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Free Field Approach to the Dilute A_L Models

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Abstract

We construct a free field realization of vertex operators of the dilute A_L models along with the Felder complex. For $L = 3$, we also study an E_8 structure in terms of the deformed Virasoro currents.

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1 Introduction

The dilute A_L model [1, 2] is an integrable lattice model obtained by an RSOS restriction of the face model of type $A_2^{(2)}$ [3]. It possesses several intriguing features. Among others, we are interested in the following points.

i) At criticality, the dilute A_L model in regime 2^+ is described by conformal field theory (CFT) which belongs to the Virasoro minimal unitary series with the central charge $c = 1 - \frac{6}{L(L+1)}$. The Andrews-Baxter-Forrester (ABF) model in regime III is known to be a different off-critical lattice model having the same critical behavior. While the ABF model corresponds to the $(1,3)$-perturbation of the minimal unitary CFT, the dilute A_L model corresponds to the $(1,2)$-perturbation of the same CFT.

ii) In the particular case of $L = 3$, the model falls within the same universality class as the two-dimensional Ising model in a magnetic field. The elliptic nome in the Boltzmann weights plays the role of a magnetic field, as opposed to the usual role as a temperature-like variable. In the field theory limit, the scattering process of particles exhibits an E_8 structure.

In [4], bosonization of the ABF model in regime III was achieved. By 'bosonization' we mean a free field realization of vertex operators (VO's) of the model as formulated in [5]. It was also found that the deformed Virasoro algebra (DVA) proposed earlier in [6] arises naturally, in such a way that the VO's play the role of deformed chiral primary fields for DVA. (More specifically, the VO's of type I and type II correspond to the simplest primary fields ϕ_{21} and ϕ_{12}, respectively. Analogs of general ϕ_{mn} are obtained by a fusion procedure [7]. For an interpretation of the VO's as intertwiners for elliptic algebras, see [8].) The BRST resolution of Fock spaces which singles out irreducible representations of the Virasoro algebra carries over to the deformed version as well. Thus the work [4] presents an off-critical lattice version of the $(1,3)$-perturbation of the minimal unitary CFT in the free field picture. In this paper we study a similar problem for the $(1,2)$-perturbed CFT, by bosonizing the dilute A_L model. For this purpose we adapt the construction of [5, 4, 9] to the case of $A_2^{(2)}$.

In the trigonometric limit, bosonization of VO's has been given in [10, 11] using representation theory of the quantum affine algebra $U_q(A_2^{(2)})$. In principle, we are to follow its elliptic analog on the basis of the face type elliptic algebra [12, 8]. Because of some technical difficulties in dealing with the latter, we take here a more pedestrian way and solve the exchange relations directly to obtain the bosonic realizations of VO's. In the course we use the elliptic Drinfeld currents obtained by a 'dressing' procedure [8].

In view of the points i), ii) above, it is natural to expect that a different deformation of the Virasoro algebra arises from the dilute A_L models. Such a deformation was found in [13] through bosonization of the $A_2^{(2)}$ affine Toda field theory. (See [14, 15] for more general deformed W algebras including the case $A_{2l}^{(2)}$.) To make distinction from the original DVA of [6], we use the symbol $\mathcal{V}_{x,r}(A_2^{(2)})$ to denote the DVA of [13]. In the original case, the generating function of DVA (hereafter referred to as the 'DVA current') can also be obtained from 'fusion' of the VO's [16, 17, 18]. In the same way, we reproduce the current of $\mathcal{V}_{x,r}(A_2^{(2)})$ by taking residues of products of the bosonized VO's in the present case.

For the dilute A_L models in regime 2^+, the space of states of the corner transfer matrix is an analog of the minimal unitary representation [2] for $\mathcal{V}_{x,r}(A_2^{(2)})$. In order to obtain them from the Fock spaces, we consider a Felder type resolution using the elliptic currents of type $A_2^{(2)}$ as screening currents. Unlike the case of the ABF models, the BRST charges are not simply a power of screening operators. Such a complication seems to be common in the higher rank situations [19]. We prove the nilpotency of BRST charges with the help of the Feigin-Odesskii algebra [19]. Assuming a cohomological property of the resulting complex, we write down integral expressions for the two-point local height probabilities (LHPs) and traces of general product of the VO's.

In the case of the dilute A_3 model, it is of some interest to see how Zamolodchikov's E_8 structure of scattering process [20] looks like in the free field picture. What plays the role of the Zamolodchikov-Faddeev operator for creation/annihilation of bound states is the DVA current [16]. Specializing to $L = 3$, we introduce eight kinds of DVA currents by suitably fusing the elementary current of $\mathcal{V}_{x,r}(A_2^{(2)})$. We then find a curious similarity between the operator product expansions of these currents and the so-called T-system of E_8 type for the transfer matrix [21, 22]. At present we do not understand its proper meaning.

This paper is organized as follows. In section 2, we review the definition of the dilute A_L model and give a brief description of the vertex operator approach. In section 3, we give bosonization of the VO's. We derive the current for $\mathcal{V}_{x,r}(A_2^{(2)})$ from them and state a conjecture for the Kac determinant formula for $\mathcal{V}_{x,r}(A_2^{(2)})$. We also present a Felder type BRST complex of the Fock spaces. Section 4 is devoted to an application of the bosonization to the calculation of the LHP. In section 5, regarding the $\mathcal{V}_{x,r}(A_2^{(2)})$ current as the ZF operator for the particles in the A_3 model, we discuss a similarity between the T-system and the 'bootstrap' of the ZF operators. Appendix A is a summary of the operator product expansion formulae for the elliptic currents and VO's. In appendix B, we prove the nilpotency of the BRST charges. Appendix C is devoted to an exposition of the fusion properties of the deformed W currents for $A_{N-1}^{(1)}$, which should be compared with those for the DVA currents associated with the dilute A_3 model in section 5.

2 The dilute A_L models

2.1 Boltzmann weights

Throughout this paper we fix a positive integer $L \geq 3$. In the dilute A_L model, the local fluctuation variables $a, b, \cdots$ take one of the L states $1, 2, \cdots, L$, and those on neighboring lattice sites are subject to the condition $a - b = 0, \pm 1$. The Boltzmann weights can be found in [2], eq.(3.1). For our purpose it is convenient to use the parametrization given in Appendix A of [2], which is suitable in the 'low-temperature' regime. With some change of notation we recall the formula below.

Let $x = e^{-2\pi\lambda/\varepsilon}$, $r = \pi/(2\lambda)$ and $u = -u_{orig}/(2\lambda)$, where λ, ε are the variables used in [2] and u_{orig} stands for 'u' there. We shall restrict ourselves to the 'regime 2^+' defined by

$$0 < x < 1, \quad r = 2\,\frac{L+1}{L+2}, \quad -\frac{3}{2} < u < 0. \tag{2.1}$$

Along with the variable u, we often use the multiplicative variable

$$z = x^{2u}.$$

Changing an overall scalar factor we put the Boltzmann weights in the form

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \rho(u)\overline{W}\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right),$$

where $\rho(u)$ will be specified in (2.4) below. To give the formula for the $\overline{W}$ factors, let us set

$$[u] = x^{\frac{u^2}{r} - u}\Theta_{x^{2r}}(x^{2u}), \qquad [u]_+ = x^{\frac{u^2}{r} - u}\Theta_{x^{2r}}(-x^{2u}), \tag{2.2}$$

where

$$\Theta_p(z) = (z;p)_\infty (pz^{-1};p)_\infty (p;p)_\infty,$$
$$(z;p_1,\cdots,p_k)_\infty = \prod_{n_1,\cdots,n_k=0}^{\infty} (1 - p_1^{n_1}\cdots p_k^{n_k} z).$$

Then we have

$$\overline{W}\left(\begin{matrix} a\pm 1 & a \\ a & a\mp 1 \end{matrix}\middle| u\right) = 1,$$

$$\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a & a\pm 1 \end{matrix}\middle| u\right) = \overline{W}\left(\begin{matrix} a\pm 1 & a\pm 1 \\ a & a \end{matrix}\middle| u\right) = -\left(\frac{[\pm a + 3/2]_+[\pm a - 1/2]_+}{[\pm a + 1/2]_+^2}\right)^{1/2} \frac{[u]}{[1+u]},$$

$$\overline{W}\left(\begin{matrix} a\pm 1 & a \\ a & a \end{matrix}\middle| u\right) = \overline{W}\left(\begin{matrix} a & a \\ a & a\pm 1 \end{matrix}\middle| u\right) = \frac{[\pm a + 1/2 + u]_+}{[\pm a + 1/2]_+}\frac{[1]}{[1+u]},$$

$$\overline{W}\left(\begin{matrix} a & a\mp 1 \\ a\pm 1 & a \end{matrix}\Big| u\right) = \left(G_a^+ G_a^-\right)^{1/2} \frac{[1/2+u]}{[3/2+u]}\frac{[u]}{[1+u]},$$
$$\overline{W}\left(\begin{matrix} a & a \\ a\pm 1 & a \end{matrix}\Big| u\right) = \overline{W}\left(\begin{matrix} a & a\pm 1 \\ a & a \end{matrix}\Big| u\right) = -\left(G_a^{\pm}\right)^{1/2} \frac{[\pm a-1-u]_+}{[\pm a+1/2]_+}\frac{[1]}{[1+u]}\frac{[u]}{[3/2+u]},$$
$$\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\Big| u\right) = \frac{[\pm 2a+1-u]}{[\pm 2a+1]}\frac{[1]}{[1+u]} - G_a^{\pm}\frac{[\pm 2a-1/2-u]}{[\pm 2a+1]}\frac{[u]}{[3/2+u]}\frac{[1]}{[1+u]},$$
$$\overline{W}\left(\begin{matrix} a & a \\ a & a \end{matrix}\Big| u\right) = \frac{[3+u]}{[3]}\frac{[1]}{[1+u]}\frac{[3/2-u]}{[3/2+u]} + H_a\frac{[1]}{[3]}\frac{[u]}{[1+u]}.$$

Here

$$G_a^{\pm} = \frac{S(a\pm 1)}{S(a)}, \quad S(a) = (-1)^a\frac{[2a]}{[a]_+}, \quad H_a = G_a^+\frac{[a-5/2]_+}{[a+1/2]_+} + G_a^-\frac{[a+5/2]_+}{[a-1/2]_+}. \qquad (2.3)$$

We choose $\rho(u)$ so that the partition function per site of the model equals to 1. Explicitly it is given by [2]

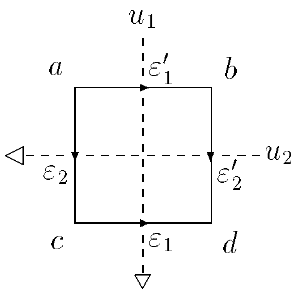
$$z^{\frac{r-1}{r}}\rho(u) = \frac{\rho_+(u)}{\rho_+(-u)}, \quad \rho_+(u) = \frac{(x^2z, x^3z, x^{2r+3}z, x^{2r+4}z; x^6, x^{2r})_\infty}{(x^5z, x^6z, x^{2r}z, x^{2r+1}z; x^6, x^{2r})_\infty}, \qquad (2.4)$$

where $z = x^{2u}$, and we have introduced the notation

$$(a_1, \cdots, a_n; p_1, \cdots, p_k)_\infty = \prod_{j=1}^{n}(a_j; p_1, \cdots, p_k)_\infty. \qquad (2.5)$$

Graphically we represent the Boltzmann weights as follows:

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u_1-u_2\right) \quad =$$



$$, \qquad \begin{aligned} b &= a+\varepsilon_1', \\ c &= a+\varepsilon_2, \\ d &= b+\varepsilon_2' = c+\varepsilon_1. \end{aligned}$$

For definiteness we list below the basic properties of the Boltzmann weights.

Yang-Baxter equation

$$\sum_g W\left(\begin{matrix} a & b \\ g & c \end{matrix}\Big| u\right) W\left(\begin{matrix} a & g \\ f & e \end{matrix}\Big| v\right) W\left(\begin{matrix} g & c \\ e & d \end{matrix}\Big| u+v\right)$$
$$= \sum_g W\left(\begin{matrix} a & b \\ f & g \end{matrix}\Big| u+v\right) W\left(\begin{matrix} b & c \\ g & d \end{matrix}\Big| v\right) W\left(\begin{matrix} f & g \\ e & d \end{matrix}\Big| u\right),$$

unitarity

$$\sum_g W\left(\begin{matrix} a & b \\ g & c \end{matrix}\middle| u\right) W\left(\begin{matrix} a & g \\ d & c \end{matrix}\middle| -u\right) = \delta_{bd},$$

crossing symmetry

$$W\left(\begin{matrix} b & d \\ a & c \end{matrix}\middle| -\frac{3}{2} - u\right) = \sqrt{\frac{S(a)S(d)}{S(b)S(c)}} W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right),$$

initial condition

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| 0\right) = \delta_{bc}.$$

2.2 Vertex operators

Hereafter we assume that L is odd. The model has ground states labeled by odd integers $l = 1, 3, \cdots, L-2$ [2]. They are characterized as configurations in which all heights take the same value b. If $L = 4m \pm 1$, then the possible values are $b = l$ ($1 \le l \le 2m-1$, l : odd) or $b = l+1$ ($2m+1 \le l \le L-2$, l : odd).

Consider the corner transfer matrices $A(z), B(z), C(z), D(z)$ corresponding to the NW, SW, SE, NE quadrants, respectively. In the infinite volume limit, we have

$$C(z) = A(z) = z^{-\mathcal{H}}, \quad B(z) = D(z) = \sqrt{S(k)}\, x^{3\mathcal{H}} z^{\mathcal{H}},$$

with k denoting the value of the central height. The operator $\mathcal{H}$ (the corner Hamiltonian) is independent of z. We denote by $\mathcal{L}_{l,k}$ the space of eigenstates of $\mathcal{H}$ in the sector where the central height is fixed to k and the boundary heights are in the ground state l. It was found in [2] that the generating function of the spectrum of $\mathcal{H}$ coincides with the character of the Virasoro minimal unitary series. Namely

$$\mathrm{tr}_{\mathcal{L}_{l,k}}(q^{\mathcal{H}}) = \chi_{l,k}(q), \tag{2.6}$$

where

$$\chi_{l,k}(q) = \frac{q^{\Delta_{l,k} - c/24}}{(q;q)_\infty} \sum_{j\in\mathbb{Z}} \left(q^{L(L+1)j^2 + ((L+1)l - Lk)j} - q^{L(L+1)j^2 + ((L+1)l + Lk)j + lk} \right),$$
$$c = 1 - \frac{6}{L(L+1)}, \quad \Delta_{l,k} = \frac{((L+1)l - Lk)^2 - 1}{4L(L+1)}.$$

Consider next the half-infinite transfer matrix extending to infinity in the north. We denote it by

$$\Phi^{(k,k+\varepsilon)}(z) : \mathcal{L}_{l,k+\varepsilon} \to \mathcal{L}_{l,k} \qquad (\varepsilon = 0, \pm 1).$$

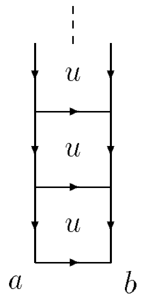
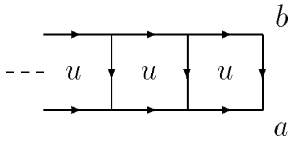
Likewise we denote by

$$\Phi^{*(k+\varepsilon,k)}(z^{-1}) : \mathcal{L}_{l,k} \to \mathcal{L}_{l,k+\varepsilon} \qquad (\varepsilon = 0, \pm 1)$$

the half-infinite transfer matrix extending to infinity in the west. We shall also write

$$\Phi^{(k,k+\varepsilon)}(z) = \Phi_\varepsilon(z), \quad \Phi^{*(k+\varepsilon,k)}(z) = \Phi^*_\varepsilon(z),$$

and call them vertex operators (VO's) of type I.

$\Phi^{(a,b)}(z) =$  , $\Phi^{*(a,b)}(z^{-1}) =$  .

Intuitive graphical arguments based on the properties of the Boltzmann weights lead to the following formulas. For the details we refer the reader to [23, 4].

$$\Phi^{(a,c)}(z_2)\Phi^{(c,d)}(z_1) = \sum_g W\left(\begin{matrix} a & g \\ c & d \end{matrix}\middle| u_1 - u_2\right) \Phi^{(a,g)}(z_1)\Phi^{(g,d)}(z_2) \qquad (z_j = x^{2u_j}),$$

$$w^{\mathcal{H}}\Phi^{(a,b)}(z)w^{-\mathcal{H}} = \Phi^{(a,b)}(wz), \qquad \Phi^{*(b,a)}(z) = \sqrt{\frac{S(a)}{S(b)}}\Phi^{(b,a)}(x^{-3}z),$$

$$\sum_\varepsilon \Phi^{*(a,g)}(z)\Phi^{(g,b)}(z) = \delta_{ab}, \qquad \Phi^{(a,b)}(z)\Phi^{*(b,c)}(z) = \delta_{ac}.$$

As explained in [5], multi-point local height probabilities are expressed as traces of VO's. Consider neighboring $n+1$ lattice sites in a row. Let $P_l(a_0, \cdots, a_n)$ denote the probability of finding these local variables to be $(a_0, \cdots, a_n)$. Then we have

$$\begin{aligned} &P_l(a_0, \cdots, a_n) \\ &= Z_l^{-1} S(a_n) \operatorname{tr}_{\mathcal{L}_{l,a_n}} \left(x^{6\mathcal{H}} \Phi^{*(a_n,a_{n-1})}(z) \cdots \Phi^{*(a_1,a_0)}(z) \Phi^{(a_0,a_1)}(z) \cdots \Phi^{(a_{n-1},a_n)}(z) \right). \end{aligned} \quad (2.7)$$

Here the nomalization factor Z_l is

$$Z_l = \sum_{k=1}^{L} S(k)\chi_{l,k}(x^6),$$

which can be expressed in product of theta functions with conjugate modulus [2]. In the simplest case $n = 0$, the one-point function $P_l(k)$ is given by $P_l(k) = Z_l^{-1} S(k)\chi_{l,k}(x^6)$.

