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Beyond CFT :
Deformed Virasoro and Elliptic Algebras[1]

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Abstract

In this lecture we discuss 'beyond CFT' from symmetry point of view. After reviewing the Virasoro algebra, we introduce deformed Virasoro algebras and elliptic algebras. These algebras appear in solvable lattice models and we study them by free field approach.

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Contents

1 Introduction

The conformal field theory (CFT) is a theory which is invariant under the conformal transformation. CFT in 2 dimensions [1] can be applied to the string theory as a worldsheet theory and statistical critical phenomena in 2 dimensional space. This theory has made a remarkable progress contacting with various branches of mathematics [2, 3]. The main reason is that in two dimensional space (or 1+1 dimensional spacetime), the group of conformal transformations is infinite dimensional. Its algebra is known as the Virasoro algebra in the field theory realization, and this symmetry is very powerful. By using its detailed representation theory one can determine the spectrum and even calculate correlation functions. Statistical critical phenomena in 2 dimensional space are understood systematically by CFT and the list of critical exponents is obtained by the representation theory of the Virasoro algebra. In the first superstring revolution (middle 80's), the string theory was regarded as CFT on a worldsheet, and the knowledge obtained in the study of CFT developed string theories very much. Recent progress in string theory (the second superstring revolution (middle 90's)) is based on spacetime symmetry consideration, e.g., duality, D-brane, and also AdS/CFT correspondence, but importance of worldsheet symmetries remains unchanged.

Quantum field theory and critical phenomena are the systems with infinite degrees of freedom. Consequently they are difficult to treat. However we can sometimes solve some models, so-called solvable models. Here we loosely use the word 'solvable' if some physical quantities of that model can be calculated exactly, for example, 2D Ising model, XYZ spin chain, solvable lattice model, CFT, 4D super Yang-Mills theory, etc. Although solvable models themselves are interesting for us, a main purpose for physicists to study solvable models is to create new idea and concept and to develop them through the study of solvable models. Physics should explain the real world. The matter in the real world is very complicated and we need some approximations to study it. An approximation is not a bad thing if it grasps the essence of the problem. Physicists have developed a variety of concepts, approximation methods, calculation techniques, etc. through the study of solvable models, in order to apply them to the real world.

Symmetry is one of the main idea of modern physics. If the model has some symmetry, its analysis becomes much simpler by using the representation theory of the symmetry algebra. By reversing this direction, symmetry is also used for a model building. For example, the general relativity and the gauge theory are constructed by imposing the invariance under the general coordinate transformation and the gauge transformation respectively. From symmetry point of view, 'solvable' in the system with infinite degrees of freedom is stated as the following 'equation':

$$\begin{aligned} &\frac{\text{System of } \textbf{infinite} \text{ degrees of freedom}}{\textbf{Infinite} \text{ dimensional symmetry}} \\ =\;& \text{System can be described by } \textbf{finite} \text{ degrees of freedom.} \end{aligned} \tag{1.1}$$

This is the reason why we are interested in infinite dimensional symmetries.

Common feature of many solvable models is the factorization of the scattering S matrix [4], in other words the Yang-Baxter equation [5, 6, 7]. Solutions of the Yang-Baxter equation are related to the Lie algebras [8, 9, 10], and three types of solutions are known; rational, trigonometric and elliptic. Associated for each type of solution (R matrix), algebras are defined [11],

$$\begin{array}{lcl} \text{rational} & \to & \text{Yangian,} \\ \text{trigonometric} & \to & \text{quantum group (quantum algebra),} \\ \text{elliptic} & \to & \text{elliptic quantum group (elliptic algebra).} \end{array}$$

This elliptic algebra is one of the topics in this lecture.

Since CFT is invariant under the scale transformation, CFT has no scale, in other words, it is a massless theory. If we add to CFT the perturbation which breaks the conformal symmetry, then the theory becomes massive. General massive theories are very difficult. So we restrict ourselves to its subset, massive integrable models (MIM). If we perturb CFT in a 'good' manner (for example $(1,3)$ or $(1,2)$ perturbation [12, 13, 14, 15]), infinitely many conserved quantities survive. In the terminology of statistical mechanics, CFT corresponds to on-critical theory, and perturbation corresponds to off-critical procedure, and a lattice analogue of MIM is a solvable lattice model. CFT is controlled by the Virasoro symmetry, but MIM is massive, therefore there is no Virasoro symmetry.

$$\begin{array}{cccc} \text{CFT} & \text{on-critical} & \text{massless} & \text{Virasoro} \\ \big\downarrow \text{good perturbation} & \big\downarrow \text{off-critical} & \big\downarrow & \big\downarrow \\ \text{massive} & \text{solvable} & \text{massive} & \text{no} \\ \text{integrable model} & \text{lattice model} & & \text{Virasoro} \end{array}$$

A natural question arises:

> What symmetry ensures the integrability or infinitely many conserved quantities of MIM or solvable lattice model ?

We would like to answer this question. This is our main motivation for recent study.

In some cases the Yangian or the quantum group symmetry plays an important role. Kyoto group investigated the XXZ spin chain and clarified its symmetry, the quantum affine Lie algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [16]. They studied XXZ spin chain from the representation theory point of view and developed vertex operator calculation technique. But naively one expected some deformation of the Virasoro algebra. Such algebras, deformed Virasoro algebra (DVA) and deformed W_N algebras (DWA), were constructed in different points of view [17, 18, 19, 20], using a correspondence of singular vectors and multivariable orthogonal symmetric polynomials or using the Wakimoto realization at the critical level. Later it was shown that this deformed Virasoro algebra appears in the Andrews-Baxter-Forester (ABF) model as a symmetry [21]. This DVA corresponds to $A_1^{(1)}$ algebra. DVA corresponding to $A_2^{(2)}$ was obtained in [22].

Another possibility is elliptic quantum groups (elliptic algebras). Corresponding to the two types of elliptic solutions of the Yang-Baxter equation, there are two type of elliptic quantum groups [23, 24]. These two elliptic quantum groups have a common structure [25]; They are quasi-Hopf algebras [11]. Along this line, explicit formulas for the twistors were presented and the vertex type algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ and the face type algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ were defined in [26].

In this lecture we would like to (i) introduce the deformed Virasoro algebras and elliptic algebras, (ii) present free field approach and vertex operator calculation technique for the solvable lattice models. Contents of this lecture is presented in previous pages. In section 2 we review the Virasoro algebra which is needed to understand the deformed case. The deformed Virasoro algebra of type $A_1^{(1)}$ is defined and its properties are given in section 3. In section 4 we review solvable lattice models and introduce the elliptic quantum groups. Section 5 is devoted to an application of these idea to the ABF model. ABF model in regime III corresponds to the $(1,3)$-perturbation of the minimal unitary CFT. Vertex operators are bosonized and local height probabilities (LHP's) are calculated. Another deformed Virasoro algebra DVA$(A_2^{(2)})$ is defined in section 6 and the dilute A_L models are studied by free field approach. Dilute A_L model in regime 2^+ corresponds to the $(1,2)$-perturbation of the minimal unitary CFT. In section 7 we mention other topics that are not treated in this lecture. Appendix A is a summary of notations and formulas used throughout this lecture.

2 Conformal Field Theory and Virasoro Algebra

2.1 Conformal field theory

The conformal field theory (CFT) is a theory which is invariant under the conformal transformation. In two dimensional space (or 1+1 dimensional spacetime) the conformal transformation is any holomorphic map $z \mapsto w = w(z)$ where z is a complex coordinate of the space. Therefore the conformal group is infinite dimensional.

An important property of the CFT (bulk theory) is the factorization into a holomorphic part (z, left mover) and an antiholomorphic part ($\bar{z}$, right mover). We can treat them independently. Usually we treat z part (chiral part) only. To get final physical quantities, however, we have to glue chiral and antichiral parts with appropriate physical conditions.

As a quantum field theory, infinitesimal conformal transformation,

$$l_n \ : \ z \mapsto z + \epsilon z^{n+1}, \tag{2.1}$$

is generated by the chiral part of the energy momentum tensor, $L(z)$. The algebra generated by this $L(z)$ is called the Virasoro algebra. Invariance under the conformal transformation imposes that correlation functions satisfy the conformal Ward identity. Correlation functions are severely controlled by this infinite dimensional Virasoro symmetry.

General review of the CFT is not the aim of this lecture. Since many good review and books are available now, for various topics of the CFT, see [1, 2, 3].

2.2 Virasoro algebra

In this and next subsections we review some properties of the Virasoro algebra in order to compare them with the deformed one in the next section.

2.2.1 definition and consistency

Definition The Virasoro algebra is a Lie algebra over $\mathbb{C}$ generated by L_n ($n \in \mathbb{Z}$) and c, and their relation is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}, \qquad [L_n, c] = 0. \tag{2.2}$$

In terms of the Virasoro current $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, this relation is equivalent to the following operator product expansion (OPE),

$$L(z)L(w) = \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w} + \text{reg.} \tag{2.3}$$

As a formal power series this can be written as

$$[L(z), L(w)] = \frac{1}{z^4}\delta'''\Big(\frac{w}{z}\Big)\frac{c}{12} + \frac{1}{z^2}\delta'\Big(\frac{w}{z}\Big)2L(w) + \frac{1}{z^1}\delta\Big(\frac{w}{z}\Big)\partial L(w), \tag{2.4}$$

or

$$[z^2L(z), w^2L(w)] = \Big(\frac{w}{z}\Big)^2\delta'''\Big(\frac{w}{z}\Big)\frac{c}{12} + \frac{w}{z}\delta'\Big(\frac{w}{z}\Big)\Big(z^2L(z) + w^2L(w)\Big), \tag{2.5}$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ (see appendix A.2). Meaning of (2.5) is that the coefficient of $z^n w^m$ in (2.5) gives (2.2).

