EFI-97-07
DPSU-97-1
January 1997

q-Difference Realization of $U_q(sl(M|N))$ and Its Application to Free Boson Realization of $U_q(\widehat{sl}(2|1))$

Hidetoshi AWATA[*1], Satoru ODAKE[2] and Jun'ichi SHIRAISHI[3]

[1] *James Frank Institute and Enrico Fermi Institute, University of Chicago, 5640 S. Ellis Ave., Chicago, IL 60637, U.S.A.*

[2] *Department of Physics, Faculty of Science Shinshu University, Matsumoto 390, Japan*

[3] *Institute for Solid State Physics, University of Tokyo, Tokyo 106, Japan*

Abstract

We present a q-difference realization of the quantum superalgebra $U_q(sl(M|N))$, which includes Grassmann even and odd coordinates and their derivatives. Based on this result we obtain a free boson realization of the quantum affine superalgebra $U_q(\widehat{sl}(2|1))$ of an arbitrary level k.

q-alg/9701032

[*] JSPS fellow
[1] e-mail address : awata@rainbow.uchicago.edu
[2] e-mail address : odake@azusa.shinshu-u.ac.jp
[3] e-mail address : shiraish@momo.issp.u-tokyo.ac.jp

1 Introduction

Free field realization is the most useful tool for studying representation theories of infinite dimensional algebras and their application to physics such as calculation of correlation functions. The Virasoro and affine Lie algebras are investigated extensively by this method[3]. Quantum deformation (q-deformation) of these algebras are also studied [7, 6, 4, 2]. The aim of this article and its continuation[9] is to generalize these to superalgebra cases, which will help us to study physical systems with superalgebra symmetries such as supersymmetric t-J model, to construct a supersymmetric quantum deformed Virasoro algebra, etc. A free boson realization of $U_q(\widehat{sl}(M|N))$ has been obtained for a level one[5] but not yet for an arbitrary level k. To obtain a free boson realization of $U_q(\widehat{sl}(M|N))$ with an arbitrary level (Wakimoto module), we first consider a q-difference realization of $U_q(sl(M|N))$, because for non-superalgebra case the q-difference realization of $U_q(sl_N)$ on the flag manifold[1] helped us to construct the free boson realization of $U_q(\widehat{sl}_N)$ [2]. Next we try to construct a free boson realization of $U_q(\widehat{sl}(M|N))$. At present we have obtained it for $M+N\le 3$. In this article the result for $(M,N)=(2,1)$ is given for simplicity.

After recalling the definitions of $U_q(sl(M|N))$ and $U_q(\widehat{sl}(M|N))$ in section 2, we present a q-difference realization of $U_q(sl(M|N))$ in section 3. In section 4 we give a free boson realization of $U_q(\widehat{sl}(2|1))$ with an arbitrary level k.

While preparing our draft, we received ref.[10], where Kimura obtained a q-difference realization of $U_q(sl(M|N))$. We will comment on it at the end of section 3.

2 $U_q(sl(M|N))$ and $U_q(\widehat{sl}(M|N))$

First let us recall the definition of $U_q(sl(M|N))$ $(M,N,M+N-2\ge 0)$[8]. We take the Cartan matrix of $sl(M|N)$ as $a_{ij}=(\nu_i+\nu_{i+1})\delta_{i,j}-\nu_i\delta_{i,j+1}-\nu_{i+1}\delta_{i+1,j}$ $(1\le i,j\le M+N-1)$, where ν_i is 1 for $1\le i\le M$, -1 for $M+1\le i\le M+N$. The quantum Lie superalgebra $U_q(sl(M|N))$ is defined by the Chevalley generators $t_i=q^{h_i}$ (invertible), $e_i^+=e_i$, $e_i^-=f_i$ $(i=1,\cdots,M+N-1)$ with the relations,

$$t_it_j=t_jt_i \quad ([h_i,h_j]=0), \tag{1}$$

$$t_ie_j^{\pm}t_i^{-1}=q^{\pm a_{ij}}e_j^{\pm} \quad ([h_i,e_j^{\pm}]=\pm a_{ij}e_j^{\pm}), \tag{2}$$

$$[e_i,f_j]=\delta_{i,j}[h_i], \tag{3}$$

$$[e_i^{\pm},[e_i^{\pm},e_j^{\pm}]_{q^{-1}}]_q=0 \quad \text{for } |a_{ij}|=1, i\ne M, \tag{4}$$

$$[e_M^{\pm},[e_{M+1}^{\pm},[e_M^{\pm},e_{M-1}^{\pm}]_{q^{-1}}]_q]=0, \tag{5}$$

where $[x]=(q^x-q^{-x})/(q-q^{-1})$, $[A,B]_\xi=AB-(-1)^{|A|\cdot|B|}\xi BA$, and $[A,B]=[A,B]_1$. Here $|\cdot|$ stands for $\mathbb{Z}_2$-grading (Grassmann parity) ($|e^{\pm}_M|=1$ and 0 for otherwise).

