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Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$: Drinfeld currents and vertex operators

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Abstract

We investigate the structure of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ introduced earlier by one of the authors. Our construction is based on a new set of generating series in the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, which are elliptic analogs of the Drinfeld currents. They enable us to identify $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and a Heisenberg algebra generated by P, Q with $[Q,P]=1$. In terms of these currents, we construct an L operator satisfying the dynamical RLL relation in the presence of the central element c. The vertex operators of Lukyanov and Pugai arise as 'intertwiners' of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ for the level one representation, in the sense to be elaborated on in the text. We also present vertex operators with higher level/spin in the free field representation.

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1 Introduction

1.1 Vertex operators in SOS models

The principle of infinite dimensional symmetry has seen an impressive success in conformal field theory (CFT). With the aim of understanding non-critical lattice models in the same spirit, the method of algebraic analysis [1, 2, 3] has been developed. In this approach, a central role is played by the notion of vertex operators (VO's). There are two kinds of VO's with distinct physical significance: the type I VO, which describes the operation of adding one lattice site, and the type II VO, which plays the role of particle creation/annihilation operators. In the most typical example of the XXZ spin chain, these VO's have a clear mathematical meaning as intertwiners [4] of certain modules over the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

An important class of CFT is the minimal unitary series [5]. Their lattice counterpart are the solvable models of Andrews-Baxter-Forrester (ABF) [6]. These are 'solid-on-solid' (SOS, or 'face') models whose Boltzmann weights are expressed by elliptic functions. Their Lie theoretic generalizations have also been studied extensively [7, 8, 9]. The vertex operator approach to the ABF models and their fusion hierarchy was formulated in [10] by a coset-type construction. In [10], $U_q(\widehat{\mathfrak{sl}}_2)$ was used only as an auxiliary tool to define the VO's, and its role as a symmetry algebra was somewhat indirect. In [11], Lukyanov and Pugai constructed a free boson realization of type I VO's for the ABF models. (The formulas for type II VO's can be found in [12].) They have shown further that these VO's commute with the action of the deformed Virasoro algebra (DVA) [13], making clear the parallelism with CFT. However, unlike the case of CFT, the VO's did not allow for direct interpretation as intertwiners, because DVA lacks a coproduct[1)]. It has remained an open problem to understand the conceptual meaning of VO's.

In [14], one of the authors introduced an elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and proposed it as an algebra of screening currents of conjectural extended DVA associated with the fusion SOS models. The aim of the present article is to continue the study of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, and to show that it offers a characterization of the VO's for SOS models in close analogy with the XXZ model.

1.2 Face type elliptic algebras

Through an attempt to understand integrable models based on elliptic Boltzmann weights, various versions of 'elliptic quantum groups' [15, 16, 17, 18, 19] have been introduced. According to Frønsdal [18, 19], elliptic quantum groups are nothing but quantum affine

[1)]The usual coproduct for the Virasoro algebra has no non-trivial deformation.

algebras $U_q(\mathfrak{g})$ equipped with a coproduct different from the original one. The resulting objects are quasi-Hopf algebras in the sense of Drinfeld [20]. Throughout this paper, we restrict our attention to the elliptic algebra of face type associated with $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, denoted as $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ in [21].

In Frønsdal's approach, the quasi-Hopf structures are defined by twistors given as formal series in the deformation parameters. An explicit construction for the twistors was given in [21]. (A very similar construction was presented independently in [22].) The L-operators and the VO's for the elliptic algebra can be obtained by 'dressing' those of $U_q(\widehat{\mathfrak{sl}}_2)$ with the twistor (up to some subtleties about the fractional powers which will be discussed shortly). From this point of view, the construction of the VO's in bosonic representations is reduced to the determination of the image of the twistors. However, the solution of this issue is not known to us at this moment.

For the bosonic realization of quantum affine algebras, the best suited presentation is in terms of the Drinfeld currents. In this paper we aim at an alternative construction of L operators and VO's based on an elliptic analog of Drinfeld currents. These operators satisfy the same relations as those derived from the quasi-Hopf approach [21]. Though the precise relation is not known, we expect that these two methods give equivalent answers. Our construction is inspired by the work of Enriquez and Felder [17], who introduced Drinfeld-type currents defined on an elliptic curve and constructed the twistor by a quantum factorization method. The algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in [14] and $U_\hbar\mathfrak{g}(\tau)$ in [17] are both central extensions of the same algebra, but there are significant differences. We shall discuss more about this in section 6.2.

1.3 Outline of the results

Let us describe the content of this paper. Our starting point is to introduce a new set of currents of $U_q(\widehat{\mathfrak{sl}}_2)$ carrying a parameter r (the elliptic modulus), obtained by modifying the usual Drinfeld currents. We shall refer to them as 'elliptic currents'. They satisfy commutation relations with coefficients written in infinite products. The latter are essentially the Jacobi theta functions but not quite so, since the elliptic currents, and hence these coefficients, comprise only integral powers in the Fourier mode expansions. In order to have relations written in theta functions alone, we need to supply fractional powers. For this purpose we introduce 'by hand' a pair of generators P, Q which commute with $U_q(\widehat{\mathfrak{sl}}_2)$ and satisfy $[Q, P] = 1$. Adjoining P, Q to the elliptic currents, we obtain 'total currents' whose commutation relations coincide with the defining relations of the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [14] (see (3.34)-(3.43)). In other words, we can identify $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and the Heisenberg algebra generated by P, Q. The algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ mentioned above is the subalgebra of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$, and is equipped with a coproduct defined via the twistor. However, this coproduct does *not* seem to extend naturally to the full algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. That is, $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ is a quasi-Hopf algebra while (to our knowledge) $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is not. We emphasize that the intertwining relations for VO's will be based on the quasi-Hopf structure of the former.

A characteristic feature of the elliptic algebras is that, in the presence of the central element c, we are forced to deal with two different elliptic moduli r and $r^* = r - c$ simultaneously [15]. From Frønsdal's point of view, it is an effect of the quasi-Hopf twisting. The appearance of two different curves makes it difficult to apply the geometric method of [17]. Instead, we take a more pedestrian approach. Motivated by similar formulas in [17], we introduce 'half currents' as certain contour integrals of the total currents. They have an advantage that the coefficients of the commutation relations can be written solely in terms of theta functions (as opposed to the delta functions appearing in the relations for the total currents). We then borrow the idea of the Gauß decomposition [23] to compose an L operator out of the half currents, and show that it satisfies the expected (dynamical) RLL relation (Proposition 4.4, 4.5).

The construction of the L operator allows us to study the VO's in the bosonic representation. Let us first consider $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. As is clear from the construction, the elliptic currents can be realized in the same bosonic Fock spaces as with the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_2)$. (We regard $p = q^{2r}$ as a formal parameter.) The VO's for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ are a family of intertwiners $\Phi(z,s)$, $\Psi^*(z,s)$ carrying a parameter s, and their intertwining relations involve a shift of s (see (2.9)-(2.10)). With the adjunction of P, Q, the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ has an enlarged Fock module. It has the decomposition $\mathcal{F} = \bigoplus_s \mathcal{F}_s$ into eigenspaces of P, each eigenspace $\mathcal{F}_s$ being a Fock module for $U_q(\widehat{\mathfrak{sl}}_2)$. Accordingly we modify further the VO's with Q,

$$\widehat{\Phi}(u,s) = z^{\frac{1}{2r}(\frac{1}{2}h^{(2)2}+(s+h^{(1)})h^{(2)})}\Phi(q^c z, s), \tag{1.1}$$

$$\widehat{\Psi}^*(u,s) = \Psi^*(z,s) z^{-\frac{1}{2r^*}(\frac{1}{2}h^{(1)2}+sh^{(1)})} e^{Qh^{(1)}}, \tag{1.2}$$

where $z = q^{2u}$, $h^{(1)} = h \otimes 1, h^{(2)} = 1 \otimes h$, h being the 'Cartan' generator of $U_q(\widehat{\mathfrak{sl}}_2)$. Solving the intertwining relations for level one, we find that the VO's of Lukyanov and Pugai arise in the form (1.1), (1.2), apart from certain signs in the intertwining relations for $\Phi(z,s)$ and $\Psi^*(z,s)$. (For the discussion of the signs, see subsections 5.2 and 5.4.) We also calculate formulas for VO's associated with higher spin representations.

1.4 Plan of the text

The text is organized as follows.

In section 2, we recall some results of [21] which are relevant to the following sections. In Section3, we introduce the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_2)$, and discuss its relation to $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

In section 4, we introduce the 'half currents' and derive their commutation relations. We then arrange them in the form of a Gauß decomposition to define the L operator. In section 5, we describe the VO's in the bosonic representation. In Section 6 we discuss the connection to other works and mention some open problems. The text is followed by four appendices. In appendix A, we give the elliptic currents for the general non-twisted affine Lie algebra $\mathfrak{g}$. In Appendix B we discuss an elliptic analog of the Drinfeld coproduct, and show that it also arises as a quasi-Hopf twist from the usual Drinfeld coproduct. In Appendix C we study the evaluation modules and R matrix in the spin $l/2$ representation. Finally, in Appendix D we review the free field realization of the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

While preparing this manuscript, we became aware of the paper by Hou et al. [24] which has some overlap with the content of the present paper.

2 RLL and intertwining relations

The purpose of this section is to set up the form of the RLL and intertwining relations which we are going to study.

2.1 Previous results

In order to fix the notation, let us recall the results of [21] relevant to the present paper. We consider the quantum affine algebra $U_q = U_q(\widehat{\mathfrak{sl}}_2)$ with standard generators e_i, f_i, h_i $(i = 0, 1)$ and d. The canonical central element is $c = h_0 + h_1$. We retain the convention of [21] for the coproduct Δ, though the details are not necessary here.

Henceforth we shall write $h = h_1$. In [21], we have constructed a twistor $F(\lambda) \in U_q^{\otimes 2}$. Changing slightly the notation, let us set $\lambda = (r^* + 2)d + (s+1)\frac{1}{2}h$ and write $F(\lambda)$ as $F(r^*, s)$. Then $F(r^*, s)$ is a formal power series in $q^{2(r^*-s)}$ and q^{2s}, satisfying the shifted cocycle condition

$$F^{(12)}(r^*, s)\,(\Delta \otimes \mathrm{id})\,F(r^*, s) = F^{(23)}(r^* + c^{(1)}, s + h^{(1)})\,(\mathrm{id} \otimes \Delta)\,F(r^*, s). \tag{2.1}$$

We obtain the quasi-Hopf algebra $\mathcal{B}_{q,\lambda} = \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ by twisting U_q via this F. Here and after, the superscripts refer to the tensor components; for instance, $F^{(23)} = 1 \otimes F$, $h^{(1)} = h \otimes 1 \otimes 1$. Let $\mathcal{R}$ be the universal R matrix of U_q. The 'dressed' R matrix $\mathcal{R}(r^*, s) = F^{(21)}(r^*, s)\mathcal{R}F^{(12)}(r^*, s)^{-1}$ of $\mathcal{B}_{q,\lambda}$ satisfies the dynamical YBE,

$$\begin{aligned}&\mathcal{R}^{(12)}(r^* + c^{(3)}, s + h^{(3)})\mathcal{R}^{(13)}(r^*, s)\mathcal{R}^{(23)}(r^* + c^{(1)}, s + h^{(1)})\\ &\quad = \mathcal{R}^{(23)}(r^*, s)\mathcal{R}^{(13)}(r^* + c^{(2)}, s + h^{(2)})\mathcal{R}^{(12)}(r^*, s).\end{aligned} \tag{2.2}$$

Let (π_V, V) be a finite dimensional U_q'-module where U_q' is the subalgebra generated by e_i, f_i, h_i $(i = 0, 1)$. Let $(\pi_{V,z}, V_z)$ denote the evaluation module

$$\pi_{V,z}(a) = \pi_V \circ \mathrm{Ad}(z^d)(a) \quad (a \in U_q'), \qquad \pi_{V,z}(d) = z\frac{d}{dz}, \qquad V_z = V[z, z^{-1}].$$

Setting

$$R^+_{VW}(z_1/z_2, r^*, s) = (\pi_{V,z_1} \otimes \pi_{W,z_2})\,\mathcal{R}(r^*, s), \tag{2.3}$$

$$L^+_V(z, r^*, s) = (\pi_{V,z} \otimes \mathrm{id})\, q^{c\otimes d + d\otimes c}\mathcal{R}(r^*, s), \tag{2.4}$$

we have the dynamical RLL relation for $\mathcal{B}_{q,\lambda}$,

$$\begin{aligned} &R^{+(12)}_{VW}(z_1/z_2, r^* + c, s + h)L^{+(1)}_V(z_1, r^*, s)L^{+(2)}_W(z_2, r^*, s + h^{(1)}) \\ &\quad = L^{+(2)}_W(z_2, r^*, s)L^{+(1)}_V(z_1, r^*, s + h^{(2)})R^{+(12)}_{VW}(z_1/z_2, r^*, s). \end{aligned} \tag{2.5}$$

Hereafter we shall write

$$r = r^* + c, \qquad R^+_{VW}(z, s) = R^+_{VW}(z, r, s), \quad R^{*+}_{VW}(z, s) = R^+_{VW}(z, r^*, s), \tag{2.6}$$

and normally suppress the r^*-dependence. In this paper we will not consider the L^- operator since it can be obtained from L^+, see Proposition 4.3 in [21].

Let now $\mathcal{F}, \mathcal{F}'$ be highest weight U_q-modules on which c acts as a scalar k. Suppose we have intertwiners of U_q-modules

$$\Phi_V(z) : \mathcal{F} \longrightarrow \mathcal{F}' \otimes V_z,$$

$$\Psi^*_V(z) : V_z \otimes \mathcal{F} \longrightarrow \mathcal{F}',$$

which we refer to as vertex operators (VO's) of type I and type II, respectively. Then the 'dressed' VO's for $\mathcal{B}_{q,\lambda}$

$$\Phi_V(z, s) = (\mathrm{id} \otimes \pi_{V,z})\, F(r^*, s) \circ \Phi_V(z), \tag{2.7}$$

$$\Psi^*_V(z, s) = \Psi^*_V(z) \circ (\pi_{V,z} \otimes \mathrm{id})\, F(r^*, s)^{-1} \tag{2.8}$$

satisfy the following intertwining relations with the L operators:

$$\Phi_W(q^k z_2, s)L^+_V(z_1, s) = R^+_{VW}(z_1/z_2, s + h)L^+_V(z_1, s)\Phi_W(q^k z_2, s + h^{(1)}), \tag{2.9}$$

$$L^+_V(z_1, s)\Psi^*_W(z_2, s + h^{(1)}) = \Psi^*_W(z_2, s)L^+_V(z_1, s + h^{(2)})R^{*+}_{VW}(z_1/z_2, s). \tag{2.10}$$

We note that all the operators (2.3), (2.4), (2.7), (2.8) are formal Laurent series comprising only integral powers of z.

Remark. In the present paper, we shall adopt the universal R matrix $\mathcal{R} = \mathcal{R}'^{(21)^{-1}}$, where $\mathcal{R}'$ is given in (2.8) of [21]. This is purely a matter of convention. The properties (2.11)-(2.14) holds equally well for $\mathcal{R}$ and $\mathcal{R}'$, and hence the same construction applies.

2.2 Fractional powers

The RLL-relation (2.5) is unchanged under the transformation of the form

$$L^{+'}_V(z,s)=\mu_V(s,h^{(1)})\mu(s+h^{(1)},h)L^+_V(z,s)\mu^*_V(s+h^{(2)},h^{(1)})^{-1}\mu(s,h)^{-1}, \quad (2.11)$$

$$R'^{+}_{VW}(z_1/z_2,s)=\mu_V(s,h^{(1)})\mu_W(s+h^{(1)},h^{(2)})$$
$$\times R^+_{VW}(z_1/z_2,s)\mu_V(s+h^{(2)},h^{(1)})\mu_W(s,h^{(2)}). \quad (2.12)$$

Here $\mu_V(s,h)$, $\mu_W(s,h)$, $\mu(s,h)$ are functions possibly depending on z and r, and $\mu^*_V(s,h)$ means $\mu_V(s,h)|_{r\to r^*}$. This corresponds to the freedom of changing the twistor by 'shifted coboundary'. Exploiting this freedom, we modify the R matrix by a fractional power of z so that it can be expressed in terms of the Jacobi theta functions.

Consider the image of the R matrix in the evaluation modules $(\pi_{l,z},V_{l,z})$ of spin $l/2$ (see Appendix C)

$$R^+_{lm}(z_1/z_2,s)=(\pi_{l,z_1}\otimes\pi_{m,z_2})\,\mathcal{R}(r,s).$$

In Appendix C, we show that it has the form $R^+_{lm}(z,s)=\rho^+_{lm}(z,p)\overline{R}_{lm}(z,s)$, where $\rho^+_{lm}(z,p)$ is a scalar factor given in (C.10) and $\overline{R}_{lm}(z,s)$ has the transformation property (C.11). Write $z=q^{2u}$ and set

$$\widetilde{R}^+_{lm}(u,s)=z^{\frac{1}{2r}(\frac{1}{2}h^{(1)2}+(s+h^{(2)})h^{(1)})}R^+_{lm}(z,s)z^{-\frac{1}{2r}(\frac{1}{2}h^{(1)2}+sh^{(1)})}. \quad (2.13)$$

The R matrix (2.13) comprises fractional powers of z, but (up to a scalar factor) becomes completely periodic,

$$\frac{1}{\rho^+_{lm}(pz,p)}\widetilde{R}^+_{lm}(u+r,s)=\frac{1}{\rho^+_{lm}(z,p)}\widetilde{R}^+_{lm}(u,s).$$

It turns out that the R matrix (2.13) is expressible in terms of the Jacobi theta functions. An explicit expression for the case $l=m=1$ will be given in (4.18) below.

In accordance with (2.13), we modify the L operator and VO's as

$$\widetilde{L}^+_m(u,s)=z^{\frac{1}{2r}(\frac{1}{2}h^{(1)2}+(s+h)h^{(1)})}L^+_{V_m}(z,s)z^{-\frac{1}{2r^*}(\frac{1}{2}h^{(1)2}+sh^{(1)})}, \quad (2.14)$$

$$\widetilde{\Phi}_l(u,s)=z^{\frac{1}{2r}(\frac{1}{2}h^{(2)2}+(s+h^{(1)})h^{(2)})}\Phi_{V_l}(q^cz,s), \quad (2.15)$$

$$\widetilde{\Psi}^*_l(u,s)=\Psi^*_{V_l}(z,s)z^{-\frac{1}{2r^*}(\frac{1}{2}h^{(1)2}+sh^{(1)})}. \quad (2.16)$$

We shall focus attention to the L operators $\widetilde{L}^+(u,s)=\widetilde{L}^+_1(u,s)$ associated with the spin 1/2 representation. The following are consequences of (2.5), (2.9), (2.10):

$$\widetilde{R}^{+(12)}_{11}(u_1-u_2,s+h)\widetilde{L}^{+(1)}(u_1,s)\widetilde{L}^{+(2)}(u_2,s+h^{(1)})$$
$$=\widetilde{L}^{+(2)}(u_2,s)\widetilde{L}^{+(1)}(u_1,s+h^{(2)})\widetilde{R}^{*+(12)}_{11}(u_1-u_2,s), \quad (2.17)$$

$$\widetilde{\Phi}_l(u_2,s)\widetilde{L}^+(u_1,s)=\widetilde{R}^+_{1l}(u_1-u_2,s+h)\widetilde{L}^+(u_1,s)\widetilde{\Phi}_l(u_2,s+h^{(1)}), \quad (2.18)$$

$$\widetilde{L}^+(u_1,s)\widetilde{\Psi}^*_l(u_2,s+h^{(1)})=\widetilde{\Psi}^*_l(u_2,s)\widetilde{L}^+(u_1,s+h^{(2)})\widetilde{R}^{*+}_{1l}(u_1-u_2,s). \quad (2.19)$$

We shall study the relations (2.17)–(2.19) in the following sections.

