

Vertex Operators of the q -Virasoro Algebra; Defining Relations, Adjoint Actions and Four Point Functions

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Abstract

Primary fields of the q -deformed Virasoro algebra are constructed. Commutation relations among the primary fields are studied. Adjoint actions of the deformed Virasoro current on the primary fields are represented by the shift operator $\Theta_\xi f(x) = f(\xi x)$. Four point functions of the primary fields enjoy the connection formula associated with the Boltzmann weights of the fusion Andrews-Baxter-Forrester model.

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1. Introduction One of the interesting topics in two dimensional systems including the conformal field theory (CFT) [1] and solvable lattice models, is to develop methods for calculating correlation functions of physical observables. In these solvable models, infinite dimensional symmetries play an essential role. The most fundamental symmetries of the CFT are the Virasoro algebra, the Kac-Moody algebra or their supersymmetric counterparts. For the six vertex model and XXZ quantum spin chain, it is well recognized that the quantum affine algebra $U_q(\widehat{sl}_2)$ takes place. However, applications of this approach to lattice models had been restricted to a class of models which are defined by trigonometric solutions of the Yang-Baxter equation. For some models, this is because we lack the techniques of bosonizing the vertex operators (intertwining operators of the symmetry algebras) and for others, the characterization of their infinite symmetries are still missing.

Recently the symmetry of the Andrews-Baxter-Forrester (ABF) model [2] was proposed in the work [3, 8]. On the other hand, some of the authors found a Virasoro-like symmetry in a q -deformation of the Calogero-Sutherland model associated with the Macdonald symmetric polynomials [4] (see also [6, 5]). A bosonization formula of the deformed Virasoro generator and the screening operators was also constructed to study the structure of the highest weight modules of the deformed Virasoro algebra. It is astonishing that this deformed Virasoro algebra was identified with the symmetry of the ABF model in [7, 8]. The important objects in the calculation of the correlation functions of the ABF model are the vertex operators which can be regarded as q -deformations of the $(2, 1)$ and $(1, 2)$ operators $\phi_{2,1}(z), \phi_{1,2}(z)$ of the Virasoro minimal model.

In this letter, we will study a natural way to deform $(\ell + 1, k + 1)$ operators. For this end, $(\ell + 1, 1)$ and $(1, \ell + 1)$ ($\ell = 1, 2, \dots$) operators will be considered in detail. Using that, some properties of the $(\ell + 1, k + 1)$ operators will be derived. One of our requirements for deformed vertex operators is that some simple cases of their four point functions can be expressed by the q -hypergeometric functions, that makes it possible to have their connection matrices give the Boltzmann weights of the fusion ABF model. This is a straightforward extension of the idea in [8]. This assumption, however, does not fix answers to this problem uniquely. Therefore, another principle must be needed to fix our goal. In the CFT, we can define an adjoint action of the energy-momentum tensor on the primary fields, and that gives us the $c = 0$ action of the Virasoro algebra: $L_n = -z^{n+1} \frac{d}{dz}$. We expect that *we are also able to define an adjoint action of the deformed Virasoro algebra on the deformed vertex operators*. So as to obtain this property, it is desirable to have the commutation relation between the vertex operator $V(z)$ and the deformed Virasoro current $T(z)$ as follows: $g(w/z)T(z)V(w) - V(w)T(z)g(z/w) = \sum_i c_i \delta(\alpha_i w/z) V(\beta_i w)$, where $g(x)$ is a structure function and c_i, α_i and β_i represent the coefficients, the place of

the singularity and the shift of the vertex operator respectively. Our solution for $(\ell + 1, 1)$ and $(1, \ell + 1)$ operator have this property, but for general $(\ell + 1, k + 1)$ operators ($\ell, k \geq 1$), this does not hold. For these operators, on the other hand, we have commutation relations: $g(w/z)T(z)V(w) - V(w)T(z)g(z/w) = \sum_i c_i \delta(\alpha_i w/z)(V(w)T(z)\tilde{g}_i(w/z))$, where $\tilde{g}(x)$'s are some functions.

The plan of this letter is the following. In Section 2, a brief summary of the deformed Virasoro algebra and the definition of the vertex operators are given. Commutation relations between the deformed Virasoro current and these vertex operators are studied. A shift operator representation of the deformed Virasoro current is derived by studying a deformed adjoint action acting on the vertex operators of type $(\ell + 1, 1)$ and $(1, \ell + 1)$. In Section 3, four-point functions of the vertex operators with one screening operator are calculated explicitly. In section 4 is devoted to discussion.