Next recall the definition of the quantum affine superalgebra $U_q(\widehat{sl}(M|N))$ in terms of Drinfeld generators, $E^{+,i}_n=E^i_n$, $E^{-,i}_n=F^i_n$ ($n\in\mathbb{Z}$), H^i_n ($n\in\mathbb{Z}_{\neq 0}$), invertible K_i ($1\le i\le M+N-1$) and invertible γ [8]. Their $\mathbb{Z}_2$ gradings are 1 for $E^{\pm,M}_n$ and zero otherwise. The relations are

$$\gamma:\ \text{central element}, \tag{6}$$

$$[K_i,H^j_n]=0,\quad K_iE^{\pm,j}_nK_i^{-1}=q^{\pm a_{ij}}E^{\pm,j}_n, \tag{7}$$

$$[H^i_n,H^j_m]=\frac{1}{n}[a_{ij}n]\frac{\gamma^n-\gamma^{-n}}{q-q^{-1}}\delta_{n+m,0}, \tag{8}$$

$$[H^i_n,E^{\pm,j}_m]=\pm\frac{1}{n}[a_{ij}n]\gamma^{\mp\frac{1}{2}|n|}E^{\pm,j}_{n+m}, \tag{9}$$

$$[E^{+,i}_n,E^{-,j}_m]=\frac{\delta^{ij}}{q-q^{-1}}\Big(\gamma^{\frac{1}{2}(n-m)}\psi^i_{+,n+m}-\gamma^{-\frac{1}{2}(n-m)}\psi^i_{-,n+m}\Big), \tag{10}$$

$$[E^{\pm,i}_{n+1},E^{\pm,j}_m]_{q^{\pm a_{ij}}}+[E^{\pm,j}_{m+1},E^{\pm,i}_n]_{q^{\pm a_{ij}}}=0, \tag{11}$$

$$[E^{\pm,i}_n,E^{\pm,j}_m]=0\quad\text{for } a_{ij}=0, \tag{12}$$

$$[E^{\pm,i}_{n_1},[E^{\pm,i}_{n_2},E^{\pm,j}_m]_{q^{-1}}]_q+(n_1\leftrightarrow n_2)=0\quad\text{for } |a_{ij}|=1, i\neq M, \tag{13}$$

$$[E^{\pm,M}_{n_1},[E^{\pm,M+1}_m,[E^{\pm,M}_{n_2},E^{\pm,M-1}_\ell]_{q^{-1}}]_q]+(n_1\leftrightarrow n_2)=0, \tag{14}$$

where $\psi^i_{\pm,n}$ is defined by

$$\sum_{n\in\mathbb{Z}}\psi^i_{\pm,n}z^{-n}=K_i^{\pm1}\exp\Big(\pm(q-q^{-1})\sum_{\pm n>0}H^i_nz^{-n}\Big). \tag{15}$$

Let H^i_0 be defined by

$$K_i=\exp\Big((q-q^{-1})\tfrac{1}{2}H^i_0\Big), \tag{16}$$

then above relations hold for H^i_n ($n\in\mathbb{Z}$) (in the case of $n=0$, $\frac{1}{n}*$ should be understood as $\lim_{n\to0}\frac{1}{n}*$). Fields $H^i(z)$, $E^{\pm,i}(z)$ and $\psi^i_\pm(z)$ are defined by

$$H^i(z)=\sum_{n\in\mathbb{Z}}H^i_nz^{-n-1},\quad E^{\pm,i}(z)=\sum_{n\in\mathbb{Z}}E^{\pm,i}_nz^{-n-1},\quad \psi^i_\pm(z)=\sum_{n\in\mathbb{Z}}\psi^i_{\pm,n}z^{-n}. \tag{17}$$

3 q-Difference realization of $U_q(sl(M|N))$

In ref.[1] $U_q(sl_N)$ was realized by q-difference operators in flag coordinates. We extend it to the super case.