3 Elliptic currents and $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section, we introduce Drinfeld-type currents of $U_q = U_q(\widehat{\mathfrak{sl}}_2)$ satisfying 'elliptic' commutation relations. We then relate them to the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ of [14] by adjoining a pair of generators P, Q with $[Q, P] = 1$.

3.1 Elliptic currents of $U_q(\widehat{\mathfrak{sl}}_2)$

First let us recall the Drinfeld currents of U_q [25]. Hereafter we fix a complex number $q \neq 0$, $|q| < 1$. We use the standard symbols

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Let $x_n^{\pm}$ $(n \in \mathbb{Z})$, a_n $(n \in \mathbb{Z}_{\neq 0})$, h, c, d denote the Drinfeld generators of U_q. In terms of the generating functions

$$x^{\pm}(z) = \sum_{n \in \mathbb{Z}} x_n^{\pm} z^{-n}, \tag{3.1}$$

$$\psi(q^{c/2}z) = q^h \exp\left((q - q^{-1}) \sum_{n>0} a_n z^{-n}\right), \tag{3.2}$$

$$\varphi(q^{-c/2}z) = q^{-h} \exp\left(-(q - q^{-1}) \sum_{n>0} a_{-n} z^{n}\right), \tag{3.3}$$

the defining relations read as follows:

$$\begin{aligned}
&c : \text{ central },\\
&[h, d] = 0, \quad [d, a_n] = n a_n, \quad [d, x_n^{\pm}] = n x_n^{\pm},\\
&[h, a_n] = 0, \qquad [h, x^{\pm}(z)] = \pm 2 x^{\pm}(z),\\
&[a_n, a_m] = \frac{[2n][cn]}{n} q^{-c|n|} \delta_{n+m,0},\\
&[a_n, x^{+}(z)] = \frac{[2n]}{n} q^{-c|n|} z^n x^{+}(z),\\
&[a_n, x^{-}(z)] = -\frac{[2n]}{n} z^n x^{-}(z),\\
&(z - q^{\pm 2} w) x^{\pm}(z) x^{\pm}(w) = (q^{\pm 2} z - w) x^{\pm}(w) x^{\pm}(z),\\
&[x^{+}(z), x^{-}(w)] = \frac{1}{q - q^{-1}} \left(\delta(q^{-c}\frac{z}{w}) \psi(q^{c/2} w) - \delta(q^{c}\frac{z}{w}) \varphi(q^{-c/2} w)\right).
\end{aligned}$$

We now introduce a new parameter p and modify (3.1)-(3.3) to define another set of currents. For notational convenience, we will frequently write

$$p = q^{2r}, \qquad p^* = p q^{-2c} = q^{2r^*} \qquad (r^* = r - c).$$

Let us introduce two currents $u^{\pm}(z,p)\in U_q$ depending on p by

$$u^{+}(z,p)=\exp\left(\sum_{n>0}\frac{1}{[r^{*}n]}a_{-n}(q^{r}z)^{n}\right), \tag{3.4}$$

$$u^{-}(z,p)=\exp\left(-\sum_{n>0}\frac{1}{[rn]}a_{n}(q^{-r}z)^{-n}\right). \tag{3.5}$$

Definition 3.1 (Elliptic currents) *We define the currents $e(z,p)$, $f(z,p)$, $\psi^{\pm}(z,p)$ by*

$$e(z,p)=u^{+}(z,p)x^{+}(z), \tag{3.6}$$

$$f(z,p)=x^{-}(z)u^{-}(z,p), \tag{3.7}$$

$$\psi^{+}(z,p)=u^{+}(q^{c/2}z,p)\psi(z)u^{-}(q^{-c/2}z,p), \tag{3.8}$$

$$\psi^{-}(z,p)=u^{+}(q^{-c/2}z,p)\varphi(z)u^{-}(q^{c/2}z,p). \tag{3.9}$$

We will often drop p, and write $e(z,p)$ as $e(z)$ and so forth.

The merit of these currents is that they obey the following 'elliptic' commutation relations.

Proposition 3.2

$$\psi^{\pm}(z)\psi^{\pm}(w)=\frac{\Theta_p\left(q^{-2}z/w\right)}{\Theta_p\left(q^{2}z/w\right)}\frac{\Theta_{p^*}\left(q^{2}z/w\right)}{\Theta_{p^*}\left(q^{-2}z/w\right)}\psi^{\pm}(w)\psi^{\pm}(z), \tag{3.10}$$

$$\psi^{+}(z)\psi^{-}(w)=\frac{\Theta_p\left(pq^{-c-2}z/w\right)}{\Theta_p\left(pq^{-c+2}z/w\right)}\frac{\Theta_{p^*}\left(p^{*}q^{c+2}z/w\right)}{\Theta_{p^*}\left(p^{*}q^{c-2}z/w\right)}\psi^{-}(w)\psi^{+}(z), \tag{3.11}$$

$$\psi^{\pm}(z)e(w)\psi^{\pm}(z)^{-1}=q^{-2}\frac{\Theta_{p^*}(q^{\pm c/2+2}z/w)}{\Theta_{p^*}(q^{\pm c/2-2}z/w)}e(w), \tag{3.12}$$

$$\psi^{\pm}(z)f(w)\psi^{\pm}(z)^{-1}=q^{2}\frac{\Theta_{p}(q^{\mp c/2-2}z/w)}{\Theta_{p}(q^{\mp c/2+2}z/w)}f(w), \tag{3.13}$$

$$[e(z),f(w)]=\frac{1}{q-q^{-1}}\left(\delta(q^{-c}\frac{z}{w})\psi^{+}(q^{c/2}w)-\delta(q^{c}\frac{z}{w})\psi^{-}(q^{-c/2}w)\right). \tag{3.14}$$

Here we have used the standard symbols

$$\Theta_p(z)=(z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty,$$
$$(z;t_1,\cdots,t_k)_\infty=\prod_{n_1,\cdots,n_k\geq 0}(1-zt_1^{n_1}\cdots t_k^{n_k}).$$

It will become convenient later to consider also the current

$$k(z)=\exp\left(\sum_{n>0}\frac{[n]}{[2n][r^{*}n]}a_{-n}(q^{c}z)^{n}\right)\exp\left(-\sum_{n>0}\frac{[n]}{[2n][rn]}a_{n}z^{-n}\right). \tag{3.15}$$

The $\psi^{\pm}(z)$ are related to $k(z)$ by the formula

$$\psi^{\pm}(p^{\mp(r-\frac{c}{2})}z) = \kappa q^{\pm h} k(qz)k(q^{-1}z), \tag{3.16}$$

$$\kappa = \frac{\xi(z;p^*,q)}{\xi(z;p,q)}\bigg|_{z=q^{-2}}, \tag{3.17}$$

where the function

$$\xi(z;p,q) = \frac{(q^2z;p,q^4)_\infty(pq^2z;p,q^4)_\infty}{(q^4z;p,q^4)_\infty(pz;p,q^4)_\infty} \tag{3.18}$$

is a solution of the difference equation

$$\xi(z;p,q)\xi(q^2z;p,q) = \frac{(q^2z;p)_\infty}{(pz;p)_\infty}.$$

We have the commutation relations supplementing (3.10)-(3.14),

$$k(z)k(w) = \frac{\xi(w/z;p,q)}{\xi(w/z;p^*,q)}\frac{\xi(z/w;p^*,q)}{\xi(z/w;p,q)}k(w)k(z), \tag{3.19}$$

$$k(z)e(w)k(z)^{-1} = \frac{\Theta_{p^*}\left(p^{*1/2}qz/w\right)}{\Theta_{p^*}\left(p^{*1/2}q^{-1}z/w\right)}e(w), \tag{3.20}$$

$$k(z)f(w)k(z)^{-1} = \frac{\Theta_p\left(p^{1/2}q^{-1}z/w\right)}{\Theta_p\left(p^{1/2}qz/w\right)}f(w). \tag{3.21}$$

The commutation relations (3.10)-(3.14) have been proposed earlier in [26], but the direct connection with the usual Drinfeld currents was not known. In Appendix B we discuss also the Drinfeld type coproduct for the elliptic currents (3.6)-(3.9), (3.15).

Remark. Strictly speaking, the currents (3.4)-(3.9), (3.15) are generating series whose coefficients belong to a completion of $U_q(\widehat{\mathfrak{sl}}_2)\otimes\mathbb{C}[[p]]$. At this level, p should be treated as an indeterminate. However, in the concrete representations we are going to discuss, such as evaluation modules and Fock modules, these currents have also analytical meaning. (For a formula for the currents in spin $l/2$ evaluation modules, see Appendix C.) We will not go into this point any further, and later treat p, p^* as complex numbers satisfying $|p|, |p^*| < 1$.

3.2 Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in [14] is very similar to the algebra of the elliptic currents (3.10)-(3.14), (3.19)-(3.21). In the former, the coefficients of the relations are written in terms of the Jacobi elliptic theta function, which differs from $\Theta_p(z)$ used in the latter by a simple factor (see (3.22) below). Let us discuss the precise connection between the two algebras.

For this purpose, it is more convenient to work with the 'additive' notation. Following [14], we use the parameterization

$$
\begin{aligned}
&q = e^{-\pi i/r\tau},\\
&p = e^{-2\pi i/\tau}, \quad p^* = e^{-2\pi i/\tau^*} \qquad (r\tau = r^*\tau^*),\\
&z = q^{2u} = e^{-2\pi i u/r\tau}.
\end{aligned}
$$

We also use the Jacobi theta functions

$$
\theta(u) = q^{\frac{u^2}{r}-u}\frac{\Theta_p(q^{2u})}{(p;p)_\infty^3}, \quad \theta^*(u) = q^{\frac{u^2}{r^*}-u}\frac{\Theta_{p^*}(q^{2u})}{(p^*;p^*)_\infty^3}. \tag{3.22}
$$

The function $\theta(u)$ has a zero at $u=0$, enjoys the quasi-periodicity property

$$
\theta(u+r) = -\theta(u), \qquad \theta(u+r\tau) = -e^{-\pi i\tau-\frac{2\pi i u}{r}}\theta(u),
$$

and is so normalized that

$$
\oint_{C_0}\frac{dz}{2\pi i z}\frac{1}{\theta(-u)} = 1,
$$

where C_0 is a simple closed curve in the u-plane encircling $u=0$ counterclockwise. The same holds for $\theta^*(u)$, with r and τ replaced by r^* and τ^* respectively.

Now let us introduce new generators P, Q such that

$$
[Q,P] = 1, \qquad Q, P \text{ commute with } U_q(\widehat{\mathfrak{sl}}_2). \tag{3.23}
$$

With the aid of them, we define the 'total' currents obtained by modifying the elliptic currents of the previous subsection 3.1. Below we shall use the notation for the conformal weight

$$
\Delta_{l,r} = \frac{l(l+2)}{4r}. \tag{3.24}
$$

Definition 3.3 (Total currents) *We define the currents $K(u), E(u), F(u), H^{\pm}(u)$ by*

$$
K(u) = k(z)e^{Q}z^{\Delta_{-P-h-1}-\Delta_{-P-h}-\Delta^*_{-P-1}+\Delta^*_{-P}}, \tag{3.25}
$$

$$
E(u) = e(z)e^{2Q}z^{-\Delta^*_{-P-1}+\Delta^*_{-P+1}}, \tag{3.26}
$$

$$
F(u) = f(z)z^{\Delta_{-P-h-1}-\Delta_{-P-h+1}}, \tag{3.27}
$$

$$
H^{\pm}(u) = \psi^{\pm}(z)e^{2Q}\left(q^{\pm\bar{r}}z\right)^{\Delta_{-P-h-1}-\Delta_{-P-h+1}-\Delta^*_{-P-1}+\Delta^*_{-P+1}}. \tag{3.28}
$$

Here we have set $z = q^{2u}$, $\Delta_l = \Delta_{l,r}$, $\Delta^*_l = \Delta_{l,r^*}$, and $\bar{r} = r - \frac{c}{2}$.

The currents $K(u)$ and $H^{\pm}(u)$ are related by

$$H^{\pm}(u) = \kappa K\left(u \pm \frac{\bar{r}}{2} + \frac{1}{2}\right) K\left(u \pm \frac{\bar{r}}{2} - \frac{1}{2}\right), \tag{3.29}$$

$$H^{-}(u) = H^{+}(u - \bar{r}), \tag{3.30}$$

with the same κ as in (3.17). We shall refer to (3.25)-(3.28) as total currents.

From the commutation relations of the elliptic currents, we can derive those of the total currents. Let us introduce a function $\rho(u)$ by

$$\rho(u) = \frac{\rho^{+*}(u)}{\rho^{+}(u)}, \tag{3.31}$$

where

$$\rho^{+}(u) = z^{\frac{1}{2r}} \rho^{+}_{11}(z,p) = z^{\frac{1}{2r}} q^{\frac{1}{2}} \frac{\{pq^2 z\}^2}{\{pz\}\{pq^4 z\}} \frac{\{z^{-1}\}\{q^4 z^{-1}\}}{\{q^2 z^{-1}\}^2}, \tag{3.32}$$

$$\{z\} = (z; p, q^4)_{\infty}, \tag{3.33}$$

$\rho^{+}_{lm}(z,p)$ is given in (C.10), and $\rho^{+*}(u) = \rho^{+}(u)|_{r\to r^*}$.

Proposition 3.4 *The following commutation relations hold:*

$$K(u)K(v) = \rho(u-v)K(v)K(u), \tag{3.34}$$

$$K(u)E(v) = \frac{\theta^*(u - v + \frac{1-r^*}{2})}{\theta^*(u - v - \frac{1+r^*}{2})} E(v)K(u), \tag{3.35}$$

$$K(u)F(v) = \frac{\theta(u - v - \frac{1+r}{2})}{\theta(u - v + \frac{1-r}{2})} F(v)K(u), \tag{3.36}$$

$$E(u)E(v) = \frac{\theta^*(u-v+1)}{\theta^*(u-v-1)} E(v)E(u), \tag{3.37}$$

$$F(u)F(v) = \frac{\theta(u-v-1)}{\theta(u-v+1)} F(v)F(u), \tag{3.38}$$

$$[E(u), F(v)] = \frac{1}{q-q^{-1}} \left(\delta(u - v - \frac{c}{2}) H^{+}(u - \frac{c}{4}) - \delta(u - v + \frac{c}{2}) H^{-}(v - \frac{c}{4})\right), \tag{3.39}$$

$$H^{+}(u)H^{-}(v) = \frac{\theta(u - v - \frac{c}{2} - 1)}{\theta(u - v - \frac{c}{2} + 1)} \frac{\theta^*(u - v + \frac{c}{2} + 1)}{\theta^*(u - v + \frac{c}{2} - 1)} H^{-}(v)H^{+}(u), \tag{3.40}$$

$$H^{\pm}(u)H^{\pm}(v) = \frac{\theta(u-v-1)}{\theta(u-v+1)} \frac{\theta^*(u-v+1)}{\theta^*(u-v-1)} H^{\pm}(v)H^{\pm}(u), \tag{3.41}$$

$$H^{\pm}(u)E(v) = \frac{\theta^*(u - v \pm \frac{c}{4} + 1)}{\theta^*(u - v \pm \frac{c}{4} - 1)} E(v)H^{\pm}(u), \tag{3.42}$$

$$H^{\pm}(u)F(v) = \frac{\theta(u - v \mp \frac{c}{4} - 1)}{\theta(u - v \mp \frac{c}{4} + 1)} F(v)H^{\pm}(u). \tag{3.43}$$

Here $\delta(u)$ means $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ $(z = q^{2u})$.

It is in this form that the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was presented in [14]. Thus we arrive at the following interpretation:

Definition 3.5 (Algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$) *We define the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ to be the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and a Heisenberg algebra with generators Q, P (3.23).*

We note that, when $c = 0$, the commutation relations (3.34)-(3.43) also coincide with those of Enriquez-Felder [17] with $K = 0$.

In the bosonization of Appendix D, the elements P, h and Q are given as follows ($c = k$):

$$P - 1 = \widehat{\Pi} = \sqrt{\frac{2r(r-k)}{k}}P_0 - \frac{r-k}{k}P_2,$$
$$P + h - 1 = \widehat{\Pi}' = \sqrt{\frac{2r(r-k)}{k}}P_0 - \frac{r}{k}P_2,$$
$$Q = -\sqrt{\frac{k}{2r(r-k)}}iQ_0 = -\sqrt{2}\alpha_0 iQ_0,$$

where $\widehat{\Pi}, \widehat{\Pi}'$ are the notation of [14]. The physical meaning of the quantity $2\alpha_0$ is the anomalous charge of the boson field ϕ_0. We note also that, using the notation (3.24) of the conformal weight, the L_0 operator in [14] can be written as

$$L_0 = -d + \Delta_{-P+1,r-k} - \Delta_{-P-h+1,r}. \tag{3.44}$$

The elliptic currents of $U_q(\widehat{\mathfrak{sl}}_2)$ and the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ are naturally extended to those associated with arbitrary non-twisted affine Lie algebras. In Appendix A, we give a summary of the results and discuss their significance.

4 The RLL relations

One of the goals of the present paper is to describe the vertex operators (VO's) $\Phi_V(z,s)$, $\Psi^*_V(z,s)$ in the bosonic representation of the algebras $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ given in Appendix D. The intertwining relations for the VO's (2.9),(2.10) are based on the operator $L^+_V(z,r^*,s)$ defined in (2.4). In order to compute the VO's, therefore, we need the image of the 'dressed' universal R matrix $\mathcal{R}(r^*,s)$ in the Fock space. The latter is given as an infinite product of the universal R matrix for $U_q(\widehat{\mathfrak{sl}}_2)$ [21], but we do not know how to calculate it at this moment.

In this section, we take an alternative approach. Namely we utilize the elliptic currents to construct a 2×2 matrix operator $L^+(u,P)$ (see (4.17), (4.22)), and show that it satisfies the same RLL-relation (2.5) as for $L^+_V(z,r^*,s)$ with V being the spin 1/2 representation. Though we do not know a proof, from this construction we expect that (modulo perhaps some base change) this $L^+(u,P)$ is the same as $L^+_V(z,r^*,s)$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ (with $s = P$).

4.1 Half currents

The commutation relations of the total currents $E(u)$ and $F(u)$ involve delta functions. We are going to modify them so as to have commutation relations involving only 'ordinary' functions. Motivated by a similar construction in [17], we define the half currents of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as follows.