Consistency Mathematically the Virasoro algebra is a one dimensional central extension of diff(S^1), Lie algebra of diffeomorphism group of a circle S^1. Vector fields on S^1 form a Lie algebra,

$$[l_n, l_m] = (n-m)l_{n+m}, \tag{2.6}$$

where $l_n = -z^{n+1}\frac{d}{dz} = ie^{in\theta}\frac{d}{d\theta}$ ($z = e^{i\theta}$ is a coordinate of S^1). Let us consider its central extension,

$$[L_n, L_m] = (n-m)L_{n+m} + cf(n,m), \tag{2.7}$$

where c is a central element and $f(n,m)$ is a number. From antisymmetry of bracket [,] and the Jacobi identity, $f(n,m)$ should satisfy

$$f(n,m) = -f(m,n), \tag{2.8}$$

$$(n-m)f(l,n+m) + (m-l)f(n,m+l) + (l-n)f(m,l+n) = 0. \tag{2.9}$$

Its nontrivial solution has the following form,

$$f(n,m) = \text{const}\cdot n^3\delta_{n+m,0}. \tag{2.10}$$

Proof. From (2.9) with $l=0$, we have $f(n,m) = \frac{n-m}{n+m}f(n+m,0)$ for $n+m \neq 0$, which satisfies eqs.(2.8,2.9) with $n+m+l \neq 0$. By setting $L'_n = L_n + \frac{c}{n}f(n,0)$ ($n \neq 0$) and $L'_0 = L_0$, (2.7) becomes

$$[L'_n, L'_m] = (n-m)L'_{n+m} + c\delta_{n+m,0}f(n,-n).$$

Hence we are enough to determine $a_n = f(n,-n)$. Eq.(2.9) with $l = -n-m$ implies

$$-(n-m)a_{n+m} + (2m+n)a_n - (2n+m)a_m = 0.$$

For $m=1$, we obtain

$$a_n = \frac{1}{6}(n^3-n)a_2 + (n - \frac{1}{6}(n^3-n))a_1,$$

and this solution satisfies eqs.(2.8, 2.9) for all cases. The term proportional to n in a_n, say bn, can be deleted by $L''_n = L'_n + \frac{c}{2}b\delta_{n,0}$. □

The central term in Virasoro algebra is chosen so that it vanishes for $n = 1, 0, -1$.

2.2.2 representation theory

We consider the highest representation of the Virasoro algebra. The highest weight state $|h\rangle$ ($h \in \mathbb{C}$) is characterized by

$$L_n|h\rangle = 0 \quad (n>0), \qquad L_0|h\rangle = h|h\rangle, \tag{2.11}$$

and the Verma module is

$$M = \bigoplus_{l\geq 0} \bigoplus_{n_1\geq\cdots\geq n_l>0} \mathbb{C} L_{-n_1}\cdots L_{-n_l}|h\rangle. \tag{2.12}$$

Descendant $L_{-n_1}\cdots L_{-n_l}|h\rangle$ is also an eigenstate of L_0 with eigenvalue $h+\sum_{i=1}^{l} n_i$. $\sum_{i=1}^{l} n_i$ is called a level of the state. Since L_0 corresponds to energy (exactly speaking Hamiltonian is $L_0+\bar{L}_0$), this representation has energy bounded below. The highest weight state $|h\rangle$ is created by a primary field with conformal weight h, $\phi_h(z)$, from the vacuum $|\mathbf{0}\rangle$ ($L_n|\mathbf{0}\rangle = 0$ for $n\geq -1$); $|h\rangle = \lim_{z\to 0}\phi_h(z)|\mathbf{0}\rangle$.

At level N there are $p(N)$ independent states, $L_{-n_1}\cdots L_{-n_l}|\lambda\rangle$ ($n_1\geq\cdots\geq n_l>0$, $\sum_{i=1}^{l} n_i = N$). Here $p(N)$ is the number of partition and its generating function is given by

$$\sum_{N=0}^{\infty} p(N)y^N = \prod_{n=1}^{\infty}\frac{1}{1-y^n}. \tag{2.13}$$

Let us number these states by the reverse lexicographic ordering for $(n_1,\cdots,n_l)$, i.e., $|h;N,1\rangle = L_{-N}|\lambda\rangle$, $|h;N,2\rangle = L_{-N+1}L_{-1}|\lambda\rangle$, $\cdots$, $|h;N,p(N)\rangle = L_{-1}^{N}|\lambda\rangle$.

The first problem of representation theory is whether M is irreducible or not, namely M has Virasoro invariant subspaces or not. If M has a singular vector at level N, $|\chi\rangle$, which is defined by

$$L_n|\chi\rangle = 0 \quad (n>0), \qquad L_0|\chi\rangle = (h+N)|\chi\rangle, \tag{2.14}$$

then $|\chi\rangle$ generates an invariant subspace and we have to quotient out it from M in order to get an irreducible module $\mathcal{L}$. Existence of singular vectors can be detected by considering an 'inner product' (bilinear form). Let introduce dual module M^* on which the Virasoro algebra act as $L_n^\dagger = L_{-n}$. M^* is generated by $\langle h|$ which satisfies $\langle h|L_n = 0$ ($n<0$), $\langle h|L_0 = h\langle h|$ and $\langle h|h\rangle = 1$. At level N there are $p(N)$ states $\langle h;N,1| = \langle h|L_N$, $\langle h;N,2| = \langle h|L_1L_{N-1}$, $\cdots$, $\langle h;N,p(N)| = \langle h|L_1^N$. A state $|\psi\rangle\in M$ is called a null state if it is orthogonal to all states, i.e., $\langle\psi'|\psi\rangle = 0$ for $\forall\langle\psi'|\in M^*$. Descendants of $|\chi\rangle$ are null state, because (i) $\langle h|\chi\rangle = 0$ (consider $\langle h|L_0|\chi\rangle$) and (ii) $\langle h|\cdots L_{m_2}L_{m_1}\cdot L_{-n_1}L_{-n_2}\cdots|\chi\rangle = \sum(\cdots)\langle h|\chi\rangle = 0$. Physical meaning of quotienting out invariant subspaces is projecting out null states which decouple from all the states.

We give an example of inner product at level 1, 2, 3.

$$\begin{aligned}
&\langle h|L_1L_{-1}|h\rangle = 2h,\\
&\begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}^2|h\rangle \\ \langle h|L_1^2L_{-2}|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h+\frac{1}{2}c & 6h \\ 6h & 4h(2h+1) \end{pmatrix},
\end{aligned}$$

$$\begin{pmatrix} \langle h|L_3L_{-3}|h\rangle & \langle h|L_3L_{-2}L_{-1}|h\rangle & \langle h|L_3L_{-1}^3|h\rangle \\ \langle h|L_1L_2L_{-3}|h\rangle & \langle h|L_1L_2L_{-2}L_{-1}|h\rangle & \langle h|L_1L_2L_{-1}^3|h\rangle \\ \langle h|L_1^3L_{-3}|h\rangle & \langle h|L_1^3L_{-2}L_{-1}|h\rangle & \langle h|L_1^3L_{-1}^3|h\rangle \end{pmatrix}$$
$$= \begin{pmatrix} 6h+2c & 10h & 24h \\ 10h & h(8h+8+c) & 12h(3h+1) \\ 24h & 12h(3h+1) & 24h(h+1)(2h+1) \end{pmatrix}.$$

A determinant of this matrix is called the Kac determinant and its zeros indicate existence of null states. The Kac determinant at level 2 is $2h(16h^2+2(c-5)h+c)$ which vanishes for $h=0$ and $h=h_\pm=\frac{1}{16}(5-c\pm\sqrt{(1-c)(c-25)})$. For $h=0$, $L_{-1}^2|h\rangle$ is a null state which is a descendant of the singular vector at level 1 $L_{-1}|h\rangle$. For $h=h_\pm$, there exists a new singular vector,

$$|\chi_\pm\rangle = \Big(L_{-2} - \frac{3}{2(2h_\pm+1)}L_{-1}^2\Big)|h_\pm\rangle. \tag{2.15}$$

At general level N, the Kac determinant is given by [27]

$$\det\Big(\langle h;N,i|h;N,j\rangle\Big)_{1\le i,j\le p(N)} = \prod_{\substack{l,k\ge1\\ lk\le N}}\Big(2lk(h-h_{l,k})\Big)^{p(N-lk)}, \tag{2.16}$$

Here we have parametrized the central charge c and conformal weight $h_{l,k}$ by a parameter β,

$$c \;=\; 1-6\alpha_0^2, \tag{2.17}$$
$$h_{l,k} \;=\; \frac{1}{4}\bigg(\Big(\sqrt{\beta}l-\frac{1}{\sqrt{\beta}}k\Big)^2-\alpha_0^2\bigg), \tag{2.18}$$

where α_0 is

$$\alpha_0 = \sqrt{\beta}-\frac{1}{\sqrt{\beta}}. \tag{2.19}$$

These β and α_0 will be used throughout this lecture. Remark that c and $h_{l,k}$ are invariant under $\beta\to\beta^{-1}$.

Detailed study of the Kac determinant shows that representations of the Virasoro algebra are classified into several classes. The most interesting class is so-called minimal series [1]. In minimal series β is a rational number (we take $\beta>1$.),

$$\beta=\frac{p''}{p'},\quad p',p''\in\mathbb{Z}_{>0},\quad p''>p',\quad (p',p'')=1, \tag{2.20}$$

and in this case eqs.(2.17, 2.18) become

$$c \;=\; 1-6\frac{(p''-p')^2}{p'p''}, \tag{2.21}$$
$$h_{l,k} \;=\; \frac{(p''l-p'k)^2-(p''-p')^2}{4p'p''}. \tag{2.22}$$

An operator algebra of $\phi_h(z)$ closes for the following finite number of $h = h_{l,k}$,

$$1 \le l \le p' - 1, \quad 1 \le k \le p'' - 1, \quad p''l - p'k > 0. \tag{2.23}$$

If one prefers $0 < \beta < 1$, it is achieved by $p' \leftrightarrow p''$ ($\beta \leftrightarrow \beta^{-1}$), and then $h_{l,k} \leftrightarrow h_{k,l}$.