Remark. We follow mostly the notation of [9] but there are minor changes. The $\Phi(\zeta^{-1})$ in [9] corresponds to $\Phi(z)$ in the present notation. We have also reversed the orientation of edges of the Boltzmann weights.

3 Bosonization of vertex operators

3.1 Bosons

In this section we present a bosonic realization of vertex operators. The working closely follows [4, 9]. We set

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}},$$

and introduce the oscillators α_n $(n \neq 0)$ and P, Q satisfying the commutation relations

$$\begin{aligned} [\alpha_n, \alpha_m] &= \frac{[n]_x\,([2n]_x - [n]_x)}{n} \frac{[rn]_x}{[(r-1)n]_x} \delta_{n+m,0}, \\ [P, iQ] &= 1. \end{aligned} \tag{3.1}$$

Notice that $[2n]_x - [n]_x = [3n]_x[n/2]_x/[3n/2]_x$. We shall also use

$$\alpha'_n = (-1)^n \frac{[(r-1)n]_x}{[rn]_x} \alpha_n.$$

We denote by

$$\mathcal{F}_{l,k} = \mathbb{C}[\alpha_{-1}, \alpha_{-2}, \cdots]|l, k\rangle$$

the Fock space generated by

$$|l, k\rangle = e^{p_{l,k} iQ}|0, 0\rangle, \quad P|l, k\rangle = p_{l,k}|l, k\rangle,$$

where $p_{l,k}$ is

$$p_{l,k} = -\frac{l}{2}\sqrt{\frac{r}{r-1}} + k\sqrt{\frac{r-1}{r}} = -l\sqrt{\frac{L+1}{2L}} + k\sqrt{\frac{L}{2(L+1)}}. \tag{3.2}$$

(Recall that $r = 2(L+1)/(L+2)$.) These Fock spaces are graded by

$$d = \sum_{n=1}^{\infty} \frac{n^2}{[n]_x([2n]_x - [n]_x)} \frac{[(r-1)n]_x}{[rn]_x} \alpha_{-n}\alpha_n + \frac{1}{2}P^2 - \frac{1}{24},$$

which satisfies $[d, \alpha_n] = -n\alpha_n$, $[d, iQ] = P$ and $d|l, k\rangle = (\Delta_{l,k} - c/24)|l, k\rangle$. For later use, we define operators $\hat{l}, \hat{k} : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k}$ by

$$\hat{l}\,|_{\mathcal{F}_{l,k}} = l \times \mathrm{id}_{\mathcal{F}_{l,k}}, \quad \hat{k}\,|_{\mathcal{F}_{l,k}} = k \times \mathrm{id}_{\mathcal{F}_{l,k}}.$$

3.2 Vertex operators

In the works [10, 11], a bosonic realization of the level-one representation of the quantum affine algebra $U_q(A_2^{(2)})$ and associated vertex operators have been obtained. We shall consider their elliptic counterparts.

The elliptic version of the Drinfeld currents are constructed from the trigonometric ones by a 'dressing' procedure described in [8]. Applying it to the present case of $U_q(A_2^{(2)})$, we obtain

$$
\begin{aligned}
& x_+(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l-2,k}, \quad x_-(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k-1}, \\
& x_+(z) = : \exp\left(-\sum_{n\neq 0} \frac{\alpha_n}{[n]_x} z^{-n} \right) : \times e^{\sqrt{\frac{r}{r-1}} iQ} z^{\sqrt{\frac{r}{r-1}}P + \frac{r}{2(r-1)}}, \qquad (3.3) \\
& x_-(z) = : \exp\left(\sum_{n\neq 0} \frac{\alpha'_n}{[n]_x} z^{-n} \right) : \times e^{-\sqrt{\frac{r-1}{r}} iQ} z^{-\sqrt{\frac{r-1}{r}}P + \frac{r-1}{2r}}. \qquad (3.4)
\end{aligned}
$$

The elliptic version of VO's (of type I and type II) are defined in terms of their trigonometric ones and a 'twistor' given by an infinite product of the universal R matrix [24]. They satisfy the commutation relations of the type (3.11)-(3.13) in the next subsection. As we do not know how to evaluate the twistor in the bosonic realization, we have solved the relations (3.11)-(3.13) directly for $\Phi_\varepsilon(z), \Psi^*_\varepsilon(z)$. We obtain the following.

Type I:

$$
\begin{aligned}
& \Phi_\varepsilon(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k-\varepsilon}, \\
\Phi_-(z) &= : \exp\left(-\sum_{n\neq 0} \frac{\alpha'_n}{[2n]_x - [n]_x} z^{-n} \right) : \times e^{\sqrt{\frac{r-1}{r}} iQ} z^{\sqrt{\frac{r-1}{r}}P + \frac{r-1}{2r}}, \qquad (3.5) \\
\Phi_0(z) &= x^{\frac{1-r}{2r}} \oint_{C_0} \underline{dz}_1 \Phi_-(z) x_-(z_1) \frac{1}{\sqrt{[\hat{k}+1/2]_+ [\hat{k}-1/2]_+}} \frac{[u-u_1+\hat{k}]_+}{[u-u_1+1/2]}, \qquad (3.6) \\
\Phi_+(z) &= x^{\frac{1-r}{r}} \oint\oint_{C_+} \underline{dz}_1 \underline{dz}_2 \Phi_-(z) x_-(z_1) x_-(z_2) \\
& \times \sqrt{\frac{S(\hat{k}-1)}{S(\hat{k})}} \frac{1}{[\hat{k}-1/2]_+ [2\hat{k}-2]} \frac{[u-u_1+2\hat{k}-3/2]}{[u-u_1+1/2]} \frac{[u_1-u_2+\hat{k}]_+}{[u_1-u_2+1/2]}. \qquad (3.7)
\end{aligned}
$$

Type II:

$$
\begin{aligned}
& \Psi^*_\varepsilon(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l-2\varepsilon,k}, \\
\Psi^*_-(z) &= : \exp\left(\sum_{n\neq 0} \frac{\alpha_n}{[2n]_x - [n]_x} z^{-n} \right) : \times e^{-\sqrt{\frac{r}{r-1}} iQ} z^{-\sqrt{\frac{r}{r-1}}P + \frac{r}{2(r-1)}}, \qquad (3.8) \\
\Psi^*_0(z) &= i x^{\frac{r}{2(r-1)}} \oint_{C^*_0} \underline{dz}_1 \Psi^*_-(z) x_+(z_1) \frac{1}{\sqrt{[(\hat{l}+1)/2]^*_+ [(\hat{l}-1)/2]^*_+}} \frac{[u-u_1-\hat{l}/2]^*_+}{[u-u_1-1/2]^*}, \qquad (3.9)
\end{aligned}
$$

$$\Psi^*_+(z) = x^{\frac{r}{r-1}} \oint\oint_{C^*_+} \underline{dz_1}\underline{dz_2}\Psi^*_-(z)x_+(z_1)x_+(z_2)$$
$$\times\sqrt{\frac{S^*(\hat{l}/2-1)}{S^*(\hat{l}/2)}}\frac{1}{[(\hat{l}-1)/2]^*_+[\hat{l}-2]^*}\frac{[u-u_1-\hat{l}+3/2]^*}{[u-u_1-1/2]^*}\frac{[u_1-u_2-\hat{l}/2]^*_+}{[u_1-u_2-1/2]^*}. \quad (3.10)$$

Here $z=x^{2u}$, $z_j=x^{2u_j}$, $\underline{dz}_j = dz_j/(2\pi i z_j)$ and

$$[u]^* = x^{\frac{u^2}{r-1}-u}\Theta_{x^{2r-2}}(x^{2u}), \quad [u]^*_+ = x^{\frac{u^2}{r-1}-u}\Theta_{x^{2r-2}}(-x^{2u}), \quad S^*(a) = (-1)^a\frac{[2a]^*}{[a]^*_+}.$$

The poles of the integrand of (3.6)–(3.10) and the integration contours are listed in the following table ($n=0,1,2,\cdots$). For example, C_0 is a simple closed contour that encircles $x^{1+2rn}z$ ($n\geq 0$) but not $x^{-1-2rn}z$ ($n\geq 0$).

	inside	outside
C_0	$z_1 = x^{1+2rn}z$	$z_1 = x^{-1-2rn}z$
C_+	$z_1 = x^{1+2rn}z$	$z_1 = x^{-1-2rn}z$
	$z_2 = x^{1+2rn}z_1$	$z_2 = x^{-1-2rn}z,\, x^{-1-2rn}z_1,\, x^{2-2r(n+1)}z_1$
C^*_0	$z_1 = x^{-1+2(r-1)n}z$	$z_1 = x^{1-2(r-1)n}z$
C^*_+	$z_1 = x^{-1+2(r-1)n}z$	$z_1 = x^{1-2(r-1)n}z$
	$z_2 = x^{-1+2(r-1)n}z_1$	$z_2 = x^{1-2(r-1)n}z,\, x^{1-2(r-1)n}z_1,\, x^{-2-2(r-1)(n+1)}z_1$

3.3 Commutation relations and inversion identities

The VO's given above satisfy the following commutation relations.

$$\Phi_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1) = \sum_{\substack{\varepsilon'_1,\varepsilon'_2 \\ \varepsilon'_1+\varepsilon'_2=\varepsilon_1+\varepsilon_2}} W\left(\begin{matrix}\hat{k} & \hat{k}+\varepsilon'_1 \\ \hat{k}+\varepsilon_2 & \hat{k}+\varepsilon_1+\varepsilon_2\end{matrix}\middle| u_1-u_2\right)\Phi_{\varepsilon'_1}(z_1)\Phi_{\varepsilon'_2}(z_2), \quad (3.11)$$

$$\Psi^*_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \sum_{\substack{\varepsilon'_1,\varepsilon'_2 \\ \varepsilon'_1+\varepsilon'_2=\varepsilon_1+\varepsilon_2}} W^*\left(\begin{matrix}\hat{l}/2 & \hat{l}/2+\varepsilon_1 \\ \hat{l}/2+\varepsilon'_2 & \hat{l}/2+\varepsilon_1+\varepsilon_2\end{matrix}\middle| u_1-u_2\right)\Psi^*_{\varepsilon'_2}(z_2)\Psi^*_{\varepsilon'_1}(z_1), \quad (3.12)$$

$$\Phi_{\varepsilon_2}(z_2)\Psi^*_{\varepsilon_1}(z_1) = \tau(u_1-u_2)\Psi^*_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2). \quad (3.13)$$

Here we have set (for $z=x^{2u}$)

$$W^*\left(\begin{matrix}a & b \\ c & d\end{matrix}\middle| u\right) = \overline{W}\left(\begin{matrix}a & b \\ c & d\end{matrix}\middle| u\right)\Bigg|_{r\to r-1}\times\rho^*(u), \quad (3.14)$$

$$z^{-\frac{r}{r-1}}\rho^*(u) = \frac{\rho^*_+(u)}{\rho^*_+(-u)}, \quad \rho^*_+(u) = \frac{(x^3z, x^4z, x^{2r}z, x^{2r+1}z; x^6, x^{2r-2})_\infty}{(z, xz, x^{2r+3}z, x^{2r+4}z; x^6, x^{2r-2})_\infty}, \quad (3.15)$$

$$\tau(u) = z\frac{\Theta_{x^6}(-xz^{-1})\Theta_{x^6}(-x^2z^{-1})}{\Theta_{x^6}(-xz)\Theta_{x^6}(-x^2z)}. \quad (3.16)$$

Note that

$$\rho^*(u) = -\rho(u)\Big|_{r\to r-1} \times z\frac{\Theta_{x^6}(xz^{-1})\Theta_{x^6}(x^2z^{-1})}{\Theta_{x^6}(xz)\Theta_{x^6}(x^2z)}.$$

We do not present the tedious but straightforward verification of (3.11)-(3.13).

For the description of correlation functions we need also the 'dual' VO's. Define

$$\Phi^*_\varepsilon(z) = g\sqrt{S(\hat{k})}^{-1}\Phi_{-\varepsilon}(x^{-3}z)\sqrt{S(\hat{k})}, \tag{3.17}$$

$$\Psi_\varepsilon(z) = g^{*-1}\sqrt{S^*(\hat{l}/2)}\Psi^*_{-\varepsilon}(x^{-3}z)\sqrt{S^*(\hat{l}/2)}^{-1}, \tag{3.18}$$

where

$$g^{-1} = \frac{(x;x^{2r})_\infty}{(x^2;x^{2r})^2_\infty(x^{2r-1};x^{2r})_\infty(x^{2r};x^{2r})^4_\infty}\frac{(x^5,x^6,x^{2r},x^{2r+1};x^6,x^{2r})_\infty}{(x^2,x^3,x^{2r+3},x^{2r+4};x^6,x^{2r})_\infty},$$

$$g^* = \frac{(x^{-1};x^{2r-2})_\infty}{(x^{-2};x^{2r-2})^2_\infty(x^{2r-1};x^{2r-2})_\infty(x^{2r-2};x^{2r-2})^5_\infty}\frac{(x^3,x^4,x^{2r},x^{2r+1};x^6,x^{2r-2})_\infty}{(x,x^6,x^{2r+3},x^{2r+4};x^6,x^{2r-2})_\infty}.$$

Then we have

$$\Phi_{\varepsilon_2}(z)\Phi^*_{\varepsilon_1}(z) = \delta_{\varepsilon_1,\varepsilon_2}\times \mathrm{id}, \quad \Psi_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \frac{\delta_{\varepsilon_1,\varepsilon_2}}{1-z_1/z_2}+\cdots, \quad (z_1\to z_2), \tag{3.19}$$

$$\sum_\varepsilon \Phi^*_\varepsilon(z)\Phi_\varepsilon(z) = \mathrm{id}, \qquad \sum_\varepsilon \Psi^*_\varepsilon(z_2)\Psi_\varepsilon(z_1) = \frac{1}{1-z_1/z_2}+\cdots, \quad (z_1\to z_2). \tag{3.20}$$

3.4 Deformed Virasoro algebra

Brazhnikov and Lukyanov [13] pointed out that one can associate to the algebra $A^{(2)}_2$ a deformed Virasoro algebra (DVA) which is different from the one found in [6]. The original DVA of [6], associated with $A^{(1)}_1$, arises also as a 'fusion' of VO's [16]. Let us discuss this point in the present case of $A^{(2)}_2$.

Let

$$\begin{aligned}
\Lambda_\pm(z) &= :\exp\Big(\pm\sum_{n\neq 0}\lambda_n(x^{\pm 3/2}z)^{-n}\Big):\times x^{\pm 2\sqrt{r(r-1)}P},\\
\Lambda_0(z) &= -\frac{[r-1/2]_x}{[1/2]_x}:\exp\Big(\sum_{n\neq 0}\lambda_n(x^{-n/2}-x^{n/2})z^{-n}\Big):,\\
T(z) &= \Lambda_+(z)+\Lambda_0(z)+\Lambda_-(z),
\end{aligned}\tag{3.21}$$

where

$$\lambda_n = (-1)^n(x-x^{-1})\frac{[(r-1)n]_x}{[2n]_x-[n]_x}\alpha_n = (x-x^{-1})\frac{[rn]_x}{[2n]_x-[n]_x}\alpha'_n,$$

$$[\lambda_n,\lambda_m] = (x-x^{-1})^2\frac{1}{n}\frac{[n]_x[rn]_x[(r-1)n]_x}{[2n]_x-[n]_x}\delta_{m+n,0}.$$

Then $T(z)$ is obtained from VO's by fusing them,

$$\begin{aligned}&\Phi_{\varepsilon_2}(x^{r+3/2}z')\Phi^*_{\varepsilon_1}(x^{-r+3/2}z)\\ =&\left(1-\frac{z}{z'}\right)(-1)^{\varepsilon_1+1}\delta_{\varepsilon_1,\varepsilon_2}T(z)\cdot x^{1-r}\frac{(x,x^6,x^{5-2r},x^{6-2r};x^6)_\infty}{(x^3,x^4,x^{2-2r},x^{3-2r};x^6)_\infty}+\cdots,\quad (z'\to z).\end{aligned}\tag{3.22}$$

The $T(z)$ satisfies the DVA of [13]

$$\begin{aligned}&f\left(\frac{z_2}{z_1}\right)T(z_1)T(z_2)-f\left(\frac{z_1}{z_2}\right)T(z_2)T(z_1)\\ =&\ (x-x^{-1})\frac{[r+1/2]_x[r]_x[r-1]_x[r-3/2]_x}{[1/2]_x[3/2]_x}\left(\delta\Big(x^3\frac{z_2}{z_1}\Big)-\delta\Big(x^{-3}\frac{z_2}{z_1}\Big)\right)\\ &+(x-x^{-1})\frac{[r]_x[r-1/2]_x[r-1]_x}{[1/2]_x}\left(\delta\Big(x^2\frac{z_2}{z_1}\Big)T(xz_2)-\delta\Big(x^{-2}\frac{z_2}{z_1}\Big)T(x^{-1}z_2)\right),\end{aligned}\tag{3.23}$$

where $f(z)$ is

$$\begin{aligned}f(z)&=\exp\Big(-\sum_{n>0}(x-x^{-1})^2\frac{1}{n}\frac{[n]_x[rn]_x[(r-1)n]_x}{[2n]_x-[n]_x}z^n\Big)\\ &=\frac{1}{1-z}\frac{(x^{2-2r}z,x^{3-2r}z,x^4z,x^5z,x^{2r}z,x^{2r+1}z;x^6)_\infty}{(x^{5-2r}z,x^{6-2r}z,xz,x^2z,x^{2r+3}z,x^{2r+4}z;x^6)_\infty}.\end{aligned}\tag{3.24}$$

The notation of [13] is related to ours by $x_{BL}=x^{3/2}$, $b/Q=r$, $1/(Qb)=1-r$, $g(z)=f(z)$, $\mathbf{V}(z)=T(z)$. In what follows we call this algebra $\mathcal{V}_{x,r}(A_2^{(2)})$. The relation (3.23) is invariant under

$$r\mapsto 1-r,\quad x\mapsto x,\quad T(z)\mapsto -T(z).\tag{3.25}$$

We remark that $\widetilde{T}(z)=-\Lambda_+(z)+\Lambda_0(z)-\Lambda_-(z)$ also satisfies (3.23), which is obtained from type II VO's,

$$\begin{aligned}&\Psi_{\varepsilon_1}(x^{r-1+3/2}z')\Psi^*_{\varepsilon_2}(x^{-(r-1)+3/2}z)\\ =&\ \frac{1}{1-z'/z}(-1)^{\varepsilon_1+1}\delta_{\varepsilon_1,\varepsilon_2}\widetilde{T}(-z)\cdot(-x^{-r})\frac{(x^2,x^3,x^{5-2r},x^{6-2r};x^6)_\infty}{(x^5,x^6,x^{2-2r},x^{3-2r};x^6)_\infty}+\cdots,\quad (z'\to z).\end{aligned}$$

Let us discuss some features of $\mathcal{V}_{x,r}(A_2^{(2)})$.