Definition 4.1 (Half currents) *We set*

$$K^+(u) = K(u + \tfrac{r+1}{2}), \tag{4.1}$$
$$E^+(u) = a^* \oint_{C^*} E(u') \frac{\theta^*(u - u' + c/2 - P + 1)}{\theta^*(u - u' + c/2)} \frac{\theta^*(1)}{\theta^*(P-1)} \frac{dz'}{2\pi i z'}, \tag{4.2}$$
$$F^+(u) = a \oint_C F(u') \frac{\theta(u - u' + P + h - 1)}{\theta(u - u')} \frac{\theta(1)}{\theta(P + h - 1)} \frac{dz'}{2\pi i z'}. \tag{4.3}$$

Here the contours are

$$C^* \quad : \quad |p^* q^c z| < |z'| < |q^c z|, \tag{4.4}$$
$$C \quad : \quad |pz| < |z'| < |z|, \tag{4.5}$$

and the constants a, a^* are chosen to satisfy

$$\frac{a^* a \theta^*(1) \kappa}{q - q^{-1}} = 1.$$

We have to be careful about the ordering of P and $E(u)$, $F(u)$ in (4.2)–(4.3), since they do not commute. In fact we have the following commutation relations.

$$[K(u), P] = K(u), \quad [E(u), P] = 2E(u), \quad [F(u), P] = 0,$$
$$[K(u), P + h] = K(u), \quad [E(u), P + h] = 0, \quad [F(u), P + h] = 2F(u).$$

The specification of the contour (4.5) should be understood as an abbreviation of the prescription "C is a simple closed curve encircling the poles $z' = p^n z$ ($n \geq 1$) of the integrand, but not containing $z' = p^n z$ ($n \leq 0$) inside". Similarly for (4.4).

The half currents (4.2)–(4.3) can also be written in terms of the Fourier modes of the elliptic currents (3.6)–(3.7),

$$e(z, p) = \sum_{n \in \mathbb{Z}} e_n z^{-n}, \qquad f(z, p) = \sum_{n \in \mathbb{Z}} f_n z^{-n}.$$

Substituting the Laurent expansion

$$\frac{\theta(u + s)}{\theta(u)\theta(s)} = -\sum_{n \in \mathbb{Z}} \frac{1}{1 - q^{-2s} p^n} z^{-n + \frac{s}{r}} \qquad (z = q^{2u})$$

valid in the domain $1 < |z| < |p^{-1}|$, we obtain

$$E^{+}(u) = e^{2Q} a^{*} \theta^{*}(1) \sum_{n \in \mathbb{Z}} e_{n} \frac{1}{1 - q^{2(P-1)} p^{*n}} (q^{c} z)^{-n - \Delta^{*}_{-P-1} + \Delta^{*}_{-P+1}}, \tag{4.6}$$

$$F^{+}(u) = -a\theta(1) \sum_{n \in \mathbb{Z}} f_{n} \frac{1}{1 - q^{-2(P+h-1)} p^{n}} z^{-n + \Delta_{-P-h-1} - \Delta_{-P-h+1}}. \tag{4.7}$$

Readers who prefer the formal series language may take (4.6), (4.7) as the definition of the half currents.

We remark that a change of contours leads to different definitions. For instance, we can define another pair of currents $E^{-}(u)$, $F^{-}(u)$ by the formulas (4.2)-(4.3), with C^{*}, C changed respectively to $C^{*}_{-}\ :\ |q^{c}z| < |z'| < |p^{*-1}q^{c}z|$ and $C_{-}\ :\ |z| < |z'| < |p^{-1}z|$. Then we have

$$\begin{aligned} -a^{*}\theta^{*}(1) E(u) &= E^{+}(u) - E^{-}(u), \\ -a\theta(1) F(u) &= F^{+}(u) - F^{-}(u). \end{aligned}$$

This looks similar to the decomposition of the total currents to 'positive' and 'negative' parts in [17]. Notice however that in our case all the Fourier components e_n, f_n appear in $E^{+}(u)$, $F^{+}(u)$, and hence the 'half' currents already generate the full algebra. For this reason we will not consider $E^{-}(u)$, $F^{-}(u)$ and the analog of the L^{-}-operator in [17].

From the commutation relations (3.34)–(3.43) for the total currents, we can obtain the relations for the half currents. Recall the function $\rho(u)$ in (3.31)-(3.32), which satisfies

$$\begin{aligned} &\rho(0) = 1, \quad \rho(1) = \frac{\theta^{*}(1)}{\theta(1)}, \\ &\rho(u)\rho(-u) = 1, \quad \rho(u)\rho(u+1) = \frac{\theta^{*}(u+1)}{\theta^{*}(u)} \frac{\theta(u)}{\theta(u+1)}. \end{aligned}$$

Proposition 4.2 *Set $u = u_1 - u_2$. Then the following commutation relations hold:*

$$K^{+}(u_1) K^{+}(u_2) = \rho(u) K^{+}(u_2) K^{+}(u_1), \tag{4.8}$$

$$K^{+}(u_1) E^{+}(u_2) K^{+}(u_1)^{-1} = E^{+}(u_2) \frac{\theta^{*}(1+u)}{\theta^{*}(u)} - E^{+}(u_1) \frac{\theta^{*}(1)}{\theta^{*}(P)} \frac{\theta^{*}(P+u)}{\theta^{*}(u)}, \tag{4.9}$$

$$K^{+}(u_1)^{-1} F^{+}(u_2) K^{+}(u_1) = \frac{\theta(1+u)}{\theta(u)} F^{+}(u_2) - \frac{\theta(1)}{\theta(P+h)} \frac{\theta(P+h-u)}{\theta(u)} F^{+}(u_1), \tag{4.10}$$

$$\begin{aligned} &\frac{\theta^{*}(1-u)}{\theta^{*}(u)} E^{+}(u_1) E^{+}(u_2) + \frac{\theta^{*}(1+u)}{\theta^{*}(u)} E^{+}(u_2) E^{+}(u_1) \\ &\quad = E^{+}(u_1)^{2} \frac{\theta^{*}(1)}{\theta^{*}(P-2)} \frac{\theta^{*}(P-2+u)}{\theta^{*}(u)} + E^{+}(u_2)^{2} \frac{\theta^{*}(1)}{\theta^{*}(P-2)} \frac{\theta^{*}(P-2-u)}{\theta^{*}(u)}, \end{aligned} \tag{4.11}$$

$$\frac{\theta(1+u)}{\theta(u)} F^{+}(u_1) F^{+}(u_2) + \frac{\theta(1-u)}{\theta(u)} F^{+}(u_2) F^{+}(u_1) \tag{4.12}$$

$$= F^+(u_1)^2 \frac{\theta(1)}{\theta(P+h-2)} \frac{\theta(P+h-2-u)}{\theta(u)} + F^+(u_2)^2 \frac{\theta(1)}{\theta(P+h-2)} \frac{\theta(P+h-2+u)}{\theta(u)},$$

$$[E^+(u_1), F^+(u_2)] = K^+(u_2-1)K^+(u_2) \frac{\theta^*(P-1-u)}{\theta^*(u)} \frac{\theta^*(1)}{\theta^*(P-1)}$$

$$-K^+(u_1)K^+(u_1-1) \frac{\theta(P+h-1-u)}{\theta(u)} \frac{\theta(1)}{\theta(P+h-1)}. \tag{4.13}$$

Proof. These relations can be proven by reducing them to identities of theta functions. Let us show (4.9). From the definition of the half currents (4.2) and the commutation relation (3.35), we have

$$K^+(u_1)E^+(u_2)K^+(u_1)^{-1}$$

$$= a^* \oint_{C^*} E(u') \frac{\theta^*(u_1 - u' + \frac{1-r^*}{2})}{\theta^*(u_1 - u' - \frac{1+r^*}{2})} \frac{\theta^*(u_2 - u' - P + 1)}{\theta^*(u_2 - u')} \frac{\theta^*(1)}{\theta^*(P-1)} \frac{dz'}{2\pi i z'}. \tag{4.14}$$

Set

$$\eta_{s,t}(u) = \frac{\theta(u+s)\theta(t)}{\theta(u)\theta(s)}.$$

Then the following identity holds:

$$\frac{\theta(u_1+t)}{\theta(u_1)} \eta_{s,t}(u_2) = \frac{\theta(u_1-u_2+t)}{\theta(u_1-u_2)} \eta_{s+t,t}(u_2) + \eta_{s,t}(u_2-u_1)\eta_{s+t,t}(u_1). \tag{4.15}$$

We obtain (4.9) by applying (4.15) to the integrand of (4.14) with the replacement $\theta(u) \to \theta^*(u)$, $a \to 1$, $s \to -P+1$, $u \to u'$ and $u_1 \to u_1 - (1+r^*)/2$.

Likewise, (4.11) leads to an equality between two-fold integrals. It can be shown by symmetrizing the integration variables and applying the identity

$$\psi(u_1, u_2; u'_1, u'_2) + \psi(u_1, u_2; u'_2, u'_1) \times \frac{\theta(u'_2 - u'_1 + t)}{\theta(u'_2 - u'_1 - t)} = (u_1 \longleftrightarrow u_2),$$

where

$$\psi(u_1, u_2; u'_1, u'_2) = \frac{\theta(u_1 - u_2 - t)}{\theta(u_1 - u_2)} \eta_{s+t,t}(u_1 - u'_1)\eta_{s-t,t}(u_2 - u'_2)$$

$$-\eta_{s,t}(u_2 - u_1)\eta_{s+t,t}(u_1 - u'_1)\eta_{s-t,t}(u_1 - u'_2). \tag{4.16}$$

The proofs of (4.10), (4.12) are similar.

Finally let us show (4.13). Integrating the delta function in (3.39), we obtain

$$(a^*a)^{-1}(q-q^{-1})[E^+(u_1), F^+(u_2)]$$

$$= \oint_{C_1} H^+(u' + \frac{c}{4}) \frac{\theta^*(u_1 - u' - P + 1)\theta^*(1)}{\theta^*(u_1 - u')\theta^*(P-1)} \frac{\theta(u_2 - u' + P + h - 1)\theta(1)}{\theta(u_2 - u')\theta(P+h-1)} \frac{dz'}{2\pi i z'}$$

$$- \oint_{C_2} H^-(u' - \frac{c}{4}) \frac{\theta^*(u_1 - u' + c - P + 1)\theta^*(1)}{\theta^*(u_1 - u' + c)\theta^*(P-1)} \frac{\theta(u_2 - u' + P + h - 1)\theta(1)}{\theta(u_2 - u')\theta(P+h-1)} \frac{dz'}{2\pi i z'}.$$

Here the contours are

$$
\begin{aligned}
C_1 &: \quad |p^* z_1|, |pz_2| < |z'| < |z_1|, |z_2|, \\
C_2 &: \quad |pz_1|, |pz_2| < |z'| < |q^{2c} z_1|, |z_2|.
\end{aligned}
$$

Change variables $z' \to pz'$ in the second term and use the periodicity of $\theta(u)$ along with the relation $H^-(u' - c/4) = H^+(u' - r + c/4)$. We see that the integrand becomes the same as the first, whereas the contour becomes

$$
C_2' \quad : \quad |z_1|, |z_2| < |z'| < |p^{-1} q^{2c} z_1|, |p^{-1} z_2|.
$$

Picking the residues at $z' = z_1, z_2$ we find (4.13). $\square$

4.2 Gauß decomposition

Our next task is to rewrite the commutation relations (4.8)–(4.13) into an 'RLL'-form. Following the idea of the Gauß decomposition of Ding-Frenkel [23], let us introduce the L-operator as follows.

Definition 4.3 (L-operator) *We define the operator $\widehat{L}^+(u) \in \mathrm{End}(V) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with $V = \mathbb{C}^2$, by*

$$
\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_1^+(u) & 0 \\ 0 & K_2^+(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}, \tag{4.17}
$$

where

$$
K_1^+(u) = K^+(u-1), \qquad K_2^+(u) = K^+(u)^{-1}.
$$

Note that

$$
[\widehat{L}^+(u), h^{(1)} + h] = 0, \qquad P\widehat{L}^+(u) = \widehat{L}^+(u)(P - h^{(1)}),
$$

where $h^{(1)}$ and h mean $h \otimes 1$ and $1 \otimes h \in \mathrm{End}(V) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_2)$, respectively.

We also need the formula for the R matrix (2.13) for $l = m = 1$. With a further transformation of the form (2.12), $R^+(u,s) = \widetilde{R}^+_{11}(u,s)$ takes the form

$$
R^+(u,s) = \rho^+(u) \begin{pmatrix} 1 & & & \\ & b(u,s) & c(u,s) & \\ & \bar{c}(u,s) & \bar{b}(u,s) & \\ & & & 1 \end{pmatrix}. \tag{4.18}
$$

Here $\rho^+(u)$[2)] is given in (3.32), and

$$b(u,s)=\frac{\theta(s+1)\theta(s-1)}{\theta(s)^2}\frac{\theta(u)}{\theta(1+u)},\quad c(u,s)=\frac{\theta(1)}{\theta(s)}\frac{\theta(s+u)}{\theta(1+u)},\tag{4.19}$$

$$\bar{c}(u,s)=\frac{\theta(1)}{\theta(s)}\frac{\theta(s-u)}{\theta(1+u)},\qquad \bar{b}(u,s)=\frac{\theta(u)}{\theta(1+u)}.\tag{4.20}$$

Up to a scalar factor, this is the same R matrix as eq.(93) in [17].

Proposition 4.4 *The relations (4.8)-(4.13) are equivalent to the following RLL relation*

$$\begin{aligned}&R^{+(12)}(u_1-u_2,P+h)\widehat{L}^{+(1)}(u_1)\widehat{L}^{+(2)}(u_2)\\&=\widehat{L}^{+(2)}(u_2)\widehat{L}^{+(1)}(u_1)R^{*+(12)}(u_1-u_2,P).\end{aligned}\tag{4.21}$$

Proposition 4.4 can be shown by a direct computation. (A little care has to be taken since the entries of $\widehat{L}^+(u)$ do not commute with those of $R^+(u,P)$.) Since the calculation is tedious but straightforward, we omit the details.

The above RLL-relation is equivalent to the dynamical RLL relation (2.17). To see this, let us 'strip off' the operator e^Q from the half currents and define

$$\begin{aligned}k_1^+(u,P)&=k(q^{r-1}z)\times(q^{r-1}z)^{-\frac{r-r^*}{4rr^*}(2P+1)+\frac{1}{2r}h}=K^+(u-1)e^{-Q},\\ k_2^+(u,P)&=k(q^{r+1}z)^{-1}\times(q^{r+1}z)^{\frac{r-r^*}{4rr^*}(2P-1)-\frac{1}{2r}h}=K^+(u)^{-1}e^{Q},\\ e^+(u,P)&=a^*\theta^*(1)\sum_{n\in\mathbb{Z}}e_n\frac{1}{1-q^{2P}p^{*n}}(q^cz)^{-n-\frac{P}{r^*}}=e^{-Q}E^+(u)e^{-Q},\\ f^+(u,P)&=-a\theta(1)\sum_{n\in\mathbb{Z}}f_n\frac{1}{1-q^{-2(P+h-1)}p^n}z^{-n+\frac{P+h-1}{r}}=F^+(u).\end{aligned}$$

These currents all commute with P. We can regard them as currents in $U_q(\widehat{\mathfrak{sl}}_2)$ having P as a parameter (P plays the same role as λ used in [17]). We set

$$\begin{aligned}L^+(u,P)&=\widehat{L}^+(u)\begin{pmatrix}e^{-Q}&0\\0&e^{Q}\end{pmatrix}\\&=\begin{pmatrix}1&f^+(u,P)\\0&1\end{pmatrix}\begin{pmatrix}k_1^+(u,P)&0\\0&k_2^+(u,P)\end{pmatrix}\begin{pmatrix}1&0\\e^+(u,P)&1\end{pmatrix}.\end{aligned}\tag{4.22}$$

Then Proposition 4.4 is equivalently rephrased as follows.

Proposition 4.5 *The L-operator (4.22) satisfies the dynamical RLL relation*

$$\begin{aligned}&R^{+(12)}(u_1-u_2,P+h)L^{+(1)}(u_1,P)L^{+(2)}(u_2,P+h^{(1)})\\&=L^{+(2)}(u_2,P)L^{+(1)}(u_1,P+h^{(2)})R^{*+(12)}(u_1-u_2,P).\end{aligned}\tag{4.23}$$

[2)] This scalar factor differs from (3.16) in [21], see the remark at the end of section 2.1.

5 Vertex operators

In this section we shall study the VO's for $U_{q,p} = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and compare them with those of Lukyanov and Pugai [11].

5.1 Intertwining relations

As in section 2, we start with VO's in the sense of (2.7),(2.8) (or their modification by fractional powers (2.15),(2.16)) acting on some highest weight modules $\mathcal{F}_J$ over $U_q(\widehat{\mathfrak{sl}}_2)$, where J is a label for the highest weight. Our main concern will be the Fock modules described in Appendix D. However, for the general considerations till the end of the next subsection, we do not need the details of $\mathcal{F}_J$. For this purpose it is convenient to consider the VO's as acting on the sum of the Fock spaces $\mathcal{F} = \oplus_J \mathcal{F}_J$.

We define the VO's for $U_{q,p}$ acting on the total Fock space $\hat{\mathcal{F}} = \oplus_\mu \mathcal{F} \otimes e^{\mu Q}$ by

$$\widehat{\Phi}_l(v) = \widetilde{\Phi}_l(v, P) \ : \ \hat{\mathcal{F}} \longrightarrow \hat{\mathcal{F}} \otimes V_{l,v}, \tag{5.1}$$

$$\widehat{\Psi}^*_l(v) = \widetilde{\Psi}^*_l(v, P) e^{h \otimes Q} \ : \ V_{l,v} \otimes \hat{\mathcal{F}} \longrightarrow \hat{\mathcal{F}}. \tag{5.2}$$

Here, just as in $\widehat{L}^+(u)$, the P in $\widehat{\Phi}_l(v)$ and $\widehat{\Psi}^*_l(v)$ is regarded as an operator on $\hat{\mathcal{F}}$. For the notation about the spin $l/2$ representation $V_{l,v}$, see Appendix C.1.

The basic relations we are going to investigate are the dynamical intertwining relations in (2.18) and (2.19). We shall solve them by replacing $\widetilde{R}^+_{1l}(u, P)$ and $\widetilde{L}^+(u, P)$ with $R^+_{1l}(u, P)$ in (C.17) and $L^+(u, P)$ in (4.22) respectively. Substituting (4.22), (5.1) and (5.2) into (2.18) and (2.19) and writing $u_1 = u, \ u_2 = v$, we get the following for $\widehat{\Phi}_l(v)$ and $\widehat{\Psi}^*_l(v)$:

$$\widehat{\Phi}_l(v) \widehat{L}^+(u) = R^{+(13)}_{1l}(u - v, P + h) \widehat{L}^+(u) \widehat{\Phi}_l(v), \tag{5.3}$$

$$\widehat{L}^+(u) \widehat{\Psi}^*_l(v) = \widehat{\Psi}^*_l(v) \widehat{L}^+(u) R^{*+(12)}_{1l}(u - v, P - h^{(1)} - h^{(2)}). \tag{5.4}$$

It should be noted that a natural coproduct for $U_{q,p}$ is not known, and hence the meaning of intertwining relations for it is not clear. Eqs. (5.3),(5.4) should be regarded as a compact way of writing the family of intertwining relations for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. With this understanding, we shall sometimes refer to (5.3),(5.4) (somewhat loosely) as 'intertwining relations for $U_{q,p}$'.