Let $x_{i,j}$ ($1\le i<j\le M+N$) be variables with Grassmann parity $\nu_i\nu_j$ and set $\vartheta_{i,j}=x_{i,j}\frac{\partial}{\partial x_{i,j}}$. For $1\le i,j\le M+N-1$, we define h_i, $e_{i,i}$, $e_{i,i'}$ ($1\le i'\le i-1$), $f^1_{j,j'}$

$(1 \le j' \le j-1)$, $f^2_{j,j}$ and $f^3_{j,j'}$ $(j+2 \le j' \le M+N)$ as follows:

$$\begin{aligned}
h_i &= -\sum_{j=1}^{i-1}(\nu_{i+1}\vartheta_{j,i+1}-\nu_i\vartheta_{j,i})+\lambda_i-(\nu_i+\nu_{i+1})\vartheta_{i,i+1}-\sum_{j=i+2}^{M+N}(\nu_i\vartheta_{i,j}-\nu_{i+1}\vartheta_{i+1,j}), &(18)\\
e_{i,i} &= \frac{1}{x_{i,i+1}}[\vartheta_{i,i+1}]q^{-\sum_{\ell=1}^{i-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})}, &(19)\\
e_{i,i'} &= x_{i',i}\frac{1}{x_{i',i+1}}[\vartheta_{i',i+1}]q^{-\sum_{\ell=1}^{i'-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})}, &(20)\\
f^1_{j,j'} &= x_{j',j+1}\frac{1}{x_{j',j}}[\vartheta_{j',j}] \\
&\quad \times q^{\sum_{m=j'+1}^{j-1}(\nu_{j+1}\vartheta_{m,j+1}-\nu_j\vartheta_{m,j})-\lambda_j+(\nu_j+\nu_{j+1})\vartheta_{j,j+1}+\sum_{m=j+2}^{M+N}(\nu_j\vartheta_{j,m}-\nu_{j+1}\vartheta_{j+1,m})}, &(21)\\
f^2_{j,j} &= x_{j,j+1}\Big[\lambda_j-\nu_j\vartheta_{j,j+1}-\sum_{m=j+2}^{M+N}(\nu_j\vartheta_{j,m}-\nu_{j+1}\vartheta_{j+1,m})\Big], &(22)\\
f^3_{j,j'} &= x_{j,j'}\frac{1}{x_{j+1,j'}}[\vartheta_{j+1,j'}]q^{\lambda_j-\sum_{m=j'}^{M+N}(\nu_j\vartheta_{j,m}-\nu_{j+1}\vartheta_{j+1,m})}, &(23)
\end{aligned}$$

where, for Grassmann odd variable x, the expression $\frac{1}{x}$ stands for the derivative $\frac{1}{x}=\frac{\partial}{\partial x}$.

Proposition 1 *The following* (i) *and* (ii) *realize $U_q(sl(M|N))$:*

$$\text{(i)} \quad h_i, \quad e_i = e_{i,i}+\nu_i\sum_{i'=1}^{i-1}e_{i,i'}, \quad f_j=\sum_{j'=1}^{j-1}f^1_{j,j'}+f^2_{j,j}-\sum_{j'=j+2}^{M+N}f^3_{j,j'}, \tag{24}$$

$$\text{(ii)} \quad h_i, \quad e_i = e_{i,i}+\sum_{i'=1}^{i-1}e_{i,i'}, \quad f_j=\nu_j\sum_{j'=1}^{j-1}f^1_{j,j'}+f^2_{j,j}-\nu_{j+1}\sum_{j'=j+2}^{M+N}f^3_{j,j'}. \tag{25}$$

