} \Big(-\beta {\lambda'_i}^2 + (\beta N_0 + 2i - 1)\lambda'_i \Big). \tag{6}$$

The eigenvalue of the momentum P is $|\lambda|$. Corresponding eigenvalues of H_{CS} and P_{CS} are $\sum_{i=1}^{N_0} \frac{1}{2} k_i^2$ and $\sum_{i=1}^{N_0} k_i$, respectively, where

$$k_i = \frac{2\pi}{L} \Big(\lambda_i + \frac{\beta}{2}(N_0 + 1 - 2i) \Big). \tag{7}$$

This formula gives the relation between the Young diagram and the momentum distribution of pseudo-particles. Since λ_i is a decreasing set of positive numbers, there is a constraint for the neighboring occupied momentum, $k_i - k_{i+1} \geq \beta \frac{2\pi}{L}$. This is a realization of the generalized exclusion principle in the momentum space.

On the other hand, the second formula in (6) shows that the total energy is alternatively expressed as

$$constant - \frac{\beta}{2} \sum_{i \geq 1} \tilde{k}_i^2$$

where

$$\tilde{k}_i = \frac{2\pi}{L} \Big(\lambda'_i - \frac{1}{2\beta}(\beta N_0 - 1 + 2i) \Big). \tag{8}$$

One may recognize that $\tilde{k}_i$'s are regarded as the momenta of pseudo-holes. They are constrained by $\tilde{k}_i - \tilde{k}_{i+1} \geq \frac{2\pi}{\beta L}$. By these observations, a Young diagram with n rows (columns) is regarded as describing a state with a excitation of n pseudo-particles (pseudo-holes). Conjugating a Young diagram is physically interpreted as interchanging the pseudo-particles with parameter β and pseudo-holes with $1/\beta$.

In order to construct explicit form of the Jack polynomial, it is important to understand the mathematical structure of the Hilbert space. It is identified with the ring of symmetric functions, which has several basis, e.g., the power-sum symmetric functions, the monomial symmetric functions and so on. The power-sum symmetric function $p_\lambda(x)$ is defined by $p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_M}(x)$, where $p_n(x) = \sum_{i=1}^{N_0} x_i^n$. The monomial symmetric function $m_\lambda(x)$ is defined by $m_\lambda(x) = \sum_\sigma x_1^{\lambda_{\sigma(1)}} \cdots x_{N_0}^{\lambda_{\sigma(N_0)}}$, where the summation is over all distinct permutations of $(\lambda_1, \cdots, \lambda_{N_0})$.

The Jack symmetric polynomial $J_\lambda(x) = J_\lambda(x;\beta) = J_\lambda(x_1, \cdots, x_{N_0};\beta)$ is uniquely specified by the following two properties and normalization,

$$\text{(i)} \qquad J_\lambda(x;\beta) = \sum_{\mu \le \lambda} v_{\lambda,\mu}(\beta) m_\mu(x), \quad v_{\lambda,\lambda}(\beta) = 1, \tag{9}$$

$$\text{(ii)} \qquad H_\beta J_\lambda(x;\beta) = \epsilon_{\beta,\lambda} J_\lambda(x;\beta). \tag{10}$$

In (i), we used the dominance partial ordering on the Young diagrams defined as $\lambda \ge \mu \Leftrightarrow |\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \ge \mu_1 + \cdots + \mu_i$ for all i.

We introduce an inner-product on the Hilbert space in the following manner [16],

$$\langle p_1^{k_1} \cdots p_n^{k_n}, p_1^{\ell_1} \cdots p_m^{\ell_m} \rangle_\beta = \delta_{\vec{k},\vec{\ell}} \beta^{-\sum_{i=1}^n k_i} \prod_{i=1}^n i^{k_i} k_i!, \tag{11}$$

for all $n, m \ge 1$. This definition of inner-product is compatible with the bosonization in the next section. With this inner-product, the condition (ii) can be replaced by the orthogonality condition,

$$\text{(ii)}' \qquad \langle J_\lambda(x;\beta), J_\mu(x;\beta) \rangle_\beta \propto \delta_{\lambda,\mu}. \tag{12}$$

In section 6, we discuss another type of inner-product.

3 Collective field method in the Calogero-Sutherland Model

We will study H_β by a collective field approach (bosonization). Since $J_\lambda(x)$ is a symmetric function in x_i, it can be written out using the power-sum polynomials p_n. Therefore H_β can be expressed in terms of creation and annihilation of power-sums. In conventional collective filed method, power-sum appears as $p_n = \int dx\, x^n \rho(x)$, where $\rho(x)$ is a density operator, $\rho(x) = \sum_{i=1}^{N_0} \delta(x - x_i)$.

To realize creation and annihilation of power-sums, we introduce a free boson field,

$$\begin{aligned} &\phi(z) = \hat{q} + a_0 \log z - \sum_{n \ne 0} \frac{1}{n} a_n z^{-n}, \quad \phi_-(z) = \sum_{n>0} \frac{1}{n} a_{-n} z^n, \\ &[a_n, a_m] = n \delta_{n+m,0}, \quad [a_0, \hat{q}] = 1. \end{aligned} \tag{13}$$

Its operator product expansion is $\phi(z)\phi(w) \sim \log(z-w)$. The normal ordering $: a_n a_m :$ is defined by $a_n a_m$ for $n \le m$, $a_m a_n$ for $n > m$ and $: \hat{q} a_0 : = : a_0 \hat{q} := \hat{q} a_0$. The boson Fock space $\mathcal{F}_\alpha$ is generated over oscillators of negative mode by the state $|\alpha\rangle$ such that

$$a_n |0\rangle = 0 \quad (n \ge 0), \qquad |\alpha\rangle = e^{\alpha \hat{q}} |0\rangle. \tag{14}$$

$\langle\alpha|$ is similarly defined, with the normalization $\langle\alpha|\alpha'\rangle = \delta_{\alpha,\alpha'}$.

We consider the following map from a state $|f\rangle$ into $\mathcal{F}_\alpha$ to a symmetric function $f(x)$,

$$\begin{aligned} |f\rangle \mapsto f(x) &= \langle\alpha|C_{\beta'}|f\rangle, \\ C_{\beta'} &= \exp\Bigl(\beta' \sum_{n>0} \tfrac{1}{n} a_n p_n\Bigr), \quad p_n = \sum_i x_i^n, \end{aligned} \tag{15}$$

where β' is a parameter. Under this correspondence, a_{-n} and a_n are interpreted as the creation and annihilation operator of power-sum, $\beta' p_n$ and $\frac{n}{\beta'}\frac{\partial}{\partial p_n}$, respectively, because $\langle\alpha|C_{\beta'}a_{-n} = \beta' p_n \langle\alpha|C_{\beta'}$ and $\langle\alpha|C_{\beta'}a_n = \frac{n}{\beta'}\frac{\partial}{\partial p_n}\langle\alpha|C_{\beta'}$. We remark that after rescaling $a_n \to \sqrt{\beta} a_n$ the inner-product on the boson Fock space agrees with that on the ring of symmetric functions (11).

Hamiltonian and momentum can be expressed in terms of boson oscillators as follows:

$$H_\beta \langle\alpha|C_{\beta'} = \langle\alpha|C_{\beta'}\hat{H}_\beta, \quad P\langle\alpha|C_{\beta'} = \langle\alpha|C_{\beta'}\hat{P}, \tag{16}$$

where

$$\hat{H}_\beta = \sum_{n,m>0} \Bigl(\beta' a_{-n-m} a_n a_m + \frac{\beta}{\beta'} a_{-n} a_{-m} a_{n+m}\Bigr) + \sum_{n>0} a_{-n} a_n \Bigl((1-\beta)n + N_0\beta\Bigr), \tag{17}$$

and $\hat{P} = \sum_{n>0} a_{-n} a_n$. We remark that $\hat{H}_\beta$ and $\hat{P}$ are independent of α. Now the problem of finding the Jack polynomials is translated to that of finding the eigenstates of $\hat{H}_\beta$ and $\hat{P}$ in $\mathcal{F}_\alpha$. In the rest of this section, the next section and appendix A, we set β' as

$$\sqrt{2}\beta' = \sqrt{\beta}, \tag{18}$$

and define $\alpha_\pm$ as

$$\frac{\alpha_+}{\sqrt{2}} = \sqrt{\beta}, \quad \frac{\alpha_-}{\sqrt{2}} = \frac{-1}{\sqrt{\beta}}. \tag{19}$$

The eigenstates for a single pseudo-particle (-hole) excitation, or the Jack polynomials of the Young diagram with single row (-column), are expressed by a single vertex operator [16] in a boson language. Its generating function is

$$e^{\alpha_\pm \phi_-(z)} = \sum_{n=0}^{\infty} \hat{J}_n^\pm z^n, \tag{20}$$

namely $\hat{J}_n^\pm|\alpha\rangle$ is the eigenstate of $\hat{H}_\beta$ which corresponds to a Young diagram with single row(+) or single column(−) with n boxes, respectively. By (15), we have

$$\langle\alpha|C_{\beta'} e^{\alpha_\pm \phi_-(z)}|\alpha\rangle = \sum_{n=0}^{\infty} J_n^\pm(x) z^n = \prod_i (1 - x_i z)^{-\beta' \alpha_\pm}. \tag{21}$$

In the appendix A, we discuss diagonalization of the Hamiltonian by the operators $\hat{J}_n^\pm$.

4 Virasoro singular vectors and Jack polynomials

The Virasoro singular vector is represented by the Jack polynomial with the rectangular Young diagram [20](see also [18]). In this section we give another proof, which will be generalized in the next section.

Using a free boson (13), the Virasoro algebra with the central charge c is realized as follows:

$$\begin{aligned} T(z) &= \sum_n L_n z^{-n-2} = \tfrac{1}{2} :\partial\phi(z)\partial\phi(z): + \alpha_0 \partial^2\phi(z), \\ c &= 1 - 12\alpha_0^2, \quad 2\alpha_0 = \alpha_+ + \alpha_-. \end{aligned} \tag{22}$$

The vertex operator $:e^{\alpha\phi(z)}:$ is a primary field of the Virasoro algebra, and it creates the highest weight state of the Virasoro algebra from the vacuum, $|\alpha\rangle = :e^{\alpha\phi(0)}:|0\rangle$, whose conformal weight is

$$h(\alpha) = \frac{1}{2}\Big((\alpha - \alpha_0)^2 - \alpha_0^2\Big). \tag{23}$$

We define $\alpha_{r,s}$ as

$$\alpha_{r,s} = \tfrac{1}{2}(1+r)\alpha_+ + \tfrac{1}{2}(1+s)\alpha_- \tag{24}$$

and remark that

$$h(\alpha_{\mp r,\pm s}) = h(\alpha_{r,s}) + rs. \tag{25}$$

In the Virasoro representation space with the highest weight state $|\alpha_{r,s}\rangle$, we have a singular vector $|\chi^+_{r,s}\rangle$ at level rs. By using a screening current $:e^{\alpha_+\phi(z)}:$, $|\chi^+_{r,s}\rangle$ is given as follows [21][22]:

$$\begin{aligned} |\chi^+_{r,s}\rangle &= \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i} \cdot \prod_{i=1}^{r} :e^{\alpha_+\phi(z_i)}: |\alpha_{-r,s}\rangle \\ &= \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1 \\ i<j}}^{r} (z_i - z_j)^{2\beta} \cdot \prod_{i=1}^{r} z_i^{(1-r)\beta - s} \cdot \prod_{j=1}^{r} e^{\alpha_+\phi_-(z_j)} |\alpha_{r,s}\rangle, \end{aligned} \tag{26}$$

where the integration contour is shown in Figure 2(b), which reduces to the contour in Figure 2(a) for a positive integer β. We note that this is just the form in (98). In [20], they acted with H_β on the polynomial $\langle\alpha_{r,s}|C_{\beta'}|\chi^+_{r,s}\rangle$ directly and showed that it is an eigenfunction. Here we will use $\hat{H}_\beta$.