After finishing this work, a paper by Kadeishvili [9] appeared. His vertex operators $V_{\ell,k}(z)$ are different from ours. While we were seeking 'good' definitions of the vertex operators, we also found a similar object as his and some algebraic structure of that. In this work, however, we will discuss another possibility, because we are interested in the adjoint action of the deformed Virasoro current.

2. Deformed vertex operators

2.1. Definition First we recall the defining relation of the q -Virasoro algebra [4] having two parameters q and t . Set $\beta = \log t / \log q$ and $p = qt^{-1}$. The relation is

$$f\left(\frac{w}{z}\right)T(z)T(w) - T(w)T(z)f\left(\frac{z}{w}\right) = -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{\delta\left(p\frac{w}{z}\right) - \delta\left(p^{-1}\frac{w}{z}\right)}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}, \quad (1)$$

where the structure function $f(x)$ is

$$f(x) = \exp\left(-\sum_{n>0} \frac{1}{n} \frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{p^{\frac{n}{2}} + p^{-\frac{n}{2}}} x^n\right),$$

and the delta function is defined by $\delta(x) = \sum_{n \in \mathbf{Z}} x^n$. The relation (1) is invariant under the following transformations,

$$(I) \quad (q, t) \rightarrow (q^{-1}, t^{-1}), \quad (II) \quad q \leftrightarrow t \quad . \quad (2)$$

In the following, we respect these symmetries in bosonization formulas.

2.2. Bosonization Let us introduce the fundamental Heisenberg algebra¹ h_n ($n \in \mathbf{Z}$), Q_h having the commutation relations

$$[h_n, h_m] = \frac{1}{n} \frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{p^{\frac{n}{2}} + p^{-\frac{n}{2}}} \delta_{n+m,0}, \quad [h_n, Q_h] = \frac{1}{2} \delta_{n,0}. \quad (3)$$

¹ The bosons a_n, Q in [4] are related to h_n, Q_h as $a_n = -n \frac{1}{1-t^n} h_n$ ($n > 0$), $a_{-n} = n \frac{1+p^n}{1-t^{-n}} h_{-n}$ ($n > 0$), $a_0 = \frac{1}{\sqrt{\beta}} h_0$, $Q = \frac{2}{\sqrt{\beta}} Q_h$, and h_n^1 in [5] is $h_n = h_n^1 p^{-\frac{n}{2}}$.

The symmetries (2) are taken into account by the invariance of (3) under the isomorphisms of the Heisenberg algebra:

$$\begin{aligned} \text{(I')} \quad \theta : (q, t) &\mapsto (q^{-1}, t^{-1}), \quad h_n \mapsto -h_n \ (n \neq 0), \quad h_0 \mapsto h_0, \quad Q_h \mapsto Q_h, \\ \text{(II')} \quad \omega : \quad q &\leftrightarrow t \quad , \quad h_n \mapsto -h_n, \quad Q_h \mapsto -Q_h. \end{aligned} \quad (4)$$

The q -Virasoro generator $T(z) = \Lambda^+(z) + \Lambda^-(z)$ and screening currents² $S_{\pm}(z)$ are bosonized as follows:

$$\Lambda^+(z) = : \exp\left(\sum_{n \neq 0} h_n p^{\frac{n}{2}} z^{-n}\right) : q^{\sqrt{\beta} h_0} p^{\frac{1}{2}}, \quad \Lambda^-(z) = \theta \cdot \Lambda^+(z) = \omega \cdot \Lambda^+(z), \quad (5)$$

$$S_+(z) = : \exp\left(-\sum_{n \neq 0} \frac{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} h_n z^{-n}\right) : e^{2\sqrt{\beta} Q_h} z^{2\sqrt{\beta} h_0}, \quad S_-(z) = \omega \cdot S_+(z), \quad (6)$$

where $\omega \cdot \beta$ should be understood as $1/\beta$. We also have $\theta \cdot S_{\pm}(z) = S_{\pm}(z)$. The screening current $S_+(z)$ enjoys the commutation relation

$$[T(z), S_+(w)] = -(1-q)(1-t^{-1}) \frac{d_q}{d_q w} \left(w \delta(q^{-\frac{1}{2}} \frac{w}{z}) : \Lambda^-(q^{-\frac{1}{2}} w) S_+(w) : \right), \quad (7)$$

where the difference operator is defined by $\frac{d_{\xi}}{d_{\xi} w} F(w) = \frac{F(w) - F(\xi w)}{(1-\xi)w}$. A similar difference formula for $S_-(z)$ can be derived from (7) by applying ω .