Let denote the Verma module with $h = h_{l,k}$ as $M_{l,k}$. From the Kac determinant and the property of $h_{l,k}$,

$$h_{l,k} = h_{-l,-k} = h_{l+p'n,k+p''n} \quad (n \in \mathbb{Z}), \tag{2.24}$$

$$h_{l,-k} = h_{l,k} + lk, \tag{2.25}$$

there are two basic singular vectors in $M_{l,k}$. One is at level lk ($h_{l,-k} = h_{l,k} + lk$, $h_{l,-k} = h_{-l,k}$) and the other is at level $(p' - l)(p'' - k)$ ($h_{l,k} = h_{p'-l,p''-k}$, $h_{-p'+l,p''-k} = h_{l,k} + (p' - l)(p'' - k)$, $h_{-p'+l,p''-k} = h_{-l+2p',k}$). Therefore we have

$$M_{l,k} \supset (M_{-l+2p',k} + M_{-l,k}). \tag{2.26}$$

To get an irreducible module we have to factor out invariant subspace, $M_{l,k}/(M_{-l+2p',k} + M_{-l,k})$. But the story has not ended yet because $M_{-l+2p',k} + M_{-l,k}$ does not coincide with $M_{-l+2p',k} \oplus M_{-l,k}$.

From the Kac determinant, $M_{-l,k}$ also has two singular vectors,

$$\begin{aligned} M_{-l,k} &= M_{p'-l,p''+k} \supset M_{-p'+l,p''+k} = M_{l-2p',k} \\ &= M_{p'+l,p''-k} \supset M_{-p'-l,p''-k} = M_{l+2p',k}. \end{aligned}$$

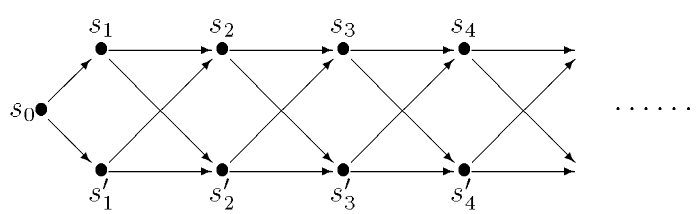
Here indices $(p' - l, p'' + k)$ and $(p' + l, p'' - k)$ are positive and minimal with respect to translation of (p', p'') (see (2.24)). Similarly

$$\begin{aligned} M_{-l+2p',k} &= M_{2p'-l,k} \supset M_{-2p'+l,k} = M_{l-2p',k} \\ &= M_{l,2p''-k} \supset M_{-l,2p''-k} = M_{l+2p',k}. \end{aligned}$$

These submodules of $M_{-l+2p',k}$ and $M_{-l,k}$ coincide, because if they do not coincide it implies more zeros of the Kac determinant. Therefore $M_{-l+2p',k}$ and $M_{-l,k}$ share two invariant submodules and we have

$$M_{-l,k} + M_{-l+2p',k} = (M_{-l,k} \oplus M_{-l+2p',k})/(M_{l-2p',k} + M_{l+2p',k}).$$

General embedding pattern is illustrated in the following figure



. (2.27)

Conformal weights of singular vectors are given by

$$
\begin{aligned}
s_0 \quad &: \quad L_0 = h_{l,k} = A(0) + \tfrac{c-1}{24}, && (2.28)\\
s'_{2m-1} \quad &: \quad L_0 = h_{-l+2mp',k} = B(-m) + \tfrac{c-1}{24} \quad (m \geq 1), && (2.29)\\
s_{2m+1} \quad &: \quad L_0 = h_{-l-2mp',k} = B(m) + \tfrac{c-1}{24} \quad (m \geq 0), && (2.30)\\
s'_{2m} \quad &: \quad L_0 = h_{l-2mp',k} = A(-m) + \tfrac{c-1}{24} \quad (m \geq 1), && (2.31)\\
s_{2m} \quad &: \quad L_0 = h_{l+2mp',k} = A(m) + \tfrac{c-1}{24} \quad (m \geq 1), && (2.32)
\end{aligned}
$$

where $A(m)$ and $B(m)$ are given by

$$
A(m) = \frac{(p''l - p'k + 2mp'p'')^2}{4p'p''}, \quad B(m) = \frac{(p''l + p'k + 2mp'p'')^2}{4p'p''}. \tag{2.33}
$$

Irreducible module $\mathcal{L}_{l,k}$ is

$$
\mathcal{L}_{l,k} = M_{l,k} - (M_{-l+2p',k} \oplus M_{-l,k}) + (M_{l-2p',k} \oplus M_{l+2p',k}) - \cdots. \tag{2.34}
$$

This information is encoded into the character.

The character of a representation is a tool that counts the number of states at each level,

$$
ch = \mathrm{tr}\ q^{L_0 - \frac{c}{24}} = \sum_{N=0}^{\infty} \dim\{\text{level } N \text{ states}\} \cdot q^{h+N-\frac{c}{24}}, \tag{2.35}
$$

where $\frac{c}{24}$ is included for the modular transformation property. For the Verma module M with h we have

$$
\mathrm{tr}_M q^{L_0 - \frac{c}{24}} = q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{2.36}
$$

From this and (2.34), the character of the irreducible Virasoro module $\mathcal{L}_{l,k}$ is [28]

$$
\chi_{l,k}(q) = \mathrm{tr}_{\mathcal{L}_{l,k}} q^{L_0 - \frac{c}{24}} = \frac{1}{\eta(\tau)} \sum_{m \in \mathbb{Z}} \Big(q^{A(m)} - q^{B(m)} \Big), \tag{2.37}
$$

where the Dedekind eta function $\eta(\tau)$ is

$$
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \quad q = e^{2\pi i \tau}. \tag{2.38}
$$

The representation is called unitary if the inner product is positive definite. The unitary minimal series is [29]

$$
\beta = \frac{m}{m-1} \text{ or } \frac{m-1}{m}, \quad c = 1 - \frac{6}{m(m-1)} \quad (m = 3, 4, \cdots). \tag{2.39}
$$

This is shown by drawing the vanishing line of the Kac determinant in (c, h) plane. Use

$$
\begin{aligned}
\beta \ &= \ \frac{1}{12}\Big(13 - c \pm \sqrt{(1-c)(25-c)}\Big) \gtrless 1, && (2.40)\\
h_{l,k} \ &= \ \frac{1-c}{24}(lk-1) + \frac{13-c}{48} \pm \frac{l^2-k^2}{48}\sqrt{(1-c)(25-c)}. && (2.41)
\end{aligned}
$$

2.2.3 primary fields and correlation functions

Primary fields are fundamental fields in CFT. Under the conformal transformation, a primary field with the conformal weight h, $\phi_h(z)$, transforms as a rank h form $(\phi_h(z)dz^h)$. We have the OPE,

$$L(z)\phi_h(w) = \frac{h\phi_h(w)}{(z-w)^2} + \frac{\partial\phi_h(w)}{z-w} + \text{reg.}, \tag{2.42}$$

or in mode

$$[L_n, \phi_{h,m}] = \Big((h-1)n - m\Big)\phi_{h,n+m}, \tag{2.43}$$

where $\phi_h(z) = \sum_{n\in\mathbb{Z}-h} \phi_{h,n} z^{-n-h}$. ($\phi_h(z)$ is called a quasiprimary field if (2.43) holds for $n = 1, 0, -1$.) We remark that the zero mode of a $h = 1$ field commutes with the Virasoro algebra, $[L_n, \phi_{1,0}] = 0$.

States are created by fields from the vacuum, $|\psi\rangle = \lim_{z\to 0}\psi(z)|\mathbf{0}\rangle$, and they are one-to-one correspondence. The highest weight state corresponds to the primary field and descendant states correspond to secondary fields. For example,

$$\begin{aligned} |h\rangle &\leftrightarrow \phi_h(z), \\ L_{-1}|h\rangle &\leftrightarrow \partial\phi_h(z), \\ L_{-3}L_{-2}|h\rangle &\leftrightarrow \hat{L}_{-3}\hat{L}_{-2}\phi_h(z), \end{aligned}$$

where $\hat{L}_n$ is

$$\hat{L}_{-n}\psi(z) = \oint_z \frac{dy}{2\pi i}(y-z)^{1-n}L(y)\psi(z). \tag{2.44}$$

In quantum field theory physical information is obtained from the Green functions (correlation functions). In CFT conformal symmetry imposes that correlation functions obey the conformal Ward identity [1], which enables us to reduce calculation of correlation function of secondary fields to that of primary fields. Moreover if null states exist, the correlation functions satisfy differential equations [1]. We illustrate this by taking an example. For $h = h_{1,2}$ or $h_{2,1}$, from (2.15), we have a null field

$$\chi(z) = \Big(\hat{L}_{-2} - \frac{3}{2(2h+1)}\hat{L}_{-1}^2\Big)\phi_h(z). \tag{2.45}$$

Since a null field decouples from all the fields, correlation functions including it vanish,

$$\begin{aligned} 0 &= \langle\chi(z)X\rangle \\ &= \Big(\mathcal{L}_{-2}(z) - \frac{3}{2(2h+1)}\mathcal{L}_{-1}(z)^2\Big)\langle\phi_h(z)X\rangle \\ &= \left(\sum_{i=1}^{N}\Big(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\Big) - \frac{3}{2(2h+1)}\partial_z^2\right)\langle\phi_h(z)X\rangle, \end{aligned} \tag{2.46}$$

where $X=\phi_{h_1}(z)\cdots\phi_{h_N}(z)$. Here we have used

$$\begin{aligned}
\langle \hat{L}_{-n}\phi_h(z)X\rangle &= \oint_z \frac{dy}{2\pi i}(y-z)^{1-n}\langle L(y)\phi_h(z)X\rangle \qquad (n\geq 1)\\
&= -\sum_{i=1}^{N}\oint_{z_i}\frac{dy}{2\pi i}(y-z)^{1-n}\Big(\frac{h_i}{(y-z_i)^2}\langle\phi_h(z)X\rangle+\frac{1}{y-z_i}\partial_{z_i}\langle\phi_h(z)X\rangle\Big)\\
&= \mathcal{L}_{-n}(z)\langle\phi_h(z)X\rangle, \qquad (2.47)\\
\mathcal{L}_{-n}(z) &= \sum_{i=1}^{N}\Big(\frac{(n-1)h_i}{(z_i-z)^n}-\frac{1}{(z_i-z)^{n-1}}\partial_{z_i}\Big). \qquad (2.48)
\end{aligned}$$

Due to translational invariance, we have $\mathcal{L}_{-1}(z)=\partial_z$. By solving this differential equation the correlation function can be calculated. Another method for calculation of correlation functions is free field realization, see [30, 2].