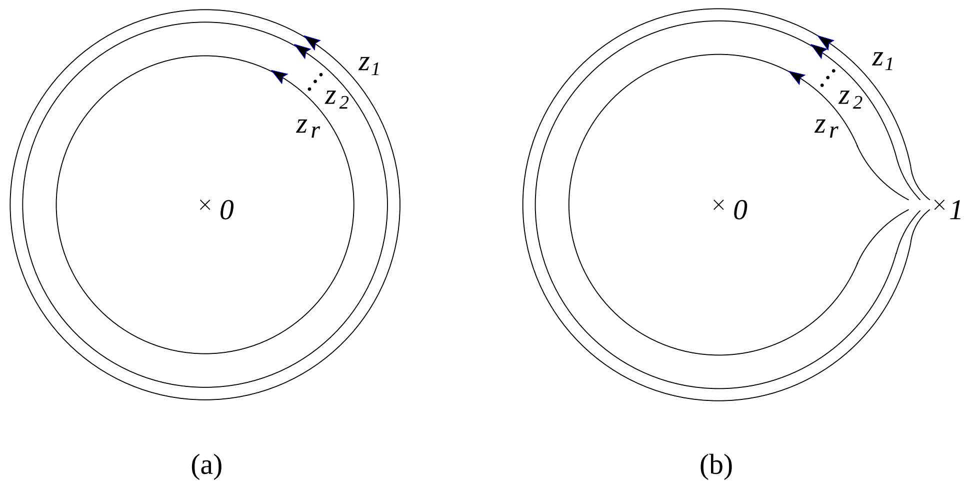


Figure 2: Integration contour

Under the choice of β' (18), the cubic term in H_β can be expressed by using the Virasoro generators. This is the key point of our argument. We have

$$\hat{H}_\beta = \sqrt{2\beta}\sum_{n>0} a_{-n}L_n + \sum_{n>0} a_{-n}a_n(N_0\beta + \beta - 1 - \sqrt{2\beta}a_0). \tag{27}$$

Since a singular vector $|\chi^+_{r,s}\rangle$ is annihilated by L_n $(n > 0)$, we have

$$\begin{aligned}\hat{H}_\beta|\chi^+_{r,s}\rangle &= \sum_{n>0} a_{-n}a_n(N_0\beta + \beta - 1 - \sqrt{2\beta}\alpha_{r,s})|\chi^+_{r,s}\rangle \\ &= rs\Big((N_0 - r)\beta + s\Big)|\chi^+_{r,s}\rangle. \end{aligned}\tag{28}$$

This eigenvalue is just $\epsilon_{\beta,\lambda}$ with the rectangular Young diagram $\lambda = (s, s, \cdots, s) = (s^r)$. Therefore we obtain an integral representation of the Jack polynomial with the Young diagram $\lambda = (s^r)$,

$$\begin{aligned}\mathcal{N}^+_{r,s}\mathcal{N}^+_{(s^r)}J_{(s^r)}(x) &= \langle\alpha_{r,s}|C_{\beta'}|\chi^+_{r,s}\rangle \\ &= \oint\prod_{j=1}^{r}\frac{dz_j}{2\pi i z_j}\cdot\prod_{\substack{i,j=1\\ i<j}}^{r}(z_i - z_j)^{2\beta}\cdot\prod_{i=1}^{r} z_i^{(1-r)\beta - s}\cdot\prod_i\prod_{j=1}^{r}(1 - x_i z_j)^{-\beta}, \end{aligned}\tag{29}$$

where the normalization constants $\mathcal{N}^+_\lambda$ [16] and $\mathcal{N}^+_{r,s}$ (see appendix B) are given by

$$\mathcal{N}^+_\lambda = \prod_{s\in\lambda}\frac{(\ell_\lambda(s) + 1)\beta + a_\lambda(s)}{\ell_\lambda(s)\beta + a_\lambda(s) + 1}, \qquad \mathcal{N}^+_{r,s} = \frac{1}{r!}\prod_{j=1}^{r}\frac{\sin\pi j\beta}{\sin\pi\beta}\cdot\frac{\Gamma(r\beta + 1)}{\Gamma(\beta + 1)^r}. \tag{30}$$

Here for each box $s = (i, j)$ (i-th row and j-th column) in the Young diagram λ, the arm-length $a_\lambda(s)$ and leg-length $\ell_\lambda(s)$ are defined by $\lambda_i - j$ and $\lambda'_j - i$, respectively. For a positive integer β, this $\mathcal{N}^+_{r,s}$ becomes $(-1)^{\frac{1}{2}r(r-1)\beta}(r\beta)!/(\beta!)^r$.

There exists another screening current $: e^{\alpha_-\phi(z)} :$. Using this, the singular vector is expressed in another form,

$$|\chi^-_{r,s}\rangle = \oint \prod_{j=1}^{s} \frac{dz_j}{2\pi i} \cdot \prod_{i=1}^{s} : e^{\alpha_-\phi(z_i)} : |\alpha_{r,-s}\rangle, \tag{31}$$

which differs from $|\chi^+_{r,s}\rangle$ only normalization factor. Similarly we can show that $|\chi^-_{r,s}\rangle$ is the eigenstate of $\hat{H}_\beta$ and obtain another integral representation of the Jack polynomial with the Young diagram (s^r),

$$\begin{aligned}
\mathcal{N}^-_{r,s}\mathcal{N}^-_{(r^s)}J_{(r^s)}(x) &= \langle\alpha_{r,s}|C_{\beta'}|\chi^-_{r,s}\rangle \\
&= \oint \prod_{j=1}^{s}\frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1\\ i<j}}^{s}(z_i - z_j)^{2/\beta} \cdot \prod_{i=1}^{s} z_i^{(1-s)/\beta - r} \cdot \prod_i \prod_{j=1}^{s}(1 - x_i z_j), \qquad (32)\\
\mathcal{N}^-_\lambda &= (-1)^{|\lambda|}, \qquad \mathcal{N}^-_{r,s} = \frac{1}{s!}\prod_{j=1}^{s}\frac{\sin\pi j\beta^{-1}}{\sin\pi\beta^{-1}} \cdot \frac{\Gamma(s\beta^{-1}+1)}{\Gamma(\beta^{-1}+1)^s}. \qquad (33)
\end{aligned}$$

To illustlate the results obtained in this section, we give explicit examples in appendix C.

5 W_N singular vectors and Jack polynomials

5.1 Review of W_N algebra

To discuss the W_N algebra, we start fixing our notation for A_{N-1}. Let $\vec{e}_i$ $(i = 1, \cdots, N)$ to be an orthonormal basis ($\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$), and the weight space of A_{N-1} to be the hyper-surface perpendicular to $\sum_{i=1}^N \vec{e}_i$. The weights of the vector representation $\vec{h}_i$ $(i = 1, \cdots, N)$, the simple roots $\vec{\alpha}^a$ $(a = 1, \cdots, N-1)$, and the fundamental weights $\vec{\Lambda}_a$ $(a = 1, \cdots, N-1)$, are given by

$$\vec{h}_i = \vec{e}_i - \tfrac{1}{N}\sum_{j=1}^{N}\vec{e}_j, \quad \vec{\alpha}^a = \vec{h}_a - \vec{h}_{a+1}, \quad \vec{\Lambda}_a = \sum_{i=1}^{a}\vec{h}_i, \tag{34}$$

and their inner-products are

$$\begin{aligned}
\vec{\alpha}^a \cdot \vec{\alpha}^b &= A^{ab} = 2\delta^{a,b} - \delta^{a,b+1} - \delta^{a,b-1}, \\
\vec{\alpha}^a \cdot \vec{\Lambda}_b &= A^a_b = \delta^{a,b}, \\
\vec{\Lambda}_a \cdot \vec{\Lambda}_b &= A^{-1}_{ab} = \tfrac{1}{N}\min(a,b)\Big(N - \max(a,b)\Big). \qquad (35)
\end{aligned}$$

Components of a vector $\vec{X}$ in the weight space are defined by

$$\vec{X} = \sum_{a=1}^{N-1} X^a \vec{\Lambda}_a = \sum_{a=1}^{N-1} X_a \vec{\alpha}^a. \tag{36}$$

They are related each other, $X_a = \sum_b A^{-1}_{ab} X^b$, $X^a = \sum_b A^{ab} X_b$ and we use the convention $X_0 = X_N = X^0 = X^N = 0$. From an orthonormal boson $\vec{\varphi}(z) = \sum_{i=1}^N \varphi_i(z)\vec{e}_i$ ($\varphi_i(z)\varphi_j(w) \sim \delta_{ij}\log(z-w)$), we define $\vec{\phi}(z) = \vec{\varphi}(z) - (\vec{\varphi}(z)\cdot\sum_{j=1}^N \vec{e}_j)\frac{1}{N}\sum_{i=1}^N \vec{e}_i = \sum_{a=1}^{N-1}\phi^a(z)\vec{\Lambda}_a = \sum_{a=1}^{N-1}\phi_a(z)\vec{\alpha}^a$, namely,

$$\begin{aligned} &\vec{\phi}(z) = \hat{\vec{q}} + \vec{a}_0 \log z - \sum_{n\neq 0}\tfrac{1}{n}\vec{a}_n z^{-n}, \quad \vec{\phi}_-(z) = \sum_{n>0}\tfrac{1}{n}\vec{a}_{-n} z^n, \\ &[a^a_n, a^b_m] = A^{ab} n\delta_{n+m,0}, \quad [a^a_0, \hat{q}^b] = A^{ab}, \end{aligned} \tag{37}$$

with operator product expansion $\phi^a(z)\phi^b(w) \sim A^{ab}\log(z-w)$. The boson Fock space $\mathcal{F}_{\vec{\lambda}}$ is generated by oscillators of negative mode on the state $|\vec{\lambda}\rangle$, which is characterized by

$$\vec{a}_n|\vec{0}\rangle = 0 \quad (n \geq 0), \qquad |\vec{\lambda}\rangle = e^{\vec{\lambda}\cdot\hat{\vec{q}}}|\vec{0}\rangle. \tag{38}$$

$\langle\vec{\lambda}|$ is similarly defined, with the normalization $\langle\vec{\lambda}|\vec{\lambda}'\rangle = \delta_{\vec{\lambda},\vec{\lambda}'}$.