2.3. Simple vertex operators Now we define vertex operators (primary fields) $V_{\ell+1,1}(z)$, $V_{1,\ell+1}(z)$ by

$$V_{\ell+1,1}(z) = : \exp\left(\sum_{n \neq 0} \frac{1}{q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}}} h_n z^{-n}\right) : e^{-\ell\sqrt{\beta} Q_h} z^{-\ell\sqrt{\beta} h_0}, \quad (8)$$

and $V_{1,\ell+1}(z) = \omega \cdot V_{\ell+1,1}(z)$ for $\ell = 1, 2, \dots$. They have the invariance $V_{1,\ell+1}(z) = \theta \cdot V_{1,\ell+1}(z)$ and $V_{\ell+1,1}(z) = \theta \cdot V_{\ell+1,1}(z)$.

The vertex operators $V_{\ell+1,1}(z)$, $V_{1,\ell+1}(z)$ are expressed as fusion of the fundamental ones $V_{2,1}(z)$, $V_{1,2}(z)$:

$$V_{\ell+1,1}(z) = : \prod_{j=1}^{\ell} V_{2,1}(q^{\frac{\ell+1-2j}{2\ell}} z) : , \quad V_{1,\ell+1}(z) = : \prod_{j=1}^{\ell} V_{1,2}(t^{\frac{\ell+1-2j}{2\ell}} z) : . \quad (9)$$

Lemma 2.1. *We obtain the fundamental commutation relation.*³

$$\begin{aligned} &g^{(\ell+1,1)}\left(\frac{w}{z}\right) \Lambda^+(z) V_{\ell+1,1}(w) - V_{\ell+1,1}(w) \Lambda^+(z) \left(-\frac{z}{w}\right)^{2-\ell} g^{(\ell+1,1)}\left(\frac{z}{w}\right) \\ &= p^{\frac{1}{2}} t^{-\frac{\ell}{2}} \prod_{j=0}^{\ell-2} \left(1 - tq^{-\frac{j}{\ell}}\right) \cdot \delta\left(p^{\frac{1}{2}} q^{-\frac{1}{2\ell}} \frac{w}{z}\right) V_{\ell+1,1}\left(q^{-\frac{1}{\ell}} w\right), \end{aligned} \quad (10)$$

² Screening currents in [4] (say, $S_{\pm}^{\text{old}}(z)$) is related to $S_{\pm}(z)$ as $S_+(z) = S_+^{\text{old}}(q^{-\frac{1}{2}} z) q^{\sqrt{\beta} h_0}$, $S_-(z) = S_-^{\text{old}}(t^{-\frac{1}{2}} z) t^{-\frac{1}{\sqrt{\beta}} h_0}$. This modification does not affect the important relations between singular vectors of the q -Virasoro algebra and the Macdonald symmetric polynomials, because both of them give the same OPE factors.

³ We use the standard convention $\prod_{j=n}^{n-1} * = 1$.

where the structure function $g^{(\ell+1,1)}(x)$ is given by

$$g^{(\ell+1,1)}(x) = \exp\left(\sum_{n>0} \frac{1}{n} g_n^{(\ell+1,1)} x^n\right), \quad (11)$$

$$g_n^{(\ell+1,1)} = -\frac{t^n - t^{-n}}{q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}}} \frac{1}{p^{\frac{n}{2}} + p^{-\frac{n}{2}}} + \frac{p^{-\frac{n}{2}} q^{\frac{n}{\ell}} - p^{\frac{n}{2}} q^{-\frac{n}{\ell}}}{q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}}}. \quad (12)$$

Note that the structure function $g^{(\ell+1,1)}(x)$ is invariant under the transformation (2)-(I). To prove (10), the identity

$$e^{\sum_{n>0} \frac{1}{n} (1-r_1^n - \dots - r_m^n) x^n} - (-x)^{m-1} \prod_{i=1}^m r_i \cdot e^{\sum_{n>0} \frac{1}{n} (1-r_1^{-n} - \dots - r_m^{-n}) x^{-n}} = \prod_{i=1}^m (1-r_i) \cdot \delta(x),$$

may be useful. The commutation relations among $T(z)$ and $V_{\ell+1,1}(z)$, $V_{1,\ell+1}(z)$ are derived from (10) by applying θ and ω .