Now using the explicit form of the R-matrix (C.17), let us write down the 'intertwining relations' (5.3) and (5.4). We shall write the entries of $\widehat{L}^+(u)$ as

$$\widehat{L}^+(u) = \begin{pmatrix} \widehat{L}^+_{++}(u) & \widehat{L}^+_{+-}(u) \\ \widehat{L}^+_{-+}(u) & \widehat{L}^+_{--}(u) \end{pmatrix}.$$

According to the Gauß decomposition (4.17), we have

$$\widehat{L}^+_{++}(u) = K^+(u-1) + F^+(u)K^+(u)^{-1}E^+(u), \quad \widehat{L}^+_{+-}(u) = F^+(u)K^+(u)^{-1},$$
$$\widehat{L}^+_{-+}(u) = K^+(u)^{-1}E^+(u), \qquad\qquad \widehat{L}^+_{--}(u) = K^+(u)^{-1}.$$

Define the components of VO's by

$$\widehat{\Phi}_l(v-\frac{1}{2}) = \sum_{m=0}^{l} \Phi_{l,m}(v) \otimes v^l_m,$$
$$\widehat{\Psi}^*_l(v-\frac{c+1}{2})\left(v^l_m \otimes \cdot\right) = \Psi^*_{l,m}(v).$$

For brevity we set $\tilde{\varphi}_l(u) = \varphi_l(u+\frac{1}{2})$, $\tilde{\varphi}^*_l(u) = \varphi^*_l(u+\frac{c+1}{2})$ and $\Lambda = P+h$. Here $\varphi_l(u)$ is given in (C.16) and $\varphi^*_l(u) = \varphi_l(u)|_{r\to r^*}$. In these notations, (5.3) reads as follows:

$$\begin{aligned}
\tilde{\varphi}_l(u-v)\Phi_{l,m}(v)\widehat{L}^+_{+\pm}(u) = &\\
&-\frac{\theta(u-v+\frac{l}{2}-m+1)\theta(\Lambda+l-m+1)\theta(\Lambda-m)}{\theta(\Lambda)\theta(\Lambda+l-2m+1)}\widehat{L}^+_{+\pm}(u)\Phi_{l,m}(v)\\
&-\frac{\theta(u-v+\Lambda+\frac{l}{2}-m+1)\theta(m)}{\theta(\Lambda+l-2m+1)}\widehat{L}^+_{-\pm}(u)\Phi_{l,m-1}(v),
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
\tilde{\varphi}_l(u-v)\Phi_{l,m}(v)\widehat{L}^+_{-\pm}(u) = &-\theta(u-v-\tfrac{l}{2}+m+1)\widehat{L}^+_{-\pm}(u)\Phi_{l,m}(v)\\
&+\frac{\theta(u-v-\Lambda-\frac{l}{2}+m+1)\theta(l-m)}{\theta(\Lambda)}\widehat{L}^+_{+\pm}(u)\Phi_{l,m+1}(v).
\end{aligned} \tag{5.6}$$

Similarly (5.4) takes the form

$$\begin{aligned}
\tilde{\varphi}^*_l(u-v)\widehat{L}^+_{\pm+}(u)\Psi^*_{l,m}(v) = &\\
&-\frac{\theta^*(u-v+\frac{l+c}{2}-m+1)\theta^*(P-l+m-1)\theta^*(P+m)}{\theta^*(P)\theta^*(P-l+2m-1)}\Psi^*_{l,m}(v)\widehat{L}^+_{\pm+}(u)\\
&+\frac{\theta^*(u-v-P+\frac{l+c}{2}-m+1)\theta^*(l-m+1)}{\theta^*(P-l+2m-1)}\Psi^*_{l,m-1}(v)\widehat{L}^+_{\pm-}(u),
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
\tilde{\varphi}^*_l(u-v)\widehat{L}^+_{\pm-}(u)\Psi^*_{l,m}(v) = &-\theta^*(u-v-\tfrac{l-c}{2}+m+1)\Psi^*_{l,m}(v)\widehat{L}^+_{\pm-}(u)\\
&-\frac{\theta^*(u-v+P-\frac{l-c}{2}+m+1)\theta^*(m+1)}{\theta^*(P)}\Psi^*_{l,m+1}(v)\widehat{L}^+_{\pm+}(u).
\end{aligned} \tag{5.8}$$

Let us investigate the relations (5.5) and (5.6) in detail. For the highest component $\Phi_{l,l}(v)$, we can immediately obtain from (5.6) the relations

$$\tilde{\varphi}_l(u-v)\Phi_{l,l}(v)K^+(u)^{-1} = -\theta(u-v+\frac{l}{2}+1)K^+(u)^{-1}\Phi_{l,l}(v), \tag{5.9}$$
$$\Phi_{l,l}(v)E^+(u) = E^+(u)\Phi_{l,l}(v). \tag{5.10}$$

Notice that $u = v - \frac{l}{2} - 1$ is a zero of $\tilde{\varphi}_l(u-v)$. Suppose that in the relation (5.5) the product of $\Phi_{l,m}(v)$ and $F^+(u)$ has no pole at this point. Then we obtain a relation which determines the components of VO's recursively:

$$\Phi_{l,m-1}(v) = F^+(v - \frac{l}{2})\frac{\theta(\Lambda + l - m)}{\theta(\Lambda)}\Phi_{l,m}(v) \qquad (m = 0, 1, .., l). \tag{5.11}$$

We will show below (Proposition 5.1) that this assumption is satisfied in the free field realization of $U_{q,p}$. Substituting (5.9), (5.11) into (5.5) and using Riemann's theta identity, we find the following relation as the sufficient condition for (5.5) with the choice of the lower sign:

$$\Phi_{l,l}(v)F(u) = \frac{\theta(u - v - \frac{l}{2})}{\theta(u - v + \frac{l}{2})}F(u)\Phi_{l,l}(v). \tag{5.12}$$

By (5.9)-(5.12), the remaining relations in (5.5)-(5.6) are reduced to those involving the highest component $\Phi_{l,l}(v)$ and $K^+(u), E^+(u), F^+(u)$. In order to ensure the existence of VO, we need to verify them. For level one ($c = 1$), we have verified that they are consequences of Proposition 4.2. In general, such a direct check seems complicated, and it would be better to invoke the fusion procedure. We do not go into this issue any further. Note however that, had we known the equivalence with the quasi-Hopf construction, there would be no need for the check because the existence of VO is clear in the latter context.

Similarly, the intertwining relations (5.7) and (5.8) for the type II vertex operator lead to the following relations as the sufficient condition for the highest component:

$$\tilde{\varphi}^*_l(u-v)K^+(u)^{-1}\Psi^*_{l,l}(v) = -\Psi^*_{l,l}(v)K^+(u)^{-1}\theta^*(u - v + \frac{l+c}{2} + 1), \tag{5.13}$$

$$\Psi^*_{l,l}(v)E(u) = E(u)\Psi^*_{l,l}(v)\frac{\theta^*(u - v + \frac{l}{2})}{\theta^*(u - v - \frac{l}{2})}, \tag{5.14}$$

$$\Psi^*_{l,l}(v)F^+(u) = F^+(u)\Psi^*_{l,l}(v), \tag{5.15}$$

and the relation for the lower component

$$\Psi^*_{l,m-1}(v) = \Psi^*_{l,m}(v)E^+(v - \frac{l+c}{2} - r^*)\frac{\theta^*(m)\theta^*(P - l + m - 2)}{\theta^*(l - m + 1)\theta^*(P - 2)} \quad (m = 0, 1, .., l). \tag{5.16}$$

We remark that in the derivation of (5.16), we took $u = v - \frac{l+c}{2} - 1 - r^*$ as a zero of $\tilde{\varphi}^*_l(u-v)$. If we chose a zero without the shift by r^*, we would have an extra term in the RHS of (5.14). The shift of u by r^* in $E^+(u)$ yields a change of the contour in (4.2). For example, we have

$$E^+(v - \frac{l+c}{2} - r^*) = a^* \oint_{\tilde{C}^*} E(u')\frac{\theta^*(v - u' - l/2 - P + 1)}{\theta^*(u - u' - l/2)}\frac{\theta^*(1)}{\theta^*(P-1)}\frac{dz'}{2\pi i z'}$$

with the contour being

$$\tilde{C}^* \quad : \quad |q^{-l}z| < |z'| < |p^{*-1}q^{-l}z|. \tag{5.17}$$

5.2 'Twisted' intertwining relations

In order to compare these results with those in [11], [12] and [14], we need a further modification by signs. In [27], the modified VO's are called 'twisted' intertwiners [3)].

In the case of $U_q(\widehat{\mathfrak{sl}}_2)$, the twisted type I $\Phi'_V(v)$ and the type II $\Psi^{*\prime}_V(v)$ VO's are the intertwiners of the same type as $\Phi_V(v)$ and $\Psi^*_V(v)$, but satisfying the following intertwining relations twisted with signs:

$$\Phi'_V(v)\iota(a) = \Delta(a)\Phi'_V(v), \tag{5.18}$$

$$\Psi^{*\prime}_V(v)\Delta(a) = \iota(a)\Psi^{*\prime}_V(v), \qquad \forall a \in U_q(\widehat{\mathfrak{sl}}_2), \tag{5.19}$$

where ι denotes the involution of $U_q(\widehat{\mathfrak{sl}}_2)$,

$$\iota(x_n^\pm) = -x_n^\pm, \qquad \iota(a_n) = a_n.$$

Analogously to the procedures (2.7), (2.8), (5.1) and (5.2), we can define the 'twisted' VO's $\widehat{\Phi}'_l(v)$ and $\widehat{\Psi}^{*\prime}_l(v)$ for $U_{q,p}$ as the operators of the same type as (5.1) and (5.2) but satisfying the following 'twisted' intertwining relations.

$$\widehat{\Phi}'_l(v)\widehat{L}^{+\prime}(u) = R^{+(13)}(u-v, P+h)\widehat{L}^+(u)\widehat{\Phi}'_l(v), \tag{5.20}$$

$$\widehat{L}^{+\prime}(u)\widehat{\Psi}^{*\prime}_l(v) = \widehat{\Psi}^{*\prime}_l(v)\widehat{L}^+(u)R^{*+(12)}(u-v, P-h^{(1)}-h^{(2)}). \tag{5.21}$$

Here the 'twisted' L-operator $\widehat{L}^{+\prime}(u)$ has the following components:

$$\widehat{L}^{+\prime}_{\pm\pm}(u) = \widehat{L}^+_{\pm\pm}(u), \qquad \widehat{L}^{+\prime}_{\pm\mp}(u) = -\widehat{L}^+_{\pm\mp}(u).$$

Defining the components of the 'twisted' VO's as

$$\widehat{\Phi}'_l(v-\frac{1}{2}) = \sum_{m=0}^{l} \Phi'_{l,m}(v)\otimes v^l_m,$$

$$\widehat{\Psi}^{*\prime}_l(v-\frac{c+1}{2})\left(v^l_m\otimes\cdot\right) = \Psi^{*\prime}_{l,m}(v),$$

we obtain the 'twisted' counterpart of (5.9)-(5.16) as follows:

$$\tilde{\varphi}_l(u-v)\Phi'_{l,l}(v)K^+(u)^{-1} = -\theta(u-v+\frac{l}{2}+1)K^+(u)^{-1}\Phi'_{l,l}(v), \tag{5.22}$$

$$\Phi'_{l,l}(v)E^+(u) = -E^+(u)\Phi'_{l,l}(v), \tag{5.23}$$

$$\Phi'_{l,l}(v)F(u) = -\frac{\theta(u-v-\frac{l}{2})}{\theta(u-v+\frac{l}{2})}F(u)\Phi'_{l,l}(v), \tag{5.24}$$

$$\Phi'_{l,m-1}(v) = F^+(v-\frac{l}{2})\frac{\theta(\Lambda+l-m)}{\theta(\Lambda)}\Phi'_{l,m}(v), \tag{5.25}$$

[3)]This terminology is not to be confused with the 'twisting' in the sense of quasi-Hopf algebras.

for type I, and

$$\tilde{\varphi}^*_l(u-v)K^+(u)^{-1}{\Psi^*}'_{l,l}(v) = -{\Psi^*}'_{l,l}(v)K^+(u)^{-1}\theta^*(u-v+\frac{l+c}{2}+1), \quad (5.26)$$

$${\Psi^*}'_{l,l}(v)E(u) = -E(u){\Psi^*}'_{l,l}(v)\frac{\theta^*(u-v+\frac{l}{2})}{\theta^*(u-v-\frac{l}{2})}, \quad (5.27)$$

$${\Psi^*}'_{l,l}(v)F^+(u) = -F^+(u){\Psi^*}'_{l,l}(v), \quad (5.28)$$

$${\Psi^*}'_{l,m-1}(v) = {\Psi^*}'_{l,m}(v)E^+(v-\frac{l+c}{2}-r^*)\frac{\theta^*(m)\theta^*(P-l+m-2)}{\theta^*(l-m+1)\theta^*(P-2)}, \quad (5.29)$$

for type II. (See the remark below (5.16).)

5.3 Free field realization

$U_{q,p}(\widehat{\mathfrak{sl}}_2)$ admits a free field representation for arbitrary level $c = k(\neq 0,-2)$ [14](see Appendix D). Using this, we obtain a realization of the VO's. We consider below the 'twisted' VO's. The non-twisted ones are obtained in a similar way.

Proposition 5.1 *Using the notation in Appendix D, the intertwining relation (5.20) has the following solution:*

$$\Phi'_{l,l}(v) = \phi_{k-l,-(k-l)}(w) : \exp\Big\{-\phi'_0(l;2,k|w)\Big\} : \qquad (l=0,1,..,k), \quad (5.30)$$

$$\Phi'_{l,m}(v) = \left(\prod_{j=1}^{l-m} a\oint_{C_j}\frac{dz_j}{2\pi i z_j}\right)\prod_{j=1}^{l-m}\frac{\theta(v-u_j-\frac{l}{2}+\Lambda-1+2j)\theta(1)\theta(\Lambda+l-m-1+2j)}{\theta(v-u_j-\frac{l}{2})\theta(\Lambda-1+2j)\theta(\Lambda+2j)}$$
$$\times\left(\prod_{j=1}^{l-m}F(u_j)\right)\Phi'_{l,l}(v) \qquad (m=0,1,..,l), \quad (5.31)$$

where $w=q^{2v},\ z_j=q^{2u_j}$, $\Lambda = P+h$ *and*

$$\phi_{l,\pm l}(w) =: \exp\Big\{-\phi_2\Big(\pm l;2,k|w;\pm\frac{k}{2}\Big)-\phi_1\Big(l;2,k+2|w;\pm\frac{k+2}{2}\Big)\Big\}:. \quad (5.32)$$

Similarly (5.21) has the solution

$${\Psi^*}'_{l,l}(v) = \phi_{k-l,k-l}(w) : \exp\Big\{\phi_0(l;2,k|z)\Big\} : \qquad (l=0,1,2,..,k), \quad (5.33)$$

$${\Psi^*}'_{l,m}(v) = \left(\prod_{j=1}^{l-m}a^*\oint_{\tilde{C}^*_j}\frac{dz_j}{2\pi i z_j}\right){\Psi^*}'_{l,l}(v)\left(\prod_{j=1}^{l-m}E(u_j)\right)$$
$$\times\prod_{j=1}^{l-m}\frac{\theta^*(v-u_j-\frac{l}{2}-P+1+2(l-m-j))\theta^*(1)\theta^*(P-j-1)\theta^*(l-j+1)}{\theta^*(v-u_j-\frac{l}{2})\theta^*(P-1-2(l-m-j))\theta^*(P-2-2(l-m-j))\theta^*(j)}$$
$$(m=0,1,..,l). \quad (5.34)$$

The contours C_j and $\tilde{C}^*_j$ $(j=1,2,..,l-m)$ are taken as

$$C_{l-m} \;:\; |q^l w|, |pq^{-l}w| \;<\; |z_{l-m}| \;<\; |q^{-l}w|, \tag{5.35}$$

$$C_j \;:\; |q^l w|, |pq^{-l}w|, |q^{-2}z_{j+1}| \;<\; |z_j| \;<\; |q^{-l}w| \quad (j=1,2,..,l-m-1), \tag{5.36}$$

$$\tilde{C}^*_{l-m} \;:\; |q^{-l}w| \;<\; |z_{l-m}| \;<\; |p^{*-1}q^{-l}w|, |q^l w|, \tag{5.37}$$

$$\tilde{C}^*_j \;:\; |q^{-l}w|, |q^2 z_{j+1}| \;<\; |z_j| \;<\; |p^{*-1}q^{-l}w|, |q^l w| \; (j=1,2,..,l-m-1). \tag{5.38}$$

Sketch of proof. First of all, we should note that the relations (5.22)-(5.24) and (5.26)-(5.28) coincide with those in Proposition 4.4 in [14] if we identify $\Phi'_{l,l}(u)$ and $\Psi^{*}{}'_{l,l}(u)$ with $\tilde{\Phi}^{(l)}_l(u)$ and $\tilde{\Psi}^{(l)*}_l(u)$ in [14], respectively. Using the bosons $a_{0,m}, a_{1,m}, a_{2,m}$ defined in Appendix D, the expressions (5.30) and (5.33) are the unique solutions of them up to a scalar.

Next, the expressions (5.31) and (5.34) are the direct consequence of (5.25) and (5.29). The contours are determined from (4.5) with the replacement $z \to q^{-l}w$, (5.17) and the following OPE's derived from (5.30), (5.33) and Proposition D.3:

$$\Phi'_{l,l}(v)F(u) = w^{\frac{l}{r}-1}\frac{(pq^{-l}z/w;p)_\infty}{(q^l z/w;p)_\infty} : \Phi'_{l,l}(v)F(u) :, \tag{5.39}$$

$$F(u)\Phi'_{l,l}(v) = z^{\frac{l}{r}-1}\frac{(pq^{-l}w/z;p)_\infty}{(q^l w/z;p)_\infty} : \Phi'_{l,l}(v)F(u) :, \tag{5.40}$$

$$\Psi^{*\prime}_{l,l}(v)E(u) = w^{-\frac{l}{r-k}-1}\frac{(p^*q^l z/w;p^*)_\infty}{(q^{-l}z/w;p^*)_\infty} : E(u)\Psi^{*\prime}_{l,l}(v) :, \tag{5.41}$$

$$E(u)\Psi^{*\prime}_{l,l}(v) = z^{-\frac{l}{r-k}-1}\frac{(p^*q^l w/z;p^*)_\infty}{(q^{-l}w/z;p^*)_\infty} : E(u)\Psi^{*\prime}_{l,l}(v) :, \tag{5.42}$$

$$\begin{aligned} E(u)E(u') &= z^{\frac{2}{r-k}}\frac{(p^*q^{-2}z'/z;p^*)_\infty}{(q^2 z'/z;p^*)_\infty} : e^{-\phi_0(k|z)}e^{-\phi_0(k|z')} : \\ &\times\frac{1}{(q-q^{-1})^2}\Big(q(1-\frac{z'}{z})(:\Psi^-_I(z)\Psi^-_I(z'): + :\Psi^-_{II}(z)\Psi^-_{II}(z'): \\ &\quad -(1-q^{-2}\frac{z'}{z})(q:\Psi^-_I(z)\Psi^-_{II}(z'): + q^{-1}:\Psi^-_{II}(z)\Psi^-_I(z'):)\Big), \end{aligned} \tag{5.43}$$

$$\begin{aligned} F(u)F(u') &= z^{-\frac{2}{r}}\frac{(pq^2 z'/z;p)_\infty}{(q^{-2}z'/z;p)_\infty} : e^{\phi'_0(k|z)}e^{\phi'_0(k|z')} : \\ &\times\frac{1}{(q-q^{-1})^2}\Big(q^{-1}(1-\frac{z'}{z})(:\Psi^+_I(z)\Psi^+_I(z'): + :\Psi^+_{II}(z)\Psi^+_{II}(z'): \\ &\quad -(1-q^2\frac{z'}{z})(q^{-1}:\Psi^+_I(z)\Psi^+_{II}(z'): + q:\Psi^+_{II}(z)\Psi^+_I(z'):)\Big), \end{aligned} \tag{5.44}$$

where $\Psi^{\pm}_{I,II}(z)$ are given in (D.12) and (D.28), and $z=q^{2u}$, $z'=q^{2u'}$, $w=q^{2v}$.