Generators of the W_N algebra, $W^k(z)$ ($k = 2,\cdots,N$), are obtained by the Miura transformation [23],

$$\begin{aligned} &:\prod_{i=1}^N\Big(\alpha_0\partial + \vec{h}_i\cdot\partial\vec{\phi}(z)\Big): = \sum_{k=0}^N W^k(z)(\alpha_0\partial)^{N-k}, \\ &\alpha_0 = \alpha_+ + \alpha_-, \quad \alpha_+ = \sqrt{\beta}, \quad \alpha_- = \frac{-1}{\sqrt{\beta}}. \end{aligned} \tag{39}$$

From this, the Virasoro generator with the central charge c is given by

$$\begin{aligned} T(z) &= -W^2(z) = \tfrac{1}{2}:\partial\vec{\phi}(z)\cdot\partial\vec{\phi}(z): + \alpha_0\vec{\rho}\cdot\partial^2\vec{\phi}(z), \\ c &= N - 1 - 12\alpha_0^2\vec{\rho}^2, \end{aligned} \tag{40}$$

where $\vec{\rho}$ is the half-sum of positive roots, $\vec{\rho} = \sum_{a=1}^{N-1}\vec{\Lambda}_a$, and $\vec{\rho}^2 = \frac{1}{12}N(N^2-1)$. The W^3 generator $W(z)$ is given by

$$\begin{aligned} W(z) &= W^3(z) = \sum_n W_n z^{-n-3} \\ &= \sum_{a=1}^{N-1} :\partial\phi_a(z)\partial\phi_a(z)\Big(\partial\phi_{a+1}(z) - \partial\phi_{a-1}(z)\Big): \\ &\quad + \alpha_0\sum_{a,b=1}^{N-1}(1-a)A^{ab}:\partial\phi_a(z)\partial^2\phi_b(z): + \alpha_0^2\sum_{a=1}^{N-1}(1-a)\partial^3\phi_a(z). \end{aligned} \tag{41}$$

The vertex operator $:e^{\vec{\lambda}\cdot\vec{\phi}(z)}:$ is a primary field of the W_N algebra and creates the highest weight state of the W_N algebra from the vacuum, $|\vec{\lambda}\rangle = :e^{\vec{\lambda}\cdot\vec{\phi}(0)}:|\vec{0}\rangle$. Its conformal weight $h(\vec{\lambda})$ and W_0-eigenvalue $w(\vec{\lambda})$ are

$$h(\vec{\lambda}) = \tfrac{1}{2}\Big((\vec{\lambda} - \alpha_0\vec{\rho})^2 - \alpha_0^2\vec{\rho}^2\Big),$$

$$
\begin{aligned}
w(\vec{\lambda}) \;=\; & \sum_{a=1}^{N-1}\Big(\lambda_a\lambda_a(\lambda_{a+1}-\lambda_{a-1}) \\
& \qquad +\alpha_0\Big(2(a-1)\lambda_a\lambda_a+(1-2a)\lambda_a\lambda_{a+1}\Big)+\alpha_0^2 2(1-a)\lambda_a\Big).
\end{aligned} \tag{42}
$$

We define $\vec{\lambda}^{\pm}_{\vec{r},\vec{s}}$ as follows[1]

$$
\vec{\lambda}^{+}_{\vec{r},\vec{s}} \;=\; \sum_{a=1}^{N-1}\Big((1+r^a-r^{a-1})\alpha_+ +(1+s^a)\alpha_-\Big)\vec{\Lambda}_a, \tag{43}
$$

$$
\vec{\lambda}^{-}_{\vec{r},\vec{s}} \;=\; \sum_{a=1}^{N-1}\Big((1+r^a)\alpha_+ +(1+s^a-s^{a-1})\alpha_-\Big)\vec{\Lambda}_a. \tag{44}
$$

We remark that

$$
h\Big(\vec{\lambda}^{\pm}_{\vec{r},\vec{s}}-\alpha_\pm\sum_{a=1}^{N-1} r^a_\pm\vec{\alpha}^a\Big)=h(\vec{\lambda}^{\pm}_{\vec{r},\vec{s}})+\sum_{a=1}^{N-1} r^a s^a, \tag{45}
$$

where $r^a_+ = r^a$ and $r^a_- = s^a$.

A singular vector $|\chi^{+}_{\vec{r},\vec{s}}\rangle$ at level $\sum_{a=1}^{N-1} r^a s^a$ in the W_N representation space with the highest weight state $|\vec{\lambda}^{+}_{\vec{r},\vec{s}}\rangle$ is expressed by using screening currents $: e^{\alpha_+\phi^a(z)} :$,

$$
\begin{aligned}
|\chi^{+}_{\vec{r},\vec{s}}\rangle \;=\; & \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r^a}\frac{dz^a_j}{2\pi i}\cdot\prod_{a=1}^{N-1}\prod_{j=1}^{r^a} : e^{\alpha_+\phi^a(z^a_j)} : |\vec{\lambda}^{+}_{\vec{r},\vec{s}}-\alpha_+\sum_{a=1}^{N-1} r^a\vec{\alpha}^a\rangle \\
\;=\; & \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r^a}\frac{dz^a_j}{2\pi i z^a_j}\cdot\prod_{a=1}^{N-1}\prod_{\substack{i,j=1\\ i<j}}^{r^a}(z^a_i-z^a_j)^{2\beta}\cdot\prod_{a=1}^{N-2}\prod_{i=1}^{r^a}\prod_{j=1}^{r^{a+1}}(z^a_i-z^{a+1}_j)^{-\beta} \\
& \times\prod_{a=1}^{N-1}\prod_{j=1}^{r^a}(z^a_j)^{(1-r^a+r^{a+1})\beta-s^a}\cdot\prod_{a=1}^{N-1}\prod_{j=1}^{r^a} e^{\alpha_+\phi^a_-(z^a_j)}|\vec{\lambda}^{+}_{\vec{r},\vec{s}}\rangle,
\end{aligned} \tag{46}
$$

where we use a similar integration contour in Figure 2(b). In the following we assume $r^1 > \cdots > r^{N-1}$.

Similarly, using another screening currents $: e^{\alpha_-\phi^a(z)} :$, we have a singular vector,

$$
|\chi^{-}_{\vec{r},\vec{s}}\rangle=\oint\prod_{a=1}^{N-1}\prod_{j=1}^{s^a}\frac{dz^a_j}{2\pi i}\cdot\prod_{a=1}^{N-1}\prod_{j=1}^{s^a} : e^{\alpha_-\phi^a(z^a_j)} : |\vec{\lambda}^{-}_{\vec{r},\vec{s}}-\alpha_-\sum_{a=1}^{N-1} s^a\vec{\alpha}^a\rangle, \tag{47}
$$

with $s^1 > \cdots > s^{N-1}$.

5.2 W_N Singular Vectors and Jack polynomials

Like as (15), we consider the map from a state $|f\rangle$ in $\mathcal{F}_{\vec{\lambda}}$ into a symmetric function $f(x)$. Since the Hilbert space of the Calogero-Sutherland model is equivalent to a single boson

[1] Usual parametrization of the weight vector is $\vec{\lambda}_{\vec{r},\vec{s}} = \sum_{a=1}^{N-1}((1+r^a)\alpha_+ + (1+s^a)\alpha_-)\vec{\Lambda}_a = \alpha_+\vec{r} + \alpha_-\vec{s} + \alpha_0\vec{\rho}$.

Fock space, we have to reduce the degree of freedoms of the $N-1$ boson Fock space of W_N algebra. To this end, we use a kind of projection to give the correspondence between the Hilbert spaces,

$$
\begin{aligned}
f(x) &= \langle\vec{\lambda}|C_{\beta'}|f\rangle, \\
C_{\beta'} &= \exp\Big(\vec{\beta}'\cdot\sum_{n>0}\tfrac{1}{n}\vec{a}_n p_n\Big), \quad \vec{\beta}'=\beta'\vec{\Lambda}_1, \quad p_n=\sum_i x_i^n,
\end{aligned} \tag{48}
$$

where β' is a parameter. In this definition, only a^1_{-n} is a creation operator of p_n and other a^a_{-n} ($a>1$, $n>1$) do not contribute this map because $\langle\vec{\lambda}|C_{\beta'}a^a_{-n}=\langle\vec{\lambda}|C_{\beta'}\delta^{a1}\beta' p_n$. The Hamiltonian and momentum can be expressed in terms of boson oscillators as follows:

$$
H_\beta\langle\vec{\lambda}|C_{\beta'}=\langle\vec{\lambda}|C_{\beta'}\hat{H}_\beta, \quad P\langle\vec{\lambda}|C_{\beta'}=\langle\vec{\lambda}|C_{\beta'}\hat{P}, \tag{49}
$$

where

$$
\hat{H}_\beta=\sum_{n,m>0}\Big(\beta' a^1_{-n-m}a_{1,n}a_{1,m}+\frac{\beta}{\beta'}a^1_{-n}a^1_{-m}a_{1,n+m}\Big)+\sum_{n>0}a^1_{-n}a_{1,n}\Big((1-\beta)n+N_0\beta\Big), \tag{50}
$$

and $\hat{P}=\sum_{n>0}a^1_{-n}a_{1,n}$. Since the above map is not one to one, $\hat{H}_\beta$ and $\hat{P}$ are determined up to the term $\sum_{n>0}\sum_{a>1}a^a_{-n}\times(\cdots)$. In the following we set β' as

$$
\beta'=\sqrt{\beta}. \tag{51}
$$

Like as the Virasoro case, a straightforward calculation shows that under the choice of β' (51), cubic term in $\hat{H}_\beta$ can be expressed by the Virasoro and W generators. We have

$$
\begin{aligned}
\hat{H}_\beta &= \frac{2}{N}\sqrt{\beta}\sum_{n>0}a^1_{-n}L_n+\sum_{n>0}\vec{a}_{-n}\cdot\vec{a}_n(N_0\beta+\beta-1-2\sqrt{\beta}a_{1,0}) \\
&\quad+\sqrt{\beta}(W_0-W_{0,zero})+\sum_{n>0}\sum_{a>1}a^a_{-n}\times(\cdots).
\end{aligned} \tag{52}
$$

Here $W_{0,zero}$ is the zero mode part of W_0,

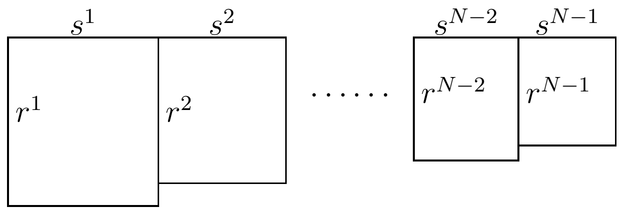
$$
\begin{aligned}
W_{0,zero} &= \sum_{a=1}^{N-1}\Big(a_{a,0}a_{a,0}(a_{a+1,0}-a_{a-1,0}) \\
&\quad+\alpha_0\Big(2(a-1)a_{a,0}a_{a,0}+(1-2a)a_{a,0}a_{a+1,0}\Big)+\alpha_0^2 2(1-a)a_{a,0}\Big).
\end{aligned} \tag{53}
$$