Proof: Direct calculation shows this proposition. For reader's convenience we present some intermediate steps;

$$[e_{i,i},f^1_{j,j'}]=[e_{i,i},f^3_{j,j'}]=[e_{i,i'},f^2_{j,j}]=0, \tag{26}$$

$$\begin{aligned}
[e_{i,i},f^2_{j,j}] &= \delta_{i,j}\Big[\lambda_i-(\nu_i+\nu_{i+1})\vartheta_{i,i+1}-\sum_{m=i+2}^{M+N}(\nu_i\vartheta_{i,m}-\nu_{i+1}\vartheta_{i+1,m})\Big]q^{-\sum_{\ell=1}^{i-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})}\\
&\quad +\delta_{i,j+1}\nu_i x_{i-1,i}\frac{1}{x_{i,i+1}}[\vartheta_{i,i+1}] &(27)\\
&\qquad \times q^{-\sum_{\ell=1}^{i-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})+\lambda_{i-1}-\nu_{i-1}\vartheta_{i-1,i}-\sum_{m=i+1}^{M+N}(\nu_{i-1}\vartheta_{i-1,m}-\nu_i\vartheta_{i,m})},\\
[e_{i,i'},f^1_{j,j'}] &= \delta_{i,j}\delta_{i',j'}\nu_i[\nu_i\vartheta_{i',i}-\nu_{i+1}\vartheta_{i',i+1}]q^{-\sum_{\ell=1}^{i'-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})+\sum_{\ell=i'+1}^{i-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})}\\
&\qquad \times q^{-\lambda_i+(\nu_i+\nu_{i+1})\vartheta_{i,i+1}+\sum_{\ell=i+2}^{M+N}(\nu_i\vartheta_{i,\ell}-\nu_{i+1}\vartheta_{i+1,\ell})}, &(28)\\
[e_{i,i'},f^3_{j,j'}] &= x_{ji}\frac{1}{x_{j+1,i+1}}[\vartheta_{j+1,i+1}]\Big(\delta_{i',j}\delta_{j',i+1}-\delta_{i',j+1}\delta_{j',i}q^{(\nu_i-\nu_j)\vartheta_{j,i}}\Big)\\
&\qquad \times q^{-\sum_{\ell=1}^{j-1}(\nu_{i+1}\vartheta_{\ell,i+1}-\nu_i\vartheta_{\ell,i})+\lambda_j-\nu_{i+1}\vartheta_{j,i+1}-\sum_{m=i+1}^{M+N}(\nu_j\vartheta_{j,m}-\nu_{j+1}\vartheta_{j+1,m})}. &(29)
\end{aligned}$$

Using these results and the formula

$$[a]q^{\sum_{i=1}^n b_i} + \sum_{i=1}^n [b_i] q^{-a+\sum_{j=1}^{i-1} b_j - \sum_{j=i+1}^n b_j} = [a + \sum_{i=1}^n b_i], \tag{30}$$

we have

$$[e_{i,i}, f^2_{j,j}] + \epsilon \sum_{i'=1}^{i-1} \sum_{j'=1}^{j-1} [e_{i,i'}, f^1_{j,j'}] - \epsilon' \sum_{i'=1}^{i-1} \sum_{j'=j+2}^{M+N} [e_{i,i'}, f^3_{j,j'}] = \delta_{i,j}[h_i], \tag{31}$$

where $\epsilon = \nu_i, \nu_j$ and $\epsilon' = \nu_i, \nu_{j+1}$ (four choices). ∎

Remark 1 (ii) is obtained from (i) by replacement $x_{i,j} \to \nu_j x_{i,j}$ and $e_i^\pm \to \nu_{i+1} e_i^\pm$.

Remark 2 Recently Kimura obtained a similar result[10]. He uses the following $f'^2_{j,j}$ instead of $f^2_{j,j}$,

$$f'^2_{j,j} = x_{j,j+1}\Big[\lambda_j - \tfrac{1}{2}(\nu_j + \nu_{j+1})\vartheta_{j,j+1} - \sum_{m=j+2}^{M+N} (\nu_j \vartheta_{j,m} - \nu_{j+1}\vartheta_{j+1,m})\Big]. \tag{32}$$

It can be checked that $f'^2_{j,j} = f^2_{j,j}$. Therefore Proposition 3 in ref.[10] agrees with our (ii) (Note that $x_{ij}^{\text{Kimura}} = x_{i,j+1}$ and we have replaced q by q^{-1} such that the realization for $N = 0$ case, $U_q(sl(M|0)) = U_q(sl_M)$, reduces to the result in [1]).

4 Free boson realization of $U_q(\widehat{sl}(2|1))$

In ref.[2], on the bases of a q-difference realization of $U_q(sl_N)$, a free boson realization of $U_q(\widehat{sl}_N)$ was obtained. Here we try to generalize it to the $U_q(\widehat{sl}(M|N))$.