Conformal limit In the conformal limit ($x=e^\hbar\to 1$, r : fixed), (3.23) admits two limits [13] related by (3.25),

$$T(z)=3-2r+\hbar^2\Big(8r(r-1)z^2L(z)+\frac{1}{6}r(r-1)(1-2r)+(2-r)^2\Big)+O(\hbar^4),\tag{3.26}$$

$$T(z)=-1-2r+\hbar^2\Big(-8r(r-1)z^2\widetilde{L}(z)+\frac{1}{6}r(r-1)(1-2r)-(1+r)^2\Big)+O(\hbar^4),\tag{3.27}$$

where $L(z),\widetilde{L}(z)$ are the Virasoro currents with the central charges $c,\widetilde{c}$ respectively,

$$c=1-\frac{3(2-r)^2}{r(r-1)}=1-\frac{6}{L(L+1)},\quad \widetilde{c}=1-\frac{3(1+r)^2}{r(r-1)}.$$

In the free boson realization (3.21), $T(z)$ and $\widetilde{T}(z)$ have 'natural' expansions (3.26) and (3.27) respectively, by the following identification:

$$
\begin{aligned}
\lambda_n &= 2\hbar\sqrt{r(r-1)}\sqrt{\frac{x^{rn}-x^{-rn}}{2\hbar rn}\frac{x^{(r-1)n}-x^{-(r-1)n}}{2\hbar(r-1)n}\frac{1}{x^n-1+x^{-n}}}a_n,\\
P &= a_0-\frac{2-r}{2\sqrt{r(r-1)}}=\widetilde{a}_0-\frac{r+1}{2\sqrt{r(r-1)}},\\
L(z) &= :\frac{1}{2}\Big(\partial\phi(z)\Big)^2:+\frac{2-r}{2\sqrt{r(r-1)}}\partial^2\phi(z),\\
\widetilde{L}(z) &= :\frac{1}{2}\Big(\partial\widetilde{\phi}(z)\Big)^2:+\frac{r+1}{2\sqrt{r(r-1)}}\partial^2\widetilde{\phi}(z),
\end{aligned}
$$

where $[a_n,a_m]=n\delta_{n+m,0}$, $\partial\phi(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}$ and $\partial\widetilde{\phi}(z)=\partial\phi(z)\Big|_{a_0\to\widetilde{a}_0}$. On the other hand, $T(z)$ has an expansion of the form (3.27) with $P=\widetilde{a}_0-\frac{r+1}{2\sqrt{r(r-1)}}+\frac{i\pi(2n+1)}{2\hbar\sqrt{r(r-1)}}$ $(n\in\mathbb{Z})$.

Kac determinant Let $T(z)=\sum_{n\in\mathbb{Z}}T_nz^{-n}$, and let $U_\pm$ be the algebra generated by $\{T_n\}_{\pm n>0}$. As usual, the Verma module of highest weight $\lambda\in\mathbb{C}$ is defined as the free left U_--module generated by a vector $|\lambda\rangle$ such that $T_n|\lambda\rangle=0$ $(n>0)$ and $T_0|\lambda\rangle=\lambda|\lambda\rangle$. Likewise the right Verma module is defined by $\langle\lambda|T_n=0$ $(n<0)$, $\langle\lambda|T_0=\lambda\langle\lambda|$, and $\langle\lambda|\lambda\rangle=1$. At level N there are $p(N)$ (the number of partition) independent states, $T_{-n_1}T_{-n_2}\cdots T_{-n_l}|\lambda\rangle$ $(n_1\geq n_2\geq\cdots\geq n_l>0,\ \sum_{i=1}^l n_i=N)$. Let us number these states by the reverse lexicographic ordering for $(n_1,n_2,\cdots,n_l)$, i.e., $|\lambda;N,1\rangle=T_{-N}|\lambda\rangle$, $|\lambda;N,2\rangle=T_{-N+1}T_{-1}|\lambda\rangle,\cdots,|\lambda;N,p(N)\rangle=T_{-1}^N|\lambda\rangle$. Similarly we define $\langle\lambda;N,1|=\langle\lambda|T_N$, $\langle\lambda;N,2|=\langle\lambda|T_1T_{N-1},\cdots,\langle\lambda;N,p(N)|=\langle\lambda|T_1^N$.

We conjecture that the Kac determinant at level N is given by

$$
\begin{aligned}
&\det\Big(\langle\lambda;N,i|\lambda;N,j\rangle\Big)_{1\leq i,j\leq p(N)}\\
&=\prod_{\substack{l,k\geq 1\\ lk\leq N}}\left((\lambda-\lambda_{l,k})(\lambda-\widetilde{\lambda}_{l,k})\frac{(x^{rl}-x^{-rl})(x^{(r-1)l}-x^{-(r-1)l})}{x^l-1+x^{-l}}\right)^{p(N-lk)},
\end{aligned}\tag{3.28}
$$

where

$$
\begin{aligned}
\lambda_{l,k} &= x^{-lr+2k(r-1)}+x^{lr-2k(r-1)}-\frac{[r-1/2]_x}{[1/2]_x},\\
\widetilde{\lambda}_{l,k} &= -x^{-2lr+k(r-1)}-x^{2lr-k(r-1)}-\frac{[r-1/2]_x}{[1/2]_x}.
\end{aligned}
$$

We remark that in the free boson realization $T_0|l,k\rangle=\lambda_{l,k}|l,k\rangle$ and $T_0e^{\alpha iQ}|2l,k/2\rangle=\widetilde{\lambda}_{l,k}e^{\alpha iQ}|2l,k/2\rangle$, where $\alpha=\frac{i\pi(2n+1)}{2\hbar\sqrt{r(r-1)}}$ $(n\in\mathbb{Z})$.

3.5 Felder complex

The Fock spaces $\mathcal{F}_{l,k}$ themselves do not give a bosonic realization of the space of states $\mathcal{L}_{l,k}$ of the corner Hamiltonian. For this we need a cohomological construction using an analog of the Felder complex [25]:

$$\begin{aligned} &\cdots \xrightarrow{X_{-2}} \mathcal{F}_{2L-l,k} \xrightarrow{X_{-1}} \mathcal{F}_{l,k} \xrightarrow{X_0} \mathcal{F}_{-l,k} \xrightarrow{X_1} \mathcal{F}_{l-2L,k} \xrightarrow{X_2} \cdots, \\ &X_j X_{j-1} = 0. \end{aligned} \tag{3.29}$$

In the case of the algebra $A_1^{(1)}$, Lukyanov and Pugai constructed the coboundary map X_j as a power of a single operator ([4], see also [26]). In our case, the formula for X_j is a little more involved.

Set

$$\begin{aligned} Q_1 &= \oint_{|z|=1} \underline{dz}\, x_+(z) \frac{[u+\hat{l}/2]^*}{[u+1/2]^*}, \\ Q_2^{(a)} &= \oint\oint_{|z_1|=|z_2|=1} \underline{dz}_1 \underline{dz}_2\, x_+(z_1) x_+(z_2) \frac{1}{[u_1+1/2]^*[u_2+1/2]^*} \\ &\quad \times \frac{[u_1-u_2]^*}{[u_1-u_2+1]^*[u_1-u_2-1/2]^*} f_2^{(a)}(u_1+\hat{l}/2, u_2+\hat{l}/2), \end{aligned}$$

where

$$\begin{aligned} f_2^{(a)}(u_1,u_2) &= [2a+1]^*[a-1/2]^*[u_1-a]^*[u_2+a-1]^*[u_1-u_2+a-1/2]^* \\ &\quad -[2a-1]^*[a+1/2]^*[u_1+a]^*[u_2-a-1]^*[u_1-u_2-a-1/2]^*. \end{aligned}$$

These operators are mutually commutative (see Lemma B.1). We define 'BRST charges' Q_l $(1 \le l \le L-1)$ as follows:

$$Q_l = \begin{cases} Q_1 Q_2^{(1)} \cdots Q_2^{(m)} & (l=2m+1), \\ Q_2^{((L+1)/2-m)} \cdots Q_2^{((L-3)/2)} Q_2^{((L-1)/2)} & (l=2m). \end{cases}$$

Note that $Q_L = Q_l Q_{L-l}$.

We prove the following propositions in appendix B (Proposition B.2, B.3).

Proposition 3.1 *Suppose l' is odd and $l' \equiv l \mod L$ $(1 \le l \le L-1)$. On the space $\mathcal{F}_{l',k}$, Q_l is expressed as*

$$Q_l = \oint\cdots\oint_{|z_1|=\cdots=|z_l|=R} \underline{dz}_1 \cdots \underline{dz}_l\, x_+(z_1)\cdots x_+(z_l)\, H_l(u_1,\cdots,u_l), \tag{3.30}$$

$$H_l(u_1,\cdots,u_l) = \pm \bar{h}_l(u_1,\cdots,u_l) \prod_{1\le i<j\le l} \frac{[u_i-u_j]^*}{[u_i-u_j+1]^*[u_i-u_j-1/2]^*}, \tag{3.31}$$

where $\bar{h}_l(u_1,\cdots,u_l)$ *is holomorphic, symmetric and satisfies* $\bar{h}_l(u_1+v,\cdots,u_l+v) = \bar{h}_l(u_1,\cdots,u_l)$*. We have*

$$H_l(u_1+r-1,\cdots,u_l) = H_l(u_1,\cdots,u_l), \tag{3.32}$$

$$H_l(u_1+\tau,\cdots,u_l) = H_l(u_1,\cdots,u_l)\, e^{-\pi i(l-1)/(r-1)}, \tag{3.33}$$

where $\tau = \pi i/\log x$*.*

Hence (3.30) does not depend on $R > 0$.

Proposition 3.2 *Under the same condition as above, we have*

$$Q_l Q_{L-l} = 0 \qquad (1 \le l \le L-1).$$

Let us call $C_{l,k}$ the cochain complex (3.29) defined by

$$\begin{aligned} X_{2j} = Q_l \ &: \ \mathcal{F}_{l-2jL,k} \longrightarrow \mathcal{F}_{-l-2jL,k}, \\ X_{2j+1} = Q_{L-l} \ &: \ \mathcal{F}_{-l-2jL,k} \longrightarrow \mathcal{F}_{l-2(j+1)L,k}. \end{aligned}$$

In the conformal limit where $x \to 1$ and $z = x^{2u}$ kept fixed, this complex formally tends to Felder's complex [25] for the minimal unitary series. In view of this, it is natural to expect that

$$H^j(C_{l,k}) = \mathrm{Ker}X_j/\mathrm{Im}X_{j-1} = 0 \qquad (j \ne 0). \tag{3.34}$$

By Euler-Poincaré principle, the 0-th cohomology $H^0(C_{l,k})$ has then the same character as the space of states $\mathcal{L}_{l,k}$ (see (2.6)),

$$\mathrm{tr}_{H^0(C_{l,k})}(q^d) = \mathrm{tr}_{\mathcal{L}_{l,k}}(q^{\mathcal{H}}).$$

Henceforth we assume (3.34) and make an identification

$$H^0(C_{l,k}) = \mathcal{L}_{l,k}, \quad d = \mathcal{H}.$$

Proposition 3.3 *Under the same assumption as in Proposition 3.1, we have on* $\mathcal{F}_{l',k}$

$$[\Phi_\varepsilon(z), Q_l] = 0 \quad (\varepsilon = 0, \pm), \tag{3.35}$$

$$[T(z), Q_l] = 0. \tag{3.36}$$

Proof. (3.36) is a consequence of (3.35) and (3.22). Let us show

$$\Phi_-(z)Q_l = Q_l\Phi_-(z).$$

The left (resp. right) hand side is well defined if we choose $R \ll 1$ (resp. $R \gg 1$) in (3.30). As meromorphic functions we have $\Phi_-(z)x_+(z_j) = x_+(z_j)\Phi_-(z)$, and the product has no poles. Since Q_l does not depend on R, the conclusion follows.

Next let us prove (3.35) with $\varepsilon = 0$. The case $\varepsilon = +$ can be shown similarly. Dropping irrelevant constants we consider

$$\Phi'_0(z) = \oint_{C_0} \underline{dz'}\Phi_-(z)x_-(z')\frac{[u-u'+\hat{k}]_+}{[u-u'+1/2]}.$$

As meromorphic functions we have

$$x_-(z)x_+(z') = x_+(z')x_-(z) = \frac{z+z'}{(z+xz')(z+x^{-1}z')} : x_-(z)x_+(z') : .$$

We use the expression (3.30) with $x^2|z| < R < x^{-2}|z|$. Taking into account the symmetry in the integration variables $z_1, \cdots, z_l$, we obtain

$$[\Phi'_0(z), Q_l] = l \oint \cdots \oint_{|z_1|=\cdots=|z_l|=R} \underline{dz_1} \cdots \underline{dz_l}\, H_l(u_1, \cdots, u_l)$$
$$\times \left(\mathrm{res}_{z'=-xz_1} + \mathrm{res}_{z'=-x^{-1}z_1}\right) \Phi_-(z)x_-(z')x_+(z_1)\cdots x_+(z_l)\frac{[u-u'+\hat{k}]_+}{[u-u'+1/2]}\underline{dz'}.$$

By noting the identity

$$: x_+(z)x_-(-x^{-1}z) := x^{-2r+1} : x_+(x^{2r-2}z)x_-(-x^{2r-1}z) :,$$

we can rewrite the right hand side as follows:

$$l \oint \cdots \oint_{|z_2|=\cdots=|z_l|=R} \underline{dz_2} \cdots \underline{dz_l} \Big(\oint_{C_1} \underline{dz_1} A(z_1, z)x_+(z_2)\cdots x_+(z_l)H_l(u_1, \cdots, u_l)$$
$$- \oint_{C_2} \underline{dz_1} A(x^{2r-2}z_1, z)x_+(z_2)\cdots x_+(z_l)H_l(u_1, \cdots, u_l)\Big), \quad (3.37)$$

where

$$\begin{aligned} A(z_1, z) &= \mathrm{res}_{z'=-xz_1}\Phi_-(z)x_-(z')x_+(z_1)\frac{[u-u'+\hat{k}]_+}{[u-u'+1/2]}\underline{dz'} \\ &= \frac{\text{holomorphic function}}{(-x^2z_1/z, -x^{2r}z/z_1; x^{2r})_\infty} : \Phi_-(z)x_-(-xz_1)x_+(z_1) : . \end{aligned}$$

The contours for z_1 are ($n \ge 0$)

	inside	outside
C_1	$z_1 = -x^{2r(n+1)}z$	$z_1 = -x^{-2-2rn}z$
C_2	$z_1 = -x^{2+2rn}z$	$z_1 = -x^{-2r(n+1)}z$

Moreover the product $: x_-(-xz_1)x_+(z_1) : x_+(z_j)$ is holomorphic in z_1 for $|x^{2r}z_j| < |z_1|$. In view of the periodicity (3.32), the two terms of (3.37) cancel out by shifting the contour $z_1 \to x^{2r-2}z_1$.

□

4 Local height probabilities

We present here a calculation of the local height probabilities (LHP) for the dilute A_L models in the regime 2^+.