2.3 Free field realization

In the previous subsection the Virasoro algebra is treated in abstract way. In this subsection we treat it more explicitly by using our familiar free boson.

2.3.1 free field realization

Let us introduce free boson oscillator a_n ($n\in\mathbb{Z}_{\neq 0}$) and zero mode a_0, Q,

$$[a_n,a_m]=2n\delta_{n+m}, \qquad [a_n,Q]=2\delta_{n,0}. \qquad (2.49)$$

Here 2 is the Cartan matrix of A_1 Lie algebra. The Fock space with momentum α, $\mathcal{F}_\alpha$, is defined by

$$\mathcal{F}_\alpha=\bigoplus_{l\geq 0}\bigoplus_{n_1\geq\cdots\geq n_l>0}\mathbb{C}a_{-n_1}\cdots a_{-n_l}|\alpha\rangle_B, \qquad (2.50)$$

where $|\alpha\rangle_B$ is characterized by

$$a_n|\alpha\rangle_B=0 \quad (n>0), \qquad a_0|\alpha\rangle_B=\alpha|\alpha\rangle_B. \qquad (2.51)$$

$|\alpha\rangle_B$ is obtained from $|0\rangle_B$ ($a_n|0\rangle_B=0$ for $n\geq 0$),

$$|\alpha\rangle_B=e^{\frac{1}{2}\alpha Q}|0\rangle_B. \qquad (2.52)$$

A boson field $\phi(z)$ is

$$\phi(z)=Q+a_0\log z-\sum_{n\neq 0}\frac{1}{n}a_nz^{-n}, \qquad \partial\phi(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}. \qquad (2.53)$$

($\phi(z)$ here and usual one used in string theory is related as $\phi(z)=i\sqrt{2}\phi^{string}(z)$. Q here is antihermitian.) Normal ordering prescription $:\ \ :$ is defined by

$$\begin{cases} \text{move} \quad a_n(n>0) \text{ and } a_0 \quad \text{to right,} \\ \text{move} \quad a_n(n<0) \text{ and } Q \quad \text{to left.} \end{cases} \tag{2.54}$$

The Virasoro algebra is realized by a free boson in the following way,

$$L(z)=\frac{1}{4}:\partial\phi(z)\partial\phi(z):+\frac{1}{2}\alpha_0\partial^2\phi(z), \tag{2.55}$$

where the back ground charge α_0 is given in (2.19). In mode it is

$$L_n=\frac{1}{4}:\sum_m a_{n-m}a_m:-\frac{1}{2}\alpha_0(n+1)a_n, \tag{2.56}$$

or more explicitly,

$$L_n = \frac{1}{4}\sum_{m\neq 0,n} a_{n-m}a_m+\frac{1}{2}a_na_0-\frac{1}{2}\alpha_0(n+1)a_n \quad (n\neq 0), \tag{2.57}$$

$$L_0 = \frac{1}{2}\sum_{m>0} a_{-m}a_m+\frac{1}{4}\Big((a_0-\alpha_0)^2-\alpha_0^2\Big). \tag{2.58}$$

This Virasoro current realizes the Virasoro algebra with the central charge (2.17). $|\alpha\rangle_B$ is the highest weight state of the Virasoro algebra with the conformal weight $h=h(\alpha)$,

$$|\alpha\rangle_B = |h(\alpha)\rangle, \tag{2.59}$$

$$h(\alpha)=\frac{1}{4}\Big((\alpha-\alpha_0)^2-\alpha_0^2\Big), \tag{2.60}$$

and the corresponding primary field is

$$V_\alpha(z)=:e^{\frac{1}{2}\alpha\phi(z)}:. \tag{2.61}$$

Due to the background charge α_0, $\partial\phi(z)$ is not a primary field,

$$[L_n,a_m]=-ma_{n+m}-\alpha_0 n(n+1)\delta_{n+m,0}. \tag{2.62}$$

The dual Fock space $\mathcal{F}_\alpha^*$ also has the Virasoro module structure by the pairing $<\ ,\ >$ $:\mathcal{F}_\alpha^*\times\mathcal{F}_\alpha\to\mathbb{C}$,

$$<{}^tAf,v>=<f,Av>, \quad f\in\mathcal{F}_\alpha^*,\ v\in\mathcal{F}_\alpha, \quad A: \text{ operator}. \tag{2.63}$$

Here ${}^tL_n$ and ${}^ta_n$ are given by

$${}^tL_n = L_{-n}, \tag{2.64}$$

$${}^ta_n = -a_{-n} \quad (n\neq 0), \tag{2.65}$$

$${}^ta_0 = -a_0+2\alpha_0, \tag{2.66}$$

and

$$< |\alpha\rangle_B^*, |\alpha\rangle_B >= 1. \tag{2.67}$$

We write the following pairings as in the last lines,

$$\begin{aligned}
& < {}^tL_{m_1} \cdots {}^tL_{m_k} |\alpha\rangle_B^*, L_{-n_1} \cdots L_{-n_l} |\alpha\rangle_B > \\
= & < |\alpha\rangle_B, L_{m_k} \cdots L_{m_1} L_{-n_1} \cdots L_{-n_l} |\alpha\rangle_B > \\
=: & \; {}_B^*\langle\alpha| L_{m_k} \cdots L_{m_1} L_{-n_1} \cdots L_{-n_l} |\alpha\rangle_B, \qquad (2.68) \\
& < {}^ta_{m_1} \cdots {}^ta_{m_k} |\alpha\rangle_B^*, a_{-n_1} \cdots a_{-n_l} |\alpha\rangle_B > \\
= & < |\alpha\rangle_B, a_{m_k} \cdots a_{m_1} a_{-n_1} \cdots a_{-n_l} |\alpha\rangle_B > \\
=: & \; {}_B^*\langle\alpha| a_{m_k} \cdots a_{m_1} a_{-n_1} \cdots a_{-n_l} |\alpha\rangle_B. \qquad (2.69)
\end{aligned}$$

By (2.65) and (2.66), $\mathcal{F}_\alpha^*$ is isomorphic to $\mathcal{F}_{2\alpha_0-\alpha}$ as a Virasoro module,

$$\begin{aligned}
\mathcal{F}_\alpha^* \; &\cong \; \mathcal{F}_{2\alpha_0-\alpha} \quad \text{(Vir. module)}. \qquad (2.70) \\
|\alpha\rangle_B^* \; &\leftrightarrow \; |2\alpha_0 - \alpha\rangle_B. \qquad (2.71)
\end{aligned}$$

Of course $h(\alpha) = h(2\alpha_0 - \alpha)$ by (2.60).

For later use we define a_0' by,

$$a_0' = a_0 - \alpha_0, \quad a_0'|\alpha\rangle_B = (\alpha - \alpha_0)|\alpha\rangle_B, \quad {}^ta_0' = -a_0'. \tag{2.72}$$

The Virasoro current (2.55) can be rewritten as

$$z^2 L(z) = \frac{1}{4} : D\phi'(z) D\phi'(z) : + \frac{1}{2}\alpha_0 D^2\phi'(z), \tag{2.73}$$

where $\phi'(z)$ and D are

$$\phi'(z) = \phi(z)\Big|_{a_0 \to a_0'}, \qquad \partial\phi(z) = \partial\phi'(z) + \alpha_0 z^{-1}, \tag{2.74}$$

$$D = z\frac{\partial}{\partial z}. \tag{2.75}$$

2.3.2 singular vectors and Kac determinant

In the case of $h = h_{l,k}$ there are singular vectors. In the free boson realization they can be expressed by using the screening currents. Screening currents $S_\pm(z)$ are primary fields with $h = 1$,

$$S_\pm(z) = V_{2\alpha_\pm} = : e^{\alpha_\pm \phi(z)} :, \quad \alpha_+ = \sqrt{\beta}, \quad \alpha_- = \frac{-1}{\sqrt{\beta}}. \tag{2.76}$$

Hence their zero modes (screening charge) commute with the Virasoro algebra,

$$\Big[L_n, \oint_0 \frac{dz}{2\pi i} S_\pm(z) \Big] = 0. \tag{2.77}$$

We will use $S_+(z)$ in what follows. $S_-(z)$ can be treated similarly. We follow the method in [31].

The representation with $h = h_{l,k}$ is realized on $\mathcal{F}_{\alpha_{l,k}}$,

$$\begin{aligned} h_{l,k} &= h(\alpha_{l,k}), && (2.78)\\ \alpha_{l,k} &= \sqrt{\beta}(1-l) - \frac{1}{\sqrt{\beta}}(1-k). && (2.79)\end{aligned}$$

In the following we write $\mathcal{F}_{\alpha_{l,k}} = \mathcal{F}_{l,k}$. To contact with our previous paper [32]($\alpha_{r,s}$ there is $\propto \alpha_{-r,-s}$ here.), we consider dual space $\mathcal{F}^*_{l,k} = \mathcal{F}_{-l,-k}$. In the Verma module with $h_{-l,-k} = h_{l,k}$, the singular vector at level lk is expressed as

$$\begin{aligned} &|\chi_{-l,-k}\rangle \\ &= \int \prod_{j=1}^{l} \frac{dz_j}{2\pi i} \cdot S_+(z_1)\cdots S_+(z_l)|\alpha_{l,-k}\rangle_B \qquad (2.80)\\ &= \int \prod_{j=1}^{l} \underline{dz}_j \cdot \prod_{1\le i<j\le l} (z_i - z_j)^{2\beta} \cdot \prod_{i=1}^{l} z_i^{\beta(1-l)-k} \cdot \prod_{j=1}^{l} \exp\Big(\sqrt{\beta}\sum_{n>0}\frac{1}{n}a_{-n}z_j^n\Big) \cdot |\alpha_{-l,-k}\rangle_B. \end{aligned}$$

Here $\underline{dz}_j$ is given in (A.14) and the integration contour is summarized in appendixA.4 (for example take I_{BMP}). Since this integral of S_+'s, which is a map from $\mathcal{F}_{l,-k}$ to $\mathcal{F}_{-l,-k}$, commutes with L_n, $|\chi_{-l,-k}\rangle$ is annihilated by L_n $(n>0)$. This state $|\chi_{-l,-k}\rangle$ is non zero because of (A.48).