Proposition 2.1. *For $V_{\ell+1,1}(z)$, we have*

$$\begin{aligned} & g^{(\ell+1,1)}\left(\frac{w}{z}\right) T(z) V_{\ell+1,1}(w) - V_{\ell+1,1}(w) T(z) \left(-\frac{z}{w}\right)^{2-\ell} g^{(\ell+1,1)}\left(\frac{z}{w}\right) \\ &= \prod_{j=0}^{\ell-2} \left(t^{-\frac{1}{2}} q^{\frac{j}{2\ell}} - t^{\frac{1}{2}} q^{-\frac{j}{2\ell}} \right) \left(p^{\frac{1}{2}} t^{-\frac{1}{2}} q^{-\frac{(\ell-1)(\ell-2)}{4\ell}} \delta\left(p^{\frac{1}{2}} q^{-\frac{1}{2\ell}} \frac{w}{z}\right) V_{\ell+1,1}\left(q^{-\frac{1}{\ell}} w\right) \right. \\ & \quad \left. - (-1)^\ell p^{-\frac{1}{2}} t^{\frac{1}{2}} q^{\frac{(\ell-1)(\ell-2)}{4\ell}} \delta\left(p^{-\frac{1}{2}} q^{\frac{1}{2\ell}} \frac{w}{z}\right) V_{\ell+1,1}\left(q^{\frac{1}{\ell}} w\right) \right). \end{aligned} \quad (13)$$

Let us define the adjoint action of the q -Virasoro generator on the vertex operator by

$$\begin{aligned} \mathcal{T}_n^\ell \cdot V_{\ell+1,1}(w) &= \oint \frac{dz}{2\pi iz} z^n \left(g^{(\ell+1,1)}\left(\frac{w}{z}\right) T(z) V_{\ell+1,1}(w) \right. \\ & \quad \left. - V_{\ell+1,1}(w) T(z) \left(-\frac{z}{w}\right)^{2-\ell} g^{(\ell+1,1)}\left(\frac{z}{w}\right) \right). \end{aligned} \quad (14)$$

Then we obtain

Theorem 2.1. *The operator \mathcal{T}_n^ℓ can be represented by the shift operator Θ_ξ defined by $\Theta_\xi f(z) = f(\xi z)$ as*

$$\begin{aligned} \mathcal{T}_n^\ell &= \prod_{j=0}^{\ell-2} \left(t^{-\frac{1}{2}} q^{\frac{j}{2\ell}} - t^{\frac{1}{2}} q^{-\frac{j}{2\ell}} \right) \cdot \left(p^{\frac{n+1}{2}} t^{-\frac{1}{2}} q^{-\frac{(\ell-1)(\ell-2)+2n}{4\ell}} w^n \Theta_{q^{-\frac{1}{\ell}}} \right. \\ & \quad \left. - (-1)^\ell p^{-\frac{n+1}{2}} t^{\frac{1}{2}} q^{\frac{(\ell-1)(\ell-2)+2n}{4\ell}} w^n \Theta_{q^{\frac{1}{\ell}}} \right), \end{aligned} \quad (15)$$

on the vertex operator $V_{\ell+1,1}(w)$.

2.4. General vertex operators The following lemma is helpful to construct general vertex operators of type $(\ell+1, k+1)$.

Lemma 2.2. *The fundamental relation (10) can be written in another way,*

$$\begin{aligned} & g^{(\ell+1,1)}\left(\frac{w}{z}\right) \Lambda^+(z) V_{\ell+1,1}(w) - V_{\ell+1,1}(w) \Lambda^+(z) \left(-\frac{z}{w}\right)^{2-\ell} g^{(\ell+1,1)}\left(\frac{z}{w}\right) \\ &= t^{-\frac{\ell}{2}} \prod_{j=0}^{\ell-2} \left(1 - tq^{-\frac{j}{\ell}} \right) \cdot \delta\left(p^{\frac{1}{2}} q^{-\frac{1}{2\ell}} \frac{z}{w}\right) \left(\tilde{g}^{(\ell+1,1)}\left(\frac{w}{z}\right) V_{\ell+1,1}(w) T(z) \right), \end{aligned} \quad (16)$$

where

$$\tilde{g}^{(\ell+1,1)}(x) = \exp \left(- \sum_{n>0} \frac{1}{n} \frac{p^{-\frac{n}{2}}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{(q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})} x^n \right). \quad (17)$$