4.1 Two-point LHP

We have already mentioned the result (2.6) about the one-point function. As the next simplest case, let us consider the probability $P_l(a-\varepsilon,a)$ of finding two neighboring local height variables to be $a-\varepsilon,\ a$ $(\varepsilon=0,\pm)$.

$$\begin{aligned} P_l(a-\varepsilon,a) &= \frac{1}{Z_l}\,S(a)\,\mathrm{tr}_{\mathcal{L}_{l,a}}\Big(x^{6\mathcal{H}}\Phi^*_\varepsilon(z)\Phi_\varepsilon(z)\Big) \\ &= \frac{1}{Z_l}\,g\sqrt{S(a)S(a-\varepsilon)}\,\mathrm{tr}_{\mathcal{L}_{l,a}}\Big(x^{6\mathcal{H}}\Phi_{-\varepsilon}(x^{-3}z)\Phi_\varepsilon(z)\Big). \end{aligned} \tag{4.1}$$

Note that $P_l(a-\varepsilon,a)$ is independent of z. From (4.1) and the property of the type I VO (3.20), we have the following relations.

$$\sum_{\varepsilon=\pm1,0} P_l(a-\varepsilon,a)=\frac{S(a)\ \chi_{l,a}(x^6)}{Z_l},\quad P_l(a-\varepsilon,a)=P_l(a,a-\varepsilon),\quad P_l(0,1)=0.$$

The evaluation of the trace yields the following expressions.

$$\begin{aligned} P_l(a-1,a) &= -\frac{S(a-1)\ x^{\frac{1-r}{r}}}{[a-\frac12]_+[2a-2]}\oint\oint_{C_+(1)}\underline{dw}_1\ \underline{dw}_2\ \mathcal{I}(w_1,w_2) \\ &\qquad\times\frac{[v_1-\frac12][v_1-2a+\frac32][v_1-v_2+a]_+}{[v_1+\frac12][v_1-\frac12][v_1-v_2+\frac12]}, \\ P_l(a,a) &= \frac{S(a)\ x^{\frac{1-r}{r}}}{[a+\frac12]_+[a-\frac12]_+}\oint_{C_0(x^{-3})}\underline{dw}_1\oint_{C_0(1)}\underline{dw}_2\ \mathcal{I}(w_1,w_2)\frac{[v_1-a+\frac32]_+[v_2-a]_+}{[v_1+1][v_2-\frac12]}, \\ P_l(a+1,a) &= -\frac{S(a)\ x^{\frac{1-r}{r}}}{[a+\frac12]_+[2a]}\oint\oint_{C_+(x^{-3})}\underline{dw}_1\ \underline{dw}_2\ \mathcal{I}(w_1,w_2) \\ &\qquad\times\frac{[v_1-2a+1][v_1-v_2+a+1]_+[v_2+\frac12]}{[v_1+1][v_1-v_2+\frac12][v_2-\frac12]}. \end{aligned}$$

Here $w_i=x^{2v_i}$ $(i=1,2)$ and

$$\begin{aligned} &\mathcal{I}(w_1,w_2) \\ &= \mathrm{tr}_{\mathcal{L}_{l,a}}\Big(x^{6\mathcal{H}}\Phi_-(x^{-3})x_-(w_1)\Phi_-(1)x_-(w_2)\Big) \\ &= \mathcal{O}_{l,k}(w_1,w_2)\frac{(x^5,x^6;x^6,x^{2r})^2_\infty}{(x^{2r+3},x^{2r+4};x^6,x^{2r})^2_\infty}\frac{(x^{2r+2}w_1,x^{2r+2}/w_1,x^{2r-1}w_2,x^{2r+2}/w_2;x^3,x^{2r})_\infty}{(x^4w_1,x^4/w_1,xw_2,x^4/w_2;x^3,x^{2r})_\infty} \\ &\quad\times(x^6,w_2/w_1,x^6w_1/w_2;x^6)_\infty\frac{G_{x^6}(x^{2r-1},w_2/w_1)G_{x^6}(x^2,w_2/w_1)}{G_{x^6}(x^{2r-2},w_2/w_1)G_{x^6}(x,w_2/w_1)}, \end{aligned}$$

where $\mathcal{O}_{l,k}(w_1,w_2)$ is the zero-mode contribution

$$\begin{aligned}\mathcal{O}_{l,k}(w_1,w_2) \;=\; & \left(x^6\right)^{\Delta_{l,a}-\frac{c}{24}}\left(x^3 w_1 w_2\right)^{\frac{l(L+1)-aL}{2(L+1)}+\frac{L}{4(L+1)}}\sum_{j\in\mathbb{Z}}(x^3 w_1 w_2)^{-Lj}\\ & \times\left(\left(x^6\right)^{L(L+1)j^2-(l(L+1)-aL)j}-\left(x^6\right)^{L(L+1)j^2+(l(L+1)+aL)j+la}\left(x^3 w_1 w_2\right)^{-l}\right),\end{aligned}$$

and

$$G_{x^6}(A,z)=(x^6A;x^6,x^{2r})^2_\infty(Az;x^6,x^{2r})_\infty(x^6A/z;x^6,x^{2r})_\infty.$$

The contours $C_+(1)$, $C_0(x^{-3})\cup C_0(1)$, $C_+(x^{-3})$ are chosen as follows $(n,m\geq 0)$; For all the contours, the poles $w_1=x^{4+3m+2rn}$, $w_2=x^{4+3m+2rn}$, $x^{4+6m+2r(n+1)}w_1$, $x^{1+6(m+1)+2rn}w_1$ are inside and the poles $w_1=x^{-4-3m-2rn}$, $w_2=x^{-1-3m-2rn}$, $x^{2-6m-2r(n+1)}w_1$, $x^{-1-6m-2rn}w_1$ are outside. In addition,

	inside	outside
$C_+(1)$	$w_1=x^{-1+2r(n+1)}$ $w_2=x^{1+2rn}w_1$	$w_1=x^{-1-2rn}$ $w_2=x^{-1-2rn},x^{-1-2rn}w_1,x^{2-2r(n+1)}w_1$
$C_0(x^{-3})\cup C_0(1)$	$w_1=x^{-2+2rn}$ $w_2=x^{1+2rn}$	$w_1=x^{-4-2rn}$ $w_2=x^{-1-2rn}$
$C_+(x^{-3})$	$w_1=x^{-2+2rn}$ $w_2=x^{1+2rn},x^{1+2rn}w_1$	$w_1=x^{-4-2rn}$ $w_2=x^{1-2r(n+1)},x^{-1-2rn}w_1,x^{2-2r(n+1)}w_1$

4.2 General case

Integral representation of the N-point correlation functions can be derived in a similar manner. It is written in terms of the traces of the type I vertex operators as in (2.7):

$$Z_l^{-1}S(k)\,\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(\Phi^*_{\varepsilon_1}(x^6z_1)\cdots\Phi^*_{\varepsilon_N}(x^6z_N)\Phi_{\varepsilon_N}(x^6z_N)\cdots\Phi_{\varepsilon_1}(x^6z_1)x^{6\mathcal{H}}\Big). \tag{4.2}$$

Here we give only the integral formula for the traces over the Fock module in a general situation

$$\mathrm{tr}_{\mathcal{F}_{l,k}}\Big(\Phi_{\varepsilon_1}(z_1)\cdots\Phi_{\varepsilon_N}(z_N)x^{6\mathcal{H}}\Big). \tag{4.3}$$

We assume $\sum_{t=1}^N\varepsilon_t=0$. Otherwise (4.3) vanishes.

First we prepare several functions.

$$F(z)=\frac{(x^{5+2r}z;x^6,x^{2r})}{(x^7z;x^6,x^{2r})},\qquad G(z)=\frac{F(z)}{F(xz)F(x^{-1}z)},$$
$$H(z)=\frac{(x^8z,x^9z,x^{9+2r}z,x^{10+2r}z;x^6,x^6,x^{2r})}{(x^{11}z,x^{12}z,x^{6+2r}z,x^{7+2r}z;x^6,x^6,x^{2r})}.$$

Define $h_{\varepsilon_n}(z_n,\{w_{n,i}\},\hat{k})$ $(\varepsilon_n=0,+)$ by normal-ordering the integrand of (3.6),(3.7),

$$\Phi_{\varepsilon_n}(z_n)=\oint\Big(\prod_{i\in I(\varepsilon)}\underline{dw}_{n,i}\Big):\Phi_-(z_n)\prod_{i\in I(\varepsilon)}x_-(w_{n,i}):\ h_{\varepsilon_n}(z_n,\{w_{n,i}\},\hat{k}),$$
$$I(0)=\{1\},\ I(+)=\{1,2\}.$$

Explicitly we have

$$h_0(z,w,k)=\frac{x^{(2u-2v+k+1/2)(k-1/2)/r^2-k+1/2}(xz^2)^{\frac{1-r}{2r}}}{\sqrt{[k+1/2]_+[k-1/2]_+}}\frac{(-x^{2k}z/w,-x^{2r-2k}w/z;x^{2r})_\infty}{(xw/z,xz/w;x^{2r})_\infty},$$

$$\begin{aligned}
&h_+(z,w_1,w_2,k)=\\
&\sqrt{\frac{S(k-1)}{S(k)}}\frac{x^{\{4(k-1)(u-v_1+k-1/2)+(k-1/2)(2v_1-2v_2+k+1/2)\}/r^2-3k+5/2}(xz^2/w_1)^{\frac{1-r}{r}}}{[k-1/2]_+[2k-2]}\\
&\times\frac{(x^{2r-1}w_2/z,x^{4k-3}z/w_1,x^{2r-4k+3}w_1/z;x^{2r})_\infty}{(xw_2/z,xw_1/z,xz/w_1;x^{2r})_\infty}\\
&\times\frac{(x^2w_2/w_1,-x^{2k}w_1/w_2,-x^{2r-2k}w_2/w_1;x^{2r})_\infty}{(xw_2/w_1,x^{2r-2}w_2/w_1,xw_1/w_2;x^{2r})_\infty}\Big(1-\frac{w_1}{w_2}\Big).
\end{aligned}$$

We use the symbol $\langle\!\langle A(z)B(w)\rangle\!\rangle$ to denote the normal ordering factors

$$A(z)B(w)=\langle\!\langle A(z)B(w)\rangle\!\rangle:A(z)B(w):.$$

(See the list in Appendix A.)

With this notation we have

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{F}_{l,k}}\Big(\Phi_{\varepsilon_1}(z_1)\cdots\Phi_{\varepsilon_N}(z_N)x^{6\mathcal{H}}\Big)\\
=&\oint\cdots\oint\prod_{\substack{1\le m\le N\\ \varepsilon_m\neq -}}\Big(\prod_{j\in I(\varepsilon_m)}\underline{dw}_{m,j}\Big)\,h_{\varepsilon_m}(z_m,\{w_{m,j}\},k+\sum_{t=1}^{m}\varepsilon_t)\\
&\times\prod_{1\le m<n\le N}\langle\!\langle\Phi_-(z_m)\Phi_-(z_n)\rangle\!\rangle\prod_{\substack{1\le m<n\le N\\ i\in I(\varepsilon_n),j\in I(\varepsilon_m)}}\langle\!\langle x_-(w_{m,j})x_-(w_{n,i})\rangle\!\rangle\qquad(4.4)\\
&\times\prod_{\substack{1\le n<m\le N\\ j\in I(\varepsilon_m)}}\langle\!\langle\Phi_-(z_n)x_-(w_{m,j})\rangle\!\rangle\prod_{\substack{1\le m<n\le N\\ j\in I(\varepsilon_m)}}\langle\!\langle x_-(w_{m,j})\Phi_-(z_n)\rangle\!\rangle\\
&\times\mathrm{tr}_{\mathcal{F}_{l,k}}\Big(:\Phi_-(z_1)\cdots\Phi_-(z_N)\prod_{\substack{1\le m\le N\\ j\in I(\varepsilon_m)}}x_-(w_{m,j}):x^{6\mathcal{H}}\Big),
\end{aligned}$$

where

$$\mathrm{tr}_{\mathcal{F}_{l,k}}\Big(:\Phi_-(z_1)\cdots\Phi_-(z_N)\prod_{\substack{1\le m\le N\\ j\in I(\varepsilon_m)}}x_-(w_{m,j}):x^{6\mathcal{H}}\Big)$$

$$= \prod_{1\le m,n\le N} H(z_n/z_m) \prod_{\substack{1\le m,n\le N\\ i\in I(\varepsilon_n), j\in I(\varepsilon_m)}} G(w_{n,i}/w_{m,j}) \prod_{\substack{1\le n,m\le N\\ j\in I(\varepsilon_m)}} F(z_n/w_{m,j})F(w_{m,j}/z_n) \tag{4.5}$$

$$\times \frac{x^{6(p_{l,k}^2/4-1/24)}}{(x^6;x^6)_\infty} \left(\Big(\prod_{\substack{1\le m\le N\\ j\in I(\varepsilon_m)}} w_{m,j}\Big) / \Big(\prod_{1\le n\le N} z_n\Big) \right)^{\sqrt{\frac{r-2}{r}}p_{l,k}} \Big(\prod_{\substack{1\le m\le N\\ j\in I(\varepsilon_m)}} w_{m,j} \prod_{1\le n\le N} z_n \Big)^{\frac{r-1}{2r}}$$

where $p_{l,k}$ is given in (3.2). The following are the list of poles of the integrand as functions of $w_{m,j}$. The contour for $\underline{dw}_{m,j}$ encircles only those denoted 'inside'$(a, b \in \mathbb{Z}_{\ge 0})$:

	inside	outside
$h_0(z_m, w_{m,1}, k)$	$w_{m,1} = x^{1+2rb} z_m$	$x^{-1-2rb} z_m$
$h_+(z_m, w_{m,1}, w_{m,2}, k)$	$w_{m,1} = x^{1+2rb} z_m$ $w_{m,2} = x^{1+2rb} w_{m,1}$	$w_{m,1} = x^{-1-2rb} z_m$ $w_{m,2} = x^{-1-2rb} w_{m,1}, x^{-1-2rb} z_m$
$\langle\langle x_-(w_{m,j})\Phi_-(z_n)\rangle\rangle$	$w_{m,j} = x^{1+2rb} z_n$	
$\langle\langle \Phi_-(z_n) x_-(w_{m,j})\rangle\rangle$		$w_{m,j} = x^{-1-2rb} z_n$
$\langle\langle x_-(w_{m,j}) x_-(w_{n,i})\rangle\rangle$	$w_{m,j} = x^{1+2rb} w_{n,i}$ $w_{m,j} = x^{-2+2r(1+b)} w_{n,i}$	
$G(w_{n,i}/w_{m,j})$	$w_{m,j} = x^{7+2rb+6a} w_{n,i}$ $w_{m,j} = x^{2r(1+b)+6(1+a)} w_{n,i}$ $w_{m,j} = x^{4+2r(1+b)+6a} w_{n,i}$	
$F(z_n/w_{m,j})$	$w_{m,j} = x^{7+2rb+6a} z_n$	$w_{m,j} = x^{-7-2rb-6a} z_n$

The formula for the N-point correlation function (4.2) can be obtained through specializing (4.5) and noting

$$H(x^3 z)H(z) = \frac{1}{F(x^2 z)F(xz)}.$$

Since the result is lengthy we do not present it here.

5 Discussion

As was discussed in the main text, the DVA for the dilute A_L model (which we have denoted by $\mathcal{V}_{x,r}(A_2^{(2)})$) exactly coincides with the one found by Brazhnikov and Lukyanov [13]. In the paper [13], $\mathcal{V}_{x,r}(A_2^{(2)})$ with $|x| = 1$ was treated as the Zamolodchikov-Faddeev (ZF) algebra for the Bullough-Dodd model ($A_2^{(2)}$ Toda field theory). We regard $\mathcal{V}_{x,r}(A_2^{(2)})$ with $0 < x < 1, r = 2(L+1)/(L+2)$ as the ZF algebra for the dilute A_L model (restricted face model), and apply the idea of bootstrap method to study the fusion of the $\mathcal{V}_{x,r}(A_2^{(2)})$ current $T(z)$.

The two-dimensional Ising model at the critical temperature $T = T_c$ is described by the $c = 1/2$ minimal CFT. Perturbing it by a magnetic field while keeping the same temperature ($T = T_c$), an off-critical integrable model is obtained [20]. A fascinating feature of this theory is that the Lie algebra E_8 appears as a hidden symmetry; one can check that the integrals of motion P_s appear at the exponents of E_8, $s = 1, 7, 11, 13, 17, 19, \cdots$, the bootstrap program closes within eight particles, the mass ratios are given by the Perron-Frobenius vector for the incidence matrix of E_8, and so on. Further discussions of the model as the $\phi_{1,2}$-perturbation of the $c = 1/2$ CFT can be found in [27]. It is argued that the dilute A_3 model is in the universality class of the magnetic-perturbed Ising model [1]. As in the case of the ABF model [4], our free field realization for the dilute A_3 model properly reduces to that of the $c = 1/2$ CFT, including the VO's, $\mathcal{V}_{x,r}(A_2^{(2)})$ and the Felder complex. Our description of the the dilute A_3 model, therefore, provides a lattice analogue of the $\phi_{1,2}$-perturbation of the $c = 1/2$ CFT.

In this section, we study an E_8-structure arising from $\mathcal{V}_{x,r}(A_2^{(2)})$ for the dilute A_3 model ($r = 8/5$). We construct eight fused DVA currents $T^{(a)}(u)$ ($a = 1, 2, \cdots, 8$) from the fundamental $\mathcal{V}_{x,r}(A_2^{(2)})$ current $T(z)$ using a bootstrap procedure. We show that these fused currents obey a set of relations which resembles the so called level-two restricted T-system of type $E_8^{(1)}$ [21].