Let us introduce oscillators and coordinates a^i_n, Q^i_a ($n \in \mathbb{Z}$, $1 \le i \le M+N-1$), b^{ij}_n, Q^{ij}_b and c^{ij}_n, Q^{ij}_c ($n \in \mathbb{Z}$, $1 \le i < j \le M+N$), which are all Grassmann even operators and satisfy the following relations,

$$[a^i_n, a^j_m] = \frac{1}{n}[(k+g)n][a_{ij}n]\delta_{n+m,0}, \quad [a^i_n, Q^j_a] = (k+g)a_{ij}\delta_{n0}, \tag{33}$$

$$[b^{ij}_n, b^{i'j'}_m] = -\nu_i\nu_j \frac{1}{n}[n]^2 \delta^{ii'}\delta^{jj'}\delta_{n+m,0}, \quad [b^{ij}_n, Q^{i'j'}_b] = -\nu_i\nu_j\delta^{ii'}\delta^{jj'}\delta_{n0}, \tag{34}$$

$$[c^{ij}_n, c^{i'j'}_m] = \frac{1}{n}[n]^2\delta^{ii'}\delta^{jj'}\delta_{n+m,0}, \quad [c^{ij}_n, Q^{i'j'}_c] = \delta^{ii'}\delta^{jj'}\delta_{n0}, \tag{35}$$

and other commutators vanish. Here $g = M - N$ and k is a complex parameter. For a pair of oscillator and coordinate (a_n, Q), we define boson fields $a(z)$ and $a_\pm(z)$ as follows:

$$a(z) = -\sum_{n\neq 0} \frac{a_n}{[n]} z^{-n} + Q + a_0 \log z, \tag{36}$$

$$a_\pm(z) = \pm\Big((q - q^{-1}) \sum_{\pm n > 0} a_n z^{-n} + a_0 \log q\Big). \tag{37}$$

Normal ordering $:\ \ :$ is defined as follows; move a_n $(n \geq 0)$ to right, a_n $(n < 0)$ and Q to left, and arrange the zero modes $e^{\pm Q_b^{ij}}$ with $\nu_i \nu_j = -1$ in lexicographic ordering for (ij). It is well known in CFT that when we bosonize many fermions we have to introduce cocycle factors such that different fermions anticommute. In our case cocycle factors are needed for $:e^{\pm b^{ij}(z)}:$ with $\nu_i\nu_j = -1$. But we suppress them by modifying the commutation relation for Q_b^{ij} with $\nu_i\nu_j = -1$; set $[Q_b^{ij}, Q_b^{i'j'}] = i\pi$, then we have a minus sign by interchanging operators $e^{\pm Q_b^{ij}}$, $e^{\epsilon Q_b^{ij}} e^{\epsilon' Q_b^{i'j'}} = -e^{\epsilon' Q_b^{i'j'}} e^{\epsilon Q_b^{ij}}$ $(\nu_i\nu_j = \nu_{i'}\nu_{j'} = -1,\ (i,j) \neq (i',j'),\ \epsilon, \epsilon' = \pm 1)$. For example, for $(M,N) = (2,1)$, $:e^{Q_b^{23}} e^{Q_b^{13}}: \ = \ :e^{Q_b^{13}} e^{Q_b^{23}}: \ = e^{Q_b^{23}} e^{Q_b^{13}} = -e^{Q_b^{13}} e^{Q_b^{23}}$.

In the following we restrict ourselves to $(M,N) = (2,1)$ case. Let us define fields $H^i(z)$, $E^i(z)$ and $F^i(z)$ $(i = 1,2)$ as follows:

$$
\begin{aligned}
H^1(z) &= \frac{1}{(q-q^{-1})z}\Big(a_+^1(q^{\frac{1}{2}}z) + b_+^{12}(q^{\frac{k}{2}}z) + b_+^{12}(q^{\frac{k}{2}+2}z) + b_+^{13}(q^{\frac{k}{2}+2}z) - b_+^{23}(q^{\frac{k}{2}+1}z) \\
&\quad - a_-^1(q^{-\frac{1}{2}}z) - b_-^{12}(q^{-\frac{k}{2}}z) - b_-^{12}(q^{-\frac{k}{2}-2}z) - b_-^{13}(q^{-\frac{k}{2}-2}z) + b_-^{23}(q^{-\frac{k}{2}-1}z)\Big), \qquad (38) \\
H^2(z) &= \frac{1}{(q-q^{-1})z}\Big(a_+^2(q^{\frac{1}{2}}z) - b_+^{12}(q^{\frac{k}{2}+1}z) - b_+^{13}(q^{\frac{k}{2}+1}z) \\
&\quad - a_-^2(q^{-\frac{1}{2}}z) + b_-^{12}(q^{-\frac{k}{2}-1}z) + b_-^{13}(q^{-\frac{k}{2}-1}z)\Big), \qquad (39) \\
E^1(z) &= \frac{-1}{(q-q^{-1})z} : \Big(e_{11} e^{b_+^{12}(z) - (b+c)^{12}(qz)} - e_{12} e^{b_-^{12}(z) - (b+c)^{12}(q^{-1}z)}\Big) :, \qquad (40) \\
E^2(z) &= : \Big(e_{21} e^{-b_+^{12}(qz) - b_+^{13}(qz) + b^{23}(qz)} + e_{22} e^{(b+c)^{12}(z) + b^{13}(z)}\Big) :, \qquad (41) \\
F^1(z) &= \frac{1}{(q-q^{-1})z} : \Big(f_{11} e^{a_+^1(q^{\frac{k+1}{2}}z) + b_+^{12}(q^{k+2}z) + b_+^{13}(q^{k+2}z) - b_+^{23}(q^{k+1}z) + (b+c)^{12}(q^{k+1}z)} \\
&\quad - f_{12} e^{a_-^1(q^{-\frac{k+1}{2}}z) + b_-^{12}(q^{-k-2}z) + b_-^{13}(q^{-k-2}z) - b_-^{23}(q^{-k-1}z) + (b+c)^{12}(q^{-k-1}z)}\Big) : \qquad (42) \\
&\quad + f_{13} : e^{a_+^1(q^{\frac{k+1}{2}}z) - b_+^{23}(q^{k+1}z) - b^{13}(q^{k+1}z) + b^{23}(q^{k+2}z)} :, \\
F^2(z) &= \frac{1}{(q-q^{-1})z} : \Big(f_{21} e^{a_+^2(q^{\frac{k+1}{2}}z) - b^{23}(q^{k+1}z)} - f_{22} e^{a_-^2(q^{-\frac{k+1}{2}}z) - b^{23}(q^{-k-1}z)} \\
&\quad - f_{23} e^{a_-^2(q^{-\frac{k+1}{2}}z) - b_-^{12}(q^{-k-1}z) - b_-^{13}(q^{-k-1}z) - (b+c)^{12}(q^{-k}z) - b^{13}(q^{-k}z)} \qquad (43) \\
&\quad + f_{24} e^{a_-^2(q^{-\frac{k+1}{2}}z) - b_+^{12}(q^{-k-1}z) - b_-^{13}(q^{-k-1}z) - (b+c)^{12}(q^{-k-2}z) - b^{13}(q^{-k}z)}\Big) :,
\end{aligned}
$$

where $e_{11}, e_{12}, e_{21}, e_{22},\ f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{24}$ are some constants. From (38)(39), we have

$$
\begin{aligned}
H_n^1 &= a_n^1 q^{-\frac{1}{2}|n|} + b_n^{12} q^{-(\frac{k}{2}+1)|n|}(q^{|n|} + q^{-|n|}) + b_n^{13} q^{-(\frac{k}{2}+2)|n|} - b_n^{23} q^{-(\frac{k}{2}+1)|n|}, \qquad (44) \\
H_n^2 &= a_n^2 q^{-\frac{1}{2}|n|} - b_n^{12} q^{-(\frac{k}{2}+1)|n|} - b_n^{13} q^{-(\frac{k}{2}+1)|n|}, \qquad (45) \\
K_1 &= q^{a_0^1 + 2b_0^{12} + b_0^{13} - b_0^{23}}, \quad K_2 = q^{a_0^2 - b_0^{12} - b_0^{13}}, \qquad (46) \\
\psi_\pm^1(q^{\pm\frac{k}{2}}z) &= e^{a_\pm^1(q^{\pm\frac{k+1}{2}}z) + b_\pm^{12}(q^{\pm k}z) + b_\pm^{12}(q^{\pm(k+2)}z) + b_\pm^{13}(q^{\pm(k+2)}z) - b_\pm^{23}(q^{\pm(k+1)}z)}, \qquad (47) \\
\psi_\pm^2(q^{\pm\frac{k}{2}}z) &= e^{a_\pm^2(q^{\pm\frac{k+1}{2}}z) - b_\pm^{12}(q^{\pm(k+1)}z) - b_\pm^{13}(q^{\pm(k+1)}z)}. \qquad (48)
\end{aligned}
$$