To obtain (16) from (10), we used the identity $V_{\ell+1,1}(q^{-\frac{1}{\ell}}w) = p^{-\frac{1}{2}} : V_{\ell+1,1}(w) \Lambda^+(p^{\frac{1}{2}}q^{-\frac{1}{2\ell}}w) :$. It may be worth to make a comment on the R.H.S. The factor $V_{\ell+1,1}(w) \Lambda^-(z)$ is regular at $z = p^{\frac{1}{2}}q^{-\frac{1}{2\ell}}w$, and $V_{\ell+1,1}(w) \Lambda^+(z)$ has a simple pole at the point. However, the function $\tilde{g}^{(\ell+1,1)}(w/z)$ has a simple zero at the point. So, the R.H.S. is well defined in totality.

Let us define the vertex operator of type $(\ell + 1, k + 1)$ by

$$V_{\ell+1,k+1}(z) = : V_{\ell+1,1}(z) V_{1,k+1}(z) :, \quad (18)$$

and introduce the structure functions: $g^{(1,k+1)}(x) = \omega \cdot g^{(k+1,1)}(x)$, $\tilde{g}^{(1,k+1)}(x) = \theta \cdot \omega \cdot \tilde{g}^{(k+1,1)}(x)$, $g^{(\ell+1,k+1)}(x) = g^{(\ell+1,1)}(x)g^{(1,k+1)}(x)$, and $\tilde{g}^{(\ell+1,k+1)}(x) = \tilde{g}^{(\ell+1,1)}(x)\tilde{g}^{(1,k+1)}(x)$. Using Lemma 2.2. and the maps θ and ω , we have the commutation relation for this vertex operator.

Proposition 2.2. *The commutation relation between the deformed Virasoro current and the vertex operator of type $(\ell + 1, k + 1)$ is given by the following relation and θ, ω operations,*

$$\begin{aligned} & g^{(\ell+1,k+1)}\left(\frac{w}{z}\right) \Lambda^+(z) V_{\ell+1,k+1}(w) - V_{\ell+1,k+1}(w) \Lambda^+(z) \left(-\frac{z}{w}\right)^{4-\ell-k} g^{(\ell+1,k+1)}\left(\frac{z}{w}\right) \\ &= V_{\ell+1,k+1}(w) T(z) \tilde{g}^{(\ell+1,k+1)}\left(\frac{z}{w}\right) q^{\frac{k}{2}} t^{-\frac{\ell}{2}} \\ & \times \left(\frac{\delta(p^{\frac{1}{2}}q^{-\frac{1}{2\ell}}\frac{w}{z})}{1 - q^{\frac{1}{2\ell}}t^{\frac{1}{2k}}} \prod_{i=0}^{\ell-2} (1 - tq^{-\frac{i}{\ell}}) \cdot \prod_{j=0}^{k-2} (1 - q^{\frac{1}{2\ell}-1}t^{\frac{1}{2k}+\frac{j}{k}}) \right. \\ & \left. + \frac{\delta(p^{\frac{1}{2}}t^{\frac{1}{2k}}\frac{w}{z})}{1 - q^{-\frac{1}{2\ell}}t^{-\frac{1}{2k}}} \prod_{j=0}^{k-2} (1 - q^{-1}t^{\frac{j}{k}}) \cdot \prod_{i=0}^{\ell-2} (1 - t^{1-\frac{1}{2k}}q^{-\frac{1}{2\ell}-\frac{i}{\ell}}) \right). \end{aligned} \quad (19)$$

In the $q \rightarrow 1$ limit, $V_{r,s}(z)$ reduces to the usual vertex operator $e^{\frac{1}{2}\alpha_{r,s}\phi(z)}$ where $\alpha_{r,s} = \sqrt{\beta}(1-r) - \frac{1}{\sqrt{\beta}}(1-s)$ and $\phi(z)\phi(w) \sim 2\log(z-w)$. This can be easily shown in the bosonized form. (We remark that there are infinitely many operators which reduce to $e^{\frac{1}{2}\alpha_{r,s}\phi(z)}$ in the $q \rightarrow 1$ limit.) However, it is rather nontrivial to derive the usual defining relation of the primary field of the Virasoro algebra from the commutation relations (19).