The T-system of type $E_8^{(1)}$ [21] is written as

$$T_m^{(a)}(u - \frac{1}{20})T_m^{(a)}(u + \frac{1}{20}) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + g_m^{(a)}(u) \prod_{b \sim a} T_m^{(b)}(u), \tag{5.1}$$

where $T_m^{(a)}(u)$ ($a = 1, 2, \cdots, 8$) denotes the eigenvalues of the transfer matrix, the symbol $b \sim a$ means that b and a are adjacent nodes in the Dynkin diagram of E_8, and $g^{(a)}(u)$'s are some functions. (We have rescaled the u variable of [21] to fit the present notation.) If the face model is restricted, then we have the truncation $1 \leq m \leq \ell$. The integer ℓ is called level. If $\ell = 2$, $T_2^{(a)}(u)$ becomes proportional to the identity, and (5.1) reduces to

$$T_1^{(a)}(u - \frac{1}{20})T_1^{(a)}(u + \frac{1}{20}) = \phi^{(a)}(u) + g_1^{(a)}(u) \prod_{b \sim a} T_1^{(b)}(u), \tag{5.2}$$

with some functions $\phi^{(a)}(u)$. This is called the level-two restricted T-system of type $E_8^{(1)}$. In the paper [22], the T-system in (5.2) is realized in terms of the 'quantum' transfer matrix for the dilute A_3 model.

Now we come back to the deformed Virasoro algebra $\mathcal{V}_{x,r}(A_2^{(2)})$ for the dilute A_3 model. Before going into the technical details, let us roughly state the type of formulas we find for $\mathcal{V}_{x,r}(A_2^{(2)})$ with $r = 8/5$.

Definition 5.1 *Define the eight DVA currents $T^{(a)}(u)$ $(a = 1, 2, \cdots, 8)$ corresponding to the simple roots of E_8 by*

$$
\begin{aligned}
T^{(a)}(u) &= T_{\overline{a}}(u) \qquad\qquad a = 1, 2, 3, 4, 5, \\
T^{(6)}(u) &= f_{1\overline{3}}(u_2 - u_1) T_1(u_1) T_{\overline{3}}(u_2) \Big|_{\substack{u_1 = u + \frac{11}{20} \\ u_2 = u - \frac{3}{20}}}, \\
T^{(7)}(u) &= T_2(u), \\
T^{(8)}(u) &= f_{1\overline{2}}(u_2 - u_1) T_1(u_1) T_{\overline{2}}(u_2) \Big|_{\substack{u_1 = u + \frac{9}{20} \\ u_2 = u - \frac{4}{20}}}.
\end{aligned}
$$

Here the fused DVA currents $T_n(u), T_{\overline{n}}(u)$ and the structure function $f_{m\overline{n}}(u)$ are defined in Definition 5.5 below.

Proposition 5.2 *The following relations hold:*

$$f^{(1)}(u_1, u_2) T^{(1)}(u_1) T^{(1)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = T^{(2)}(u), \tag{5.3}$$

$$f^{(2)}(u_1, u_2) T^{(2)}(u_1) T^{(2)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = g^{(2)}(u_1, u_2) T^{(1)}(u_1) T^{(3)}(u_2) \Big|_{\substack{u_1 = u \\ u_2 = u}}, \tag{5.4}$$

$$f^{(3)}(u_1, u_2) T^{(3)}(u_1) T^{(3)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = g^{(3)}(u_1, u_2) T^{(2)}(u_1) T^{(4)}(u_2) \Big|_{\substack{u_1 = u \\ u_2 = u}}, \tag{5.5}$$

$$f^{(4)}(u_1, u_2) T^{(4)}(u_1) T^{(4)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = g^{(4)}(u_1, u_2) T^{(3)}(u_1) T^{(5)}(u_2) \Big|_{\substack{u_1 = u \\ u_2 = u}}, \tag{5.6}$$

$$f^{(5)}(u_1, u_2) T^{(5)}(u_1) T^{(5)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = g^{(5)}(u_1, u_2, u_3) T^{(4)}(u_1) T^{(6)}(u_2) T^{(8)}(u_3) \Big|_{\substack{u_1 = u \\ u_2 = u \\ u_3 = u}}, \tag{5.7}$$

$$f^{(6)}(u_1, u_2) T^{(6)}(u_1) T^{(6)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = g^{(6)}(u_1, u_2) T^{(5)}(u_1) T^{(7)}(u_2) \Big|_{\substack{u_1 = u \\ u_2 = u}}, \tag{5.8}$$

$$f^{(7)}(u_1, u_2) T^{(7)}(u_1) T^{(7)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = T^{(6)}(u), \tag{5.9}$$

$$f^{(8)}(u_1, u_2) T^{(8)}(u_1) T^{(8)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} \\ u_2 = u - \frac{1}{20}}} = T^{(5)}(u), \tag{5.10}$$

with appropriate functions $f^{(a)}, g^{(a)}$ (see Definition 5.15). Both sides are regarded as operators on the cohomology $H^0(C_{l,k})$. Likewise we have the relations

$$[\![u_2 - u_1 - \frac{3}{2}]\!] f^{(a)}(u_1, u_2) T^{(a)}(u_1) T^{(a)}(u_2) \Big|_{\substack{u_1 = u + \frac{1}{20} - \frac{r}{2} \\ u_2 = u - \frac{1}{20} + \frac{r}{2}}} = c^{(a)} \mathrm{id}, \tag{5.11}$$

for $a = 1, 2, \cdots, 8$, where $c^{(a)}$'s are some constants and the symbol $[\![\ ]\!]$ is defined below in (5.14).

We notice that (5.3)–(5.11) for the fused $\mathcal{V}_{x,r}(A_2^{(2)})$ currents look very similar to the T-system (5.2) arising from the analytic Bethe ansatz. There is, however, an obvious discrepancy between them; while the T-system (5.2) comprises two terms, (5.3)–(5.11) consists of only one term. More precisely, the right hand side of (5.3)–(5.10) corresponds to the second term in (5.2), whereas that of (5.11) corresponds to the first term. In the left hand side, the spectral parameters $u_1 - u_2$ of (5.3)–(5.10) and (5.11) differ by r. Such a shift by r is irrelevant in the T-system (5.2), because the transfer matrix eigenvalues $T_m^{(a)}(u)$ (with appropriate normalization) are periodic, $T_m^{(a)}(u+r) = T_m^{(a)}(u)$. This is a reflection of the quasi-periodicity of the Boltzmann weights. On the other hand, the $\mathcal{V}_{x,r}(A_2^{(2)})$ currents $T^{(a)}(u)$ are *by no means* doubly quasi-periodic; we have $T^{(a)}(u + \pi i/\log x) = T^{(a)}(u)$ but $T^{(a)}(u+r) \neq T^{(a)}(u)$.

We have not understood yet the reason why we have such similarities between the T-system for the Bethe ansatz and the exchange relations for the DVA. For the purpose of comparison, we summarize in Appendix C the fusions of the DVA current and the 'T-system' for the algebra $A_{N-1}^{(1)}$.

In the rest of this section, we briefly sketch the derivation of Proposition 5.2.

5.1 OPE's

Let r be generic, for a while. We set

$$r^* = r - 1, \tag{5.12}$$

$$[u]_x = \frac{x^u - x^{-u}}{x - x^{-1}}, \tag{5.13}$$

$$[\![u]\!] = \frac{[u]_x}{[u + r^* + 1]_x} = \frac{1}{[\![-u - r^* - 1]\!]}. \tag{5.14}$$

In this section we prefer to use the additive notation and write $f(u)$ for the structure function $f(z)$ ($z = x^{2u}$) in (3.24). It satisfies the relations:

Lemma 5.3

$$(i) \quad \frac{f(u - \frac{1}{2}) f(u + \frac{1}{2})}{f(u)} = \frac{[\![u - r^* - 1/2]\!]}{[\![u - 1/2]\!]},$$

$$(ii) \quad f(u) f(u \pm \frac{3}{2}) = \frac{[\![\pm u - r^*]\!]}{[\![\pm u]\!]} \frac{[\![\pm u - r^* + 1/2]\!]}{[\![\pm u + 1/2]\!]},$$

$$(iii) \quad f(u-1) f(u) f(u+1) = \frac{[\![u - r^* - 1]\!]}{[\![u - 1]\!]} \frac{[\![u - r^* - 1/2]\!]}{[\![u - 1/2]\!]} \frac{[\![u - r^*]\!]}{[\![u]\!]}.$$

The operators $\Lambda_\pm(z), \Lambda_0(z)$ are defined by (3.21).

Lemma 5.4 *The operator product expansions (OPE's) among $\Lambda_i(u)$ are*

$$
\begin{aligned}
& f(u_2-u_1)\Lambda_i(x^{2u_1})\Lambda_j(x^{2u_2}) \\
= & :\Lambda_i(x^{2u_1})\Lambda_j(x^{2u_2}): \\
& \times \begin{cases}
1 & (i,j)=(+,+), \\
\dfrac{[\![u_1-u_2-r^*]\!]}{[\![u_1-u_2]\!]} & (i,j)=(+,0), \\
\dfrac{[\![u_1-u_2-r^*]\!]}{[\![u_1-u_2]\!]}\dfrac{[\![u_1-u_2-r^*+1/2]\!]}{[\![u_1-u_2+1/2]\!]} & (i,j)=(+,-), \\
\dfrac{[\![u_1-u_2-r^*-1]\!]}{[\![u_1-u_2-1]\!]} & (i,j)=(0,+), \\
\dfrac{[\![u_1-u_2-r^*-1/2]\!]}{[\![u_1-u_2-1/2]\!]} & (i,j)=(0,0), \\
\dfrac{[\![u_1-u_2-r^*]\!]}{[\![u_1-u_2]\!]} & (i,j)=(0,-), \\
\dfrac{[\![u_1-u_2-r^*-1]\!]}{[\![u_1-u_2-1]\!]}\dfrac{[\![u_1-u_2-r^*-3/2]\!]}{[\![u_1-u_2-3/2]\!]} & (i,j)=(-,+), \\
\dfrac{[\![u_1-u_2-r^*-1]\!]}{[\![u_1-u_2-1]\!]} & (i,j)=(-,0), \\
1 & (i,j)=(-,-).
\end{cases}
\end{aligned}
$$

5.2 Fused $\mathcal{V}_{x,r}(A_2^{(2)})$ currents $T_n(u)$ and $T_{\overline{n}}(u)$

Suggested by the bootstrap program for general r, we introduce the following fused currents of $\mathcal{V}_{x,r}(A_2^{(2)})$.

Definition 5.5 *Define the fused DVA currents $T_n(u)$ and $T_{\overline{n}}(u)$ by*

$$
\begin{aligned}
& T_0(u)=T_{\overline{0}}(u)=\mathrm{id}, \quad T_1(u)=T_{\overline{1}}(u)=T(x^{2u}), \\
& T_n(u)=\prod_{1\le i<j\le n} f(u_j-u_i)\cdot T_1(u_1)T_1(u_2)\cdots T_1(u_n)\Big|_{\substack{u_i=u+\frac{n-1}{2}r^*-(i-1)r^* \\ 1\le i\le n}}, \\
& T_{\overline{n}}(u)=\prod_{1\le i<j\le n} f(u_j-u_i)\cdot T_1(u_1)T_1(u_2)\cdots T_1(u_n)\Big|_{\substack{u_i=u+\frac{n-1}{2}(r^*-\frac{1}{2})-(i-1)(r^*-\frac{1}{2}) \\ 1\le i\le n}}.
\end{aligned}
$$

Define the structure functions for $T_n(u)$, $T_{\overline{n}}(u)$ by

$$
\begin{aligned}
& f_{mn}(u)=\prod_{i=1}^m\prod_{j=1}^n f(u+\frac{n-m}{2}r^*-(j-i)r^*), \\
& f_{\overline{m}\,\overline{n}}(u)=\prod_{i=1}^m\prod_{j=1}^n f(u+\frac{n-m}{2}(r^*-\frac{1}{2})-(j-i)(r^*-\frac{1}{2})), \\
& f_{\overline{m}\,n}(u)=f_{n\,\overline{m}}(u)=\prod_{i=1}^m\prod_{j=1}^n f(u+\frac{n+1}{2}r^*-\frac{m+1}{2}(r^*-\frac{1}{2})-jr^*+i(r^*-\frac{1}{2})).
\end{aligned}
$$

These fused DVA currents enjoy the ZF exchange relations.

Lemma 5.6 *As meromorphic functions, the following exchange relations hold.*

$$
\begin{aligned}
f_{mn}(v-u)T_m(u)T_n(v) &= f_{nm}(u-v)T_n(v)T_m(u),\\
f_{\overline{m}\,\overline{n}}(v-u)T_{\overline{m}}(u)T_{\overline{n}}(v) &= f_{\overline{n}\,\overline{m}}(u-v)T_{\overline{n}}(v)T_{\overline{m}}(u),\\
f_{m\,\overline{n}}(v-u)T_m(u)T_{\overline{n}}(v) &= f_{\overline{n}\,m}(u-v)T_{\overline{n}}(v)T_m(u).
\end{aligned}
$$

Lemma 5.7

$$
\begin{aligned}
&(i) \quad [\![u-v\pm 1]\!] f(v-u)T_1(u)T_1(v)\Big|_{u=v\mp 1} = \frac{\mp T_1(v\mp 1/2)}{[\![-1/2]\!][\![-1]\!]},\\
&(ii) \quad [\![u-v\pm 3/2]\!] f(v-u)T_1(u)T_1(v)\Big|_{u=v\mp 3/2} = \frac{\mp \mathrm{id}}{[\![1/2]\!][\![-1]\!][\![-3/2]\!]},\\
&(iii) \quad [\![u_1-u_2-1]\!][\![u_2-u_3-1]\!]\\
&\qquad \times f(u_2-u_1)f(u_3-u_1)f(u_3-u_2)T_1(u_1)T_1(u_2)T_1(u_3)\Big|_{\substack{u_2=u_1-1\\ u_3=u_2-1}}\\
&\quad = \frac{\mathrm{id}}{[\![1]\!][\![\frac{1}{2}]\!][\![-\frac{1}{2}]\!][\![-1]\!][\![-1]\!][\![-\frac{3}{2}]\!][\![-2]\!]}.
\end{aligned}
$$

We need the analyticity properties of the operator products. (Some details of the derivation are given in Appendix for the case of $A^{(1)}_{N-1}$.)

Lemma 5.8 *The product $f_{mn}(v-u)T_m(u)T_n(v)$ has poles only at*

$$
u-v = \left\{
\begin{array}{l}
\left(\frac{m+n}{2}-k\right)r^*-1,\\
\left(\frac{m+n}{2}-k\right)r^*-\frac{3}{2},\\
-\left(\frac{m+n}{2}-k\right)r^*+1,\\
-\left(\frac{m+n}{2}-k\right)r^*+\frac{3}{2},
\end{array}
\right.
\qquad k=1,2,\cdots,\min(m,n).
$$

All the poles are simple.

Lemma 5.9 *The product $f_{\overline{m}\,\overline{n}}(v-u)T_{\overline{m}}(u)T_{\overline{n}}(v)$ has poles only at*

simple pole :

$$
u-v = \left\{
\begin{array}{l}
\left(\frac{m+n}{2}-k\right)\left(r^*-\frac{1}{2}\right)-\frac{3}{2},\\
-\left(\frac{m+n}{2}-k\right)\left(r^*-\frac{1}{2}\right)+\frac{3}{2},
\end{array}
\right.
\quad k=1,2,\cdots,\min(m,n),
$$

and

poles with multiplicity $\min(\min(m,n),\min(l,m+n-l))$:

$$
u-v = \left\{
\begin{array}{l}
\left(\frac{m+n}{2}-l\right)\left(r^*-\frac{1}{2}\right)-1,\\
-\left(\frac{m+n}{2}-l\right)\left(r^*-\frac{1}{2}\right)+1,
\end{array}
\right.
\quad l=1,2,\cdots,m+n-1.
$$

Lemma 5.10 *The product $f_{\overline{m}n}(v-u)T_{\overline{m}}(u)T_n(v)$ has poles only at*

$$u-v=\begin{cases} \frac{n-1}{2}r^*+\left(\frac{m+1}{2}-l\right)\left(r^*-\frac{1}{2}\right)-1, & \\ \frac{n-1}{2}r^*+\left(\frac{m+1}{2}-k\right)\left(r^*-\frac{1}{2}\right)-\frac{3}{2}, & l=1,2,\cdots,m, \\ -\frac{n-1}{2}r^*-\left(\frac{m+1}{2}-l\right)\left(r^*-\frac{1}{2}\right)+1, & k=1,2,\cdots,\min(m,n). \\ -\frac{n-1}{2}r^*-\left(\frac{m+1}{2}-k\right)\left(r^*-\frac{1}{2}\right)+\frac{3}{2}, & \end{cases}$$

All the poles are simple.

5.3 The case of the dilute A_3 model

The parameters for the dilute A_3 model are given by

$$L=3, \qquad r=2\frac{L+1}{L+2}=\frac{8}{5}, \qquad r^*=\frac{3}{5}.$$

For $L=3$, we expect to have the following extra symmetry for $T_n(u)$.