Using these, we have

$$
\begin{aligned}
\hat{H}_\beta|\chi^+_{\vec{r},\vec{s}}\rangle &= \Big(\sum_{a=1}^{N-1}r^a s^a\times\Big(N_0\beta+\beta-1-2\sqrt{\beta}(\alpha_+r_1+\alpha_-s_1+\alpha_0\rho_1)\Big) \\
&\quad+\sqrt{\beta}\Big(w(\vec{\lambda}^+_{\vec{r},\vec{s}}-\alpha_+\sum_{a=1}^{N-1}r^a\vec{\alpha}^a)-w(\vec{\lambda}^+_{\vec{r},\vec{s}})\Big)\Big)|\chi^+_{\vec{r},\vec{s}}\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n>0}\sum_{a>1} a^a_{-n}\times(\cdots)|\chi^+_{\vec r,\vec s}\rangle \\
&= \Big(\sum_{a=1}^{N-1} r^a s^a s^a + 2\sum_{\substack{a,b=1\\ a>b}}^{N-1} r^a s^a s^b + \beta\sum_{a=1}^{N-1} r^a s^a (N_0 - r^a)\Big)|\chi^+_{\vec r,\vec s}\rangle \\
&\quad +\sum_{n>0}\sum_{a>1} a^a_{-n}\times(\cdots)|\chi^+_{\vec r,\vec s}\rangle. \qquad (54)
\end{aligned}
$$

Therefore $|\chi^+_{\vec r,\vec s}\rangle$ is the eigenstate of $\hat H_\beta$ up to the last term, which will vanish after multiplying by $\langle\vec\lambda^+_{\vec r,\vec s}|C_{\beta'}$. This eigenvalue is just $\epsilon_{\beta,\lambda}$ with $\lambda' = ((r^1)^{s^1}, (r^2)^{s^2}, \cdots, (r^{N-1})^{s^{N-1}})$, namely, $\lambda = ((s^1+\cdots+s^{N-1})^{r^{N-1}}, (s^1+\cdots+s^{N-2})^{r^{N-2}-r^{N-1}}, \cdots, (s^1)^{r^1-r^2})$,

$\lambda =$ 

.

Using the state–function correspondence (48), we obtain an integral representation of the Jack polynomial with the Young diagram λ,

$$
\begin{aligned}
J_\lambda(x) &= (\mathcal N^+_{\vec r,\vec s}\mathcal N^+_\lambda)^{-1}\langle\vec\lambda^+_{\vec r,\vec s}|C_{\beta'}|\chi^+_{\vec r,\vec s}\rangle \\
&= (\mathcal N^+_{\vec r,\vec s}\mathcal N^+_\lambda)^{-1}\oint\prod_{a=1}^{N-1}\prod_{j=1}^{r^a}\frac{dz^a_j}{2\pi i z^a_j}\cdot\prod_{a=1}^{N-1}\prod_{\substack{i,j=1\\ i<j}}^{r^a}(z^a_i - z^a_j)^{2\beta}\cdot\prod_{a=1}^{N-2}\prod_{i=1}^{r^a}\prod_{j=1}^{r^{a+1}}(z^a_i - z^{a+1}_j)^{-\beta} \\
&\qquad\times\prod_{a=1}^{N-1}\prod_{j=1}^{r^a}(z^a_j)^{(1-r^a+r^{a+1})\beta - s^a}\cdot\prod_i\prod_{j=1}^{r^1}(1-x_i z^1_j)^{-\beta}, \qquad (55)
\end{aligned}
$$

where normalization constants $\mathcal N^+_\lambda$ and $\mathcal N^+_{\vec r,\vec s}$ are in (30) and appendix B, respectively.

Similarly, using another screening currents $:e^{\alpha_-\phi^a(z)}:$, we obtain another integral representation of the Jack polynomial with the Young diagram $\lambda = ((s^1)^{r^1}, (s^2)^{r^2}, \cdots, (s^{N-1})^{r^{N-1}})$,

$$
\begin{aligned}
J_{\lambda'}(x) &= (\mathcal N^-_{\vec r,\vec s}\mathcal N^-_\lambda)^{-1}\langle\vec\lambda^-_{\vec r,\vec s}|C_{\beta'}|\chi^-_{\vec r,\vec s}\rangle \\
&= (\mathcal N^-_{\vec r,\vec s}\mathcal N^-_\lambda)^{-1}\oint\prod_{a=1}^{N-1}\prod_{j=1}^{s^a}\frac{dz^a_j}{2\pi i z^a_j}\cdot\prod_{a=1}^{N-1}\prod_{\substack{i,j=1\\ i<j}}^{s^a}(z^a_i - z^a_j)^{2/\beta}\cdot\prod_{a=1}^{N-2}\prod_{i=1}^{s^a}\prod_{j=1}^{s^{a+1}}(z^a_i - z^{a+1}_j)^{-1/\beta} \\
&\qquad\times\prod_{a=1}^{N-1}\prod_{j=1}^{s^a}(z^a_j)^{(1-s^a+s^{a+1})/\beta - r^a}\cdot\prod_i\prod_{j=1}^{s^1}(1-x_i z^1_j). \qquad (56)
\end{aligned}
$$

6 Direct derivation of the integral formulas

6.1 Integral transformations

In this section, we describe a direct method to give the integral representation of the Jack polynomial. We hope that it may give some insight on the symmetry structure of the

Calogero-Sutherland model.

Our method is based on two types of transformations which map any eigenstates of the Hamiltonian into another. The first transformation may be physically interpreted as a global Galilean transformation which describes a uniform shift of momentum of the pseudo-particles. The second one is defined as the integral transformation which changes the number of particles without touching the Young diagram of the original Jack polynomial.

The first one, the Galilean transformation G_s, is defined by,

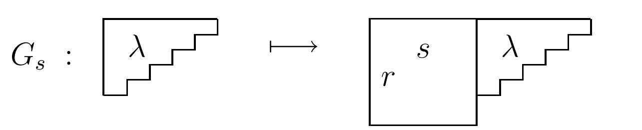
$$(G_s\psi)(x_1,\cdots,x_r)=\prod_{i=1}^{r}(x_i)^s\cdot\psi(x_1,\cdots,x_r), \tag{57}$$

for any symmetric function ψ. Recalling the definition of $x_j=e^{i\frac{2\pi}{L}q_j}$, it produces a uniform shift of the momentum of the pseudo-particles,

$$k_i\to k_i+\frac{2\pi}{L}s. \tag{58}$$

Therefore, when it operates on the Jack polynomial, G_s adds a rectangle Young diagram (s^r) to the original one from the left,

$$G_sJ_\lambda(x;\beta)=J_{\lambda+(s^r)}(x;\beta), \tag{59}$$



By the definition of the Jack polynomial, the normalization factor is one.

To define the second (integral) transformation, we prepare some notations. Let $x^a\equiv(x_1^a,\cdots,x_{r^a}^a)$, $a\in\mathbf{Z}_{\geq 0}$ be finite or infinite sequences of independent variables and denote

$$\begin{aligned}\Delta(x) &\equiv \prod_{\substack{i,j=1\\ i\neq j}}^{r}\left(1-x_i/x_j\right)\propto\Delta_{CS}(x)^2,\\ \Gamma(x^a,x^b) &\equiv \prod_{i=1}^{r^a}\prod_{j=1}^{r^b}\left(1-x_i^a/x_j^b\right).\end{aligned} \tag{60}$$

There are two types of the inner-products between the symmetric polynomial with which the Jack polynomial becomes mutually orthogonal. The first one, $\langle\ ,\ \rangle_\beta$, is defined in (11). The second one $\langle\ ,\ \rangle'_{\beta;r}$ as [29] is defined for a positive integer β by

$$\langle f(x),g(x)\rangle'_{\beta;r}=\frac{1}{r!}\oint\prod_{j=1}^{r}\frac{dx_j}{2\pi ix_j}\cdot f(1/x)\,g(x)\Delta(x)^\beta. \tag{61}$$

Here the integral $\oint \prod_{j=1}^{r} \frac{dx_j}{2\pi i x_j} f(x)$ stands for a constant part of $f(x)$. The second one appears in the computation of dynamical correlation functions [11]–[15]. These two definitions are equivalent only when $\beta = 1$. The norm of the Jack polynomial is related by [29], for a positive integer β,

$$\begin{aligned}\langle J_\lambda, J_\lambda \rangle'_{\beta;r} &= \prod_{1\le i<j\le r} \prod_{k=1}^{\beta-1} \frac{\lambda_i - \lambda_j + k + \beta(j-i)}{\lambda_i - \lambda_j - k + \beta(j-i)}, \\ &= \prod_{i=2}^{r} \binom{i\beta - 1}{\beta - 1} \cdot \prod_{(i,j)\in\lambda} \frac{j - 1 + \beta(r-i+1)}{j + \beta(r-i)} \cdot \langle J_\lambda, J_\lambda \rangle_\beta, \end{aligned} \tag{62}$$

where r is a number of variables, $\lambda = (\lambda_1 \ge \cdots \ge \lambda_r \ge 0)$ and $(i,j) \in \lambda$ is a square in the Young diagram such that $1 \le i \le \lambda'_1$ and $1 \le j \le \lambda_i$. Note that $\langle 1, 1 \rangle'_{\beta;r} = \prod_{i=2}^{r} \binom{i\beta-1}{\beta-1} = (r\beta)!/r!(\beta!)^r$.

After this preparation, we introduce the second integral transformation as

$$(N^{(\beta)}_{r^a,r^b}\psi)(x_1^a, \cdots, x_{r^a}^a) = \oint \prod_{j=1}^{r^b} \frac{dx_j^b}{2\pi i x_j^b} \cdot \Gamma(x^a, x^b)^{-\beta} \Delta(x^b)^\beta \psi(x_1^b, \cdots, x_{r^b}^b). \tag{63}$$

It transforms any eigenstate into another by the orthogonality relations of the Jack polynomials

$$\begin{aligned} \Gamma(x^a, x^b)^{-\beta} &= \sum_\lambda J_\lambda(x^a;\beta)\, J_\lambda\left(1/x^b;\beta\right) \langle J_\lambda, J_\lambda \rangle_\beta^{-1}, \\ \frac{1}{r!} \oint \prod_{j=1}^{r} \frac{dx_j}{2\pi i x_j} &\cdot J_\lambda(x;\beta) J_\mu\left(1/x;\beta\right) \Delta(x)^\beta = \delta_{\lambda,\mu} \langle J_\lambda, J_\lambda \rangle'_{\beta;r}. \end{aligned} \tag{64}$$

With the normalization factor, we get,

$$J_\lambda(x^a;\beta) = \frac{\langle J_\lambda, J_\lambda \rangle_\beta}{r^b! \langle J_\lambda, J_\lambda \rangle'_{\beta;r^b}} N^{(\beta)}_{r^a,r^b} J_\lambda(x^b;\beta). \tag{65}$$

As we discussed, it changes the number of pseudo-particles without touching the Young diagram.