To prove the Kac determinant formula, we first study the relation between states in the Verma module and Fock space. Let us introduce following notation,

$$I = \{n_1,\cdots,n_l\}, \quad n_1 \ge \cdots \ge n_l > 0, \qquad \ell(I) = l, \qquad (2.81)$$

$$L_I = L_{n_l}\cdots L_{n_1}, \quad L_{-I} = L_{-n_1}\cdots L_{-n_l}, \qquad (2.82)$$

$$a_I = a_{n_l}\cdots a_{n_1}, \quad a_{-I} = a_{-n_1}\cdots a_{-n_l}, \qquad (2.83)$$

and I is ordered by the reverse lexicographic ordering. States at level N in the Verma module are linear combinations of those of Fock space,

$$L_{-I}|\alpha\rangle_B = \sum_J C(N,\alpha)_{I,J} a_{-J}|\alpha\rangle_B. \qquad (2.84)$$

For example,

$$\begin{aligned} L_{-1}|\alpha\rangle_B &= \tfrac{1}{2}\alpha a_{-1}|\alpha\rangle_B,\\ \begin{pmatrix} L_{-2}|\alpha\rangle_B \\ cL_{-1}^2|\alpha\rangle_B \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}(\alpha+\alpha_0) & \frac{1}{4}a_{-1}^2 \\ \frac{1}{2}\alpha & \frac{1}{4}\alpha^2 \end{pmatrix} \begin{pmatrix} a_{-2}|\alpha\rangle_B \\ a_{-1}^2|\alpha\rangle_B \end{pmatrix},\\ \begin{pmatrix} L_{-3}|\alpha\rangle_B \\ cL_{-2}L_{-1}|\alpha\rangle_B \\ cL_{-1}^3|\alpha\rangle_B \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\alpha+\alpha_0 & \frac{1}{2} & 0 \\ \frac{1}{2}\alpha & \frac{1}{4}\alpha(\alpha+\alpha_0) & \frac{1}{8}\alpha \\ \alpha & \frac{3}{4}\alpha^2 & \frac{1}{8}\alpha^3 \end{pmatrix} \begin{pmatrix} a_{-3}|\alpha\rangle_B \\ a_{-2}a_{-1}|\alpha\rangle_B \\ a_{-1}^3|\alpha\rangle_B \end{pmatrix}. \end{aligned}$$

The determinant of matrix $C(N,\alpha)$ is given by

$$\det C(N,\alpha)_{I,J} = \prod_{\substack{l,k\geq 1\\ lk\leq N}} \Big(\tfrac{1}{2}(\alpha-\alpha_{l,k})\Big)^{p(N-lk)}. \tag{2.85}$$

Proof. Since $L_{-n} = \frac{1}{2}a_{-n}a_0 + \cdots$, leading term of α in $\det C(N,\alpha)$ is

$$\prod_{\substack{\{k_i\}\\ \sum_i ik_i=N}} \prod_i \Big(\frac{\alpha}{2}\Big)^{k_i} = \prod_{\substack{l,k\geq 1\\ lk\leq N}} \Big(\frac{\alpha}{2}\Big)^{p(N-lk)}.$$

Here we have used (A.30). In the last paragraph we have constructed singular vectors as same number as this order of α. □

Next we consider a dual $\mathcal{F}^*_\alpha$. With the notation,

$$({}^tL)_I = {}^t(L_I) = {}^tL_{n_1}\cdots{}^tL_{n_l}, \tag{2.86}$$

$$({}^ta)_I = {}^t(a_I) = {}^ta_{n_1}\cdots{}^ta_{n_l}, \tag{2.87}$$

we define a matrix $C'(N,\alpha)$,

$$({}^tL)_I|\alpha\rangle^*_B = \sum_J C'(N,\alpha)_{I,J}({}^ta)_J|\alpha\rangle^*_B. \tag{2.88}$$

By (2.64) and (2.65), the left and right hand sides of this equation are

$$\text{LHS} = L_{-I}|\alpha\rangle^*_B = \sum_K C(N,2\alpha_0-\alpha)_{I,K}a_{-K}|\alpha\rangle^*_B, \tag{2.89}$$

$$\text{RHS} = \sum_{J,K} C'(N,\alpha)_{I,J}D_{J,K}a_{-K}|\alpha\rangle^*_B, \tag{2.90}$$

where matrix D is

$$D_{J,K} = \delta_{J,K}(-1)^{\ell(J)}. \tag{2.91}$$

Hence C' can be expressed by C,

$$C'(N,\alpha)_{I,J} = \sum_K C(N,2\alpha_0-\alpha)_{I,K}D_{K,J}, \tag{2.92}$$

and its determinant is

$$\begin{aligned}\det C'(N,\alpha)_{I,J} &= \det C(N,2\alpha_0-\alpha)\cdot\det D\\ &= \prod_{\substack{l,k\geq 1\\ lk\leq N}} \Big(\tfrac{1}{2}(\alpha-\alpha_{-l,-k})\Big)^{p(N-lk)}.\end{aligned} \tag{2.93}$$

By gathering these results, the inner product of two states in the Verma module becomes

$$\begin{aligned}\langle h|L_I L_{-J}|h\rangle &= <({}^tL)_I|\alpha\rangle_B^*, L_{-J}|\alpha\rangle_B> \\ &= <\sum_K C'(N,\alpha)_{I,K}({}^ta)_K|\alpha\rangle_B^*, \sum_L C(N,\alpha)_{J,L}a_{-L}|\alpha\rangle_B> \\ &= \sum_{K,L} C'(N,\alpha)_{I,K}G_{K,L}C(N,\alpha)_{J,L}. \end{aligned} \tag{2.94}$$

Here $G_{K,L}$ is

$$\begin{aligned} G_{K,L} &= <({}^ta)_K|\alpha\rangle_B^*, a_{-L}|\alpha\rangle_B> \\ &= {}_B^*\langle\alpha|a_{m_k}\cdots a_{m_1}a_{-n_1}\cdots a_{-n_l}|\alpha\rangle_B \\ &= \delta_{K,L}\ {}_B^*\langle\alpha|\cdots a_2^{k_2}a_1^{k_1}a_{-1}^{k_1}a_{-2}^{k_2}\cdots|\alpha\rangle_B \\ &= \delta_{K,L}\prod_i(2i)^{k_i}k_i!, \end{aligned} \tag{2.95}$$

where $(m_1,\cdots,m_k)=1^{k_1}2^{k_2}\cdots$. By using (A.28) and (A.30), its determinant becomes

$$\det G_{K,L} = \prod_{\substack{\{k_i\} \\ \sum_i ik_i=N}}\prod_i(2i)^{k_i}k_i! = \prod_{\substack{l,k\geq 1 \\ lk\leq N}}\Big(2lk\Big)^{p(N-lk)}. \tag{2.96}$$

Therefore we obtain the Kac determinant (2.16),

$$\begin{aligned}\det\langle h|L_I L_{-J}|h\rangle &= \det C'(N,\alpha)\cdot\det G\cdot\det{}^tC(N,\alpha) \\ &= \prod_{\substack{l,k\geq 1 \\ lk\leq N}}\Big(2lk(h-h_{l,k})\Big)^{p(N-lk)}. \end{aligned} \tag{2.97}$$

Here we have used $(\alpha-\alpha_{l,k})(\alpha-\alpha_{-l,-k})=4(h-h(\alpha_{l,k}))$.

2.3.3 Felder complex

Irreducible Virasoro module $\mathcal{L}_{l,k}$ is realized on the Fock space $\mathcal{F}_{l,k}$, but this $\mathcal{F}_{l,k}$ is bigger than $\mathcal{L}_{l,k}$. So we have to discuss how $\mathcal{L}_{l,k}$ is obtained from $\mathcal{F}_{l,k}$. The Virasoro structure of $\mathcal{F}_{l,k}$ was investigated by Feigin-Fuchs [33] and Felder [34]. Here we review it following [34]. We consider minimal series eqs.(2.20)-(2.23).

The Becchi-Rouet-Stora-Tyupin (BRST) charge Q_m is defined by using the screening current $S_+(z)$,

$$\begin{aligned} Q_m &= \int\prod_{j=1}^m\frac{dz_j}{2\pi i}\cdot S_+(z_1)\cdots S_+(z_m), \qquad (2.98) \\ &= \int\prod_{j=1}^m\frac{dz_j}{2\pi i}\cdot\prod_{1\leq i<j\leq m}(z_i-z_j)^{2\beta}\cdot :S_+(z_1)\cdots S_+(z_m):, \end{aligned}$$

where the integration contour is taken as I_{BMP} in appendixA.4. On $\mathcal{F}_{l',k'}$ conditions (A.32) and (A.33) become

$$\text{(i) } m \equiv 0 \pmod{p'}, \quad \text{(ii) } m \equiv l' \pmod{p'}. \tag{2.99}$$

For example Q_l is well defined on $\mathcal{F}_{l',k}$ ($l' \equiv l \,(\bmod p')$), and the contour can be deformed to I_F type by (A.47). Importance property of the BRST charge is commutativity with the Virasoro algebra,

$$Q_l \,:\, \mathcal{F}_{l',k} \to \mathcal{F}_{l'-2l,k}, \quad l' \equiv l \pmod{p'}, \quad [L_n, Q_l] = 0, \tag{2.100}$$

$$Q_{p'-l} \,:\, \mathcal{F}_{l',k} \to \mathcal{F}_{l'-2(p'-l),k}, \quad l' \equiv -l \pmod{p'}, \quad [L_n, Q_{p'-l}] = 0, \tag{2.101}$$

and the nilpotency,

$$Q_l Q_{p'-l} = Q_{p'-l} Q_l = Q_{p'} = 0. \tag{2.102}$$

Here $Q_l Q_{p'-l} = Q_{p'}$ and $Q_{p'-l} Q_l = Q_{p'}$ are shown by I_{BMP} type contour, and $Q_{p'} = 0$ is shown by (A.47). The singular vector (2.80) can be written as $|\chi_{-l,-k}\rangle = Q_l |\alpha_{l,-k}\rangle_B$.