Conjecture 5.11 *As operators acting on the BRST cohomology $H^0(C_{l,k})$ ($k=1,2,3, l=1,2$), we have*

$$\begin{aligned} (i) \quad T_3(u) &= \frac{T_2(u)}{[\![\frac{2}{10}]\!][\![\frac{3}{10}]\!]}, \\ (ii) \quad T_4(u) &= \frac{-T_1(u)}{[\![-\frac{9}{10}]\!][\![-\frac{4}{10}]\!][\![-\frac{3}{10}]\!][\![\frac{3}{10}]\!][\![\frac{2}{10}]\!][\![\frac{8}{10}]\!]}, \\ (iii) \quad T_5(u) &= \frac{-\mathrm{id}}{[\![-\frac{15}{10}]\!][\![-\frac{10}{10}]\!][\![-\frac{9}{10}]\!][\![-\frac{4}{10}]\!][\![-\frac{3}{10}]\!][\![\frac{2}{10}]\!][\![\frac{3}{10}]\!][\![\frac{8}{10}]\!][\![\frac{9}{10}]\!][\![\frac{14}{10}]\!]}. \end{aligned}$$

One of the grounds for this conjecture 5.11 is the degeneration of the structure functions.

Lemma 5.12 *For $r^*=3/5$, we have*

$$\begin{aligned} (i) \quad f_{13}(u) &= f_{12}(u)\frac{[\![u-\frac{7}{10}]\!][\![u-\frac{12}{10}]\!]}{[\![u-\frac{4}{10}]\!][\![u-\frac{9}{10}]\!]}, \\ (ii) \quad f_{14}(u) &= f_{11}(u)\frac{[\![u-\frac{10}{10}]\!][\![u-\frac{15}{10}]\!]}{[\![u-\frac{1}{10}]\!][\![u-\frac{6}{10}]\!]}, \\ (iii) \quad f_{15}(u) &= \frac{[\![u-\frac{13}{10}]\!][\![u-\frac{18}{10}]\!]}{[\![u+\frac{2}{10}]\!][\![u-\frac{3}{10}]\!]}. \end{aligned}$$

Using Lemma 5.12, 5.8, 5.9 and 5.10, we can check that the replacement $T_m(u) \leftrightarrow T_{5-m}(u)$ will not affect the analyticity in any of the OPE's acting on the BRST cohomology space.

To obtain the correct proportionality constants in Conjecture 5.11, we calculated $\langle l,k|T_m(u)|l,k\rangle$ for $k=1,2,3, l=1,2$.

5.4 Fusions of $T_2(u)$ at $r^*=3/5$

If we study the bootstrap for E_8-symmetric particles carefully [20], we realize that it is helpful to consider the fusions of $T_2(u)$.

Lemma 5.13 *For $r^*=3/5$, the following equalities hold.*

$$
\begin{aligned}
(i)\quad & f_{22}(u_2-u_1)T_2(u_1)T_2(u_2)\Big|_{\substack{u_1=u+\frac{1}{20}\\ u_2=u-\frac{1}{20}}}\\
&=\frac{[\![-\frac{8}{10}]\!][\![-\frac{7}{10}]\!][\![-\frac{2}{10}]\!][\![\frac{3}{10}]\!]}{[\![-\frac{5}{10}]\!][\![-\frac{4}{10}]\!][\![-\frac{3}{10}]\!]}f_{1\overline{3}}(u_2-u_1)T_1(u_1)T_{\overline{3}}(u_2)\Big|_{\substack{u_1=u+\frac{11}{20}\\ u_2=u-\frac{3}{20}}}\\
&=\frac{[\![-\frac{8}{10}]\!][\![-\frac{7}{10}]\!][\![-\frac{2}{10}]\!][\![\frac{3}{10}]\!]}{[\![-\frac{5}{10}]\!][\![-\frac{4}{10}]\!][\![-\frac{3}{10}]\!]}f_{\overline{3}1}(u_2-u_1)T_{\overline{3}}(u_1)T_1(u_2)\Big|_{\substack{u_1=u+\frac{3}{20}\\ u_2=u-\frac{11}{20}}},\\
(ii)\quad & f_{22}(u_2-u_1)T_2(u_1)T_2(u_2)\Big|_{\substack{u_1=u+\frac{7}{20}\\ u_2=u-\frac{7}{20}}}\\
&=\frac{[\![-\frac{8}{10}]\!]}{[\![-\frac{5}{10}]\!][\![-\frac{3}{10}]\!][\![-\frac{2}{10}]\!]}f_{1\overline{2}}(u_2-u_1)T_1(u_1)T_{\overline{2}}(u_2)\Big|_{\substack{u_1=u+\frac{9}{20}\\ u_2=u-\frac{4}{20}}}\\
&=\frac{[\![-\frac{8}{10}]\!]}{[\![-\frac{5}{10}]\!][\![-\frac{3}{10}]\!][\![-\frac{2}{10}]\!]}f_{\overline{2}1}(u_2-u_1)T_{\overline{2}}(u_1)T_1(u_2)\Big|_{\substack{u_1=u+\frac{4}{20}\\ u_2=u-\frac{9}{20}}}.
\end{aligned}
$$

To prove these, we use

$$
\frac{5}{2}r^*-\frac{3}{2}=0,\qquad T_2(u)=[\![\frac{2}{10}]\!][\![\frac{3}{10}]\!]T_3(u),
$$

and Lemma 5.7.

5.5 $T_{\overline{5}}(u), T_{\overline{6}}(u), T_{\overline{7}}(u)$ at $r^*=3/5$

The fused DVA currents $T_{\overline{5}}(u), T_{\overline{6}}(u), T_{\overline{7}}(u)$ for $r^*=3/5$ can be rewritten as follows.

Lemma 5.14 *For $r^*=3/5$, we have*

$$
\begin{aligned}
T_{\overline{5}}(u)\ =\ & [\![-\frac{1}{2}]\!][\![-1]\!]\left(\frac{[\![-\frac{6}{10}]\!][\![-\frac{7}{10}]\!]}{[\![-\frac{12}{10}]\!][\![-\frac{13}{10}]\!]}\right)^2[\![u_2'-u_2-1]\!]\\
&\times f_{\overline{2}1}(u_2-u_1)f_{\overline{2}1}(u_2'-u_1)f_{\overline{2}\,\overline{2}}(u_3-u_1)f_{11}(u_2'-u_2)f_{1\overline{2}}(u_3-u_2')f_{1\overline{2}}(u_3-u_2)\\
&\times T_{\overline{2}}(u_1)T_1(u_2)T_1(u_2')T_{\overline{2}}(u_3)\Big|_{\substack{u_1=u+\frac{3}{20}\\ u_2=u-\frac{1}{2}\\ u_2'=u+\frac{1}{2}\\ u_3=u-\frac{3}{20}}},
\end{aligned}
$$

$$
\begin{aligned}
T_{\overline{6}}(u) \;=\;& \llbracket -\frac{1}{2}\rrbracket \llbracket -1\rrbracket \frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket} \frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket \llbracket -\frac{8}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket \llbracket -\frac{14}{10}\rrbracket} \llbracket u_2' - u_2 - 1\rrbracket \\
&\times f_{\overline{3}1}(u_2-u_1) f_{\overline{3}1}(u_2'-u_1) f_{\overline{3}\,\overline{2}}(u_3-u_1) f_{11}(u_2'-u_2) f_{1\overline{2}}(u_3-u_2') f_{1\overline{2}}(u_3-u_2) \\
&\times T_{\overline{3}}(u_1) T_1(u_2) T_1(u_2') T_{\overline{2}}(u_3)\Big|_{\substack{u_1=u+\frac{3}{20}\\ u_2=u-\frac{11}{20}\\ u_2'=u+\frac{9}{20}\\ u_3=u-\frac{4}{20}}},
\end{aligned}
$$

$$
\begin{aligned}
T_{\overline{7}}(u) \;=\;& f_{\overline{5}2}(u_2-u_1) T_{\overline{5}}(u_1) T_2(u_2)\Big|_{\substack{u_1=u\\ u_2=u}} \\
=\;& \llbracket -\frac{1}{2}\rrbracket \llbracket -1\rrbracket \left(\frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket \llbracket -\frac{8}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket \llbracket -\frac{14}{10}\rrbracket} \right)^2 \llbracket u_2' - u_2 - 1\rrbracket \\
&\times f_{\overline{3}1}(u_2-u_1) f_{\overline{3}1}(u_2'-u_1) f_{\overline{3}\,\overline{3}}(u_3-u_1) f_{11}(u_2'-u_2) f_{1\overline{3}}(u_3-u_2') f_{1\overline{3}}(u_3-u_2) \\
&\times T_{\overline{3}}(u_1) T_1(u_2) T_1(u_2') T_{\overline{3}}(u_3)\Big|_{\substack{u_1=u+\frac{4}{20}\\ u_2=u-\frac{1}{2}\\ u_2'=u+\frac{1}{2}\\ u_3=u-\frac{4}{20}}}.
\end{aligned}
$$

5.6 Structure functions

In accordance with the currents $T^{(a)}(u)$, we introduce the following structure functions.

Definition 5.15 *Define* $f^{(a)}$ $(a=1,2,\cdots,8)$, $g^{(a)}$ $(a=2,3\cdots,6)$ *by*

$$
\begin{aligned}
& f^{(a)}(u_1,u_2) = f_{\overline{a}\,\overline{a}}(u_2-u_1) \qquad (1 \le a \le 5), \\
& g^{(a)}(u_1,u_2) = f_{\overline{a-1}\,\overline{a+1}}(u_2-u_1) \qquad (2 \le a \le 4), \\
& g^{(5)}(u_1,u_2,u_3) = \llbracket -\frac{1}{2}\rrbracket \llbracket -1\rrbracket \frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket} \frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket \llbracket -\frac{8}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket \llbracket -\frac{14}{10}\rrbracket} \frac{\llbracket -\frac{4}{10}\rrbracket \llbracket -\frac{5}{10}\rrbracket \llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket}{\llbracket -\frac{10}{10}\rrbracket \llbracket -\frac{11}{10}\rrbracket \llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket} \\
&\qquad \times f_{\overline{4}\,\overline{3}}(u_2-u_1+\frac{3}{20}) f_{\overline{4}1}(u_2-u_1-\frac{11}{20}) f_{\overline{4}1}(u_3-u_1+\frac{9}{20}) f_{\overline{4}\,\overline{2}}(u_3-u_1-\frac{4}{20}) \\
&\qquad \times \llbracket u_3-u_2\rrbracket f_{\overline{3}1}(u_3-u_2+\frac{6}{20}) f_{\overline{3}\,\overline{2}}(u_3-u_2-\frac{7}{20}) \\
&\qquad \times f_{11}(u_3-u_2+1) f_{1\overline{2}}(u_3-u_2+\frac{7}{20}), \\
& f^{(6)}(u_1,u_2) = \llbracket -\frac{1}{2}\rrbracket \llbracket -1\rrbracket \left(\frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket \llbracket -\frac{8}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket \llbracket -\frac{14}{10}\rrbracket} \right)^2 \frac{[u_1-u_2-\frac{1}{10}]_x}{[\frac{16}{10}]_x} \llbracket u_1-u_2-\frac{1}{10}\rrbracket \\
&\qquad \times f_{1\overline{3}}(u_2-u_1+\frac{4}{10}) f_{\overline{3}\,\overline{3}}(u_2-u_1-\frac{3}{10}) f_{11}(u_2-u_1+\frac{11}{10}) f_{\overline{3}1}(u_2-u_1+\frac{4}{10}), \\
& g^{(6)}(u_1,u_2) = f_{\overline{5}2}(u_2-u_1), \\
& f^{(7)}(u_1,u_2) = \frac{\llbracket -\frac{5}{10}\rrbracket \llbracket -\frac{4}{10}\rrbracket \llbracket -\frac{3}{10}\rrbracket}{\llbracket -\frac{8}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket \llbracket -\frac{2}{10}\rrbracket \llbracket \frac{3}{10}\rrbracket} \times f_{22}(u_2-u_1), \\
& f^{(8)}(u_1,u_2) = \llbracket -\frac{1}{2}\rrbracket \llbracket -1\rrbracket \left(\frac{\llbracket -\frac{6}{10}\rrbracket \llbracket -\frac{7}{10}\rrbracket}{\llbracket -\frac{12}{10}\rrbracket \llbracket -\frac{13}{10}\rrbracket} \right)^2 \llbracket u_1-u_2-\frac{1}{10}\rrbracket
\end{aligned}
$$

$$\times f_{1\overline{2}}(u_2-u_1-\frac{5}{20})f_{\overline{2}\,\overline{2}}(u_2-u_1+\frac{8}{20})f_{11}(u_2-u_1-\frac{18}{20})f_{\overline{2}1}(u_2-u_1-\frac{5}{20}).$$

Collecting all the information together, we easily obtain the 'T-system' with the E_8 symmetry stated in Proposition5.2.

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A Operator product expansions

We list the normal ordering relations. For operators $A(z), B(w)$ that have the form : exp(linear in boson) :, we use the notation

$$A(z)B(w)=\langle\!\langle A(z)B(w)\rangle\!\rangle :A(z)B(w):$$

and write down only the part $\langle\!\langle A(z)B(w)\rangle\!\rangle$.

$$\langle\!\langle x_+(z_1)x_+(z_2)\rangle\!\rangle = z_1^{\frac{r}{r-1}}(1-z_2/z_1)\frac{(x^{-2}z_2/z_1,x^{2r-1}z_2/z_1;x^{2r-2})_\infty}{(x^{-1}z_2/z_1,x^{2r}z_2/z_1;x^{2r-2})_\infty},$$

$$\langle\!\langle x_-(z_1)x_-(z_2)\rangle\!\rangle = z_1^{\frac{r-1}{r}}(1-z_2/z_1)\frac{(x^{2}z_2/z_1,x^{2r-1}z_2/z_1;x^{2r})_\infty}{(xz_2/z_1,x^{2r-2}z_2/z_1;x^{2r})_\infty},$$

$$\langle\!\langle x_\pm(z_1)x_\mp(z_2)\rangle\!\rangle = z_1^{-1}\frac{1+z_2/z_1}{(1+xz_2/z_1)(1+x^{-1}z_2/z_1)},$$

$$\langle\!\langle \Phi_-(z_1)x_-(z_2)\rangle\!\rangle = z_1^{-\frac{r-1}{r}}\frac{(x^{2r-1}z_2/z_1;x^{2r})_\infty}{(xz_2/z_1;x^{2r})_\infty},$$

$$\langle\!\langle x_-(z_2)\Phi_-(z_1)\rangle\!\rangle = z_2^{-\frac{r-1}{r}}\frac{(x^{2r-1}z_1/z_2;x^{2r})_\infty}{(xz_1/z_2;x^{2r})_\infty},$$

$$\langle\!\langle \Phi_-(z_1)x_+(z_2)\rangle\!\rangle = x_+(z_2)\Phi_-(z_1)=(z_1+z_2),$$

$$\langle\!\langle \Psi^*_-(z_1)x_+(z_2)\rangle\!\rangle = z_1^{-\frac{r}{r-1}}\frac{(x^{2r-1}z_2/z_1;x^{2r-2})_\infty}{(x^{-1}z_2/z_1;x^{2r-2})_\infty},$$

$$\langle\!\langle x_+(z_2)\Psi^*_-(z_1)\rangle\!\rangle = z_2^{-\frac{r}{r-1}}\frac{(x^{2r-1}z_1/z_2;x^{2r-2})_\infty}{(x^{-1}z_1/z_2;x^{2r-2})_\infty},$$

$$\langle\!\langle \Psi^*_-(z_1)x_-(z_2)\rangle\!\rangle = \langle\!\langle x_-(z_2)\Psi^*_-(z_1)\rangle\!\rangle=(z_1+z_2),$$

$$\langle\!\langle \Phi_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{\frac{r-1}{r}}\frac{(x^2z_2/z_1,x^3z_2/z_1,x^{2r+3}z_2/z_1,x^{2r+4}z_2/z_1;x^6,x^{2r})_\infty}{(x^5z_2/z_1,x^6z_2/z_1,x^{2r}z_2/z_1,x^{2r+1}z_2/z_1;x^6,x^{2r})_\infty},$$

$$\langle\!\langle \Psi^*_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1^{\frac{r}{r-1}}\frac{(z_2/z_1,xz_2/z_1,x^{2r+3}z_2/z_1,x^{2r+4}z_2/z_1;x^6,x^{2r-2})_\infty}{(x^3z_2/z_1,x^4z_2/z_1,x^{2r}z_2/z_1,x^{2r+1}z_2/z_1;x^6,x^{2r-2})_\infty},$$

$$\langle\!\langle \Phi_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1^{-1}\frac{(-x^4z_2/z_1,-x^5z_2/z_1;x^6)_\infty}{(-xz_2/z_1,-x^2z_2/z_1;x^6)_\infty},$$

$$\langle\!\langle \Psi^*_-(z_2)\Phi_-(z_1)\rangle\!\rangle = z_2^{-1}\frac{(-x^4z_1/z_2,-x^5z_1/z_2;x^6)_\infty}{(-xz_1/z_2,-x^2z_1/z_2;x^6)_\infty}.$$

As meromorphic functions we have

$$
\begin{aligned}
x_+(z_1)x_+(z_2) &= \frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}\frac{[u_1-u_2-1/2]^*}{[-u_1+u_2-1/2]^*}x_+(z_2)x_+(z_1),\\
x_-(z_1)x_-(z_2) &= \frac{[u_1-u_2-1]}{[u_1-u_2+1]}\frac{[u_1-u_2+1/2]}{[-u_1+u_2+1/2]}x_-(z_2)x_-(z_1),\\
x_\pm(z_1)x_\mp(z_2) &= x_\mp(z_2)x_\pm(z_1),\\
\Phi_-(z_1)x_-(z_2) &= \frac{[u_1-u_2+1/2]}{[-u_1+u_2+1/2]}x_-(z_2)\Phi_-(z_1),\\
\Phi_-(z_1)x_+(z_2) &= x_+(z_2)\Phi_-(z_1),\\
\Psi^*_-(z_1)x_+(z_2) &= \frac{[u_1-u_2-1/2]^*}{[-u_1+u_2-1/2]^*}x_+(z_2)\Psi^*_-(z_1),\\
\Psi^*_-(z_1)x_-(z_2) &= x_-(z_2)\Psi^*_-(z_1),\\
\Phi_-(z_1)\Phi_-(z_2) &= \rho(u_2-u_1)\Phi_-(z_2)\Phi_-(z_1),\\
\Psi^*_-(z_1)\Psi^*_-(z_2) &= \rho^*(u_1-u_2)\Psi^*_-(z_2)\Psi^*_-(z_1),\\
\Phi_-(z_1)\Psi^*_-(z_2) &= \tau(u_2-u_1)\Psi^*_-(z_2)\Phi_-(z_1).
\end{aligned}
$$

Here $\rho(u)$, $\rho^*(u)$ and $\tau(u)$ are given respectively by (2.4),(3.15) and (3.16).