In the derivation of (5.25) and (5.29), we made an assumption that no counter poles to the zeros of $\tilde{\varphi}_l(u-v)$ and $\tilde{\varphi}^*_l(u-v)$ appear from the OPE's $\Phi'_{l,m}(v)F^+(u)$ and $E^+(u)\Psi^{*\prime}_{l,m}(v)$. The verification of this assumption is not so hard. Substitute the OPE's

(5.39)- (5.44) into the products $\Phi'_{l,m}(v)F^+(u)$ and $E^+(u)\Psi^{*'}_{l,m}(v)$, we can show that such counter poles do not appear.

We have also checked at level one ($k=1$) that, upon elimination of ${\Phi'}_{1,m}(v)$ and $\Psi^{*'}_{1,m}(v)$, the rest of the intertwining relations are consequences of the commutation relations for the half currents in Proposition 4.2. As we mentioned already, such a direct check is difficult for higher levels. However, modulo the assumption about the equivalence with the quasi-Hopf construction (where the existence of the VO's is known), the expressions (5.30)-(5.34) are unique up to a scalar factor and a choice of an equivalent set of three bosons. □

5.4 Level one case ($k=1$)

In this case, writing $\Phi_-(v)=\Phi'_{1,1}(v)$, $\Phi_+(v)=\Phi'_{1,0}(v)$, $\Psi^*_-(v)=\Psi^{*'}_{1,1}(v)$ and $\Psi^*_+(v)=\Psi^{*'}_{1,0}(v)$, we have from Proposition 5.1,

$$\Phi_-(v) =: \exp\Big\{-\phi'_0(2|w)\Big\}:, \tag{5.45}$$

$$\Phi_+(v) = a\oint_C \frac{dz'}{2\pi i z'} z'^{-\frac{r-1}{r}} \frac{(pq^{-1}w/z';p)_\infty}{(qw/z';p)_\infty} \frac{\theta(v-u'-\frac{1}{2}+\Lambda+1)\theta(1)}{\theta(v-u'-\frac{1}{2})\theta(\Lambda+1)} :F(u')\Phi_-(v):, \tag{5.46}$$

$$\Psi^*{}_-(v) =: \exp\Big\{\phi_0(2|z)\Big\}:, \tag{5.47}$$

$$\Psi^*{}_+(v) = a^*\oint_{\tilde{C}^*} \frac{dz'}{2\pi i z'} :\Psi^*_-(v)E(u'): w^{-\frac{r}{r-1}} \frac{(p^*qz'/w;p^*)_\infty}{(q^{-1}z'/w;p^*)_\infty} \frac{\theta^*(v-u'-\frac{1}{2}-P+1)\theta^*(1)}{\theta^*(v-u'-\frac{1}{2})\theta^*(P-1)}, \tag{5.48}$$

where the contours C and $\tilde{C}^*$ are given by (5.35) and (5.37) letting $l=1,\ m=0$. Since the level one parafermion theory is a trivial theory, one can neglect the parafermion currents in $E(u)$ and $F(u)$. Then, the expressions (5.45)-(5.48) agree with the results in [11] and [12].

Remark. Our notation here is related to those of [12] as follows:

$$\begin{aligned}
&x=-q, \quad \alpha_m = \frac{1}{[2m]}a_{0,m}, \quad \beta_n = \frac{1}{[2m]}a'_{0,m},\\
&Q=Q_0, \quad P=P_0,\\
&L-1 = P \text{ (in (5.48))} = \hat{\Pi}+1, \quad K-1=\Lambda=\hat{\Pi}'+1,\\
&A(z)=F(u), \quad B(z)=E(u).
\end{aligned}$$

6 Discussions

6.1 Classical limit

We have not been able so far to identify the L-operator constructed in section 4 with the one obtained by a quasi-Hopf twist [21]. In this subsection, we study the classical limit of the elliptic algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ in the RLL formulation, and compare it with the half currents (4.6), (4.7) in the quantum case.

Let $\mathfrak{a} = \mathfrak{sl}_2$, with the standard generators e, f, h. Let $(\ ,\ )$ be the invariant inner product normalized as $(h,h) = 2$, $(e,f) = (f,e) = 1$. We consider the homogeneous realization of the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$,

$$
\begin{aligned}
&\mathfrak{g} = \mathrm{span}\{e_n, f_n, h_n\ (n \in \mathbb{Z}), d, c\},\\
&[x_m, y_n] = [x,y]_{m+n} + m(x,y)\delta_{m+n,0}c,\\
&[d, x_m] = m x_m,\\
&c:\ \text{central}\ .
\end{aligned}
$$

In what follows, we identify $h_0 \in \mathfrak{g}$ with $h \in \mathfrak{a}$.

Let us recall the notion of a quasi-Lie bialgebra [20] which is the classical counterpart of a quasi-Hopf algebra. By definition, it is a triple $(\mathfrak{g}, \delta, \varphi)$ consisting of a Lie algebra $\mathfrak{g}$, a 1-cocycle (cobracket) $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ and a tensor $\varphi \in \wedge^3\mathfrak{g}$, satisfying

$$
\frac{1}{2}\mathrm{Alt}(\delta\otimes 1)\delta(x) = [x^{(1)} + x^{(2)} + x^{(3)}, \varphi], \tag{6.1}
$$
$$
\mathrm{Alt}(\delta\otimes 1\otimes 1)\varphi = 0. \tag{6.2}
$$

Here the symbol Alt stands for skew-symmetrization. In the case of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$, the corresponding quasi-Lie bialgebra structure on $\mathfrak{g}$ is given as follows [19]:

$$
\delta(x) = [x^{(1)} + x^{(2)}, r], \tag{6.3}
$$
$$
\varphi = -2\left(D^{(1)}r^{(23)} - D^{(2)}r^{(13)} + D^{(3)}r^{(12)}\right). \tag{6.4}
$$

Here r denotes the classical r matrix

$$
\begin{aligned}
r = \frac{1}{2}h\otimes h + \sum_{n\neq 0}\frac{1}{1-p^n}h_n\otimes h_{-n} + 2\sum_{n\in\mathbb{Z}}\frac{1}{1-wp^n}e_n\otimes f_{-n}\\
+2\sum_{n\in\mathbb{Z}}\frac{1}{1-w^{-1}p^n}f_n\otimes e_{-n} + c\otimes d + d\otimes c,
\end{aligned} \tag{6.5}
$$

with p, w being parameters (having the same meaning as in the body of the text), and we have set

$$
D^{(i)}r^{(jk)} = \left(c^{(i)}p\frac{\partial}{\partial p} + h^{(i)}w\frac{\partial}{\partial w}\right)r^{(jk)}.
$$

Let $\rho_z : \mathfrak{g}' \to \mathfrak{a} \otimes \mathbb{C}[z, z^{-1}]$ ($\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$) be the evaluation morphism given by $\rho_z(x_n) = xz^n$ ($x = e, f, h$), $\rho_z(c) = 0$. Setting $r' = r - c \otimes d - d \otimes c$, we define

$$\mathcal{L}^+(z) = (\rho_z \otimes \mathrm{id})\, r', \qquad r(z) = (\mathrm{id} \otimes \rho_1)\, \mathcal{L}^+(z).$$

These are formal series with values in $\mathfrak{a} \otimes \mathfrak{g}$ and $\mathfrak{a} \otimes \mathfrak{a}$, respectively. Explicitly we have

$$\begin{aligned}
\mathcal{L}^+(z) \;&=\; h \otimes \left(\frac{1}{2} h + \sum_{n \neq 0} \frac{1}{1 - p^n} z^n h_{-n} \right) \\
&\quad + e \otimes \left(\sum_{n \in \mathbb{Z}} \frac{2}{1 - wp^n} z^n f_{-n} \right) + f \otimes \left(\sum_{n \in \mathbb{Z}} \frac{2}{1 - w^{-1} p^n} z^n e_{-n} \right), \qquad (6.6) \\
r(z) \;&=\; h \otimes h \left(\frac{1}{2} + \sum_{n \neq 0} \frac{1}{1 - p^n} z^n \right) \\
&\quad + e \otimes f \left(\sum_{n \in \mathbb{Z}} \frac{2}{1 - wp^n} z^n \right) + f \otimes e \left(\sum_{n \in \mathbb{Z}} \frac{2}{1 - w^{-1} p^n} z^n \right). \qquad (6.7)
\end{aligned}$$

Up to a change of 'gauge' and the extra zero-mode operators P, Q, (6.6) agrees with the classical limit of the L operator based on the half currents (4.6), (4.7).

According to Drinfeld [20], there is a bijective correspondence between quasi-Lie bialgebras and Manin pairs. This means the following. Let $D\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ be the direct sum with the dual vector space $\mathfrak{g}^*$. Equip it with the inner product $(\ ,\ )$ requiring that it vanishes on $\mathfrak{g} \times \mathfrak{g}$, $\mathfrak{g}^* \times \mathfrak{g}^*$ and coincides with the canonical paring on $\mathfrak{g} \times \mathfrak{g}^*$. Then, for $(\mathfrak{g}, \delta, \varphi)$ a quasi-Lie bialgebra, $D\mathfrak{g}$ is endowed with a unique Lie algebra structure (the classical double) such that $(\ ,\ )$ is an invariant inner product for $D\mathfrak{g}$. Moreover the correspondence $(\mathfrak{g}, \delta, \varphi) \leftrightarrow D\mathfrak{g}$ is one-to-one.

In the present case, let us take the dual basis e_n^*, f_n^*, h_n^* ($n \in \mathbb{Z}$), d^*, c^* of $\mathfrak{g}^*$, with the dual pairing given by

$$\langle x_m, y_n^* \rangle = \langle x, y \rangle \delta_{m+n,0}, \quad \langle d, c^* \rangle = 1, \quad \langle c, d^* \rangle = 1, \quad \text{others } = 0.$$

Set[4)]

$$\begin{aligned}
\mathcal{L}^-(z) &= \frac{1}{2} h \otimes h^-(z) + e \otimes f^-(z) + f \otimes e^-(z), \qquad (6.8) \\
x^-(z) &= \sum_{n \in \mathbb{Z}} x_n^* z^{-n} \qquad (x = e, f, h).
\end{aligned}$$

Then the dual pairing takes the form

$$\langle \mathcal{L}^{+(1)}(z_1), \mathcal{L}^{-(2)}(z_2) \rangle = r^{(12)}(z_1/z_2).$$

[4)] The $\mathcal{L}^-(z)$ is a generating series in $\mathfrak{g}^*$ and is independent of $\mathcal{L}^+(z)$. It should not be confused with the classical limit of the $L^-(z)$ operator in [21] which has a simple relation with $L^+(z)$.

With the above notation, the Lie algebra structure of $D\mathfrak{g}$ can be described as follows.

$$
\begin{aligned}
&[\mathcal{L}^{\pm(1)}(z_1),\mathcal{L}^{\pm(2)}(z_2)] = -[r^{(12)}(z_1/z_2),\mathcal{L}^{\pm(1)}(z_1)+\mathcal{L}^{\pm(2)}(z_2)]\\
&\quad \pm 2w\frac{\partial}{\partial w}\left(h^{(1)}\mathcal{L}^{+(2)}(z_2)-h^{(2)}\mathcal{L}^{+(1)}(z_1)+hr^{(12)}(z_1/z_2)\right)\pm 2c\,p\frac{\partial}{\partial p}r^{(12)}(z_1/z_2),\\
&[\mathcal{L}^{+(1)}(z_1),\mathcal{L}^{-(2)}(z_2)] = -[r^{(12)}(z_1/z_2),\mathcal{L}^{+(1)}(z_1)+\mathcal{L}^{-(2)}(z_2)]\\
&\quad +2w\frac{\partial}{\partial w}\left(h^{(1)}\mathcal{L}^{+(2)}(z_2)-h^{(2)}\mathcal{L}^{+(1)}(z_1)+hr^{(12)}(z_1/z_2)\right)\\
&\quad +2cp\frac{\partial}{\partial p}r^{(12)}(z_1/z_2)-c^*z\frac{\partial}{\partial z}r^{(12)}(z_1/z_2),\\
&c,c^*:\ \text{central}\ ,\quad [d,d^*]=0,\\
&[d,\mathcal{L}^{\pm}(z)] = -z\frac{\partial}{\partial z}\mathcal{L}^{\pm}(z),\\
&[d^*,\mathcal{L}^{\pm}(z)] = \pm 2p\frac{\partial}{\partial p}\mathcal{L}^{+}(z)-z\frac{\partial}{\partial z}\widetilde{\mathcal{L}}^{-}(z).
\end{aligned}
$$

Here we have set

$$
\begin{aligned}
\widetilde{\mathcal{L}}^{-}(z) \;&=\; h\otimes\left(\frac{1}{2}h+\sum_{n\neq 0}\frac{1}{1-p^n}z^n h^*_{-n}\right)\\
&\quad +e\otimes\left(\sum_{n\in\mathbb{Z}}\frac{2}{1-wp^n}z^n f^*_{-n}\right)+f\otimes\left(\sum_{n\in\mathbb{Z}}\frac{2}{1-w^{-1}p^n}z^n e^*_{-n}\right).
\end{aligned}
$$

Notice that, because of the quasi-Lie nature of $(\mathfrak{g},\delta,\varphi)$, $\mathfrak{g}$ is a Lie subalgebra of the double $D\mathfrak{g}$ whereas $\mathfrak{g}^*$ is only a linear subspace (the Lie bracket does not close inside $\mathfrak{g}^*$).

A similar description is possible for the classical limit of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$, but we omit the details.

6.2 Comparison with Enriquez-Felder

In [17], Enriquez and Felder studied the quasi-Hopf structure of an elliptic algebra $U_\hbar\mathfrak{g}(\tau)$ associated with the face-type R matrix. This algebra $U_\hbar\mathfrak{g}(\tau)$ contains a central element K. Roughly speaking, $U_\hbar\mathfrak{g}(\tau)$ and $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ are central extensions of the same algebra, as we already mentioned in section 3.2. Let us examine the main differences between the two algebras.

The formulation of [17] starts with an elliptic curve with modulus τ and a coordinate u on it. The latter plays the role of an 'additive' spectral parameter, to be compared with our 'multiplicative' spectral parameter $z=q^{2u}$. In the classical case, the relevant Manin pair in [17] is defined by assigning 'positive' and 'negative' parts *in powers of* u. Accordingly, in the construction of the half currents, the integration contours are chosen around a point in the u-plane. In our case, the integrations are taken along a circle around the origin on the z-plane.

A more serious (perhaps related) difference arises in the quantum case. In [17], the curve is fixed throughout quantization. In contrast, in the presence of the central term c, we have to deal simultaneously with two different elliptic curves with nomes p and $p^* = pq^{-2c}$. This makes it difficult to adapt the geometric construction of [17].

The nature of the central element $K \in U_\hbar\mathfrak{g}(\tau)$ and $c \in U_{q,p}(\widehat{\mathfrak{sl}}_2)$ are quite different. Being dual to the grading element $d \in \mathfrak{g}$, K seems similar to the element $c^* \in \mathfrak{g}^*$ in the double $D\mathfrak{g}$ discussed in the previous subsection. For infinite dimensional representations and bosonization, we feel that it is more natural to consider extension by c. A similar distinction has been discussed in the context of the double Yangian [28]. In [17], $U_\hbar\mathfrak{g}(\tau)$ is initially endowed with a simple Hopf-algebra structure given by a Drinfeld-type coproduct. The quasi-Hopf structure related to the dynamical RLL relation is then obtained by constructing a suitable twist from the initial Hopf structure. Such a Drinfeld-type coproduct persists in the presence of c as well, but it provides only a quasi-Hopf structure (Appendix B).

Let us also mention a 'physical' reason why we prefer z to u. Recall Baxter's corner transfer matrix (CTM) method [29]. A CTM is composed of a product of infinitely many Boltzmann weights. For the elliptic models, the individual weights (with appropriate normalization) are doubly periodic functions in u. The most important property of CTM is that, in the infinite lattice limit, the eigenvalues are all simple integral powers of z. This means that, in passing to the infinite lattice limit, one of the periods is lost. It is because the infinite lattice limit makes sense only inside the physical region, whereas shifting by another period takes us out of that region. Intuitively the L-operators generating the elliptic algebra are also products of Boltzmann weights on a single row of the lattice. The above argument indicates that in infinite dimensional representations the currents of the algebra possess only one period, and that z is the natural variable to use.

6.3 Space of states

Let us discuss how we view the space of states of the k-fusion unrestricted ABF model in connection with the algebras $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$, $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The parameter r^* in $\lambda = (r^*+2)d + (s+1)\frac{1}{2}h$ corresponds to the elliptic nome. We shall argue below that the parameter s corresponds to the boundary height degrees of freedom.

First consider the 'low temperature' limit $p, q \to 0$. Let us recall the 'paths' of the spin $k/2$-XXZ model [30]. A vertex-path v is a semi-infinite sequence $v = (\cdots, v(2), v(1))$, where $v(l) \in \{0, 1, \cdots, k\}$. We have $k+1$ different ground state vertex-paths $\bar{v}_m$ ($m = 0, 1, \cdots, k$) given by

$$\bar{v}_m(l) = \begin{cases} m & \text{for } l \equiv 0 \bmod 2 \\ k-m & \text{for } l \equiv 1 \bmod 2. \end{cases} \tag{6.9}$$

We say v is an m-vertex-path if it satisfies the boundary condition $v(l) = \bar{v}_m(l) \quad \text{for } l \gg 1$. We assign a weight to an m-vertex-path by the formula

$$\mathrm{wt}(v) = m(\Lambda_1 - \Lambda_0) + \sum_{l>0} (\bar{v}_m(l) - v(l))\alpha_1 - h(v)(\alpha_0 + \alpha_1),$$

where $h(v)$ is the 'energy' of the path v (see [30] for definition). The collection of all m-vertex-paths can be regarded as (the low temperature limit of) the space on which the CTM of the fusion vertex model acts. Its character $\sum_v q^{h(v)} z^{(wt(v),h)}$ (sum over the m-vertex-paths) is known to be the same as the character of the integrable highest weight $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\mu_m)$ of highest weight

$$\mu_m = (k-m)\Lambda_0 + m\Lambda_1. \tag{6.10}$$

Next we consider the paths for the unrestricted k-fusion ABF model. A face-path s is a semi-infinite sequence $s = (\cdots, s(1), s(0))$ of integers $s(l) \in \mathbb{Z}$, subject to the admissibility condition $s(l) - s(l-1) \in \{k, k-2, \cdots, -k\}$ for $l \geq 1$. A face-path s is called an (m,n)-face-path ($m \in \{0, 1, \cdots, k\}$, $n \in \mathbb{Z}$) if it satisfies the boundary condition $s(l) = \bar{s}_{m,n}(l) \quad \text{for } l \gg 1$, where $\bar{s}_{m,n}$ signifies the ground state face-path

$$\begin{aligned} \bar{s}_{m,n}(0) &= n + m, \\ \bar{s}_{m,n}(l) - \bar{s}_{m,n}(l-1) &= 2\bar{v}_m(l) - k. \end{aligned}$$

From a face-path s, we can construct a vertex-path as $(\cdots, (s(2)-s(1)+k)/2, (s(1)-s(0)+k)/2)$, and conversely we obtain a unique face-path from a vertex-path up to a uniform shift $s(l) \to s(l) + a$ (for all l). Thus an (m,n)-face-path s is uniquely represented by an m-vertex-path v as

$$\begin{aligned} s(0) &= n + m + \sum_{l>0} 2(\bar{v}_m(l) - v(l)), \\ s(l) - s(l-1) &= 2v(l) - k. \end{aligned}$$

The parameter n determines the 'boundary height at infinity', and the 'bulk configuration' is described by some m-vertex-path v.