Proposition 2 *If $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{24}$ satisfy*

$$(f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{24}) = (\frac{1}{e_{11}}, \frac{1}{e_{12}}, \frac{q^{k+1}e_{21}}{e_{11}e_{22}}, \frac{q}{e_{21}}, \frac{qe_{12}}{e_{11}e_{21}}, \frac{1}{e_{22}}, \frac{e_{12}}{e_{11}e_{22}}),$$

then eqs. (38)-(43) realize $U_q(\widehat{sl}(2|1))$ with $\gamma = q^k$.

Proof: Straightforward OPE calculation shows this proposition. For useful formulas, see ref.[2] (some modifications are needed for b^{ij} with $\nu_i\nu_j = -1$.). ∎

Remark Choices $e_{11} = e_{12} = e_{21} = e_{22} = 1$ may be important. Two of them, e.g. e_{11} and e_{21}, is just a normalization of F^i's. When $e_{11} = e_{22}$, screening currents exist.

At present it is not clear that what choice of $e_{11}, e_{12}, e_{21}, e_{22}$ is most natural. Perhaps it will be clarified by finding general formula for $U_q(\widehat{sl}(M|N))$. This problem is now under investigation. For $M + N \leq 3$ realizations have been already obtained. The Cartan part $H^i(z)$ for general M and N has also been obtained[9] and we hope that we will be able to report a full answer.

Acknowledgments

We would like to thank J. Harvey, H. Kubo, E. Martinec, J. Uchiyama and H. Yamane for valuable discussion. We also would like to thank members of Nankai Institute for their hospitality since some of the results in section 3 were obtained during our stay in Tianjin 19-24 August 1996. This work is supported in part by Grant-in-Aid for Scientific Research from Ministry of Science and Culture.

References

[1] H. Awata, M. Noumi and S. Odake, "Heisenberg realization for $U_q(sl_n)$ on the flag manifold", *Lett. Math. Phys.* **30** (1994) 35-43.

[2] H. Awata, S. Odake and J. Shiraishi, "Free Boson Representation of $U_q(\widehat{sl}_3)$",*Lett. Math. Phys.* **30** (1994) 207-216; "Free Boson Realization of $U_q(\widehat{sl}_N)$",*Comm. Math. Phys.* **162** (1994) 61-83.

[3] For example, P. Bouwknegt, J. McCarthy and K. Pilch, "Free Field Approach to 2-Dimensional Conformal Field Theories", *Prog. Theor. Phys. Suppl.* **102** (1990) 67-135.

[4] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, "Diagonalization of the XXZ Hamiltonian by Vertex Operators", *Comm. Math. Phys.* **151** (1993) 89-154;

M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, "Correlation Functions of the XXZ Model for $\Delta < -1$", *Phys. Lett.* **A168** (1992) 256-263.

[5] K. Kimura, J. Shiraishi and J. Uchiyama, "A Level-one Representation of the Quantum Affine Superalgebra $U_q(\widehat{sl}(M+1|N+1))$", preprint q-alg/9605047.

[6] S. Lukyanov and Y. Pugai, "Multi-point Local Height Probabilities in the Integrable RSOS Model", *Nucl. Phys.* **B473** (1996) 631-658.

[7] J. Shiraishi, H. Kubo, H. Awata and S. Odake, "A Quantum Deformation of the Virasoro Algebra and the Macdonald Symmetric Functions", *Lett. Math. Phys.* **38** (1996) 33-51; H. Awata, H. Kubo, S. Odake and J. Shiraishi, "Quantum $\mathcal{W}_N$ Algebras and Macdonald Polynomials", *Comm. Math. Phys.* **179** (1996) 401-416; B. Feigin and E. Frenkel, "Quantum $\mathcal{W}$-Algebras and Elliptic Algebras", *Comm. Math. Phys.* **178** (1996) 653-678.

[8] H. Yamane, "On defining relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras", preprint q-alg/9603015.

[9] H. Awata, S. Odake, J. Shiraishi and J. Uchiyama.

[10] K. Kimura, "q-Differential Operator Representation of the Quantum Superalgebra $U_q(sl(M+1|N+1))$", preprint OIT-96-1, q-alg/9612036.