B BRST charges

We give here a proof of the properties of BRST charges stated in subsection 3.5. The method is a proper adaptation of the work [19] to the present situation.

B.1 Feigin-Odesskii algebra

First let us prepare the notation. Let $\tilde{A}_n$ be the set of all functions $F(u_1,\cdots,u_n)$ which is holomorphic on $\mathbb{C}^n$, symmetric in $u_1,\cdots,u_n$, and enjoys the quasi-periodicity properties ($r^*=r-1$)

$$
F(u_1+r^*,u_2,\cdots,u_n) = (-1)^n F(u_1,u_2,\cdots,u_n), \tag{B.1}
$$

$$
\begin{aligned}
F(u_1+\tau,u_2,\cdots,u_n) &= (-1)^n F(u_1,u_2,\cdots,u_n)\\
&\quad\times\exp\Bigl(\frac{2\pi i}{r^*}(nu_1-\sum_{j=2}^n u_j-\frac{n-1}{2}+n\frac{\tau}{2})\Bigr),
\end{aligned} \tag{B.2}
$$

where $\tau=\pi i/\log x$. Clearly

$$
\tilde{A}_1=\mathbb{C}f_1,\quad f_1(u)=[u]^*.
$$

We have also $\dim \tilde{A}_2 = 2$. If $F \in \tilde{A}_n$ is not identically 0, then it has n zeroes $\{u_1^{(j)}\}_{j=1}^n$ mod $\mathbb{Z}r^* \oplus \mathbb{Z}\tau$ satisfying

$$\sum_{j=1}^n u_1^{(j)} = \sum_{j=2}^n u_2 + \frac{n-1}{2}.$$

Let $F \in \tilde{A}_m$, $G \in \tilde{A}_n$. Following the line of [19], we define the $*$-product $F * G \in \tilde{A}_{m+n}$ by

$$\begin{aligned}(F*G)(u_1,\cdots,u_{m+n}) \;=\; & \mathrm{Sym}\Big(F(u_1-n,\cdots,u_m-n)G(u_{m+1},\cdots,u_{m+n}) \\ & \times \prod_{\substack{1\le i\le m \\ m+1\le j\le m+n}} \frac{[u_i-u_j+1]^*[u_i-u_j-1/2]^*}{[u_i-u_j]^*}\Big).\end{aligned}$$

Here the symbol Sym stands for the symmetrization. $\tilde{A} = \oplus_{n=0}^\infty \tilde{A}_n$ equipped with the $*$-product is an associative graded algebra with unit. We denote by $A = \oplus_{n=0}^\infty A_n$ ($A_n = \tilde{A}_n \cap A$) the subalgebra of $\tilde{A}$ generated by $\tilde{A}_1$ and $\tilde{A}_2$.

Let us say that a function $F(u_1,\cdots,u_n)$ has the property (P) if either $n = 1,2$, or $n \ge 3$ and

$$F(u_1,\cdots,u_n)\Big|_{u_j-u_i=u_k-u_j=1/2} \equiv 0 \quad \text{for any distinct } i,j,k.$$

Lemma B.1 *(i) Elements of A has the property (P).*

(ii) A is commutative.

Proof. From the definition of $*$ we can verify that, if $F \in \tilde{A}_m$ and $G \in \tilde{A}_n$ have the property (P), then so does $F * G$. Hence (i) follows.

Let $g \in A_2$ be an element linearly independent from $f_1 * f_1 \in A_2$. To see (ii), it suffices to show that $f_1 * g = g * f_1$. Set $h = f_1 * g - g * f_1$. A simple check shows that $h(u+1,u,u-1) = 0$. By symmetry and the property (P), $h(u_1,u_2,u_2-1)$ viewed as a function of u_1 has zeroes at $u_2+1, u_2-1/2, u_2-2$. Since their sum is different from $2u_2$ mod $\mathbb{Z}r^* \oplus \mathbb{Z}\tau$, we have $h(u_1,u_2,u_2-1) \equiv 0$. By symmetry, this implies that $h(u_1,u_2,u_3)$ has 4 zeroes $u_1 = u_2 \pm 1, u_3 \pm 1$. Hence $h \equiv 0$. □

B.2 BRST charges

For $F \in A_n$, we define

$$\begin{aligned}Q(F) \;=\; & \oint\cdots\oint \prod_{j=1}^n \frac{dz_j}{[u_j+\frac{1}{2}]^*}\, x_+(z_1)\cdots x_+(z_n) \\ & \times \Bigg(\prod_{1\le i<j\le n} \frac{[u_i-u_j]^*}{[u_i-u_j+1]^*[u_i-u_j-1/2]^*}\Bigg) F(u_1+\hat{l}/2,\cdots,u_n+\hat{l}/2),\end{aligned}$$

where the contour is $|z_1| = \cdots = |z_n| = 1$. Because of the quasi-periodicity (B.2), the integrand is single valued. Using the exchange relation

$$x_+(z_1)x_+(z_2) = -\frac{[u_1-u_2+1]^*}{[u_2-u_1+1]^*}\frac{[u_1-u_2-1/2]^*}{[u_2-u_1-1/2]^*}x_+(z_2)x_+(z_1)$$

along with $\hat{l}\, x_+(z) = x_+(z)(\hat{l}-2)$, we find

$$Q(F)Q(G) = Q(F*G). \tag{B.3}$$

Now set

$$\begin{aligned} f_1(u) &= [u]^*, \\ f_2^{(a)}(u_1,u_2) &= [2a+1]^*[a-1/2]^*[u_1-a]^*[u_2+a-1]^*[u_1-u_2+a-1/2]^* \\ &\quad -[2a-1]^*[a+1/2]^*[u_1+a]^*[u_2-a-1]^*[u_1-u_2-a-1/2]^*, \end{aligned}$$

and introduce $h_l \in A_l$ $(1 \le l \le L)$ by

$$\begin{aligned} h_{2m+1} &= f_1 * f_2^{(1)} * \cdots * f_2^{(m)}, \qquad \Big(0 \le m \le \frac{L-1}{2}\Big), \\ h_{2m} &= f_2^{((L+1)/2-m)} * \cdots * f_2^{((L-3)/2)} * f_2^{((L-1)/2)}, \qquad \Big(1 \le m \le \frac{L-3}{2}\Big). \end{aligned}$$

We have $h_L = h_l * h_{L-l}$ $(1 \le l \le L-1)$. We define the BRST charges by

$$Q_l = Q(h_l).$$

Propositions 3.1,3.2 reduce to the following assertions.

Proposition B.2 *$h_l(u_1,\cdots,u_l)$ can be written as*

$$h_l(u_1,\cdots,u_l) = \bar{h}_l(u_1,\cdots,u_l) \times \begin{cases} \displaystyle\prod_{i=1}^{l}\Big[u_i + \frac{-l+1}{2}\Big]^* & (l\,:\,odd) \\ \displaystyle\prod_{i=1}^{l}\Big[u_i + \frac{L-l+1}{2}\Big]^* & (l\,:\,even), \end{cases} \tag{B.4}$$

where $\bar{h}_l$ are holomorphic and satisfies

$$\begin{aligned} \bar{h}_l(u_1+r^*,\cdots,u_l) &= (-1)^{l-1}\bar{h}_l(u_1,\cdots,u_l), \\ \bar{h}_l(u_1+\tau,\cdots,u_l) &= \bar{h}_l(u_1,\cdots,u_l) \times \exp\Big(\frac{2\pi i}{r^*}\sum_{j=2}^{l}(u_1-u_j+\frac{\tau}{2})\Big). \end{aligned}$$

Moreover it is translationally invariant, i.e.,

$$\bar{h}_l(u_1+v,\cdots,u_l+v) = \bar{h}_l(u_1,\cdots,u_l).$$

Proposition B.3 *We have* $h_L \equiv 0$.

Proof of Proposition B.3. Let $0 \le m \le \frac{L-3}{2}$. In the equality $h_L = h_{2m+1} * h_{L-2m-1}$ we set $u_1 = -m-1$. Using Proposition B.2, we find that each summand in the symmetrization (B.3) vanishes. Similarly, if we set $u_1 = m$, then each summand of $h_L = h_{L-2m-1} * h_{2m+1}$ vanishes. Therefore h_L has $L-1$ zeroes $u_1 = 0, 1, \cdots, \frac{L-3}{2}, -1, -2, \cdots, -\frac{L-1}{2}$.

Suppose h_L did not vanish identically. From the quasi-periodicity, h_L has a zero at $u_1 = \sum_{j=2}^{L} u_j + L - 1$. By symmetry, $u_1 = u_2 - \sum_{j=3}^{L} u_j - (L-1)$ must also be a zero. This is a contradiction. □

In the next subsection we prove Proposition B.2.

B.3 Proof of Proposition B.2

We prove Proposition B.2 for odd $l = 2m+1$. The statement is obvious for $m = 0$. Assuming $m \ge 1$ we proceed by induction on m.

Lemma B.4 *For* $m = 1, \cdots, \frac{L-1}{2}$ *we have*

$$h_{2m+1}(m, m \pm 1, u_3, \cdots, u_{2m+1}) \equiv 0. \tag{B.5}$$

Proof. We use the property

$$f_2^{(a)}(\pm a, \pm a + 1) = 0. \tag{B.6}$$

In the definition of $h_{2m+1} = h_{2m-1} * f_2^{(m)}$, set $u_1 = m, u_2 = m+1$. Using the induction hypothesis $h_{2m-1}(m-1, \cdots) \equiv 0$ and $f_2^{(m)}(m, m+1) = 0$, we see that each summand vanishes. Similarly if we set $u_1 = m, u_2 = m-1$ in $h_{2m+1} = f_2^{(m)} * h_{2m-1}$ and use $f_2^{(m)}(-m, -m+1) = 0$, the result is zero. □

Lemma B.5 *For* $t = 2, 3, \cdots,$ *we have*

$$h_{2m+1}(m, u_2, \cdots, u_{2m+1})\Big|_{\substack{u_{2s+1} = u_{2s} + 1/2 \\ t \le s \le m}} \equiv 0. \tag{B.7}$$

Taking $t = m+1$ we obtain the first assertion of Proposition B.2.

Proof. Denote the left hand side of (B.7) by g_t. We show $g_t \equiv 0$ by induction on t.

Let $t = 2$, and consider first

$$h_{2m+1}(m, m \pm 1/2, u_3, \cdots, u_{2m+1})\Big|_{\substack{u_{2s+1} = u_{2s} + 1/2 \\ 2 \le s \le m}}. \tag{B.8}$$

As a function of u_3, (B.8) has $2m+1$ zeroes at $m+1, m-1, m\mp 1/2$ and $u_{2s}+1, u_{2s}-1/2$ $(2\leq s\leq m)$. Comparing with the quasi-periodicity, we conclude that (B.8) vanishes identically. This means that g_2 as a function of u_2 has $2m+2$ zeroes at $m\pm 1, m\pm 1/2$ and $u_{2s}+1, u_{2s}-1/2$ $(2\leq s\leq m)$. Therefore $g_2\equiv 0$.

Suppose we have shown $g_t\equiv 0$, and consider

$$g_{t+1}=h_{2m+1}(m,u_2,\cdots,u_{2t},u_{2t+1},u_{2t+2},u_{2t+2}+1/2,\cdots,u_{2m},u_{2m}+1/2). \tag{B.9}$$

From the induction hypothesis, it vanishes for $u_{2t+1}=u_{2t}+1/2$. By symmetry it vanishes for

$$u_2=u_3\pm 1/2,\cdots,u_{2t+1}\pm 1/2.$$

It also vanishes for $u_2=m\pm 1$ and $u_2=u_{2s}-1/2, u_{2s}+1$ $(t+1\leq s\leq m)$. Since the number of zeroes exceed $2m+1$, we conclude $g_{t+1}\equiv 0$. □

In the case of even l, we note that

$$f_2^{((L-1)/2)}(u_1,u_2)=[2]^*\left[\frac{L}{2}\right]^*\left[u_1+\frac{L-1}{2}\right]^*\left[u_2+\frac{L-1}{2}\right]^*\left[u_1-u_2-\frac{L}{2}\right]^*$$

has a zero at $u_1=(1-L)/2$. Using this the proof goes similarly.

Lemma B.6 *The function $\bar{h}_l$ in (B.4) is translationally invariant.*

Proof. Consider

$$\bar{h}_l(u_1+v,\cdots,u_l+v).$$

It is holomorphic and doubly periodic in v, hence is a constant. The conclusion follows by setting $v=0$. □

C Deformed W algebra for $\mathfrak{sl}_N$

We discuss here the fusion of the deformed W algebra (DWA) associated with $\mathfrak{sl}_N$ [28, 29].

C.1 Basic current

Fix complex numbers $x, r^*\in\mathbb{C}$, $0<|x|<1$. We keep the notation $[u]_x$ (5.13) and $[\![u]\!]$ (5.14). In this appendix we consider the case of 'generic' r^*, i.e., we assume that $m,n\in\mathbb{Z}$, $[m+nr^*]_x=0$ implies $m=n=0$.

In the free field realization, the simplest current of DWA for the algebra $\mathfrak{sl}_N$ is presented in the form

$$W_{(1)}(u)=\sum_{i=1}^{N}\Lambda_i(u).$$

Each $\Lambda_i(u)$ is a normally-ordered exponential of bosonic oscillators. Their explicit formula is irrelevant here (see e.g. [28], eq.(2), wherein $z = x^{2u}$, $q = x^{2r^*+2}$, $t = x^{2r^*}$ in the present notation). We need only the following normal ordering rule for their products:

$$f(u,v)\Lambda_i(u)\Lambda_j(v) = :\Lambda_i(u)\Lambda_j(v): \times \begin{cases} \dfrac{[\![u-v-1-r^*]\!]}{[\![u-v-1]\!]} & (i<j) \\ 1 & (i=j)\ , \\ \dfrac{[\![u-v-r^*]\!]}{[\![u-v]\!]} & (i>j) \end{cases} \tag{C.1}$$

where the structure function $f(u,v) = f(v-u)$ is given by

$$f(u) = \frac{1}{(1-x^{2u})} \frac{(x^{2(u+N-1)}, x^{2(u+1+r^*)}, x^{2(u-r^*)}; x^{2N})_\infty}{(x^{2(u+1)}, x^{2(u+N+r^*)}, x^{2(u+N-r^*-1)}; x^{2N})_\infty}.$$

Lemma C.1 *We have the exchange relation as meromorphic functions*

$$f(u,v)W_{(1)}(u)W_{(1)}(v) = f(v,u)W_{(1)}(v)W_{(1)}(u). \tag{C.2}$$

Both sides are regular except for simple poles at $u-v = \pm 1 \bmod \Gamma$, *where* $\Gamma = (\pi i/\log x)\mathbb{Z}$.

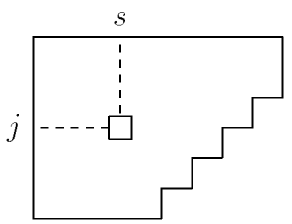
Notice that the pole $u = v$ which appears in (C.1) is canceled in (C.2). In general, each matrix element of the product

$$\prod_{1\le s<t\le m} f(u_s,u_t) \times W_{(1)}(u_1)\cdots W_{(1)}(u_m)$$

is a rational function of x^{2u_i} with at most simple poles at $u_i - u_j = \pm 1$ $(i<j)$.

C.2 Fused currents

Let $\lambda = (\lambda_1,\cdots,\lambda_l)$ $(\lambda_1 \ge \cdots \ge \lambda_l > 0)$ be a partition. We identify λ with a Young diagram. For $j, s = 1, 2, \cdots$, we attach a variable $u(j,s)$ to the box on the j-th row and s-th column of λ:

$\lambda =$ 

, $\qquad u(j,s) = u - (j-1) - (s-1)r^*.$

For partitions $\lambda = (\lambda_1,\cdots,\lambda_l)$, $\mu = (\mu_1,\cdots,\mu_m)$, we set

$$f_{\lambda,\mu}(u,v) = \prod_{\substack{1\le j\le l \\ 1\le s\le\lambda_j}} \prod_{\substack{1\le k\le m \\ 1\le t\le\mu_k}} f(u(j,s), v(k,t)). \tag{C.3}$$

We shall associate 'fused' currents $W_\lambda(u)$ with each λ. First consider the case of a single row diagram $\lambda = (m)$.