Let us consider the Felder complex $C_{l,k}$,

$$\cdots \xrightarrow{X_{-3}} C_{-2} \xrightarrow{X_{-2}} C_{-1} \xrightarrow{X_{-1}} C_0 \xrightarrow{X_0} C_1 \xrightarrow{X_1} C_2 \xrightarrow{X_2} \cdots, \tag{2.103}$$

where C_j and $X_j \,:\, C_j \to C_{j+1}$ ($j \in \mathbb{Z}$) are

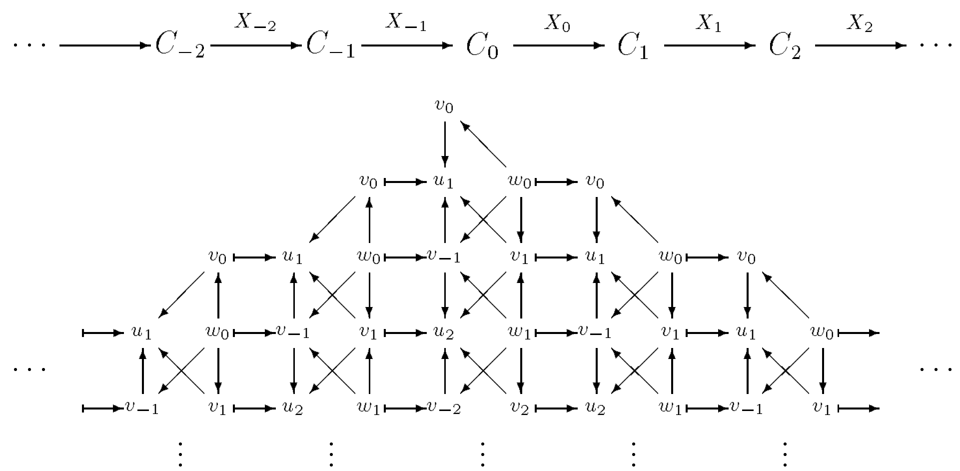
$$C_{2j} = \mathcal{F}_{l-2p'j,k}, \quad C_{2j+1} = \mathcal{F}_{-l-2p'j,k}, \tag{2.104}$$

$$X_{2j} = Q_l, \quad X_{2j+1} = Q_{p'-l}. \tag{2.105}$$

X satisfies the BRST property,

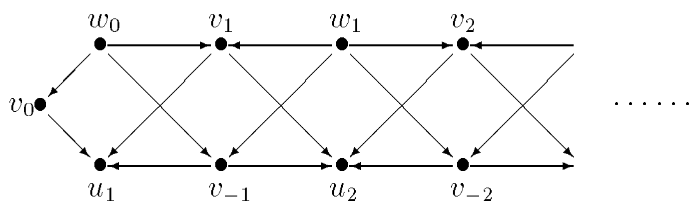
$$X_j X_{j-1} = 0. \tag{2.106}$$

Structure of the Felder complex is illustrated in the following figure:



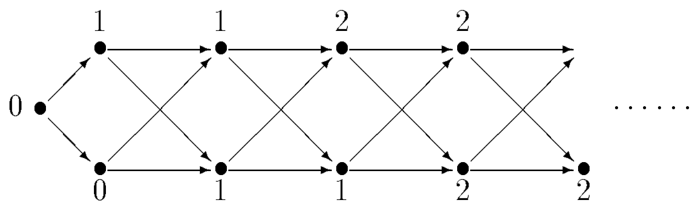
(2.107)

Each Fock space has the structure



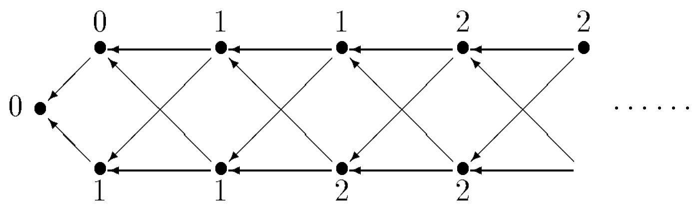
, (2.108)

which is obtained by combining the information from $C(N,\alpha)$



,

and the information from $C'(N,\alpha)$



.

v_0, u_m, w_{m-1}, v_{-m}, v_m $(m \geq 1)$ have the same conformal weight as s_0, s'_{2m-1}, s_{2m-1}, s'_{2m}, s_{2m} in (2.27) respectively. s_{2m-1} is u_m but s'_{2m-1} vanishes on the Fock space. At the level of s'_{2m-1}, there is a state which does not belong the Verma module. That state is w_{m-1}.

We write states in C_j with superfix (j). Conformal weights of v, u, w are following:

$$v_0^{(2j)} \ : \ L_0 = A(-j) + \tfrac{c-1}{24}, \tag{2.109}$$
$$v_0^{(2j+1)} \ : \ L_0 = B(j) + \tfrac{c-1}{24}, \tag{2.110}$$
$$u_m^{(2j)} \ : \ L_0 = B(-|j| - m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.111}$$
$$u_m^{(2j+1)} \ : \ L_0 = A(\tfrac{|2j+1|-1}{2} + m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.112}$$
$$v_{-m}^{(2j)} \ : \ L_0 = A(-|j| - m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.113}$$
$$v_{-m}^{(2j+1)} \ : \ L_0 = B(\tfrac{|2j+1|-1}{2} + m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.114}$$
$$v_m^{(2j)} \ : \ L_0 = A(|j| + m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.115}$$
$$v_m^{(2j+1)} \ : \ L_0 = B(-\tfrac{|2j+1|+1}{2} - m) + \tfrac{c-1}{24} \quad (m \geq 1), \tag{2.116}$$
$$w_m^{(2j)} \ : \ L_0 = B(|j| + m) + \tfrac{c-1}{24} \quad (m \geq 0), \tag{2.117}$$
$$w_m^{(2j+1)} \ : \ L_0 = A(-\tfrac{|2j+1|+1}{2} - m) + \tfrac{c-1}{24} \quad (m \geq 0), \tag{2.118}$$

where A and B are given in (2.33). BRST charge X_j maps u, v, w in the following way:

$$
\begin{aligned}
j<0 \quad : \quad & X_j w_m^{(j)} = v_{-m-1}^{(j+1)} \quad (m \geq 0), \quad X_j v_m^{(j)} = u_{m+1}^{(j+1)} \quad (m \geq 0), \\
& X_j u_m^{(j)} = 0 \quad (m \geq 1), \quad X_j v_{-m}^{(j)} = 0 \quad (m \geq 1), && (2.119) \\
j \geq 0 \quad : \quad & X_j w_m^{(j)} = v_{-m}^{(j+1)} \quad (m \geq 0), \quad X_j v_m^{(j)} = u_m^{(j+1)} \quad (m \geq 1), \\
& X_j u_m^{(j)} = 0 \quad (m \geq 1), \quad X_j v_{-m}^{(j)} = 0 \quad (m \geq 0). && (2.120)
\end{aligned}
$$

The cohomology groups of the complex $C_{l,k}$ are [34]

$$
H^j(C_{l,k}) = \text{Ker } X_j / \text{Im } X_{j-1} = \begin{cases} 0 & j \neq 0, \\ \mathcal{L}_{l,k} & j = 0. \end{cases} \tag{2.121}
$$

Using this fact, the trace of operator $\mathcal{O}$ over the irreducible Virasoro module can be converted to the alternated sum of those over the Fock spaces. If we can draw the commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{-1} \arrow[r, "X_{-1}"] \arrow[d, "\mathcal{O}^{(-1)}"] & C_0 \arrow[r, "X_0"] \arrow[d, "\mathcal{O}^{(0)}"] & C_1 \arrow[r, "X_1"] \arrow[d, "\mathcal{O}^{(1)}"] & \cdots, \\
\cdots \arrow[r] & C_{-1} \arrow[r, "X_{-1}"] & C_0 \arrow[r, "X_0"] & C_1 \arrow[r, "X_1"] & \cdots
\end{tikzcd} \tag{2.122}
$$

where $\mathcal{O}^{(j)}$ is an operator $\mathcal{O}$ realized on C_j, then by Euler-Poincaré principle, we have

$$
\mathrm{tr}_{\mathcal{L}_{l,k}} \mathcal{O} = \mathrm{tr}_{H^0(C_{l,k})} \mathcal{O}^{(0)} = \mathrm{tr}_{H^*(C_{l,k})} \mathcal{O} = \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}_{C_j} \mathcal{O}^{(j)}. \tag{2.123}
$$

For example let us calculate the character. Since the trace over the Fock space is

$$
\mathrm{tr}_{C_{2m}} q^{L_0 - \frac{c}{24}} = q^{-\frac{1}{24} + A(-m)} \prod_{n>0} \frac{1}{1-q^n}, \quad \mathrm{tr}_{C_{2m+1}} q^{L_0 - \frac{c}{24}} = q^{-\frac{1}{24} + B(m)} \prod_{n>0} \frac{1}{1-q^n},
$$

we obtain the Virasoro character (2.37),

$$
\mathrm{tr}_{\mathcal{L}_{l,k}} q^{L_0 - \frac{c}{24}} = \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}_{C_j} q^{L_0 - \frac{c}{24}} = \frac{1}{\eta(\tau)} \sum_{m \in \mathbb{Z}} \Big(q^{A(-m)} - q^{B(m)} \Big). \tag{2.124}
$$

If we use $S_-(z)$ instead of $S_+(z)$, another complex in which k' of $\mathcal{F}_{l,k'}$ changes is obtained.