These two transformations are enough to give the Jack polynomials with arbitrary Young diagrams [25]. Namely, by the Galilean transformation, we can add a rectangle to the arbitrary Young diagram. The difficulty was that the number of rows of such rectangle is constrained by the number of pseudo-particle. However, we may change it by the second transformation. Any Jack polynomial can be constructed from the trivial state, the vacuum, by the iterative use of them. We arrive at the integral representation of the Jack polynomials,

$$\begin{aligned} J_\lambda(x^0;\beta) &= N^{(\beta)}_{r^0,r^1} G_{s^1} N^{(\beta)}_{r^1,r^2} \cdot \cdots \cdot G_{s^{N-2}} N^{(\beta)}_{r^{N-2},r^{N-1}} G_{s^{N-1}} \cdot 1 \\ &= C_\lambda^+ \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} \frac{dx_j^a}{2\pi i x_j^a} \cdot \prod_{a=1}^{N-1} \Gamma(x^{a-1}, x^a)^{-\beta} \Delta(x^a)^\beta \prod_{j=1}^{r^a} \left(x_j^a\right)^{s^a}, \end{aligned} \tag{66}$$

$$C^+_\lambda = \prod_{a=1}^{N-1} \frac{\langle J_{\lambda^{(a)}}, J_{\lambda^{(a)}}\rangle_\beta}{r^a!\langle J_{\lambda^{(a)}}, J_{\lambda^{(a)}}\rangle'_{\beta;r^a}}, \tag{67}$$

with $\lambda^{(a)\prime} \equiv ((r^a)^{s^a}, (r^{a+1})^{s^{a+1}}, \cdots, (r^{N-1})^{s^{N-1}})$ and $\lambda \equiv \lambda^{(1)}$. If we replace respectively x^0_i and x^a_i $(a \neq 0)$ with x_i and $1/z^a_i$, this formula (66) reduces to the integral formula (55). The relation between the normalization constant C^+_λ and that of the previous sections is $(-1)^{\sum_a (r^a + \frac{1}{2} r^a (r^a - 1)\beta)} C^+_\lambda = (\mathcal{N}^+_\lambda \mathcal{N}^+_{\vec{r},\vec{s}})^{-1}$.

6.2 Dual transformations

We next consider the dual orthogonal relation. It is defined by the automorphism ω_β [16],

$$\omega_\beta\, p_n \equiv (-1)^{n-1}\beta^{-1} p_n, . \tag{68}$$

for $n \neq 0$. It satisfies

$$\begin{aligned} \omega^{(a)}_\beta \Gamma(x^a, x^b)^{-\beta} &= \widetilde{\Gamma}(x^a, x^b) \equiv \prod_{i,j} \left(1 + x^a_i / x^b_j\right), \\ \omega_\beta\, J_\lambda(x;\beta)\langle J_\lambda, J_\lambda\rangle^{-1}_\beta &= J_{\lambda'}(x; 1/\beta), \end{aligned} \tag{69}$$

where $\omega^{(a)}_\beta$ is acted on the variables x^a_i's. The second relation shows that it interchanges rows and columns of the Young diagram. Physically, it amounts to interchange pseudo-particles and pseudo-holes with the change of the parameter $\beta \leftrightarrow 1/\beta$. Some aspects of this transformation was discussed in [16] and [15].

Using this automorphism $\omega^{(a)}_\beta$, we get the following dual orthogonality relation,

$$\widetilde{\Gamma}(x^a, x^b) = \sum_\lambda J_{\lambda'}\,(x^a; 1/\beta)\, J_\lambda(1/x^b; \beta). \tag{70}$$

We can introduce an integral transformation which realizes the duality,

$$(\widetilde{N}^{(\beta)}_{r^a, r^b}\psi)(x^a) \equiv \oint \prod_{j=1}^{r^b} \frac{dx^b_j}{2\pi i x^b_j} \cdot \widetilde{\Gamma}(x^a, x^b) \Delta(x^b)^\beta \psi(x^b). \tag{71}$$

It maps into

$$J_{\lambda'}(x^a; 1/\beta) = \frac{1}{r^b!\langle J_\lambda, J_\lambda\rangle'_{\beta;r^b}} \widetilde{N}^{(\beta)}_{r^a, r^b} J_\lambda(x^b; \beta). \tag{72}$$

Applying the automorphism $\omega^{(0)}_\beta$ to the eq. (66) and replacing β with $1/\beta$, we obtain the dual form of the Jack polynomials:

$$\begin{aligned} J_{\lambda'}(x^0; \beta) &= C^-_\lambda \widetilde{N}^{(1/\beta)}_{r^0, r^1} G_{s^1} N^{(1/\beta)}_{r^1, r^2} \cdots G_{s^{N-2}} N^{(1/\beta)}_{r^{N-2}, r^{N-1}} G_{s^{N-1}} \cdot 1 \\ &= C^-_\lambda \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} \frac{dx^a_j}{2\pi i x^a_j} \cdot \widetilde{\Gamma}(x^0, x^1) \prod_{a=2}^{N-1} \Gamma(x^{a-1}, x^a)^{-1/\beta} \cdot \prod_{a=1}^{N-1} \Delta(x^a)^{1/\beta} \prod_{j=1}^{r^a} \left(x^a_j\right)^{s^a}, \end{aligned} \tag{73}$$

$$C_\lambda^- = \frac{\prod_{a=2}^{N-1} \langle J_{\lambda^{(a)}}, J_{\lambda^{(a)}} \rangle_{1/\beta}}{\prod_{a=1}^{N-1} r^a! \langle J_{\lambda^{(a)}}, J_{\lambda^{(a)}} \rangle'_{1/\beta; r^a}}. \tag{74}$$

If we replace respectively x_i^0 and x_i^a $(a \neq 0)$ with x_i and $-1/z_i^a$, then this formula also reduces to the our integral formula (56).

It is obvious that we may get various types of decomposition of the Young diagram into the rectangles. For example, we may obtain the (generalized) hook decomposition if we use only $\widetilde{N}^{(1/\beta)}_{r^0,r^1}$.

7 Skew-Jack polynomials

To calculate higher point dynamical correlation functions, we will need the inner product $\langle J_\lambda, J_\mu J_\nu \rangle'$ or equivalently the branching rule

$$J_\mu(x;\beta) J_\nu(x;\beta) = \sum_\lambda{}' C^\lambda_{\mu\nu}(\beta) J_\lambda(x;\beta), \tag{75}$$

where $\sum_\lambda{}' = \sum_\lambda \langle J_\lambda, J_\lambda \rangle^{-1}$. This information is encoded in the skew-Jack polynomial $J_{\lambda/\mu}(x;\beta)$ characterized by the following three equivalent definitions

$$\begin{array}{lrl} \text{(i)} & J_{\lambda/\mu}(x;\beta) = & \sum_\nu{}' C^\lambda_{\mu\nu}(\beta) J_\nu(x;\beta), \\ \text{(ii)} & \langle J_\lambda, J_\mu J_\nu \rangle_\beta = & \langle J_{\lambda/\mu}, J_\nu \rangle_\beta = C^\lambda_{\mu\nu}(\beta) \quad (\forall \nu), \\ \text{(iii)} & \sum_\lambda{}' J_{\lambda/\mu}(x) J_\lambda(y) = & \sum_\nu{}' J_\nu(x) J_\nu(y) J_\mu(y). \end{array}$$

In order to match the inner products on the boson Fock space and the ring of symmetric functions, we rescale oscillators $a_n = \sqrt{2\beta} a'_n$ and $a_{-n} = \sqrt{\beta/2} a'_{-n}$, namely

$$[a'_n, a'_m] = \frac{n}{\beta} \delta_{n+m,0}, \tag{76}$$

and define $\dagger$ operation by $a'^\dagger_n = a'_{-n}$. We set $\phi'_-(z) = \sum_{n>0} \frac{1}{n} a'_{-n} z^n$ and $\phi'^\dagger_-(z) = \sum_{n>0} \frac{1}{n} a'_n z^n$. The correspondence between a state $|f\rangle = \hat{f}|0\rangle \in \mathcal{F}_0$ (or $\langle f| = \langle 0|\hat{f}^\dagger$) and a symmetric function $f(x)$ is

$$\begin{aligned} |f\rangle \mapsto f(x) &= \langle 0|C'_\beta(x)|f\rangle, \quad C'_\beta(x) = e^{\beta \sum_{n>0} \frac{1}{n} a'_n p_n} (= C_{\beta'}) \\ &= \langle f|C'^\dagger_\beta(x)|0\rangle, \quad C'^\dagger_\beta(x) = e^{\beta \sum_{n>0} \frac{1}{n} a'_{-n} p_n}. \end{aligned} \tag{77}$$

In these notations, the two inner products agrees,

$$\langle f, g \rangle_\beta = \langle f|g \rangle. \tag{78}$$

We remark that

$$|fg\rangle = \hat{f}\hat{g}|0\rangle \mapsto f(x)g(x) = \langle 0|C'_\beta(x)|fg\rangle. \tag{79}$$

For the Young diagram $\lambda' = ((r^1)^{s^1}, (r^2)^{s^2}, \cdots, (r^{N-1})^{s^{N-1}})$, we define

$$|J_\lambda\rangle = \oint \prod_{j=1}^{r^1} \frac{dz_j^1}{2\pi i z_j^1} \cdot f_\lambda^\pm(z^1) \prod_{j=1}^{r^1} e^{\gamma_\pm \phi'_-(z_j^1)}|0\rangle, \tag{80}$$

$$f_\lambda^\pm\left(\frac{1}{z^1}\right) = C_\lambda^\pm \oint \prod_{a=2}^{N-1} \prod_j^{r^a} \frac{dz_j^a}{2\pi i z_j^a} \Gamma(z^{a-1}, z^a)^{-\beta^{\pm 1}} \cdot \prod_{a=1}^{N-1} \Delta(z^a)^{\beta^{\pm 1}} \prod_{i=1}^{r^a} (z_i^a)^{s^a}, \tag{81}$$

with $\gamma_+ = \beta$ and $\gamma_- = -1$.