Definition C.2

$$W_{(m)}(u) = \Big(\prod_{1\le s<t\le m} f(u_s, u_t) \times W_{(1)}(u_1)\cdots W_{(1)}(u_m)\Big)\Big|_{\substack{u_s=u-(s-1)r^*\\ 1\le s\le m}}.$$

In view of the remark after Lemma C.1, the right hand side is well-defined. Alternatively $W_{(m)}(u)$ can be defined inductively as

$$\begin{aligned} W_{(m)}(u) &= f_{(m-1),(1)}(u,u')W_{(m-1)}(u)W_{(1)}(u')\Big|_{u'=u-(m-1)r^*} &\text{(C.4)}\\ &= f_{(1),(m-1)}(u,u')W_{(1)}(u)W_{(m-1)}(u')\Big|_{u'=u-r^*}. &\text{(C.5)}\end{aligned}$$

Lemma C.3 *We have*

$$f_{(1),(m)}(u,v)W_{(1)}(u)W_{(m)}(v) = f_{(m),(1)}(v,u)W_{(m)}(v)W_{(1)}(u). \qquad \text{(C.6)}$$

Both sides of (C.6) are regular except for simple poles ($\bmod \Gamma$ *) at* $u-v = -1, 1-(m-1)r^*$.

Proof. The exchange relation (C.6) is obvious. Let us verify the statement about the position of poles by induction on m. The case $m = 1, 2$ can be verified by direct calculation. Suppose it is true for $m-1$. Using the expression (C.4) and (C.5) and the induction hypothesis, we see that the possible poles in u are confined to

$$\begin{aligned}&\{v-1, v+1-(m-2)r^*, v\pm 1-(m-1)r^*\} \cap \{v\pm 1, v-r^*-1, v+1-(m-1)r^*\}\\ &= \{v-1, v+1-(m-1)r^*\}.\end{aligned}$$

□

Arguing similarly, we have

Lemma C.4

$$f_{(m),(n)}(u,v)W_{(m)}(u)W_{(n)}(v) = f_{(n),(m)}(v,u)W_{(n)}(v)W_{(m)}(u). \qquad \text{(C.7)}$$

Both sides of (C.7) are regular except for simple poles ($\bmod \Gamma$*) at*

$$\begin{aligned} u-v &= 1-jr^* \qquad (\max(0,n-m)\le j\le n-1),\\ &= -1+jr^* \qquad (\max(0,m-n)\le j\le m-1).\end{aligned}$$

For a general partition $\lambda = (\lambda_1, \cdots, \lambda_l)$, we define

Definition C.5

$$W_\lambda(u) = \Bigl(\prod_{i=1}^{l-1} [\![u_i - u_{i+1} - 1]\!] \cdot \prod_{1\le i<j\le l} f_{(\lambda_i),(\lambda_j)}(u_i, u_j) \cdot W_{(\lambda_1)}(u_1) \cdots W_{(\lambda_l)}(u_l)\Bigr)\Big|_{\substack{u_i = u-(i-1)\\ 1\le i\le l}}.$$

This definition makes sense by Lemma C.3. We have

$$f_{\lambda,\mu}(u,v) W_\lambda(u) W_\mu(v) = W_\mu(v) W_\lambda(u) f_{\mu,\lambda}(v,u). \tag{C.8}$$

In the case of a single column diagram $\lambda = (1^a)$, $W_{(1^a)}(u)$ coincides with the fundamental DWA currents $W_a(z)$ in [28, 29] up to a numerical factor and a shift of u (see (C.10) below).

We remark that another fused currents can be constructed similarly by replacing $u(j,s) = u - (j-1) - (s-1)r^*$ with $u - (j-1) + (s-1)r$ $(r = r^* + 1)$.

C.3 Tableaux sum

Let λ be a partition. Denote by $SST(\lambda)$ the set of semi-standard tableaux of shape λ on the letters $\{1, 2, \cdots, N\}$. For $T \in SST(\lambda)$, we set

$$\Lambda_T(u) =: \prod_{\substack{1\le j\le l\\ 1\le s\le \lambda_j}} \Lambda_{T(j,s)}\Bigl(u(j,s)\Bigr) :,$$

where $T(j,s) \in \{1, \cdots, N\}$ signifies the letter in the (j,s)-th position of T.

The current $W_\lambda(u)$ is given explicitly as follows.

Proposition C.6 *We have*

$$W_\lambda(u) = d_\lambda \sum_{T\in SST(\lambda)} c_T \cdot \Lambda_T(u).$$

The coefficients d_λ, c_T are given by

$$\begin{aligned} d_\lambda &= \prod_{1+j<k} \frac{[\![k-j-1-\lambda_j r^*]\!]_{\lambda_k}}{[\![k-j-1]\!]_{\lambda_k}} \cdot \prod_{k=2}^{l} \frac{[\![-\lambda_{k-1} r^*]\!]_{\lambda_k}}{[\![r^*]\!]_{\lambda_k - 1}}, \\ c_T &= \prod_{j=1}^{l} \frac{\prod_{i=1}^N [\![-1]\!]_{w_{ji}}}{[\![-1]\!]_{\lambda_j}} \cdot \prod_{j<k} \frac{[\![k-j-\lambda_j r^*]\!]_{\lambda_k}}{[\![k-j-1-\lambda_j r^*]\!]_{\lambda_k}} \\ &\quad \times \prod_{j<k} \prod_{i=1}^{N} \frac{[\![k-j-1+(s_{k,i-1} - s_{j,i-1}) r^*]\!]_{w_{k,i}}}{[\![k-j+(s_{k,i-1} - s_{j,i}) r^*]\!]_{w_{k,i}}}, \end{aligned}$$

where w_{ji} is the number of the letter i in the j-th row of T $(1 \le j \le l, 1 \le i \le N)$, $s_{ji} = w_{j1} + \cdots + w_{ji}$, and

$$[\![u]\!]_n = [\![u]\!][\![u + r^*]\!] \cdots [\![u + (n-1)r^*]\!].$$

We omit the proof. Notice that $c_{T_0} = 1$ for the tableau T_0 with $T_0(j,s) = j$ for all j, s.

Example.

$$W_{(m)}(u) = \sum_{\substack{w_1,\cdots,w_N \geq 0 \\ w_1+\cdots+w_N=m}} \frac{\prod_{i=1}^{N} [\![-1]\!]_{w_i}}{[\![-1]\!]_m} \cdot \Lambda_T(u), \tag{C.9}$$

where $T = (1^{w_1}, 2^{w_2}, \cdots, N^{w_N})$.

$$W_{(1^a)}(u) = d_{(1^a)} \sum_{1 \leq i_1 < \cdots < i_a \leq N} : \Lambda_{i_1}(u) \cdots \Lambda_{i_a}(u-a+1) : . \tag{C.10}$$

C.4 $W_\lambda(u)$ in terms of $W_{(1^a)}(u)$

$W_\lambda(u)$ can also be obtained from $W_{(1^a)}(u)$. First note the following fact which can be shown similarly as Lemma C.4.

Lemma C.7 *We have*

$$\tilde{f}_{(1^a),(1^b)}(u,v) W_{(1^a)}(u) W_{(1^b)}(v) = \tilde{f}_{(1^b),(1^a)}(v,u) W_{(1^b)}(v) W_{(1^a)}(u), \tag{C.11}$$

where $\tilde{f}_{\lambda,\mu}(u,v)$ is defined similarly as in (C.3) with $f(u,v)$ replaced by

$$\tilde{f}(u,v) = \frac{[\![u-v-1]\!]}{[\![u-v-r^*]\!]} f(u,v) = \frac{[\![v-u-1]\!]}{[\![v-u-r^*]\!]} f(u,v).$$

Both sides of (C.11) are regular except for simple poles $(\bmod \Gamma)$ *at*

$$\begin{aligned} u - v &= r^* - j \qquad (\max(0, b-a) \leq j \leq b-1), \\ &= -r^* + j \qquad (\max(0, a-b) \leq j \leq a-1). \end{aligned}$$

We remark that the exchange relation (C.8) holds true with $f_{\lambda,\mu}(u,v)$ replaced by $\tilde{f}_{\lambda,\mu}(u,v)$.

Returning to the general λ, denote by $\mu_1 \geq \cdots \geq \mu_m$ its column lengths (hence the transposed diagram is $\lambda' = (\mu_1, \cdots, \mu_m)$).

Lemma C.8

$$\begin{aligned} W_\lambda(u) &= \Big(\prod_{i=2}^{m} [\![u_{i-1} - u_i - r^*]\!]^{-\mu_i+1} \\ &\quad \times \prod_{1 \leq i < j \leq l} f_{(1^{\mu_i}),(1^{\mu_j})}(u_i, u_j) \cdot W_{(1^{\mu_1})}(u_1) \cdots W_{(1^{\mu_m})}(u_m) \Big) \Big|_{\substack{u_i = u-(i-1)r^* \\ 1 \leq i \leq m}} . \end{aligned}$$

Proof. Let $\bar{\lambda}$ be the diagram obtained by removing the last column of λ so that $\lambda' = (\bar{\lambda}', \mu_m)$. We show

$$W_\lambda(u) = [\![u - v - (m-1)r^*]\!]^{-\mu_m+1} f_{\bar{\lambda},(1^{\mu_m})}(u,v) W_{\bar{\lambda}}(u) W_{(1^{\mu_m})}(v) \Big|_{v=u-(m-1)r^*} \tag{C.12}$$

by induction on μ_m. The lemma follows by repeated use of this equation.

If $\mu_m = 1$, then (C.12) is immediate from the definition. Assuming the statement is true for μ_m $(\mu_{m-1} \geq \mu_m + 1 \geq 2)$ we consider (C.12) with $\lambda' = (\bar{\lambda}', \mu_m + 1)$. We have

$$\begin{aligned}
&[\![u - v - (m-1)r^*]\!]^{-\mu_m} f_{\bar{\lambda},(1^{\mu_m+1})}(u,v) W_{\bar{\lambda}}(u) W_{(1^{\mu_m+1})}(v) \\
&= [\![u - v - (m-1)r^*]\!]^{-\mu_m+1} [\![u - v' - \mu_m - (m-1)r^*]\!]^{-1} f_{\bar{\lambda},(1^{\mu_m})}(u,v) f_{\bar{\lambda},(1)}(u,v') \\
&\quad \times [\![v - v' - \mu_m]\!] f_{(1^{\mu}_m),(1)}(v,v') W_{\bar{\lambda}}(u) W_{(1^{\mu_m})}(v) W_{(1)}(v') \Big|_{v'=v-\mu_m}. \qquad \text{(C.13)}
\end{aligned}$$

Let us verify that the right hand side of (C.13) (before specialization $v' = v - \mu_m$) is regular at $v = u - (m-1)r^*, v' = u - \mu_m - (m-1)r^*$. From the induction hypothesis,

$$[\![u - v - (m-1)r^*]\!]^{-\mu_m+1} f_{\bar{\lambda},(1^{\mu_m})}(u,v) W_{\bar{\lambda}}(u) W_{(1^{\mu_m})}(v)$$

is regular at $v = u - (m-1)r^*$, and

$$[\![v - v' - \mu_m]\!] f_{(1^{\mu}_m),(1)}(v,v') W_{(1^{\mu_m})}(v) W_{(1)}(v')$$

is regular at $v' = v - \mu_m$. Finally Lemma C.7 implies that

$$\tilde{f}_{\bar{\lambda},(1)}(u,v') W_{\bar{\lambda}}(u) W_{(1)}(v')$$

is regular at $v' = u - \mu_m - (m-1)r^*$, and

$$\begin{aligned}
&[\![u - v' - \mu_m - (m-1)r^*]\!]^{-1} \frac{f_{\bar{\lambda},(1)}(u,v')}{\tilde{f}_{\bar{\lambda},(1)}(u,v')} \\
&= [\![u - v' - \mu_m - (m-1)r^*]\!]^{-1} \prod_{\substack{1 \leq k \leq m-1 \\ 1 \leq j \leq \mu_k}} \frac{[\![u - v' - (j-1) - kr^*]\!]}{[\![u - v' - j - (k-1)r^*]\!]}
\end{aligned}$$

is also regular (since $\mu_{m-1} \geq \mu_m + 1$).

We let $v = u - (m-1)r^*$ in (C.13) and change the order of specialization. Using the induction hypothesis for the diagram $\tilde{\lambda}' = (\bar{\lambda}', \mu_m)$, we obtain

$$\begin{aligned}
&[\![u - v' - \mu_m - (m-1)r^*]\!]^{-1} [\![u - (m-1)r^* - v' - \mu_m]\!] \\
&\quad \times f_{\bar{\lambda},(1)}(u,v') f_{(1^{\mu}_m),(1)}(u - (m-1)r^*, v') W_{\tilde{\lambda}}(u) W_{(1)}(v') \Big|_{v'=u-\mu_m-(m-1)r^*} \\
&= f_{\tilde{\lambda},(1)}(u,v') W_{\tilde{\lambda}}(u) W_{(1)}(v') \Big|_{v'=u-\mu_m-(m-1)r^*} \\
&= W_{\lambda}(u).
\end{aligned}$$

□

C.5 Rectangular diagrams

For a rectangular Young diagram $\lambda = (m^a)$, we write $W^{(a)}_m(u) = W_{(m^a)}(u)$. The following relations may be viewed as an analog of the T-system for the transfer matrices discussed in [21].

Proposition C.9

$$
\begin{aligned}
& f_{(m^a),(m^a)}(u,v) W^{(a)}_m(u) W^{(a)}_m(v)\Big|_{v=u-r^*} \\
& \quad = (-1)^{a-1} f_{((m+1)^a),((m-1)^a)}(u,v) W^{(a)}_{m+1}(u) W^{(a)}_{m-1}(v)\Big|_{v=u-r^*},
\end{aligned} \tag{C.14}
$$

$$
\begin{aligned}
& \tilde{f}_{(m^a),(m^a)}(u,v) W^{(a)}_m(u) W^{(a)}_m(v)\Big|_{v=u-1} \\
& \quad = (-1)^{m-1} C^{(a)}_m \tilde{f}_{(m^{a+1}),(m^{a-1})}(u,v) W^{(a+1)}_m(u) W^{(a-1)}_m(v)\Big|_{v=u-1},
\end{aligned} \tag{C.15}
$$

where

$$
C^{(a)}_m = \prod_{1\le s,t\le m} \frac{[\![a-1-(s-t)r^*]\!]}{[\![a-(1+s-t)r^*]\!]}.
$$

Both sides of (C.14),(C.15) are well defined.

We sketch below the proof of (C.14). First we check the regularity of both sides at $v = u - r^*$. For the right hand side, this can be shown from Lemma C.4. For the left hand side, we use Lemma C.7 to find that

$$
\tilde{f}_{(m^a),(m^a)}(u,v) W^{(a)}_m(u) W^{(a)}_m(v)
$$

has poles of order at most $2(m-1)$ at $u = v - r^*$. Since

$$
\frac{f_{(m^a),(m^a)}(u,v)}{\tilde{f}_{(m^a),(m^a)}(u,v)} = O([u-v+r^*]_x^{2(m-1)}) \qquad (u \to v - r^*),
$$

the desired regularity follows. In the same way (using Lemma C.4) we see that

$$
f_{(m^a),((m-1)^a)}(u,v) W^{(a)}_m(u) W^{(a)}_{m-1}(v)
$$

has poles of order at most $(a-1)$ at $u = v - r^*$.

Consider the expression

$$
A \equiv f_{(m^a),((m-1)^a)}(u,u') f_{(m^a),(1^a)}(u,v) f_{((m-1)^a),(1^a)}(u',v) W^{(a)}_m(u) W^{(a)}_{m-1}(u') W^{(a)}_1(v).
$$

From the definition of $W_\lambda(u)$, we have

$$
\begin{aligned}
A \;=\; & [\![u'-v-(m-1)r^*]\!]^{a-1} f_{(m^a),(m^a)}(u,u') W^{(a)}_m(u) W^{(a)}_m(u') \\
& + O([\![u'-v-(m-1)r^*]\!]^a) \qquad (v \to u' - (m-1)r^*), \\
=\; & [\![u-v-mr^*]\!]^{a-1} f_{((m+1)^a),((m-1)^a)}(u,u') W^{(a)}_{m+1}(u) W^{(a)}_{m-1}(u') \\
& + O([\![u-v-mr^*]\!]^a) \qquad (v \to u - mr^*).
\end{aligned}
$$

Writing $y = u-v-mr^*$, $y' = u'-v-(m-1)r^*$ and multiplying both sides by $[\![u-u'-r^*]\!]^{a-1}$ we have the equality of the form

$$\begin{aligned}\varphi(y,y') &= [\![y]\!]^{a-1}[\![y-y']\!]^{a-1}\psi(y-y') + O(y^a) \qquad (y \to 0), \\ &= [\![y']\!]^{a-1}[\![y-y']\!]^{a-1}\psi'(y-y') + O(y'^a) \qquad (y' \to 0),\end{aligned}$$

where $\varphi(y,y')$, $\psi(y-y')$ and $\psi'(y-y')$ are regular near $y = y' = 0$. This implies that $(-1)^{a-1}\psi(0) = \psi'(0)$, and (C.14) follows.

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