Returning to the finite temperature situation $0 < |p| < |q| < 1$, let us consider the space $\mathcal{H}_{m,n}$ for the face model CTM under the boundary condition determined by m, n. As we have seen, for each fixed n, the character of $\mathcal{H}_{m,n}$ is the same as that of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V(\mu_m)$. Since $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ has the same representations as the underlying algebra $U_q(\widehat{\mathfrak{sl}}_2)$, $\mathcal{H}_{m,n}$ can also be viewed as the level k irreducible highest weight module $V(\mu_m; r^*, s)$ over $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ with $\lambda = (r^*+2)d + (s+1)\frac{1}{2}h$. Let us consider the relation between n and the parameter s. As we have discussed in the main text, the algebra

$U_{q,p}(\widehat{\mathfrak{sl}}_2)$ consists of two sectors, $U_q(\widehat{\mathfrak{sl}}_2)$ and a Heisenberg algebra generated by P, Q with $[P,Q] = -1$. A representation of this Heisenberg algebra is given by the zero mode lattice spanned by $e^{-nQ}|0\rangle$ $(n \in \mathbb{Z})$, where $P|0\rangle = 0$. The operator P takes a fixed value n on each state $e^{-nQ}|0\rangle$. Thus we can regard the direct sum

$$\bigoplus_{\substack{n\in\mathbb{Z} \\ m=0,1,\cdots,k}} V(\mu_m; r^*, n) \otimes e^{-nQ} \tag{6.11}$$

as a $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module. This suggests that it is natural to identify s with n. The above picture is consistent with the manner in which the dynamical shift appears in the RLL relation for $L^+(u,P)$.

In summary, we identify the $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module (6.11) with the space $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_{\substack{n\in\mathbb{Z} \\ m=0,1,\cdots,k}} \mathcal{H}_{m,n}, \tag{6.12}$$

on which the CTM of the k-fusion unrestricted ABF model acts under all possible boundary conditions.

6.4 Further issues

Finally let us mention some related works and open problems.

(i) In [14], the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was found by deforming the free field realization of the coset CFT $U(\widehat{\mathfrak{sl}}_2)_l \otimes U(\widehat{\mathfrak{sl}}_2)_k / U(\widehat{\mathfrak{sl}}_2)_{l+k}$. As has been pointed out in [11], in the case $k=1$, $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ appears as the algebra of screening currents for the deformed Virasoro algebra (DVA). (For the coset-type description of the RSOS model and DVA, see [10, 31].) We wish to understand the conceptual meaning of DVA from the quasi-Hopf point of view.

(ii) We note that the screening operators for the deformed W_{n+1}- algebra coincide with $E(u)$, $F(u)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ at level one up to some cocycle factors which adjust signs in the commutation relations(Appendix A). A 'higher rank extension' of DVA (the deformed W_{n+1}-algebra) and its screening currents have been studied in [32, 33]. The VO's for the $A_n^{(1)}$ face models is constructed in [34] by the use of the screening operator for the deformed W_{n+1}-algebra. The cohomological structure of the Fock module of the deformed W_{n+1}-algebra is studied in [35]. Even though not everything has yet been made clear, these works indicate that the deformed W_{n+1}-algebras play the role of the dynamical symmetry of the $A_n^{(1)}$ face models.

(iii) In [36], the deformed W-algebra $\mathcal{W}_{q,t}(\bar{\mathfrak{g}})$ associated to an arbitrary simple Lie algebra $\bar{\mathfrak{g}}$ has been proposed. It can be regarded as a quantization of the deformed Poisson W-algebra obtained from a difference analogue of the Drinfeld-Sokolov reduction [37, 38]. On the other hand, we have obtained the algebra $U_{q,p}(\mathfrak{g})$ for an arbitrary non-twisted affine Lie algebra $\mathfrak{g}$, extending the results for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as shown in Appendix A. For non-simply laced $\mathfrak{g}$, there is a considerable difference between the Drinfeld-type currents for $U_{q,p}(\mathfrak{g})$ at level one and the screening currents for $\mathcal{W}_{q,t}(\bar{\mathfrak{g}})$. It seems natural to have such a difference since these two have different CFT limits; the former originates from the coset construction $U(\mathfrak{g})_k \otimes U(\mathfrak{g})_l / U(\mathfrak{g})_{k+l}$ and the latter from the Drinfeld-Sokolov reduction of the loop group $G((z))$. We thus expect the existence of another type of deformed W-algebra for non-simply laced $\mathfrak{g}$ corresponding to $U_{q,p}(\mathfrak{g})$.

(iv) A superalgebra version of the W-algebra was proposed recently in [39]. We hope this class can be treated through the Lie superalgebra version of the quasi-Hopf structure [22]. We note also that a construction of some extended version of DVA and deformed W-algebras from the vertex-type elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_{n+1})$ is discussed in the works [40, 41, 42, 43].

(v) It is also tempting to guess that there is a 'higher level extension' of the deformed Virasoro algebra whose screening currents are given by $E(u)$, $F(u)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ at level k. Such an algebra would be a deformation of (fractional) super-Virasoro algebra in CFT (see [14] for detailed discussions). It is an open problem to find such an extended algebraic structure.

(vi) In this paper as well as in [21], we focused attention to the case where the parameters p, q are generic. In terms of lattice models, we considered only the unrestricted SOS case. The RSOS case corresponds to non-generic p and needs special treatment. We wish to understand in particular the mechanism of obtaining possible extra singular vectors.[5)] The structure of the states of the RSOS model was also approached by using the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [10], and the DVA current was constructed within this language [10, 31]. It is desirable to study the relationship between this description and the one based on the quasi-Hopf algebra.

(vii) An algebraic approach to the fusion ABF models has been presented on the basis of the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in this work. Another interesting direction is to study Baxter's eight vertex model and Belavin's

[5)] In this connection we refer to the works [44, 45] on DVA and deformed W-algebras, where a detailed study is made on the Kac-determinant and properties of extra singular vectors at roots of unity.

generalization. Recently, a remarkable bosonization formula of the type I VO for the eight vertex model was proposed by Lashkevich and Pugai [46]. They succeeded in reducing the problem to the already known bosonization for the ABF model through the use of intertwining vectors and Lukyanov's screening operators. To understand their bosonization scheme, it seems necessary to clarify the relationship between the intertwining vectors and the two twistors $F(\lambda)$ and $E(r)$, which define $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ respectively. It is also interesting to seek a more direct bosonization, which is intrinsically connected with the quasi-Hopf structure of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ and does not rely on the bosonization of the ABF model.

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A Elliptic currents for general $\mathfrak{g}$

In this appendix we give the elliptic currents and the algebra $U_{q,p}(\mathfrak{g})$ for non-twisted affine Lie algebras $\mathfrak{g}$.

A.1 $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be an affine Lie algebra of non-twisted type associated with a generalized Cartan matrix $A = (a_{ij})$ and let $\bar{\mathfrak{g}}$ be a corresponding simple finite dimensional Lie algebra. Fixing an invariant bilinear form (,) on the Cartan subalgebra $\mathfrak{h}$, we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via (,). Denoting the simple roots by α_j, we set $b_{ij} = d_i a_{ij} = b_{ji}$ with $d_i = (\alpha_i, \alpha_i)/2$. Hence $B = (b_{ij})$ is the symmetrized Cartan matrix. We also set $q_i = q^{d_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad (n)_i = \frac{q^n - q^{-n}}{q_i - q_i^{-1}}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i![m-n]_i!}. \tag{A.1}$$

Consider the quantum affine algebra $U_q(\mathfrak{g})$ of non-twisted type, and let

$$x_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} x_{i,n}^{\pm} z^{-n},$$

$$\psi_i(q^{c/2}z) = q_i^{h_i} \exp\left((q_i - q_i^{-1}) \sum_{n>0} a_{i,n} z^{-n}\right),$$

$$\varphi_i(q^{-c/2}z) = q_i^{-h_i} \exp\left(-(q_i - q_i^{-1}) \sum_{n>0} a_{i,-n} z^{n}\right)$$

be the Drinfeld currents ($i=1,2,..,\mathrm{rank}\ \bar{\mathfrak{g}}$). The defining relations for $U_q(\mathfrak{g})$ read as follows:

$$
\begin{aligned}
&c:\ \text{central}\ ,\\
&[h_i,d]=0,\quad [d,a_{i,n}]=na_{i,n},\quad [d,x^{\pm}_{i,n}]=nx^{\pm}_{i,n},\\
&[h_i,a_{j,n}]=0,\qquad [h_i,x^{\pm}_j(z)]=\pm a_{ij}x^{\pm}_j(z),\\
&[a_{i,n},a_{j,m}]=\frac{[a_{ij}n]_i[cn)_j}{n}q^{-c|n|}\delta_{n+m,0},\\
&[a_{i,n},x^{+}_j(z)]=\frac{[a_{ij}n]_i}{n}q^{-c|n|}z^n x^{+}_j(z),\\
&[a_{i,n},x^{-}_j(z)]=-\frac{[a_{ij}n]_i}{n}z^n x^{-}_j(z),\\
&(z-q^{\pm b_{ij}}w)x^{\pm}_i(z)x^{\pm}_j(w)=(q^{\pm b_{ij}}z-w)x^{\pm}_i(w)x^{\pm}_j(z),\\
&[x^{+}_i(z),x^{-}_j(w)]=\frac{\delta_{i,j}}{q_i-q_i^{-1}}\left(\delta(q^{-c}\frac{z}{w})\psi_i(q^{c/2}w)-\delta(q^{c}\frac{z}{w})\varphi_j(q^{-c/2}w)\right),\\
&\sum_{\sigma\in S_a}\sum_{l=0}^{a}(-)^l\begin{bmatrix}a\\l\end{bmatrix}_i x^{\pm}_{i,m_{\sigma(1)}}\cdots x^{\pm}_{i,m_{\sigma(l)}}x^{\pm}_{j,m}x^{\pm}_{i,m_{\sigma(l+1)}}\cdots x^{\pm}_{i,m_{\sigma(a)}}=0,\\
&\qquad (i\neq j,\ a=1-a_{ij},\ m_1,\cdots,m_a\in\mathbb{Z}).
\end{aligned}
$$

In the last line, S_a denotes the symmetric group on a letters.

A.2 Elliptic currents

Let us introduce the currents $u^{\pm}_i(z,p)\in U_q(\mathfrak{g})$ with $p=q^{2r}$ by

$$
\begin{aligned}
u^{+}_i(z,p)&=\exp\left(\sum_{n>0}\frac{1}{[r^*n)_i}a_{i,-n}(q^r z)^n\right),\\
u^{-}_i(z,p)&=\exp\left(-\sum_{n>0}\frac{1}{[rn)_i}a_{i,n}(q^{-r}z)^{-n}\right).
\end{aligned}
$$

Then the following commutation relations hold.

Lemma A.1

$$u^{+}_i(z,p)x^{+}_j(w)=\frac{(p^*q^{b_{ij}}z/w;p^*)_\infty}{(p^*q^{-b_{ij}}z/w;p^*)_\infty}x^{+}_j(w)u^{+}_i(z,p), \tag{A.2}$$

$$u^{+}_i(z,p)x^{-}_j(w)=\frac{(p^*q^{-b_{ij}+c}z/w;p^*)_\infty}{(p^*q^{b_{ij}+c}z/w;p^*)_\infty}x^{-}_j(w)u^{+}_i(z,p), \tag{A.3}$$

$$u^{-}_i(z,p)x^{+}_j(w)=\frac{(pq^{-b_{ij}-c}z/w;p)_\infty}{(pq^{b_{ij}-c}z/w;p)_\infty}x^{+}_j(w)u^{-}_i(z,p), \tag{A.4}$$

$$u^{-}_i(z,p)x^{-}_j(w)=\frac{(pq^{b_{ij}}z/w;p)_\infty}{(pq^{-b_{ij}}z/w;p)_\infty}x^{-}_j(w)u^{-}_i(z,p), \tag{A.5}$$

$$u_i^+(z,p)u_j^-(w,p)=\frac{(pq^{-c-b_{ij}}z/w;p)_\infty}{(pq^{-c+b_{ij}}z/w;p)_\infty}\frac{(p^*q^{c+b_{ij}}z/w;p^*)_\infty}{(p^*q^{c-b_{ij}}z/w;p^*)_\infty}u_j^-(w,p)u_i^+(z,p). \quad (A.6)$$

Define the 'dressed' currents $x_i^\pm(z,p)$, $\psi_i^\pm(z,p)$ in $U_q(\mathfrak{g})$ by

$$x_i^+(z,p)=u_i^+(z,p)x_i^+(z), \quad (A.7)$$
$$x_i^-(z,p)=x_i^-(z)u_i^-(z,p), \quad (A.8)$$
$$\psi_i^+(z,p)=u_i^+(q^{c/2}z,p)\psi_i(z)u_i^-(q^{-c/2}z,p), \quad (A.9)$$
$$\psi_i^-(z,p)=u_i^+(q^{-c/2}z,p)\varphi_i(z)u_i^-(q^{c/2}z,p). \quad (A.10)$$

Set $e_i(z)=x_i^+(z,p)$ and $f_i(z)=x_i^-(z,p)$. From Lemma A.1, we obtain

Proposition A.2

$$[h_i,a_{j,n}]=0,\quad [h_i,e_j(z)]=a_{ij}e_j(z),\quad [h_i,f_j(z)]=-a_{ij}f_j(z), \quad (A.11)$$
$$[d,h_i]=0,\quad [d,a_{i,n}]=na_{i,n}, \quad (A.12)$$
$$[d,e_i(z)]=-z\frac{\partial}{\partial z}e_i(z),\quad [d,f_i(z)]=-z\frac{\partial}{\partial z}f_i(z), \quad (A.13)$$
$$[a_{i,n},a_{j,m}]=\frac{[a_{ij}n]_i[cn]_j}{n}q^{-c|n|}\delta_{n+m,0}, \quad (A.14)$$
$$[a_{i,n},e_j(z)]=\frac{[a_{ij}n]_i}{n}q^{-c|n|}z^ne_j(z), \quad (A.15)$$
$$[a_{i,n},f_j(z)]=-\frac{[a_{ij}n]_i}{n}z^nf_j(z), \quad (A.16)$$
$$z\Theta_{p^*}\left(q^{b_{ij}}w/z\right)e_i(z)e_j(w)=-w\Theta_{p^*}\left(q^{b_{ij}}z/w\right)e_j(w)e_i(z), \quad (A.17)$$
$$z\Theta_p\left(q^{-b_{ij}}w/z\right)f_i(z)f_j(w)=-w\Theta_p\left(q^{-b_{ij}}z/w\right)f_j(w)f_i(z), \quad (A.18)$$
$$[e_i(z),f_j(w)]=\frac{\delta_{i,j}}{q_i-q_i^{-1}}\left(\delta(q^{-c}\frac{z}{w})\psi_i^+(q^{c/2}w,p)-\delta(q^c\frac{z}{w})\psi_i^-(q^{-c/2}w,p)\right), \quad (A.19)$$

$$\begin{aligned}&\sum_{\sigma\in S_a}\prod_{1\le k<m\le a}\frac{(p^*q^2z_{\sigma(m)}/z_{\sigma(k)};p^*)_\infty}{(p^*q^{-2}z_{\sigma(m)}/z_{\sigma(k)};p^*)_\infty}\\ &\qquad\times\sum_{l=0}^a(-)^l\begin{bmatrix}a\\ l\end{bmatrix}_i\prod_{k=1}^l\frac{(p^*q^{b_{ij}}z/z_{\sigma(k)};p^*)_\infty(p^*q^{-b_{ij}}z_{\sigma(k)}/z;p^*)_\infty}{(p^*q^{-b_{ij}}z/z_{\sigma(k)};p^*)_\infty(p^*q^{b_{ij}}z_{\sigma(k)}/z;p^*)_\infty} \qquad (A.20)\\ &\qquad\times e_i(z_{\sigma(1)})\cdots e_i(z_{\sigma(l)})e_j(z)e_i(z_{\sigma(l+1)})\cdots e_i(z_{\sigma(a)})=0 \qquad (i\neq j,\ a=1-a_{ij}),\end{aligned}$$

$$\begin{aligned}&\sum_{\sigma\in S_a}\prod_{1\le k<m\le a}\frac{(pq^{-2}z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}{(pq^2z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}\\ &\qquad\times\sum_{l=0}^a(-)^l\begin{bmatrix}a\\ l\end{bmatrix}_i\prod_{k=1}^l\frac{(pq^{b_{ij}}z/z_{\sigma(k)};p)_\infty(pq^{-b_{ij}}z_{\sigma(k)}/z;p)_\infty}{(pq^{-b_{ij}}z/z_{\sigma(k)};p)_\infty(pq^{b_{ij}}z_{\sigma(k)}/z;p)_\infty} \qquad (A.21)\\ &\qquad\times f_i(z_{\sigma(1)})\cdots f_i(z_{\sigma(l)})f_j(z)f_i(z_{\sigma(l+1)})\cdots f_i(z_{\sigma(a)})=0 \qquad (i\neq j,\ a=1-a_{ij}).\end{aligned}$$