Then Jack and skew-Jack polynomials are given by

$$J_\lambda(x;\beta) = \langle 0|C'_\beta(x)|J_\lambda\rangle = \langle J_\lambda|C'^\dagger_\beta(x)|0\rangle, \tag{82}$$

$$J_{\lambda/\mu}(x;\beta) = \langle J_\mu|C'_\beta(x)|J_\lambda\rangle = \langle J_\lambda|C'^\dagger_\beta(x)|J_\mu\rangle. \tag{83}$$

The former is proved in sections 5 and 6. The latter is done as follows;

$$\begin{aligned} \langle J_\lambda, J_\mu J_\nu\rangle_\beta &= \langle J_\lambda|J_\mu J_\nu\rangle = \langle J_\lambda|\hat{J}_\mu|J_\nu\rangle \\ &= \langle\, \langle J_\lambda|\hat{J}_\mu C'^\dagger_\beta(x)|0\rangle, \langle 0|C'_\beta(x)|J_\nu\rangle\,\rangle_\beta \\ &= \langle\, \langle J_\lambda|C'^\dagger_\beta(x)|J_\mu\rangle, \langle 0|C'_\beta(x)|J_\nu\rangle\,\rangle_\beta \\ &= \langle J_{\lambda/\mu}, J_\nu\rangle_\beta. \end{aligned} \tag{84}$$

Here we have used (79) and $[\hat{J}_\mu, C'^\dagger_\beta(x)] = 0$. □

We can give another proof;

$$\begin{aligned} \sum_\lambda{}' J_{\lambda/\mu}(x) J_\lambda(y) &= \sum_\lambda{}' \langle 0|C'_\beta(y)|J_\lambda\rangle\langle J_\lambda|C'^\dagger_\beta(x)|J_\mu\rangle = \langle 0|C'_\beta(y)C'^\dagger_\beta(x)|J_\mu\rangle \\ &= \langle 0|C'_\beta(y)|J_\mu\rangle \prod_i \prod_j (1 - x_i y_j)^{-\beta} \\ &= \sum_\nu{}' J_\nu(x) J_\nu(y) J_\mu(y). \end{aligned} \tag{85}$$

Here we have used (64) and completeness of $\{|J_\lambda\rangle\}$ in $\mathcal{F}_0$. □

Now we can write down integral representations of skew-Jack polynomials. By using (80), we have

$$J_{\lambda/\mu}(x;\beta) = \oint \prod_{j=1}^{\ell(\mu)} \frac{dw_j^1}{2\pi i w_j^1} \prod_{j=1}^{\ell(\lambda)} \frac{dz_j^1}{2\pi i z_j^1} \cdot f_\mu^a(w^1) f_\lambda^b(z^1) \Gamma(w^1, z^1)^{\frac{-\gamma_a\gamma_b}{\beta}} \Gamma(x, z^1)^{-\gamma_b}. \tag{86}$$

More generally, the skew-Jack polynomial can be written in the integral transformation $N_{N,M}^{(\beta)}$ in (63) or in the power-sum as follows:

$$\begin{aligned} J_{\lambda/\mu}(x_1, \cdots, x_N) &= \frac{\langle J_\lambda, J_\lambda\rangle_\beta}{M!\langle J_\lambda, J_\lambda\rangle'_{\beta;M}} N_{N,M}^{(\beta)} J_\lambda(t_1, \cdots, t_M) J_\mu\left(\frac{1}{t_1}, \cdots, \frac{1}{t_M}\right) \\ J_{\lambda/\mu}(p) &= J_\mu(\overline{p}) J_\lambda(p) \cdot 1, \end{aligned} \tag{87}$$

for all $M \geq \ell(\lambda)$. Here $\overline{p}_n = \frac{n}{\beta}\frac{\partial}{\partial p_n}$. When $M = \ell(\lambda)$, this reduces to (86).

8 Discussion

There are some points which may be interesting if it is clarified in the future study.

1. The methods in section 6 and 7 are applicable to the q–analogue of the Jack polynomial, that is the Macdonald polynomial. Integral representation of the (skew-)Macdonald polynomial can be obtained from that of the Jack polynomial by replacing Δ, Γ and $\widetilde{\Gamma}$ with q–deformed ones [30].

2. It is challenging to calculate dynamical n–point correlation functions for n greater than two. For this purpose, we will have to use the inner-product $\langle J_\lambda, J_\mu J_\nu \rangle'$, whose integral representation has been obtained in section 7.

3. In our construction, the choice of the W algebra depends on the form of the Young diagram. It will be natural to conjecture that there is a underlying symmetry which explains the appearance of various symmetries.

4. This is also related with Kac-Moody algebras. In fact, the operator $C_{\beta'}$ in (48) corresponds to the product of N–vertex operators of $\widehat{sl(N)}$ with fundamental representations. The level is $k + N = 1/\beta$. Hence if we decompose the Young diagram as $r^a = N - a$ and allow s^a's vanish, then the integrand of (55) is just the ϕ–boson part of that of a zero-weight N–point function.

5. The large distance behavior of correlation functions is described by $c = 1$ CFT [31]. To give the eigenfunction, as we observed, $c < 1$ CFT plays the essential role. It is interesting to understand the relation between them.

6. The situation that the only relevant states are given by the null states reminds us of the situation in the quantum gravity [26]. This fact comes from the similarity between our Hamiltonian (2) and the BRST currents when we replaced the a_{-n} by the ghost field.

7. Another analogy with the gravity is that our Hamiltonian is a deformation of that of quantum gravity considered by Ishibashi and Kawai [27]. It may be interesting to understand what quantum gravity system our Hamiltonian (or its continuum limit) describes. It is well known that the matrix model can be described by free fermions. Our construction shows that we may define similar model even if we replace these fermions with anyons.

8. One can generalize our state–function correspondence (48) to a invertible map. In fact, if we introduce $N - 1$ kinds of power-sums $p_n^{(a)}$ $(a = 1, \cdots, N - 1)$, then the

operator $C_{\beta'} \equiv \exp\left\{\beta' \sum_{n>0} \frac{1}{n} \sum_{b=1}^{N-1} \vec{\Lambda}_b \cdot \vec{a}_n \, p_n^{(b)}\right\}$ gives such a map. When $s^a = \beta(1 - r^a + r^{a-1}) - 1$ with $r^0 = 0$, the function $Z(p) \equiv \langle \vec{\lambda}^+_{\vec{r},\vec{s}} | C_{\beta'} | \chi^+_{\vec{r},\vec{s}} \rangle$ is regarded as a generalized partition function of the conformal matrix model in [32] of $\beta = 1$. However, it is still unknown what system this map describes in general.

9. The two–point function derived by Ha [13] has following form,

$$\langle 0 | \rho(x,t) \rho(0,0) | 0 \rangle = C \prod_{i=1}^{q} \int_0^\infty dx_i \prod_{j=1}^{p} \int_0^1 dy_i Q^2 F(q,p,\beta | \{x_i, y_j\}) \cos(Qx) e^{-iEt}, \tag{88}$$

where $Q = 2\pi\rho_0(\sum_{i=1}^q x_i + \sum_{j=1}^p y_j)$, $E = (2\pi\rho_0)^2(\sum_{i=1}^q \epsilon_P(x_i) + \sum_{j=1}^p \epsilon_H(y_j))$. ϵ_P and ϵ_H are the energy for pseudo-particle and hole. The form factor is given by,

$$\begin{aligned} F(q,p,\beta | \{x_i, y_j\}) &= \prod_{i=1}^{q} \prod_{j=1}^{p} (x_i + \beta y_j)^{-2} \frac{\prod_{i<j}(x_i - x_j)^{2\beta} \prod_{i<j}(y_i - y_j)^{2/\beta}}{\prod_{i=1}^q \epsilon_P(x_i)^{1-\beta} \prod_{j=1}^p \epsilon_H(y_j)^{1-1/\beta}} \\ &\propto \frac{\langle \prod_{i=1}^q : e^{\alpha_+ \phi(x_i/\alpha_+)} : \prod_{j=1}^p : e^{\alpha_- \phi(y_j/\alpha_-)} : \rangle}{\prod_{i=1}^q \epsilon_P(x_i)^{1-\beta} \prod_{j=1}^p \epsilon_H(y_j)^{1-1/\beta}}. \end{aligned} \tag{89}$$

The Vertex operators in the final expression are nothing but the screening charges we used in section 4 and 5 [33]. Although there is a definite gap between our approach and theirs, this fact may indicate that some structure we obtained in this paper survives in the thermo-dynamical limit.

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Appendix A : Construction of states with a excitation of M pseudo-particle (pseudo-hole)

A.1

Action with $\hat{H}_\beta$ on $\prod_{i=1}^M e^{\alpha_\pm \phi_-(z_i)} |\alpha\rangle$ ($|z_1| > \cdots > |z_M|$) is given by [18]

$$\hat{H}_\beta \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)} |\alpha\rangle = H'_{\beta,M} \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)} |\alpha\rangle, \tag{90}$$

where

$$
\begin{aligned}
H'_{\beta,M} &= \sum_{i=1}^{M}\Big(\frac{\sqrt{\beta}}{\alpha_\pm/\sqrt{2}}D_i^2 + \Big(N_0\beta - \sqrt{\beta}\tfrac{\alpha_\pm}{\sqrt{2}}\Big)D_i\Big) + 2\sqrt{\beta}\tfrac{\alpha_\pm}{\sqrt{2}}\sum_{\substack{i,j=1\\ i<j}}^{M}\frac{1}{1-\frac{z_j}{z_i}}\Big(\tfrac{z_j}{z_i}D_i - D_j\Big) \\
&= \frac{\sqrt{\beta}}{\alpha_\pm/\sqrt{2}}H_{\frac{1}{2}\alpha_\pm^2} + \Big(N_0\beta - M\sqrt{\beta}\tfrac{\alpha_\pm}{\sqrt{2}}\Big)P. \qquad (91)
\end{aligned}
$$

Here $D_i = z_i\frac{\partial}{\partial z_i}$, and H and P are the Hamiltonian and momentum with variable z_i $(i = 1, \cdots, M)$. Expanding this, we obtain

$$
\begin{aligned}
&\hat{H}_\beta \hat{J}^\pm_{n_1}\cdots\hat{J}^\pm_{n_M}|\alpha\rangle \\
&= \sum_{i=1}^{M}\Big(\frac{\sqrt{\beta}}{\alpha_\pm/\sqrt{2}}n_i^2 + \Big(N_0\beta - \sqrt{\beta}\tfrac{\alpha_\pm}{\sqrt{2}}(2i-1)\Big)n_i\Big)\hat{J}^\pm_{n_1}\cdots\hat{J}^\pm_{n_M}|\alpha\rangle \\
&\quad + 2\sqrt{\beta}\tfrac{\alpha_\pm}{\sqrt{2}}\sum_{\substack{i,j=1\\ i<j}}^{M}\sum_{\ell=1}^{n_j}(n_i - n_j + 2\ell)\hat{J}^\pm_{n_1}\cdots\hat{J}^\pm_{n_i+\ell}\cdots\hat{J}^\pm_{n_j-\ell}\cdots\hat{J}^\pm_{n_M}|\alpha\rangle. \qquad (92)
\end{aligned}
$$

The subspace with $\hat{P} = |\lambda|$ has basis $\hat{J}^\pm_{n_1}\cdots\hat{J}^\pm_{n_{|\lambda|}}|\alpha\rangle$ $(n_1 \geq \cdots \geq n_{|\lambda|} \geq 0, \sum_i n_i = |\lambda|)$, on which $\hat{H}_\beta$ is represented as a triangular matrix. The energy eigenvalue $\epsilon_{\beta,\lambda}$ (6) can be read from the diagonal elements. By diagonalizing this triangular matrix, the eigenstates are determined as

$$
\begin{aligned}
|J^\pm_\lambda\rangle &= \Big(\prod_{i=1}^{M}\hat{J}^\pm_{\lambda_i} + \cdots\Big)|\alpha\rangle, \quad (\lambda = (\lambda_1, \cdots, \lambda_M)), \\
\hat{H}_\beta|J^\pm_\lambda\rangle &= \epsilon_{\beta,\lambda^\pm}|J^\pm_\lambda\rangle, \qquad (\lambda^+ = \lambda, \lambda^- = \lambda'). \qquad (93)
\end{aligned}
$$