Example: $c = \frac{1}{2}$ (Ising model)
Parameters and conformal weights are

$$
\begin{aligned}
& c = \tfrac{1}{2}, \quad p' = 3, \quad p'' = 4, \quad \beta = \tfrac{4}{3}, \quad \alpha_0 = \tfrac{1}{2\sqrt{3}}, \quad \alpha_+ = \tfrac{2}{\sqrt{3}}, \quad \alpha_- = -\tfrac{\sqrt{3}}{2}, \\
& h_{1,1} = 0, \quad h_{2,1} = \tfrac{1}{2}, \quad h_{2,2} = \tfrac{1}{16}, \quad \alpha_{1,1} = 0, \quad \alpha_{2,1} = -\tfrac{2}{\sqrt{3}}, \quad \alpha_{2,2} = -\tfrac{1}{2\sqrt{3}}.
\end{aligned}
$$

$h=0,\frac{1}{2},\frac{1}{16}$ correspond to identity operator, energy operator, spin operator respectively. Remark that

$$h_{2,3}=0,\quad h_{1,3}=\tfrac{1}{2},\quad h_{1,2}=\tfrac{1}{16},\quad \alpha_{2,3}=\tfrac{1}{\sqrt{3}},\quad \alpha_{1,3}=\sqrt{3},\quad \alpha_{1,2}=\tfrac{\sqrt{3}}{2}.$$

Basic two singular vectors are

$$\begin{aligned}
h=0\ &:\ |\chi\rangle=L_{-1}|0\rangle, && (2.125)\\
&\quad\ |\chi'\rangle=(L_{-6}+\tfrac{22}{9}L_{-4}L_{-2}-\tfrac{31}{36}L_{-3}^2-\tfrac{16}{27}L_{-2}^3)|0\rangle &&\\
&\qquad +(b_1L_{-5}+b_2L_{-4}L_{-1}+b_3L_{-3}L_{-2} && (2.126)\\
&\qquad\quad +b_4L_{-3}L_{-1}^2+b_5L_{-2}^2L_{-1}+b_6L_{-2}L_{-1}^3+b_7L_{-1}^5)|\chi\rangle, &&\\
h=\frac{1}{2}\ &:\ |\chi\rangle=(L_{-2}-\tfrac{3}{4}L_{-1}^2)|\tfrac{1}{2}\rangle, && (2.127)\\
&\quad\ |\chi'\rangle=(L_{-3}-\tfrac{1}{3}L_{-1}^3)|\tfrac{1}{2}\rangle+aL_{-1}|\chi\rangle, && (2.128)\\
h=\frac{1}{16}\ &:\ |\chi\rangle=(L_{-2}-\tfrac{4}{3}L_{-1}^2)|\tfrac{1}{16}\rangle, && (2.129)\\
&\quad\ |\chi'\rangle=(L_{-4}+\tfrac{40}{31}L_{-3}L_{-1}-\tfrac{256}{465}L_{-1}^4)|\tfrac{1}{16}\rangle+(bL_{-2}+b'L_{-1}^2)|\chi\rangle, && (2.130)
\end{aligned}$$

where

$$\begin{aligned}
&b_1=-\tfrac{41}{225},\quad b_2=-\tfrac{78}{25},\quad b_3=\tfrac{14}{75},\quad b_4=\tfrac{34}{25},\quad b_5=\tfrac{172}{75},\quad b_6=-\tfrac{8}{5},\quad b_7=\tfrac{4}{25},\\
&a=-\tfrac{4}{5},\quad b=\tfrac{147}{3100},\quad b'=-\tfrac{401}{775},
\end{aligned}$$

are determined by the requirement that $|\chi'\rangle$ is orthogonal (with double zero) to descendants of $|\chi\rangle$.

In the free field realization these singular vectors become as follows. For $h=0$,

$$\begin{aligned}
|0\rangle=|\alpha_{1,1}\rangle_B\ &:\ |\chi\rangle=0, && (2.131)\\
&\quad\ |\chi'\rangle=-\tfrac{1}{216\sqrt{3}}(a_{-6}+6\sqrt{3}a_{-5}a_{-1}-32\sqrt{3}a_{-4}a_{-2}-51a_{-4}a_{-1}^2 &&\\
&\qquad +\tfrac{51\sqrt{3}}{2}a_{-3}^2+75a_{-3}a_{-2}a_{-1}-18\sqrt{3}a_{-3}a_{-1}^3-\tfrac{97}{3}a_{-2}^3 && (2.132)\\
&\qquad +\tfrac{31\sqrt{3}}{2}a_{-2}^2a_{-1}^2+6a_{-2}a_{-1}^4+2\sqrt{3}a_{-1}^6)|\alpha_{1,1}\rangle_B, &&\\
|0\rangle=|\alpha_{2,3}\rangle_B\ &:\ |\chi\rangle=\tfrac{1}{2\sqrt{3}}a_{-1}|\alpha_{2,3}\rangle_B, && (2.133)\\
&\quad\ |\chi'\rangle=0, && (2.134)
\end{aligned}$$

for $h=\frac{1}{2}$,

$$\begin{aligned}
|\tfrac{1}{2}\rangle=|\alpha_{2,1}\rangle_B\ &:\ |\chi\rangle=0, && (2.135)\\
&\quad\ |\chi'\rangle=\tfrac{1}{6\sqrt{3}}(a_{-3}+\sqrt{3}a_{-2}a_{-1}+\tfrac{2}{3}a_{-1}^3)|\alpha_{2,1}\rangle_B, && (2.136)\\
|\tfrac{1}{2}\rangle=|\alpha_{1,3}\rangle_B\ &:\ |\chi\rangle=\tfrac{5}{8\sqrt{3}}(a_{-2}-\tfrac{\sqrt{3}}{2}a_{-1}^2)|\alpha_{1,3}\rangle_B, && (2.137)\\
&\quad\ |\chi'\rangle=0, && (2.138)
\end{aligned}$$

and for $h=\frac{1}{16}$,

$$|\tfrac{1}{16}\rangle = |\alpha_{2,2}\rangle_B \quad : \quad |\chi\rangle = \tfrac{1}{3\sqrt{3}}(a_{-2}+\tfrac{2}{\sqrt{3}}a_{-1}^2)|\alpha_{2,2}\rangle_B, \tag{2.139}$$

$$|\chi'\rangle = 0, \tag{2.140}$$

$$|\tfrac{1}{16}\rangle = |\alpha_{1,2}\rangle_B \quad : \quad |\chi\rangle = 0, \tag{2.141}$$

$$|\chi'\rangle = -\tfrac{\sqrt{3}}{310}(a_{-4}-8\sqrt{3}a_{-3}a_{-1}+\tfrac{37}{2\sqrt{3}}a_{-2}^2-2a_{-2}a_{-1}^2 +2\sqrt{3}a_{-1}^4)|\alpha_{1,3}\rangle_B. \tag{2.142}$$

On $\mathcal{F}_\alpha$, BRST charges are

$$\begin{aligned} Q_1 &= \int_{C_{KM}}\frac{dz_1}{2\pi i}S_+(z_1) \\ &= \oint_0 \frac{dz}{2\pi i}\exp\Big(\frac{2}{\sqrt{3}}\sum_{n>0}\frac{1}{n}a_{-n}z^n\Big)\exp\Big(-\frac{2}{\sqrt{3}}\sum_{n>0}\frac{1}{n}a_nz^{-n}\Big)e^{\frac{2}{\sqrt{3}}Q}z^{\frac{2}{\sqrt{3}}a_0}, \end{aligned} \tag{2.143}$$

$$\begin{aligned} Q_2 &= \frac{1}{2}(e^{2\pi i\alpha_+^2}+1)(e^{2\pi i\alpha_+\alpha}-1)\int_{C_{KM}}\frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}S_+(z_1)S_+(z_2) \\ &= \frac{1}{2}(e^{2\pi i\frac{1}{3}}+1)(e^{2\pi i\frac{2}{\sqrt{3}}\alpha}-1)\frac{1}{2\pi i}\oint_0\frac{dz}{2\pi iz}\int_0^1 du\, z^{\frac{14}{3}}(1-u)^{\frac{8}{3}}e^{\frac{4}{\sqrt{3}}Q}z^{\frac{4}{\sqrt{3}}a_0}u^{\frac{2}{\sqrt{3}}a_0}, \\ &\quad \times\exp\Big(\frac{2}{\sqrt{3}}\sum_{n>0}\frac{1}{n}a_{-n}z^n(1+u^n)\Big)\exp\Big(-\frac{2}{\sqrt{3}}\sum_{n>0}\frac{1}{n}a_nz^{-n}(1+u^{-n})\Big), \end{aligned} \tag{2.144}$$

where we have taken I_{KM} contour (see appendix A.4) and changed integration variables $z_1=z$, $z_2=zu$. We give lower level examples. We perform $\oint_0 dz$ first, and next use $\int_0^1 duu^{a-1}(1-u)^{b-1}=B(a,b)$ for 'arbitrary' values a,b (analytic continuation). For $h=\frac{1}{2}$ and $|\frac{1}{2}\rangle=|\alpha_{2,1}\rangle_B$,

$$L_{-2}|\alpha_{2,1}\rangle_B = \tfrac{3}{4}L_{-1}^2|\alpha_{2,1}\rangle_B = -\tfrac{\sqrt{3}}{4}(a_{-2}-\tfrac{1}{\sqrt{3}}a_{-1}^2)|\alpha_{2,1}\rangle_B, \tag{2.145}$$

$$Q_1|\alpha_{4,1}\rangle_B = \tfrac{2}{3\sqrt{3}}(a_{-3}+\sqrt{3}a_{-2}a_{-1}+\tfrac{2}{3}a_{-1}^3)|\alpha_{2,1}\rangle_B, \tag{2.146}$$

$$Q_2(Aa_{-2}+Ba_{-1}^2)|\alpha_{2,1}\rangle_B = -\tfrac{120}{7\pi}\tfrac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})}(A+\sqrt{3}B)|\alpha_{-2,1}\rangle_B, \tag{2.147}$$

$$\begin{aligned} &Q_2(A'a_{-3}+B'a_{-2}a_{-1}+C'a_{-1}^3)|\alpha_{2,1}\rangle_B \\ &\quad = -\tfrac{90\sqrt{3}}{7\pi}\tfrac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})}(A'+\tfrac{1}{\sqrt{3}}B'-3C')a_{-1}|\alpha_{-2,1}\rangle_B, \end{aligned} \tag{2.148}$$

therefore we have

$$Q_1|\alpha_{4,1}\rangle_B = 4|\chi'\rangle, \quad Q_2L_{-2}|\alpha_{2,1}\rangle_B = Q_2L_{-1}^2|\alpha_{2,1}\rangle_B = 0, \quad Q_2Q_1|\alpha_{4,1}\rangle_B = 0. \tag{2.149}$$

For the complex $C_{2,1}$, we have

$$v_0^{(-1)} = |\alpha_{4,1}\rangle_B, \quad v_0^{(0)} = |\alpha_{2,1}\rangle_B, \quad v_0^{(1)} = |\alpha_{-2,1}\rangle_B, \tag{2.150}$$

$$u_1^{(0)} = Q_1v_0^{(-1)} = \tfrac{2}{3\sqrt{3}}(a_{-3}+\sqrt{3}a_{-2}a_{-1}+\tfrac{2}{3}a_{-1}^3)|\alpha_{2,1}\rangle_B, \tag{2.151}$$

$$Q_2w_0^{(0)} = v_0^{(1)}, \quad w_0^{(0)} = -\tfrac{7\pi}{300}\tfrac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})^2}(a_{-2}+\tfrac{\sqrt{3}}{2}a_{-1}^2)|\alpha_{2,1}\rangle_B. \tag{2.152}$$

Remark that the last equation is the dual of the following state,

$$Q_2|\alpha_{2,-1}\rangle_B = \tfrac{30}{7\pi}\tfrac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{3})}(a_{-2} - \tfrac{\sqrt{3}}{2}a_{-1}^2)|\alpha_{-2,-1}\rangle_B. \tag{2.153}$$

For more examples and explicit expressions of the Virasoro singular vectors, see [2, 35, 36].