A.3 $U_{q,p}(\mathfrak{g})$

Let us introduce further a set of generators of the Heisenberg algebra $\{P_i, Q_i\}$ ($i = 1, 2, .., \mathrm{rank}\ \bar{\mathfrak{g}}$) which commute with $U_q(\mathfrak{g})$ and satisfy

$$[P_i, e^{Q_j}] = -\frac{a_{ij}}{2} e^{Q_j}. \tag{A.22}$$

Setting $\tilde{P}_i = d_i P_i$, $\tilde{h}_i = d_i h_i$ and

$$\begin{aligned}
&E_i(u) = e_i(z) e^{2Q_i} z^{-\frac{\tilde{P}_i - 1}{r^*}},\\
&F_i(u) = f_i(z) z^{\frac{\tilde{P}_i + \tilde{h}_i - 1}{r}},\\
&H_i^{\pm}(z) = \psi_i^{\pm}(z) e^{2Q_i} (q^{\pm(r - \frac{c}{2})} z)^{-\frac{r - r^*}{r r^*}\tilde{P}_i + \frac{1}{r}\tilde{h}_i},\\
&\hat{d} = d - \Delta^* + \Delta,\\
&\Delta^* = \frac{1}{2r^*}\sum_{i,j} (B^{-1})_{ij} (\tilde{P}_i - 1)(\tilde{P}_j - 3), \quad \Delta = \frac{1}{2r}\sum_{i,j} (B^{-1})_{ij} (\tilde{P}_i + \tilde{h}_i - 1)(\tilde{P}_j + \tilde{h}_j - 3),
\end{aligned}$$

with $z = q^{2u}$, we have

$$c: \ \text{central}, \tag{A.23}$$

$$[h_i, a_{j,n}] = 0, \quad [h_i, E_j(u)] = a_{ij} E_j(u), \quad [h_i, F_j(u)] = -a_{ij} F_j(u), \tag{A.24}$$

$$[\hat{d}, h_i] = 0, \quad [\hat{d}, a_{i,n}] = n a_{i,n}, \tag{A.25}$$

$$[\hat{d}, E_i(u)] = \left(-z\frac{\partial}{\partial z} + \frac{1}{r^*}\right) E_i(u), \quad [\hat{d}, F_i(u)] = \left(-z\frac{\partial}{\partial z} + \frac{1}{r}\right) F_i(u), \tag{A.26}$$

$$[a_{i,n}, a_{j,m}] = \frac{[a_{ij} n]_i [cn]_j}{n} q^{-c|n|} \delta_{n+m,0}, \tag{A.27}$$

$$[a_{i,n}, E_j(u)] = \frac{[a_{ij} n]_i}{n} q^{-c|n|} z^n E_j(u), \tag{A.28}$$

$$[a_{i,n}, F_j(u)] = -\frac{[a_{ij} n]_i}{n} z^n F_j(u), \tag{A.29}$$

$$\theta^*\left(u - v - \frac{b_{ij}}{2}\right) E_i(u) E_j(v) = \theta^*\left(u - v + \frac{b_{ij}}{2}\right) E_j(v) E_i(u), \tag{A.30}$$

$$\theta\left(u - v + \frac{b_{ij}}{2}\right) F_i(u) F_j(v) = \theta\left(u - v - \frac{b_{ij}}{2}\right) F_j(v) F_i(u), \tag{A.31}$$

$$[E_i(u), F_j(v)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c}\frac{z}{w}) H_i^+(q^{c/2} w) - \delta(q^{c}\frac{z}{w}) H_i^-(q^{-c/2} w)\right), \tag{A.32}$$

$$\begin{aligned}
&\sum_{\sigma \in S_a} \prod_{1 \le k < m \le a} z_{\sigma(k)}^{-\frac{2}{r^*}} \frac{(p^* q^2 z_{\sigma(m)}/z_{\sigma(k)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(m)}/z_{\sigma(k)}; p^*)_\infty}\\
&\quad \times \sum_{l=0}^{a} (-)^l \begin{bmatrix} a \\ l \end{bmatrix}_i \prod_{k=1}^{l} \left(\frac{z}{z_{\sigma(k)}}\right)^{\frac{b_{ij}}{r^*}} \frac{(p^* q^{b_{ij}} z/z_{\sigma(k)}; p^*)_\infty (p^* q^{-b_{ij}} z_{\sigma(k)}/z; p^*)_\infty}{(p^* q^{-b_{ij}} z/z_{\sigma(k)}; p^*)_\infty (p^* q^{b_{ij}} z_{\sigma(k)}/z; p^*)_\infty}\\
&\quad \times E_i(u_{\sigma(1)}) \cdots E_i(u_{\sigma(l)}) E_j(u) E_i(u_{\sigma(l+1)}) \cdots E_i(u_{\sigma(a)}) = 0 \qquad (i \neq j, \ a = 1 - a_{ij}),
\end{aligned} \tag{A.33}$$

$$
\begin{aligned}
&\sum_{\sigma\in S_a}\prod_{1\le k<m\le a} z_{\sigma(k)}^{\frac{2}{r}}\frac{(pq^{-2}z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}{(pq^{2}z_{\sigma(k)}/z_{\sigma(m)};p)_\infty}\\
&\quad\times\sum_{l=0}^{a}(-)^l\begin{bmatrix} a\\ l\end{bmatrix}_i\prod_{k=1}^{l}\left(\frac{z}{z_{\sigma(k)}}\right)^{\frac{b_{ij}}{r}}\frac{(pq^{b_{ij}}z/z_{\sigma(k)};p)_\infty(pq^{-b_{ij}}z_{\sigma(k)}/z;p)_\infty}{(pq^{-b_{ij}}z/z_{\sigma(k)};p)_\infty(pq^{b_{ij}}z_{\sigma(k)}/z;p)_\infty} \qquad (A.34)\\
&\quad\times F_i(u_{\sigma(1)})\cdots F_i(u_{\sigma(l)})F_j(u)F_i(u_{\sigma(l+1)})\cdots F_i(u_{\sigma(a)})=0 \qquad (i\neq j,\ a=1-a_{ij}).
\end{aligned}
$$

These are the generalizations of the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Free field realizations of $U_{q,p}(\mathfrak{g})$ for $\mathfrak{g}=A_n^{(1)},\ B_n^{(1)}$ and $D_n^{(1)}$ are easily obtained. We will report on this subject in a future publication.

Remark. Comparing $U_{q,p}(\widehat{\mathfrak{sl}}_n)$ at level one with the relations among the screening currents of the deformed W_n-algebra[32], we have a difference by a sign factor $(-)^{a_{ij}}$. However, such discrepancy always occurs in the free field realization of (quantum) affine Lie algebras and is known to be adjusted by using cocycle factors[47] or by using a procedure of central extension of the group algebra of the weight lattice. After such an adjustment, one can regard the algebra $U_{q,p}(\widehat{\mathfrak{sl}}_n)$ at level one as the algebra of the screening currents of the deformed W_n-algebra.

B Drinfeld coproduct

Besides the standard Hopf algebra structure, the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is also endowed with the so-called Drinfeld coproduct:

$$\Delta_\infty x = x\otimes 1+1\otimes x \qquad (x=h,c,d), \tag{B.1}$$

$$\Delta_\infty a_n = a_n\otimes 1+q^{-c^{(1)}|n|}\otimes a_n, \tag{B.2}$$

$$\Delta_\infty x^+(z) = x^+(q^{-c^{(2)}}z)\otimes\psi^+(q^{-c^{(2)}/2}z)+1\otimes x^+(z), \tag{B.3}$$

$$\Delta_\infty x^-(z) = x^-(z)\otimes 1+\psi^-(q^{-c^{(1)}/2}z)\otimes x^-(q^{-c^{(1)}}z). \tag{B.4}$$

The universal R matrix $\mathcal{R}_\infty$ associated with this coproduct is given in [48]. We have also an elliptic analog of the Drinfeld coproduct given as follows:

$$\Delta_{p,\infty} x = x\otimes 1+1\otimes x \qquad (x=h,c,d), \tag{B.5}$$

$$\Delta_{p,\infty} a_n = \begin{cases} a_n\otimes 1+\dfrac{[rn]}{[(r-c^{(1)})n]}\otimes a_n & (n>0),\\[2ex] \dfrac{[(r-c^{(1)}-c^{(2)})n]}{[(r-c^{(1)})n]}a_n\otimes q^{c^{(2)}n}+q^{c^{(1)}n}\otimes a_n & (n<0),\end{cases} \tag{B.6}$$

$$\Delta_{p,\infty} e(z,p) = e(q^{-c^{(2)}}z,p)\otimes\psi^+(q^{-c^{(2)}/2}z,pq^{-2c^{(1)}})+1\otimes e(z,pq^{-2c^{(1)}}), \tag{B.7}$$

$$\Delta_{p,\infty} f(z,p) = f(z,p)\otimes 1+\psi^-(q^{-c^{(1)}/2}z,p)\otimes f(q^{-c^{(1)}}z,pq^{-2c^{(1)}}). \tag{B.8}$$

In terms of $\psi^{\pm}(z,p)$ we have

$$\Delta_{p,\infty}\psi^{\pm}(z,p) = \psi^{\pm}(q^{\mp c^{(2)}/2}z,p) \otimes \psi^{\pm}(q^{\pm c^{(1)}/2}z, pq^{-2c^{(1)}}). \tag{B.9}$$

As it turns out, this coproduct is obtained by a twist of (B.1)-(B.4). Set

$$F_{\infty}(p) = \exp\left(-\sum_{n>0}\frac{n}{[(r-c^{(1)})n][2n]}q^{rn}a_{-n}\otimes a_n\right). \tag{B.10}$$

Then a simple computation shows the following.

Proposition B.1 *The twistor (B.10) enjoys the shifted cocycle property*

$$F_{\infty}^{(12)}(p)\,(\Delta_{\infty}\otimes id)\,F_{\infty}(p) = F_{\infty}^{(23)}(pq^{-2c^{(1)}})\,(id\otimes\Delta_{\infty})\,F_{\infty}(p),$$

and satisfies

$$\Delta_{p,\infty}(a) = F_{\infty}(p)\cdot\Delta_{\infty}(a)\cdot F_{\infty}(p)^{-1} \qquad \forall a\in U_q(\widehat{\mathfrak{sl}}_2).$$

The universal R matrix associated with this coproduct $\Delta_{p,\infty}$ is given by

$$\mathcal{R}_{\infty}(p) = F_{\infty}^{(21)}(p)\mathcal{R}_{\infty}F_{\infty}(p)^{-1}. \tag{B.11}$$

The classical limit of (B.11) reads

$$r_{\infty}(p) = \frac{1}{2}h\otimes h + 2\sum_{n>0}\frac{p^n}{1-p^n}h_n\wedge h_{-n} + 2\sum_{n} e_n\otimes f_{-n} + c\otimes d + d\otimes c.$$

Upon skew-symmetrization, it gives the limiting case $w\to 0$ of the classical r matrix (6.5). Note that the elliptic parameter p enters only via the 'Cartan' part. A similar classical r-matrix appeared also in the Drinfeld-Sokolov reduction [37].

C Evaluation modules

C.1 Spin $l/2$ modules

Let l be a non-negative integer. We recall here the evaluation module of $U_q(\widehat{\mathfrak{sl}}_2)$ based on the spin $l/2$ representation.

Let $V_l = \oplus_{m=0}^{l}\mathbb{C}v_m^l$, $V_{l,z} = V_l[z,z^{-1}]$. Define operators $h, S^{\pm}$ on V_l by

$$hv_m^l = (l-2m)v_m^l, \qquad S^{\pm}v_m^l = v_{m\mp 1}^l, \tag{C.1}$$

where by convention we set $v_m^l = 0$ for $m < 0$ or $m > l$. In terms of the Drinfeld generators, the evaluation module $(\pi_{l,z}, V_{l,z})$ is defined by the following formulas:

$$\pi_{l,z}(c) = 0, \qquad \pi_{l,z}(d) = z\frac{d}{dz}, \tag{C.2}$$

$$\pi_{l,z}(a_n) = \frac{z^n}{n}\frac{1}{q-q^{-1}}\left((q^n+q^{-n})q^{nh} - (q^{(l+1)n}+q^{-(l+1)n})\right), \tag{C.3}$$

$$\pi_{l,z}(x^{\pm}(z')) = S^{\pm}\Big[\frac{\pm h+l+2}{2}\Big]\delta\left(q^{h\pm 1}\frac{z}{z'}\right). \tag{C.4}$$

In (C.4), $[x]$ means $(q^x - q^{-x})/(q-q^{-1})$.

The images of the elliptic currents (3.6)-(3.9), (3.15) are then given as follows:

$$\pi_{l,z}(k(z')) = \frac{\{q^{r-l}\frac{z}{z'}\}\{q^{r+l+2}\frac{z}{z'}\}}{\{q^{r-l+2}\frac{z}{z'}\}\{q^{r+l+4}\frac{z}{z'}\}}\frac{\{q^{r-l}\frac{z'}{z}\}\{q^{r+l+2}\frac{z'}{z}\}}{\{q^{r-l+2}\frac{z'}{z}\}\{q^{r+l+4}\frac{z'}{z}\}}\frac{(p;p)_\infty}{\Theta_p(q^{r-h}\frac{z'}{z})}, \tag{C.5}$$

$$\pi_{l,z}(\psi^+(z',p)) = q^h\frac{\Theta_p\left(q^{-l-1}\frac{z}{z'}\right)\Theta_p\left(q^{l+1}\frac{z}{z'}\right)}{\Theta_p\left(q^{-1+h}\frac{z}{z'}\right)\Theta_p\left(q^{1+h}\frac{z}{z'}\right)}, \tag{C.6}$$

$$\pi_{l,z}(\psi^-(z',p)) = q^{-h}\frac{\Theta_p\left(q^{-l-1}\frac{z'}{z}\right)\Theta_p\left(q^{l+1}\frac{z'}{z}\right)}{\Theta_p\left(q^{1-h}\frac{z'}{z}\right)\Theta_p\left(q^{-1-h}\frac{z'}{z}\right)}, \tag{C.7}$$

$$\pi_{l,z}(e(z',p)) = S^+\frac{q^{(-h-l)/2}}{1-q^2}\frac{(q^{h+l+2};p)_\infty(pq^{h-l};p)_\infty}{(p;p)_\infty(pq^{-2};p)_\infty}\delta\left(q^{h+1}\frac{z}{z'}\right), \tag{C.8}$$

$$\pi_{l,z}(f(z',p)) = S^-\frac{q^{(h-l)/2}}{1-q^2}\frac{(q^{-h+l+2};p)_\infty(pq^{-h-l};p)_\infty}{(p;p)_\infty(pq^{2};p)_\infty}\delta\left(q^{h-1}\frac{z}{z'}\right), \tag{C.9}$$

where $\{z\} = (z;p,q^4)_\infty$. In the text, we shall also write $V_{l,z}$ as $V_{l,u}$ with $z = q^{2u}$.

C.2 R matrix for spin $l/2$ representation

Let $\mathcal{R}^+ = \mathcal{R}$, $\mathcal{R}^- = \mathcal{R}^{(21)^{-1}}$ be the universal R matrices [6)] of $U_q(\widehat{\mathfrak{sl}}_2)$, and let $\mathcal{R}^+(r,s) = \mathcal{R}(r,s)$, $\mathcal{R}^-(r,s) = \mathcal{R}^{(21)}(r,s)^{-1}$ be the elliptic counterparts. We consider their images in $V_{l,z_1}\otimes V_{m,z_2}$,

$$R^{\pm}_{lm}(z_1/z_2) = (\pi_{l,z_1}\otimes\pi_{m,z_2})\mathcal{R}^{\pm}, \qquad R^{\pm}_{lm}(z_1/z_2,s) = (\pi_{l,z_1}\otimes\pi_{m,z_2})\mathcal{R}^{\pm}(r,s).$$

The former has the form

$$R^{\pm}_{lm}(z) = \rho^{\pm}_{lm}(z)\overline{R}_{lm}(z),$$

where $\rho^{\pm}_{lm}(z) = \rho_{lm}(z^{\pm 1})^{\pm 1}$ is a scalar factor and $\overline{R}_{lm}(z)$ is normalized as $\overline{R}_{lm}(z)v_0^l\otimes v_0^m = v_0^l\otimes v_0^m$. From the formula (3.10) of [30] for $\rho_{lm}(z)$, we find that

$$R^{\pm}_{lm}(z,s)v_0^l\otimes v_0^m = \rho^{\pm}_{lm}(z,p)v_0^l\otimes v_0^m,$$

$$\rho^{+}_{lm}(z,p) = q^{lm/2}\frac{\{pq^{l-m+2}z\}\{pq^{-l+m+2}z\}}{\{pq^{l+m+2}z\}\{pq^{-l-m+2}z\}}\frac{\{q^{l+m+2}z^{-1}\}\{q^{-l-m+2}z^{-1}\}}{\{q^{l-m+2}z^{-1}\}\{q^{-l+m+2}z^{-1}\}}. \tag{C.10}$$

[6)] Our $\mathcal{R}$ here is $\mathcal{R}^{(21)^{-1}}$ in [21], see the remark at the end of section 2.1.

Set further $R^{\pm}_{lm}(z,s)=\rho^{\pm}_{lm}(z,p)\overline{R}_{lm}(z,s)$. Noting that $\overline{R}_{lm}(z)$ is a rational function in z, we find the following relation from (4.8) of [21]:

$$\overline{R}_{lm}(pz,s)=q^{-\frac{1}{2}h^{(1)2}-(s+h^{(2)})h^{(1)}}\cdot\overline{R}_{lm}(z,s)\cdot q^{\frac{1}{2}h^{(1)2}+sh^{(1)}}. \tag{C.11}$$

Let us consider the image of the L^{+} operator (4.22) in the spin $l/2$ representation (C.5)–(C.9). With a suitable base change of the form $v^l_m\to g(h)v^l_m$, we find the following expression:

$$\pi_v(e^{+}(u,s)) = -S^{+}\frac{\theta(\frac{l+h+2}{2})\theta(u-v-\frac{h+1}{2}-s)}{\theta(u-v-\frac{h+1}{2})\theta(s)}b(q^{h+1}w)^{-\frac{s}{r}}q^{-\frac{1}{r}\left(\frac{l+h+2}{2}\right)^2+\frac{3}{4r}}, \tag{C.12}$$

$$\pi_v(f^{+}(u,s)) = S^{-}\frac{\theta(\frac{l-h+2}{2})\theta(u-v+\frac{h-1}{2}+s)}{\theta(u-v-\frac{h-1}{2})\theta(s+h-1)}b^{-1}(q^{h-1}w)^{\frac{s+h-1}{r}}q^{-\frac{1}{r}\left(\frac{l-h+2}{2}\right)^2+\frac{1}{4r}}, \tag{C.13}$$

$$\pi_v(k^{+}_1(u,s)) = -\frac{\varphi_l(u-v-1)}{\theta(u-v-\frac{h+1}{2})}w^{\frac{h}{2r}}q^{\frac{h^2-l(l+2)}{4r}}, \tag{C.14}$$

$$\pi_v(k^{+}_2(u,s)) = -\frac{\theta(u-v-\frac{h-1}{2})}{\varphi_l(u-v)}w^{-\frac{h}{2r}}q^{-\frac{h^2-l(l+2)}{4r}}. \tag{C.15}$$

Here b is a constant, $w=q^{2v}$, and

$$\varphi_l(u)=-z^{-\frac{l}{2r}}\theta(u+\frac{l+1}{2})\rho^{+}_{1l}(z,p)^{-1}. \tag{C.16}$$

We note the relation

$$\varphi_l(u)\varphi_l(u-1)=\theta(u-\frac{l+1}{2})\theta(u+\frac{l+1}{2}).$$

We can get rid of the powers of w and q appearing in (C.12)-(C.15) by the transformation (2.11). Choosing

$$\mu(s,h)=w^{\frac{1}{2r}h(s+\frac{h}{2})}q^{\frac{h^2-l(l+2)}{4r}s+\frac{h^3+(l+1)h^2-(l+1)^2h}{8r}}$$

and $b=1$ we set

$$R^{+}_{1l}(u-v,s)=(\mathrm{id}\otimes\pi_v)L^{+'}(u,s).$$

The result is as follows:

$$R^{+}_{1l}(u,s)=\frac{1}{\varphi_l(u)}\begin{pmatrix}R_{++}(u) & R_{+-}(u)\\ R_{-+}(u) & R_{--}(u)\end{pmatrix}. \tag{C.17}$$

Here $\varphi_l(u)$ is given in (C.16), and the entries $R_{\varepsilon,\varepsilon'}(u)\in\mathrm{End}(V_{l,v})$ are given by

$$R_{++}(u)=-\frac{\theta(u+\frac{h+1}{2})\theta(\frac{h-l}{2}+s)\theta(\frac{h+l}{2}+s+1)}{\theta(s+h+1)\theta(s)}, \tag{C.18}$$

$$R_{+-}(u) = -S^{-}\frac{\theta(\frac{l-h+2}{2})\theta(u+\frac{h-1}{2}+s)}{\theta(s+h-1)}, \tag{C.19}$$

$$R_{-+}(u) = S^{+}\frac{\theta(\frac{l+h+2}{2})\theta(u-\frac{h+1}{2}-s)}{\theta(s)}, \tag{C.20}$$

$$R_{--}(u) = -\theta(u-\frac{h-1}{2}). \tag{C.21}$$

In the simplest case $l=1$, this R-matrix coincides with (4.18) constructed from the image of twistors.