For example, $M = 2$ case was explicitly solved,

$$
\begin{aligned}
|J^\pm_{(\lambda_1,\lambda_2)}\rangle &= \sum_{\ell=0}^{\lambda_2}c^\pm(\lambda_1-\lambda_2,\ell)\hat{J}^\pm_{\lambda_1+\ell}\hat{J}^\pm_{\lambda_2-\ell}|\alpha\rangle, \\
c^\pm(\lambda,\ell) &= \frac{\lambda+2\ell}{\lambda+\ell}\prod_{j=1}^{\ell}\frac{\lambda+j}{j}\cdot\prod_{i=1}^{\ell}\frac{-\sqrt{\beta}\frac{\alpha_\pm}{\sqrt{2}} + \frac{\sqrt{\beta}}{\alpha_\pm/\sqrt{2}}(i-1)}{\sqrt{\beta}\frac{\alpha_\pm}{\sqrt{2}} + \frac{\sqrt{\beta}}{\alpha_\pm/\sqrt{2}}(\lambda+i)}. \qquad (94)
\end{aligned}
$$

The Jack polynomial is obtained by the state–function correspondence (15), $J^\pm_\lambda(x) = \langle\alpha|C_{\beta'}|J^\pm_\lambda\rangle$. The normalization between $J^\pm_\lambda(x)$ and $J_\lambda(x)$ are given by [16]

$$
J^+_\lambda(x;\beta) = \mathcal{N}^+_\lambda J_\lambda(x;\beta), \qquad \mathcal{N}^+_\lambda = \prod_{s\in\lambda}\frac{(\ell_\lambda(s)+1)\beta + a_\lambda(s)}{\ell_\lambda(s)\beta + a_\lambda(s) + 1}, \qquad (95)
$$

$$
J^-_\lambda(x;\beta) = \mathcal{N}^-_\lambda J_{\lambda'}(x;\beta), \qquad \mathcal{N}^-_\lambda = (-1)^{|\lambda|}. \qquad (96)
$$

A.2

Instead of using mode expansion, next we will consider the eigenstates in the following form,

$$|J_\lambda^\pm\rangle = \oint \prod_{j=1}^{M} \frac{dz_j}{2\pi i z_j} \cdot \prod_{i=1}^{M} z_i^{-\lambda_i} \cdot f_\lambda^\pm(z_1,\cdots,z_M) \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)}|\alpha\rangle. \tag{97}$$

Here the integration contour is shown in Figure 2(a). From the argument of mode expansion, $f_\lambda^\pm(z_1,\cdots,z_M)$ is a finite sum of terms $\prod_{i<j}(\frac{z_j}{z_i})^{n_{ij}}$ $(n_{ij} \geq 0)$ and has a constant term 1. In the case of $\beta = 1$ when the Jack polynomial reduces to the Schur polynomial, $f_\lambda^\pm$ is independent of λ and given by $f_\lambda^\pm(z_1,\cdots,z_M) = \prod_{i<j}(1-\frac{z_j}{z_i})$, which can be written in a determinant form, $\hat{J}_\lambda^\pm = \det(\hat{J}^\pm_{\lambda_i - i + j})_{1\leq i,j\leq M}$.

Although $f_\lambda^\pm$ has only finite number of terms, we can add to $f_\lambda^\pm$ the terms that do not contribute to the integral. This freedom may give us the possibility to write down solutions for general M. In fact integral representations of such solutions are given in section 5,6. In the following, we allow that $f_\lambda^\pm$ may have infinitely many terms. We will give a sufficient condition for such $f_\lambda^\pm$. By setting $f_\lambda^\pm \prod_i z_i^{-\lambda_i} = F_\lambda^\pm \prod_{i<j}(z_i - z_j)^{\alpha_\pm^2}$, $|J_\lambda^\pm\rangle$ is rewritten as

$$\begin{aligned} |J_\lambda^\pm\rangle &= \oint \prod_{j=1}^{M} \frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1\\ i<j}}^{M} (z_i - z_j)^{\alpha_\pm^2} \cdot F_\lambda^\pm(z_1,\cdots,z_M) \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)}|\alpha\rangle \\ &= \oint \prod_{j=1}^{M} \frac{dz_j}{2\pi i z_j} \cdot F_\lambda^\pm(z_1,\cdots,z_M) \prod_{i=1}^{M} z_i^{-\alpha_\pm(\alpha - M\alpha_\pm)} \prod_{i=1}^{M} :e^{\alpha_\pm \phi(z_i)}: |\alpha - M\alpha_\pm\rangle, \end{aligned} \tag{98}$$

where we may have to choose appropriate integration contour. By using (90) and integration by parts,

$$\begin{aligned} &\hat{H}_\beta |J_\lambda^\pm\rangle = \epsilon_{\beta,\lambda^\pm}|J_\lambda^\pm\rangle \\ &= \oint \prod_{j=1}^{M} \frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1\\ i<j}}^{M} (z_i - z_j)^{\alpha_\pm^2} \cdot F_\lambda^\pm(z_1,\cdots,z_M) H'_{\beta,M} \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)}|\alpha\rangle \\ &= \oint \prod_{j=1}^{M} \frac{dz_j}{2\pi i z_j} \cdot (H'_{\beta,M})^\dagger \Big(\prod_{\substack{i,j=1\\ i<j}}^{M} (z_i - z_j)^{\alpha_\pm^2} \cdot F_\lambda^\pm(z_1,\cdots,z_M) \Big) \cdot \prod_{i=1}^{M} e^{\alpha_\pm \phi_-(z_i)}|\alpha\rangle, \end{aligned}$$

we obtain the sufficient condition for $F_\lambda^\pm$; this $F_\lambda^\pm(z_1,\cdots,z_M)$ is homogeneous and the eigenfunction of H with variable z_i,

$$H_{\frac{1}{2}\alpha_\pm^2} F_\lambda^\pm = \epsilon_{\frac{1}{2}\alpha_\pm^2,\tilde{\lambda}} F_\lambda^\pm, \quad \tilde{\lambda}_i = -\lambda_i - (M-i)\alpha_\pm^2. \tag{99}$$

Note a symmetry $\epsilon_{\beta,\lambda} = \epsilon_{\beta,\overline{\lambda}}$ with $\overline{\lambda}_i = -\lambda_i + \beta(2i-1)$. Also note that $F_\lambda^\pm$ may not be a polynomial.

We remark that the following property of H_β. Let $\psi_\lambda(x_1,\cdots,x_{N_0})$, which is symmetric and homogeneous, be the eigenfunction of H_β, $H_\beta\psi_\lambda=\epsilon_{\beta,\lambda}\psi_\lambda$. Since the original Hamiltonian H_{CS} is symmetric with respect to the transformation $\beta\leftrightarrow 1-\beta$, the function $\tilde{\psi}_{\tilde{\lambda}}=\psi_\lambda\prod_{i<j}(x_i-x_j)^{2\beta-1}$ is also the eigenfunction of H,

$$H_{\tilde{\beta}}\tilde{\psi}_{\tilde{\lambda}}=\epsilon_{\tilde{\beta},\tilde{\lambda}}\tilde{\psi}_{\tilde{\lambda}},\quad \tilde{\beta}=1-\beta,\quad \tilde{\lambda}_i=\lambda_i+(2\beta-1)(N_0-i). \tag{100}$$

Setting $\tilde{F}^{\pm}_{\lambda}=F^{\pm}_{\lambda}\prod_{i<j}(z_i-z_j)^{\alpha_\pm^2-1}$, then $|J^{\pm}_{\lambda}\rangle$ is rewritten as

$$|J^{\pm}_{\lambda}\rangle=\oint\prod_{j=1}^{M}\frac{dz_j}{2\pi i z_j}\cdot\prod_{\substack{i,j=1\\ i<j}}^{M}(z_i-z_j)\cdot\tilde{F}^{\pm}_{\lambda}(z_1,\cdots,z_M)\prod_{i=1}^{M}e^{\alpha_\pm\phi_-(z_i)}|\alpha\rangle. \tag{101}$$

Due to above property, the sufficient condition is reexpressed as follows; $\tilde{F}^{\pm}_{\lambda}(z_1,\cdots,z_M)$ is homogeneous and the eigenfunction of H with variable z_i,

$$H_{1-\frac{1}{2}\alpha_\pm^2}\tilde{F}^{\pm}_{\lambda}=\epsilon_{1-\frac{1}{2}\alpha_\pm^2,\tilde{\lambda}}\tilde{F}^{\pm}_{\lambda},\quad \tilde{\lambda}_i=-\lambda_i-(M-i). \tag{102}$$

For $M=2$, we can easily find a solution of this equation. Let us set $\tilde{F}^{\pm}_{\lambda}=\tilde{f}^{\pm}_{\lambda}\prod_i z_i^{-\lambda_i-(M-i)}$. Substituting the form $\tilde{f}^{\pm}_{(\lambda_1,\lambda_2)}(z_1,z_2)=\sum_{n=0}^{\infty}c_n^{\pm}(\frac{z_2}{z_1})^n$ ($c_0^{\pm}=1$) in (102), we obtain the result that $\tilde{f}^{\pm}_{(\lambda_1,\lambda_2)}$ is given by the hypergeometric function,

$$\tilde{f}^{\pm}_{(\lambda_1,\lambda_2)}(z_1,z_2)={}_2F_1\left[\begin{matrix}\lambda_1-\lambda_2+1,\ -\frac{1}{2}\alpha_\pm^2+1\\ \lambda_1-\lambda_2+\frac{1}{2}\alpha_\pm^2+1\end{matrix};\frac{z_2}{z_1}\right]. \tag{103}$$

$f^{\pm}_{(\lambda_1,\lambda_2)}(z_1,z_2)=(1-\frac{z_2}{z_1})\tilde{f}^{\pm}_{(\lambda_1,\lambda_2)}(z_1,z_2)$ agrees with the generating function for $c^{\pm}(\lambda,\ell)$ of (94),

$$\sum_{l=0}^{\infty}c^{\pm}(\lambda_1-\lambda_2,\ell)\Big(\frac{z_2}{z_1}\Big)^{\ell}=f^{\pm}_{(\lambda_1,\lambda_2)}(z_1,z_2). \tag{104}$$

For $M=3$, we can derive the recurrence formula for the coefficients of the Taylor series of $\tilde{F}^{\pm}_{\lambda}$, which has a property of the root system of sl_3. However it is hard to obtain a general solution for $\tilde{F}^{\pm}_{\lambda}$.