3. Correlation functions First we recall standard notations; $(a; q_1, \dots, q_\ell)_n = \prod_{k_1, \dots, k_\ell=0}^{n-1} (1 - aq_1^{k_1} \dots q_\ell^{k_\ell})$, $(a_1, \dots, a_m; q_1, \dots, q_\ell)_n = \prod_{j=1}^m (a_j; q_1, \dots, q_\ell)_n$, $\Gamma_q(z) = (1-q)^{1-z} (q; q)_\infty / (q^z; q)_\infty$, $B_q(x, y) = \Gamma_q(x)\Gamma_q(y)/\Gamma_q(x+y)$, $\vartheta_q(z) = (q; q)_\infty (z; q)_\infty (qz^{-1}; q)_\infty$. The Jackson integral is defined by $\int_0^a d_q z f(z) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n$, $\int_0^{a\infty} d_q z f(z) = a(1-q) \sum_{n=-\infty}^{\infty} f(aq^n) q^n$, $\int_A^B d_q z f(z) = \int_0^B d_q z f(z) - \int_0^A d_q z f(z)$. The q -hypergeometric

function ${}_2\phi_1$ is defined by

$${}_2\phi_1(a, b; c; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n. \quad (20)$$

In the following we abbreviate $V_{\ell+1,1}(z)$ as $V_\ell(z)$. We will calculate the following four point functions,

$$U_+(z, w) = \int_{t^{\frac{1}{2}} q^{\frac{1}{2\ell}} z}^{t^{\frac{1}{2}} q^{\frac{1}{2\ell}} z \infty} d_{q^{\frac{1}{\ell}}} \mu \langle * | S_+(\mu) V_\ell(z) V_\ell(w) V_L(0) | 0 \rangle, \quad (21)$$

$$U_-(z, w) = \int_0^{t^{-\frac{1}{2}} q^{\frac{1}{2\ell}} w} d_{q^{\frac{1}{\ell}}} \mu \langle * | V_\ell(z) V_\ell(w) S_+(\mu) V_L(0) | 0 \rangle, \quad (22)$$

where the momentum of the bra state $*$ is chosen such that $\langle * | : S_+(\mu) V_\ell(z) V_\ell(w) V_L(0) : | 0 \rangle = 1$. These four point functions are expressed as

$$\begin{aligned} & U_+(z, w) / \langle V_\ell(z) V_\ell(w) \rangle \\ &= (zw)^{\frac{1}{2}\ell L\beta} B_{q^{\frac{1}{\ell}}}(2a - b, 1 - a) (t^{\frac{1}{2}} q^{-\frac{1}{2\ell}} z)^{b-2a} {}_2\phi_1(q^{\frac{a}{\ell}}, q^{\frac{2a-b}{\ell}}; q^{\frac{1+a-b}{\ell}}; q^{\frac{1}{\ell}}, q^{\frac{1-a}{\ell}} \frac{w}{z}), \end{aligned} \quad (23)$$

$$\begin{aligned} & U_-(z, w) / \langle V_\ell(z) V_\ell(w) \rangle \\ &= (zw)^{\frac{1}{2}\ell L\beta} B_{q^{\frac{1}{\ell}}}(b, 1 - a) (zw)^{-a} (t^{-\frac{1}{2}} q^{\frac{1}{2\ell}} w)^b {}_2\phi_1(q^{\frac{a}{\ell}}, q^{\frac{b}{\ell}}; q^{\frac{1-a+b}{\ell}}; q^{\frac{1}{\ell}}, q^{\frac{1-a}{\ell}} \frac{w}{z}), \end{aligned} \quad (24)$$

where $a = \ell\beta$, $b = 1 - L\beta$ and

$$\langle V_\ell(z) V_\ell(w) \rangle = z^{\ell^2\beta/2} \prod_{j=1}^{\ell} \frac{(t^{-1} q^{j/\ell} w/z, p q^{j/\ell} w/z; p^2, q^{1/\ell})_{\infty}}{(q^{j/\ell} w/z, t^{-1} p q^{j/\ell} w/z; p^2, q^{1/\ell})_{\infty}}. \quad (25)$$