D Free field representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we review the free field representation of $U_q(\widehat{\mathfrak{sl}}_2)$. We then construct a free field representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following the prescription of section 3.

D.1 Bosons

Let a_n be the bosons in section 3.1. In addition to them, we introduce two more kinds of bosons $a_{j,m}$ ($m \in \mathbb{Z}_{\neq 0}$ $j=1,2$) satisfying the commutation relations

$$[a_{1,m}, a_{1,n}] = \frac{[2m][(c+2)m]}{m}\delta_{m+n,0}, \tag{D.1}$$

$$[a_{2,m}, a_{2,n}] = -\frac{[2m][cm]}{m}\delta_{m+n,0}. \tag{D.2}$$

We need also the zero-mode operators Q_j and P_j ($j=0,1,2$) satisfying

$$[P_0, Q_0] = -i, \qquad [P_1, Q_1] = 2(c+2), \qquad [P_2, Q_2] = -2c. \tag{D.3}$$

It is also convenient to introduce the notation

$$\alpha_+ = \sqrt{\frac{r}{cr^*}}, \qquad \alpha_- = -\sqrt{\frac{r^*}{cr}}, \qquad 2\alpha_0 = \alpha_+ + \alpha_- = \sqrt{\frac{c}{rr^*}}. \tag{D.4}$$

Let us set

$$h = -P_2, \qquad \alpha = \frac{1}{c}Q_2, \qquad \beta = -\sqrt{2}\alpha_- iQ_0. \tag{D.5}$$

Then $[h, \alpha] = 2$.

We define the Fock space $\mathcal{F}_{J,\tilde{M}}$ by

$$\mathcal{F}_{J,\tilde{M}} = \bigoplus_{m,m'\in\mathbb{Z}} \mathcal{F}_{J,\tilde{M};m,m'}, \tag{D.6}$$

$$\begin{aligned}\mathcal{F}_{J,\tilde{M};m,m'} &= \mathbb{C}[a_{-1}, a_{-2}, ..; a_{j,-1}, a_{j,-2}, ..(j=1,2)]\\ &\quad \otimes \mathbb{C}e^{\frac{J}{2(c+2)}Q_1} \otimes \mathbb{C}e^{m\beta} \otimes \mathbb{C}e^{\frac{\tilde{M}}{2}\alpha + m'\alpha}.\end{aligned} \tag{D.7}$$

D.2 Bosonization of total currents

Let us introduce the generating functions of bosons (boson fields),

$$\phi_j(A;B,C|z;D) = -\frac{A}{BC}(Q_j + P_j \log z) + \tilde{\phi}_j(A;B,C|z;D), \tag{D.8}$$

$$\tilde{\phi}_j(A;B,C|z;D) = \sum_{m\neq 0} \frac{[Am]}{[Bm][Cm]} a_{j,m} z^{-m} q^{D|m|}, \tag{D.9}$$

and

$$\phi_j^{(\pm)}(A;B|z;C) = \frac{P_j}{2}\log q + (q-q^{-1}) \sum_{m>0} \frac{[Am]}{[Bm]} a_{j,\pm m} z^{\mp m} q^{Cm} \quad (j=1,2). \tag{D.10}$$

We sometimes use the abridgment

$$\phi_j(C|z;D) = \phi_j(A;A,C|z;D), \qquad \phi_j(C|z) = \phi_j(C|z;0). \tag{D.11}$$

Now let us define the 'parafermion fields' $\tilde{\Psi}(z)$ and $\tilde{\Psi}^\dagger(z)$ by $\tilde{\Psi}(z) = \tilde{\Psi}^-(z)$, $\tilde{\Psi}^\dagger(z) = \tilde{\Psi}^+(z)$, with

$$\tilde{\Psi}^\pm(z) = \mp \frac{1}{(q-q^{-1})} \Big(\tilde{\Psi}_I^\pm(z) - \tilde{\Psi}_{II}^\pm(z) \Big), \tag{D.12}$$

$$\tilde{\Psi}_I^\pm(z) =: \exp\Big\{ \pm\tilde{\phi}_2(c|z;\pm\frac{c}{2}) \Big\} \exp\Big\{ -\phi_2^{(+)}\Big(1;2|z;\mp\frac{c+2}{2}\Big) \pm \phi_1^{(+)}\Big(1;2|z;\mp\frac{c}{2}\Big) \Big\} :,$$

$$\tilde{\Psi}_{II}^\pm(z) =: \exp\Big\{ \pm\tilde{\phi}_2(c|z;\pm\frac{c}{2}) \Big\} \exp\Big\{ \phi_2^{(-)}\Big(1;2|z;\mp\frac{c+2}{2}\Big) \mp \phi_1^{(-)}\Big(1;2|z;\mp\frac{c}{2}\Big) \Big\} : .$$

Then we have

Proposition D.1 *The following currents $x^\pm(z)$ and operator d with h, c give a representation of $U_q(\widehat{\mathfrak{sl}}_2)$ on $\mathcal{F}_J = \mathcal{F}_{J,J}$:*

$$x^+(z) = \tilde{\Psi}(z) \ : \exp\Big\{ -\sum_{n\neq 0} \frac{1}{[cn]} a_n z^{-n} \Big\} : e^\beta \ e^\alpha, \tag{D.13}$$

$$x^-(z) = \tilde{\Psi}^\dagger(z) \ : \exp\Big\{ \sum_{n\neq 0} \frac{q^{c|n|}}{[cn]} a_n z^{-n} \Big\} : e^{-\beta} \ e^{-\alpha}, \tag{D.14}$$

$$d = d_{1,2} + d_a, \tag{D.15}$$

where

$$d_{1,2} = -\sum_{m>0} \frac{m^2}{[2m][(c+2)m]} a_{1,-m} a_{1,m} + \sum_{m>0} \frac{m^2}{[2m][cm]} a_{2,-m} a_{2,m} - \frac{P_1(P_1+2)}{4(c+2)}, \tag{D.16}$$

$$d_a = -\sum_{m>0} \frac{m^2 q^{cm}}{[2m][cm]} a_{-m} a_m. \tag{D.17}$$

Note that this representation is slightly different from the one obtained by Matsuo[49]. The main difference is in the identification of the Cartan operator h. See the discussion in subsection D.3.

Note also that $\mathcal{F}_J$ gives a level k highest weight representation of $U_q(\widehat{\mathfrak{sl}}_2)$ for $c=k$ with the highest weight state

$$|J\rangle = 1\otimes e^{\frac{J}{2(k+2)}Q_1}\otimes 1\otimes e^{\frac{J}{2}\alpha}.$$

With a substitution of (D.13) and (D.14) into (3.6) and (3.7), the boson a_n $(n\in\mathbb{Z}_{\neq 0})$, the currents $e(z,p), f(z,p)$ and $h,\ c,\ d$ give a representation of the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_2)$ on $\mathcal{F}_J$. Explicitly, we have

Proposition D.2

$$e(z,p)=\tilde{\Psi}(z)\ \ :\exp\Big\{-\sum_{n\neq 0}\frac{1}{[cn]}a_{0,n}z^{-n}\Big\}:e^{\beta}\ e^{\alpha}, \tag{D.18}$$

$$f(z,p)=\tilde{\Psi}^{\dagger}(z)\ \ :\exp\Big\{\sum_{n\neq 0}\frac{q^{c|n|}}{[cn]}a'_{0,n}z^{-n}\Big\}:e^{-\beta}\ e^{-\alpha}. \tag{D.19}$$

Here we introduced 'dressed' bosons $a_{0,n}$ and $a'_{0,n}$ by

$$a_{0,n}=\begin{cases} a_n & \textit{for } n>0\\ \dfrac{[rn]}{[r^*n]}q^{c|n|}a_n & \textit{for } n<0,\end{cases} \tag{D.20}$$

$$a'_{0,n}=\frac{[r^*n]}{[rn]}a_{0,n} \tag{D.21}$$

satisfying $[a_{0,m},a_{0,n}]=\frac{[2m][cm]}{m}\frac{[rm]}{[r^*m]}\delta_{m+n,0}$ *and* $[a'_{0,m},a'_{0,n}]=\frac{[2m][cm]}{m}\frac{[r^*m]}{[rm]}\delta_{m+n,0}$.

Let us next introduce the Heisenberg algebra generated by P and Q. We realize them as

$$P-1=\sqrt{\frac{2rr^*}{c}}P_0+\frac{r^*}{c}h,\qquad Q=-\sqrt{2}\alpha_0 iQ_0. \tag{D.22}$$

It is easy to check that $[Q,P]=1$ and that P and Q commute with $U_q(\widehat{\mathfrak{sl}}_2)$.

Accordingly, we modify the Fock space $\mathcal{F}_J$ by e^Q to $\hat{\mathcal{F}}_J$,

$$\hat{\mathcal{F}}_J=\bigoplus_{\mu\in\mathbb{Z}}\hat{\mathcal{F}}_{J,\mu},\qquad \hat{\mathcal{F}}_{J,\mu}=\mathcal{F}_J\otimes e^{\mu Q}. \tag{D.23}$$

Now we define the currents $K(z)$, $E(z)$ and $F(z)$ by (3.25)-(3.27) replacing $e(z)$ and $f(z)$ with (D.18) and (D.19), respectively. Let us define also

$$\hat{d}=d-\Delta_{-P+1,r^*}+\Delta_{-P-h+1,r}. \tag{D.24}$$

Then we have[14]

Proposition D.3 *The currents $K(z)$, $E(z)$ and $F(z)$ and h, c, $\hat{d}$ give a representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\hat{\mathcal{F}}_J$. Explicitly, these currents are given by*

$$K(z) =: \exp\Big\{-\phi_0(1;2,r|z)\Big\} :, \tag{D.25}$$

$$E(z) = \Psi(z) \ : \exp\Big\{-\phi_0(c|z)\Big\} :, \tag{D.26}$$

$$F(z) = \Psi^\dagger(z) \ : \exp\Big\{\phi_0'(c|z)\Big\} :, \tag{D.27}$$

where

$$\left.\begin{array}{c}\Psi^\dagger(z)\\ \Psi(z)\end{array}\right\} = \mp\frac{1}{(q-q^{-1})}\Big(\Psi_I^\pm(z) - \Psi_{II}^\pm(z)\Big), \tag{D.28}$$
$$\Psi_{I,II}^\pm(z) = \tilde{\Psi}_{I,II}^\pm(z)\ e^{\mp\alpha}\ z^{\pm\frac{1}{c}h},$$

and

$$\phi_0(A;B,C|z;D) = \frac{A}{BC}\sqrt{\frac{2cr}{r^*}}(iQ_0 + P_0\log z) + \sum_{m\neq 0}\frac{[Am]}{[Bm][Cm]}a_{0,m}z^{-m}q^{D|m|}, \tag{D.29}$$

$$\phi_0'(A;B,C|z;D) = \phi_0(A;B,C|z;D) \quad \text{with } r\leftrightarrow r^*,\ a_{0,m}\to a_{0,m}'. \tag{D.30}$$

Using the field $\phi_2(A|z;D)$, the boson expression for the parafermion current $\Psi(z)$ (resp. $\Psi^\dagger(z)$) is obtained from the one for $\tilde{\Psi}(z)$ (resp. $\tilde{\Psi}^\dagger(z)$) by replacing the field $\tilde{\phi}_2(c|z;-c/2)$ with $\phi_2(c|z;-c/2)$ (resp. $\tilde{\phi}_2(c|z;c/2)$ with $\phi_2(c|z;c/2)$).

Remark. The parameterization of the vacuum charges of the Fock space $\hat{\mathcal{F}}_{J,\tilde{M};m,m',\mu} = \mathcal{F}_{J,\tilde{M};m,m'} \otimes e^{\mu Q}$ is related to those of $\mathcal{F}_{J,M;n'n}$ in [14] as follows. Let us set $\alpha_{n',n} = \frac{1-n'}{2}\alpha_- + \frac{1-n}{2}\alpha_+$. Then

$$\tilde{M} + 2m' = M, \qquad m\beta + \mu Q = -\sqrt{2}\alpha_{n'n}iQ_0 \tag{D.31}$$

with $1-n' = 2m+\mu$ and $1-n = \mu$.

D.3 An alternative form

There is another way of constructing $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ from $U_q(\widehat{\mathfrak{sl}}_2)$ in terms of free bosons. Let us set

$$\bar{h} = -\sqrt{2c}P_0, \qquad \bar{\alpha} = -\sqrt{\frac{2}{c}}iQ_0. \tag{D.32}$$

Then $[\bar{h},\bar{\alpha}] = 2$.

Define the Fock space $\bar{\mathcal{F}}_{J,\bar{M}}$ by

$$\bar{\mathcal{F}}_{J,\bar{M}} = \bigoplus_{m,\bar{m}'\in\mathbb{Z}} \bar{\mathcal{F}}_{J,\bar{M};m,\bar{m}'}, \tag{D.33}$$

$$\bar{\mathcal{F}}_{J,\bar{M};m,\bar{m}'} = \mathbb{C}[a_{-1},a_{-2},..;a_{j,-1},a_{j,-2},..(j=1,2)] \otimes \mathbb{C}e^{\frac{J}{2(c+2)}Q_1} \otimes \mathbb{C}e^{\bar{m}'\alpha} \otimes \mathbb{C}e^{\frac{\bar{M}}{2}\bar{\alpha}+m\bar{\alpha}}. \tag{D.34}$$

Proposition D.4 *The following currents $x^{\pm}(z)$ and operator $\bar{d}$ with $\bar{h}$, c give a representation of $U_q(\widehat{\mathfrak{sl}}_2)$ on $\bar{\mathcal{F}}_J = \bar{\mathcal{F}}_{J,J}$[49]:*

$$x^+(z) = \Psi(z)\ :\exp\Big\{-\sum_{n\neq 0}\frac{1}{[cn]}a_n z^{-n}\Big\}: e^{\bar{\alpha}}z^{\frac{1}{c}\bar{h}}, \tag{D.35}$$

$$x^-(z) = \Psi^\dagger(z)\ :\exp\Big\{\sum_{n\neq 0}\frac{q^{c|n|}}{[cn]}a_n z^{-n}\Big\}: e^{-\bar{\alpha}}z^{-\frac{1}{c}\bar{h}}, \tag{D.36}$$

$$\bar{d} = \bar{d}_{1,2} + \bar{d}_a, \tag{D.37}$$

where

$$\bar{d}_{1,2} = -\sum_{m>0}\frac{m^2}{[2m][(c+2)m]}a_{1,-m}a_{1,m} + \sum_{m>0}\frac{m^2}{[2m][cm]}a_{2,-m}a_{2,m} - \frac{P_1(P_1+2)}{4(c+2)} + \frac{P_2^2}{4c}, \tag{D.38}$$

$$\bar{d}_a = -\sum_{m>0}\frac{m^2q^{cm}}{[2m][cm]}a_{-m}a_m - \frac{\bar{h}^2}{4c}. \tag{D.39}$$

Then we have

Proposition D.5 *Dressing $x^{\pm}(z)$ by the procedure (3.6) and (3.7), we have the following currents $e(z,p)$, $f(z,p)$ with which the boson a_n $(n\in\mathbb{Z}_{\neq 0})$ and h, c, d give a representation of the elliptic currents of $U_q(\widehat{\mathfrak{sl}}_2)$ on $\bar{\mathcal{F}}_J$.*

$$e(z,p) = \Psi(z)\ :\exp\Big\{-\sum_{n\neq 0}\frac{1}{[cn]}a_{0,n}z^{-n}\Big\}: e^{\bar{\alpha}}z^{\frac{1}{c}\bar{h}}, \tag{D.40}$$

$$f(z,p) = \Psi^\dagger(z)\ :\exp\Big\{\sum_{n\neq 0}\frac{q^{c|n|}}{[cn]}a'_{0,n}z^{-n}\Big\}: e^{-\bar{\alpha}}z^{-\frac{1}{c}\bar{h}}. \tag{D.41}$$

In this case, we can obtain $U_{q,p}$ by dressing the elliptic currents via $\bar{\alpha}$ and $\bar{h}$ instead of adjoining P and Q. This is a procedure of turning on the anomalous background charge $2\alpha_0$ in ϕ_0. In conformal field theory, this corresponds to the twist of the energy-momentum tensor by the Cartan operator. Then, the zero-mode lattice associated with $\bar{\alpha}$ gains one additional dimension and becomes 2-dimensional. Hence the Fock space $\bar{\mathcal{F}}_{J,\bar{M}}$ is changed to

$$\bar{\mathcal{F}}'_J = \bigoplus_{\bar{m}',\bar{n},\bar{n}'\in\mathbb{Z}} \bar{\mathcal{F}}_{J;\bar{m}',\bar{n},\bar{n}'}, \tag{D.42}$$

$$\begin{aligned}\bar{\mathcal{F}}'_{J;\bar{m}',\bar{n},\bar{n}'} &= \mathbb{C}[a_{-1}, a_{-2}, ..; a_{j,-1}, a_{j,-2}, ..(j=1,2)] \\ &\quad \otimes \mathbb{C}e^{\frac{J}{2(c+2)}Q_1} \otimes \mathbb{C}e^{\bar{m}'\alpha} \otimes \mathbb{C}e^{\left(\frac{\bar{n}}{2}\sqrt{\frac{r}{r^*}}+\frac{\bar{n}'}{2}\sqrt{\frac{r^*}{r}}\right)\bar{\alpha}}.\end{aligned} \tag{D.43}$$

Proposition D.6 *The following currents $\bar{K}(z)$, $\bar{E}(z)$, $\bar{F}(z)$ and $\tilde{d}$ with h, c give a representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ on $\bar{\mathcal{F}}'_J$:*

$$\bar{K}(z) = e^{\sqrt{c}\alpha_0\bar{\alpha}}\ k(z)\ z^{\frac{1}{2\sqrt{rr^*}}\bar{h}}, \tag{D.44}$$

$$\bar{E}(z) = e^{-(1-\sqrt{\frac{r}{r^*}})\bar{\alpha}}\ e(z,p)\ z^{-\frac{1}{c}(1-\sqrt{\frac{r}{r^*}})\bar{h}}, \tag{D.45}$$

$$\bar{F}(z) = e^{(1-\sqrt{\frac{r^*}{r}})\bar{\alpha}}\ f(z,p)\ z^{\frac{1}{c}(1-\sqrt{\frac{r^*}{r}})\bar{h}}, \tag{D.46}$$

$$\tilde{d} = \bar{d} - \frac{\alpha_0}{\sqrt{c}}\bar{h}. \tag{D.47}$$

Expressing P, $\bar{h}$ by P_j $(j=0,1,2)$ and $\bar{\alpha},\gamma$ by Q_0, Q_2, the resultant $\bar{K}(z)$, $\bar{E}(z)$, $\bar{F}(z)$ and $\tilde{d}$ coincide with $K(z)$, $E(z)$, $F(z)$ and $\hat{d}$ in Proposition D.3, respectively. The Fock space $\bar{\mathcal{F}}'_{J;\bar{m}',\bar{n},\bar{n}'}$ is isomorphic to $\hat{\mathcal{F}}_{J,\tilde{M};m,m',\mu}$ by

$$\bar{m}' = \frac{\tilde{M}}{2} + m', \quad \bar{n} = \mu, \quad \bar{n}' = -2m - \mu.$$

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