Appendix B

We consider the analytic continuation of the following integral [28, 22],

$$I=\int\prod_{j=1}^{r}dz_j\cdot\prod_{\substack{i,j=1\\ i<j}}^{r}(z_i-z_j)^{2\alpha}\cdot\prod_{i=1}^{r}z_i^{\alpha'}\cdot g(z_1,\cdots,z_r). \tag{105}$$

Let I_1 to be I with the integration region $0 < z_r < \cdots < z_1 < 1$, and I_2 to be I with the integration contour shown in Figure 2(b). We assume that $g(z_1, \cdots, z_r)$ has no poles at $z_i = z_j$. Then I_1 and I_2 are related as follows:

$$I_2 = (-1)^r \prod_{j=1}^{r} (1 - a' a^{j-1}) \cdot \prod_{j=1}^{r} (1 + a + \cdots + a^{j-1}) \cdot I_1, \tag{106}$$

where a and a' are

$$a = e^{2\pi i \alpha}, \quad a' = e^{2\pi i \alpha'}. \tag{107}$$

Therefore we have

$$\frac{1}{(2\pi i)^r} I_2 = e^{\pi i r((r-1)\alpha + \alpha')} \prod_{j=1}^{r} \frac{\sin \pi j \alpha}{\sin \pi \alpha} \frac{\sin \pi((j-1)\alpha + \alpha')}{\pi} \cdot I_1. \tag{108}$$

When $g(z_1, \cdots, z_r) = \prod_{i=1}^{r} (1 - z_i)^{\alpha''}$, this integral is known as the Selberg integral [24],

$$I_1 = \prod_{j=1}^{r} \frac{\Gamma(j\alpha)}{\Gamma(\alpha)} \frac{\Gamma((j-1)\alpha + \alpha' + 1)\Gamma((j-1)\alpha + \alpha'' + 1)}{\Gamma((r-2+j)\alpha + \alpha' + \alpha'' + 2)}. \tag{109}$$

The normalization constant $\mathcal{N}^+_{r,s}$ in (29) is given by the coefficient of $(J^+_s)^r$ in $\langle \alpha_{r,s} | C_{\beta'} | \chi^+_{r,s} \rangle$,

$$\mathcal{N}^+_{r,s} = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1 \\ i<j}}^{r} (z_i - z_j)^{2\beta} \cdot \prod_{i=1}^{r} z_i^{(1-r)\beta}, \tag{110}$$

where the integration contour is shown in Figure 2(b). Using (108) and (109), setting $\alpha = \beta$, $\alpha'' = 0$ and tending to a limit $\alpha' \to (1-r)\beta - 1$, we obtain

$$\mathcal{N}^+_{r,s} = \frac{1}{r!} \prod_{j=1}^{r} \frac{\sin \pi j \beta}{\sin \pi \beta} \cdot \frac{\Gamma(r\beta + 1)}{\Gamma(\beta + 1)^r}. \tag{111}$$

Here we have used $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$. The other one $\mathcal{N}^-_{r,s}$ is obtained from $\mathcal{N}^+_{r,s}$ by replacing β, r, s with $\frac{1}{\beta}$, s, r.

The normalization constant $\mathcal{N}^+_{\vec{r},\vec{s}}$ in (55) is given by the coefficient of $(J^+_{s^1+\cdots+s^{N-1}})^{r^{N-1}}$ $(J^+_{s^1+\cdots+s^{N-2}})^{r^{N-2}-r^{N-1}} \cdots (J^+_{s^1})^{r^1-r^2}$ in $\langle \vec{\lambda}^+_{\vec{r},\vec{s}} | C_{\beta'} | \chi^+_{\vec{r},\vec{s}} \rangle$,

$$\begin{aligned} \mathcal{N}^+_{\vec{r},\vec{s}} = & \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} \frac{dz^a_j}{2\pi i z^a_j} \cdot \prod_{a=1}^{N-1} \prod_{\substack{i,j=1 \\ i<j}}^{r^a} (z^a_i - z^a_j)^{2\beta} \cdot \prod_{a=1}^{N-2} \prod_{i=1}^{r^a} \prod_{j=1}^{r^{a+1}} (z^a_i - z^{a+1}_j)^{-\beta} \\ & \times \prod_{a=1}^{N-1} \prod_{j=1}^{r^a} (z^a_j)^{(1-r^a+r^{a+1})\beta - s^a} \cdot \prod_{a=1}^{N-1} \frac{1}{(r^a - r^{a+1})!} \cdot \sum_{\sigma} \prod_{j=1}^{r^1} (z^1_j)^{\lambda_{\sigma(j)}}, \end{aligned} \tag{112}$$

where $\sum_\sigma$ stands for the summation over all permutations of r^1 objects and λ_i is $(\lambda_1, \cdots, \lambda_{r^1}) = ((s^1 + \cdots + s^{N-1})^{r^{N-1}}, (s^1 + \cdots + s^{N-2})^{r^{N-2}-r^{N-1}}, \cdots, (s^1)^{r^1-r^2})$. The other one $\mathcal{N}^-_{\vec{r},\vec{s}}$ is obtained from $\mathcal{N}^+_{\vec{r},\vec{s}}$ by replacing β, $\vec{r}$ and $\vec{s}$ with $\frac{1}{\beta}$, $\vec{s}$ and $\vec{r}$, respectively. At present we have not obtained the explicit form of $\mathcal{N}^\pm_{\vec{r},\vec{s}}$. For a positive integer β, the normalization constants are explicitly given in section 6, (67) and (62).

Appendix C : Explicit examples

The null state at level n is defined as,

$$L_n|\chi\rangle = 0 \quad (n>0), \qquad L_0|\chi\rangle = (h+n)|\chi\rangle \tag{113}$$

By a standard argument, it has null norm with any states in the Verma module,

$$\langle *|\chi\rangle = 0. \tag{114}$$

The existence of such states depend crucially on the choice of parameter c and h. Cerebrated Kac formula shows that if they are explicitly parametrized as,

$$c = 1 - \frac{6(\beta-1)^2}{\beta} \qquad h_{rs} = \frac{(\beta r - s)^2 - (\beta-1)^2}{4\beta}, \tag{115}$$

for an arbitrary parameter β, there exists null state at level rs. Some of the lower lying states can be explicitly obtained by solving the conditions (113). Let us introduce the notation $|\chi_{rs}\rangle$ as the null state that occurs in the highest module over the vacuum $|h_{rs}\rangle$ at level rs. We obtain,

$$\begin{aligned}
|\chi_{11}\rangle &= L_{-1}|h_{11}\rangle \\
|\chi_{21}\rangle &= (L_{-2} - \frac{1}{\beta}L_{-1}^2)|h_{21}\rangle \\
|\chi_{12}\rangle &= (L_{-2} - \beta L_{-1}^2)|h_{12}\rangle \\
|\chi_{31}\rangle &= \left((1-2\beta)L_{-3} + 2L_{-2}L_{-1} - \frac{1}{2\beta}L_{-1}^3\right)|h_{31}\rangle \\
|\chi_{13}\rangle &= \left((1-2/\beta)L_{-3} + 2L_{-2}L_{-1} - \frac{\beta}{2}L_{-1}^3\right)|h_{13}\rangle \\
|\chi_{41}\rangle &= \left((1-4\beta+6\beta^2)L_{-4} + \frac{5-12\beta}{3}L_{-3}L_{-1}\right. \\
&\quad \left. -\frac{3\beta}{2}L_{-2}^2 + \frac{5}{3}L_{-2}L_{-1}^2 - \frac{1}{6\beta}L_{-1}^4\right)|h_{41}\rangle \\
|\chi_{14}\rangle &= |\chi_{41}\rangle|_{\beta\to 1/\beta,\ h_{41}\to h_{14}} \\
|\chi_{22}\rangle &= \left(L_{-4} + \frac{2(\beta^2-3\beta+1)}{3(\beta-1)^2}L_{-3}L_{-1} - \frac{(\beta+1)^2}{3\beta}L_{-2}^2\right. \\
&\quad \left. + \frac{2(\beta^2+1)}{3(\beta-1)^2}L_{-2}L_{-1}^2 - \frac{\beta}{3(\beta-1)^2}L_{-1}^4\right)|h_{22}\rangle.
\end{aligned} \tag{116}$$

The duality $\beta \leftrightarrow 1/\beta$ is realized as a symmetry $r \leftrightarrow s$.

These null states can be regarded as the Jack polynomial once we use the bosonic representation. In mode expansion, Virasoro charges are replaced as,

$$L_n = \frac{1}{2}\sum_{m\in\mathbf{Z}} : a_{n+m}a_{-m} : -\alpha_0(n+1)a_n. \tag{117}$$

The central charge (115) is obtained by choosing $\alpha_0 = \frac{1}{\sqrt{2}}(\sqrt{\beta} - \sqrt{1/\beta})$. The highest weight state $|h_{rs}\rangle$ can be replaced by the Fock space vacuum $|\alpha_{rs}\rangle$, with

$$\alpha_{rs} = \frac{1}{\sqrt{2}}\left((1+r)\sqrt{\beta} - (1+s)\sqrt{1/\beta}\right).$$

In terms of free boson oscillators, the null states (116) are written as (up to overall normalization),

$$\begin{aligned}
|\chi_{11}\rangle &\sim a_{-1}|\alpha_{11}\rangle \\
|\chi_{21}\rangle &\sim \left(a_{-2} - \sqrt{2/\beta}a_{-1}^2\right)|\alpha_{21}\rangle \\
|\chi_{12}\rangle &\sim \left(a_{-2} + \sqrt{2\beta}a_{-1}^2\right)|\alpha_{12}\rangle \\
|\chi_{31}\rangle &\sim \left(a_{-3} - \frac{3}{\sqrt{2\beta}}a_{-2}a_{-1} + \frac{1}{\beta}a_{-1}^3\right)|\alpha_{31}\rangle \\
|\chi_{13}\rangle &\sim \left(a_{-3} + 3\sqrt{\frac{\beta}{2}}a_{-2}a_{-1} + \beta a_{-1}^3\right)|\alpha_{13}\rangle \\
|\chi_{41}\rangle &\sim \left(a_{-4} - \frac{8}{3\sqrt{2\beta}}a_{-3}a_{-1} - \frac{1}{\sqrt{2\beta}}a_{-2}^2 + \frac{2}{\beta}a_{-2}a_{-1}^2 - \frac{\sqrt{2}}{3\beta\sqrt{\beta}}a_{-1}^4\right)|\alpha_{41}\rangle \\
|\chi_{22}\rangle &\sim \left(a_{-4} + \frac{4\sqrt{2\beta}}{1-\beta}a_{-3}a_{-1} - 2\frac{1+\beta+\beta^2}{\sqrt{2\beta}(1-\beta)}a_{-2}^2 - 4a_{-2}a_{-1}^2 - \frac{2\sqrt{2\beta}}{1-\beta}a_{-1}^4\right)|\alpha_{22}\rangle
\end{aligned} \tag{118}$$

The state $|\chi_{s,r}\rangle$ is obtained from $|\chi_{r,s}\rangle$ by $\beta \leftrightarrow 1/\beta$ and $a_n \leftrightarrow -a_n$.

To translate these expressions into symmetric functions, one can apply the rule,

$$a_{-n} \to \sqrt{\frac{\beta}{2}}p_n(x), \qquad |\alpha_{rs}\rangle \to 1, \qquad p_n = \sum_i (x_i)^n, \tag{119}$$

which gives the Jack polynomials for the rectangular Young diagram $\lambda = \{r^s\}$.

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