We have used the formulas,

$$S_+(\mu) V_\ell(z) = : S_+(\mu) V_\ell(z) : \mu^{-\ell\beta} \frac{(t^{1/2} q^{1/2\ell} z/\mu; q^{1/\ell})_{\infty}}{(t^{-1/2} q^{1/2\ell} z/\mu; q^{1/\ell})_{\infty}}, \quad (26)$$

$$V_\ell(z) S_+(\mu) = : V_\ell(z) S_+(\mu) : z^{-\ell\beta} \frac{(t^{1/2} q^{1/2\ell} \mu/z; q^{1/\ell})_{\infty}}{(t^{-1/2} q^{1/2\ell} \mu/z; q^{1/\ell})_{\infty}}. \quad (27)$$

We note that the ratio of the coefficients in the R.H.S.'s of (26) and (27) is a pseudo-constant with respect to the shift $z/\mu \rightarrow q^{1/\ell} z/\mu$.

Introduce the notation $x = q^{\frac{1}{2r\ell}}$ and the definition $[u] = \sqrt{2\pi r/\varepsilon} x^{r/4} x^{u(u-r)/r} \vartheta_{x^{2r}}(x^{2u})$ where $\varepsilon = -2\pi^2/\log x$.

Proposition 3.1. *The connection formula for the four point functions $U_{\pm}(z, w)$ can be written as*

$$\begin{pmatrix} U_+(z, w) \\ U_-(z, w) \end{pmatrix} = \frac{\langle V_\ell(z) V_\ell(w) \rangle}{\langle V_\ell(w) V_\ell(z) \rangle} \begin{pmatrix} \frac{[\ell][-u+\ell+L]}{[\ell+L][u+\ell]} & \frac{[L][-u]}{[\ell+L][u+\ell]} \\ \frac{[2\ell+L][-u]}{[\ell+L][u+\ell]} & \frac{[\ell][u+\ell+L]}{[\ell+L][u+\ell]} \end{pmatrix} \begin{pmatrix} U_+(w, z) \\ U_-(w, z) \end{pmatrix}, \quad (28)$$

where u is defined by $\frac{w}{z} = x^{2u}$.

The elements of the connection matrix are identified with some of the Boltzmann weights of the $\ell \times \ell$ fusion RSOS model [2]. We conjecture that the connection matrices for the four point functions having arbitrary numbers of screening operators are expressible in the same way by the Boltzmann weights of the $\ell \times \ell$ fusion ABF model.

4. Discussion

We constructed the vertex operators (primary fields) of the deformed Virasoro algebra. Using the vertex operators, we obtained the adjoint action of the deformed Virasoro current acting on the vertex operator of type $(\ell + 1, 1)$, $(1, \ell + 1)$. This kind of adjoint action is found only for this simple cases, so far. Thus it is desirable having more general definition of this kind of adjoint action.

Applications of the vertex operators might be possible to integrable systems; integrable massive field theories, solvable lattice models and so on. One of these candidates is the calculation of the correlation functions of the fusion ABF model, since we get some of the Boltzmann weights of the $\ell \times \ell$ fusion ABF model in the connection formula of the four point functions derived in Section 3. As is discussed by Lukyanov and Pugai [3, 8], the vertex operators $V_{2,1}(z)$ and $V_{1,2}(z)$ act on the physical space of the ABF model. Then, it may be natural to consider that our vertex operators $V_{\ell+1,1}(z)$ and $V_{1,\ell+1}(z)$ ($\ell \geq 2$) act on some fusion ABF model. The situation is, however, slightly complicated i.e. the central charge of this model in the regime III, which is derived from the corner transfer matrix method, is greater than one except for the special case: #of states = #of fusion + 2. Thus it seems impossible to represent the physical space of the general fusion ABF model by a Fock space of single bosonic field. Therefore, the first task should be to identify the model which our vertex operators are associated with, if there is any. To this end, it might be suggestive to study the factor $\frac{\langle V_\ell(z)V_\ell(w) \rangle}{\langle V_\ell(w)V_\ell(z) \rangle}$, which appears in the connection matrix (28). In the case $\ell = 1$, it reduces to $\frac{[u+1]/[1]}{\kappa(u)}$, where $\kappa(u)$ is the free energy of the ABF model in the regime III.

In the definition of the correlation functions given in Section 3, the cycle of these Jackson integrals are given by hand depending on the ordering of the vertex operators and screening operators. In the conformal field theory, however, the screened vertex operator is defined by the contour integral whose contour is independent of this kind of ordering [11]. Thus, it would be desirable to obtain the screened vertex operator by a contour integral [3, 8, 9].

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