2.3.4 Calogero-Sutherland model and Jack symmetric polynomial

In this subsection we review the relation between the Virasoro singular vector and the Calogero-Sutherland model (CSM) [37, 32]. Calogero-Sutherland model is a many body quantum mechanical system on a circle with length L under the $1/r^2$ potential. Its Hamiltonian and momentum are

$$H_{CS} = \sum_{j=1}^{N_0} \frac{1}{2m}\hat{p}_j^2 + \frac{\bar{\beta}(\bar{\beta}-\hbar)}{m}\left(\frac{\pi}{L}\right)^2 \sum_{1\le i<j\le N_0} \frac{1}{\sin^2\frac{\pi}{L}(q_i-q_j)}, \tag{2.154}$$

$$P_{CS} = \sum_{j=1}^{N_0}\hat{p}_j, \quad \hat{p}_j = \frac{\hbar}{i}\frac{\partial}{\partial q_j}, \tag{2.155}$$

where q_j is a coordinate of j-th particle, p_j is its momentum, N_0 is a number of particles, m is a mass of particles, $\bar{\beta}$ is a (dimensionful) coupling constant and $\hbar$ is a Planck constant. Since this Hamiltonian can be rewritten as

$$H_{CS} = \sum_j \frac{1}{2m}\Pi_j^\dagger\Pi_j + \frac{\bar{\beta}^2}{m}\left(\frac{\pi}{L}\right)^2\frac{N_0^3-N_0}{6}, \tag{2.156}$$

$$\Pi_j = \hat{p}_j + i\bar{\beta}\frac{\pi}{L}\sum_{k\ne j}\cot\frac{\pi}{L}(q_j-q_k), \tag{2.157}$$

the ground state Ψ_0 is determined by $\Pi_j\Psi_0 = 0$,

$$\Psi_0 = \mathcal{N}\left(\prod_{i<j}\sin\frac{\pi}{L}(q_i-q_j)\right)^\beta \propto \left(\prod_{i<j}\sqrt{\frac{x_i}{x_j}}\left(1-\frac{x_j}{x_i}\right)\right)^\beta, \tag{2.158}$$

where $\mathcal{N}$ is a normalization constant, and β and x_j are dimensionless quantities,

$$\beta = \frac{\bar{\beta}}{\hbar}, \quad x_j = e^{2\pi i\frac{q_j}{L}}. \tag{2.159}$$

Under the interchange of particle, this ground state gives a phase $(-1)^\beta$ which means that this system obeys fractional statistics. Excited states that we seek for have the form $\Psi = \psi\Psi_0$, where ψ is a symmetric function of x_j because Ψ has same statistical property as Ψ_0. By removing the contribution from the ground state,

$$\Psi_0^{-1}\circ H_{CS}\circ\Psi_0 = \frac{1}{2m}\left(\frac{2\pi\hbar}{L}\right)^2\Big(H_\beta + \tfrac{1}{12}\beta^2(N_0^3-N_0)\Big), \tag{2.160}$$

$$\Psi_0^{-1}\circ P_{CS}\circ\Psi_0 = \frac{2\pi\hbar}{L}P, \tag{2.161}$$

Hamiltonian and momentum that act directly ψ are

$$H_\beta = \sum_{i=1}^{N_0} D_i^2 + \beta \sum_{1\le i<j\le N_0} \frac{x_i+x_j}{x_i-x_j}(D_i - D_j), \tag{2.162}$$

$$P = \sum_{i=1}^{N_0} D_i, \quad D_i = x_i \frac{\partial}{\partial x_i}. \tag{2.163}$$

Here we recall basic definitions of symmetric polynomials [38]:

partition: $\lambda = (\lambda_1, \lambda_2, \cdots) = 1^{m_1} 2^{m_2} \cdots, \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge 0, \quad m_i \ge 0,$

$$|\lambda| = \sum_i \lambda_i = \sum_i i m_i, \quad \ell(\lambda) = \max_i \{\lambda_i > 0\}, \tag{2.164}$$

dominance (partial)ordering: $\lambda \ge \mu$

$$\Leftrightarrow |\lambda| = |\mu|, \quad \lambda_1 + \cdots + \lambda_i = \mu_1 + \cdots + \mu_i \quad (\forall i), \tag{2.165}$$

monomial symmetric function: $m_\lambda = \sum_\alpha x^\alpha, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots,$

$$\alpha : \text{all distinct permutation of } \lambda = (\lambda_1, \lambda_2, \cdots), \tag{2.166}$$

power sum symmetric function: $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}, \quad p_n = \sum_i x_i^n,$ (2.167)

inner product: $\langle p_\lambda, p_\mu \rangle_\beta = \delta_{\lambda,\mu} \prod_i i^{m_i} m_i! \cdot \beta^{-\sum_i m_i},$ (2.168)

The Jack symmetric polynomial $J_\lambda = J_\lambda(x;\beta)$ is uniquely determined by the following two conditions [39, 38],

$$\text{(i)} \quad J_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} m_\mu(x), \quad u_{\lambda,\lambda} = 1, \tag{2.169}$$

$$\text{(ii)} \quad \langle J_\lambda, J_\mu \rangle_\beta = 0 \quad \text{if } \lambda \ne \mu. \tag{2.170}$$

The condition (ii) can be replaced by (ii)',

$$\text{(ii)'} \quad H_\beta J_\lambda = \varepsilon_{\beta,\lambda} J_\lambda, \quad \varepsilon_{\beta,\lambda} = \sum_{i=1}^{N_0} \Big(\lambda_i^2 + \beta(N_0 + 1 - 2i)\lambda_i \Big). \tag{2.171}$$

Therefore excited states of Calogero-Sutherland model are described by the Jack symmetric polynomials. Using properties of the Jack symmetric polynomial, some dynamical correlation functions were calculated [40].

There exists another inner product,

$$\langle f, g \rangle'_{l,\beta} = \frac{1}{l!} \oint \prod_{j=1}^{l} \underline{dx}_j \cdot \bar{\Delta}(x) f(\bar{x}) g(x), \tag{2.172}$$

$$\bar{\Delta}(x) = \prod_{i\ne j} \Big(1 - \frac{x_j}{x_i}\Big)^\beta, \quad f(\bar{x}) = f\Big(\frac{1}{x_1}, \frac{1}{x_2}, \cdots\Big), \quad \underline{dx}_j = \frac{dx_j}{2\pi i x_j}. \tag{2.173}$$

This is a usual inner product in quantum mechanics. These two inner products are proportional,

$$\langle\ ,\ \rangle'_{l,\beta} \propto \langle\ ,\ \rangle_\beta. \tag{2.174}$$

Let us introduce two transformations for symmetric polynomials,

$$\mathcal{G}_k \quad : \quad \Big(\mathcal{G}_k f\Big)(x_1, \cdots, x_l) = \prod_{i=1}^{l} x^k \cdot f(x_1, \cdots, x_l), \tag{2.175}$$

$$\mathcal{N}_{l',l} \quad : \quad \Big(\mathcal{N}_{l',l} f\Big)(x'_1, \cdots, x'_{l'}) = \oint \prod_{j=1}^{l} \underline{dx}_j \cdot \bar{\Pi}(x', \bar{x})\bar{\Delta}(x) f(x_1, \cdots, x_l), \tag{2.176}$$

$$\bar{\Pi}(x, y) = \prod_i \prod_j (1 - x_i y_j)^{-\beta}. \tag{2.177}$$

Then the Jack symmetric polynomial satisfies the following two properties,

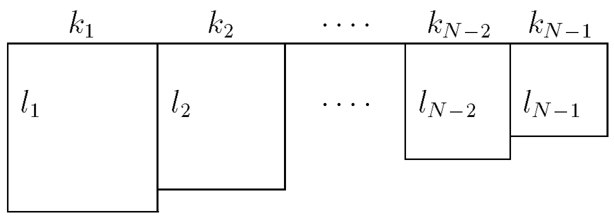
$$J_{(k^l)+\lambda} = \mathcal{G}_k J_\lambda \quad (l \text{ variables}), \tag{2.178}$$

$$J_\lambda = \frac{\langle J_\lambda, J_\lambda\rangle_\beta}{l!\langle J_\lambda, J_\lambda\rangle'_{l,\beta}} \mathcal{N}_{l',l} J_\lambda \quad \begin{array}{l} (J_\lambda \text{ in LHS} : l' \text{ variables}) \\ (J_\lambda \text{ in RHS} : l \text{ variables}) \end{array}. \tag{2.179}$$

By successive action of these two transformations to $J_\phi(x) = 1$, an integral representation of the Jack symmetric polynomial can be obtained [37, 32],

$$J_\lambda(x) \propto \mathcal{N}_{l,l_1}\mathcal{G}_{k_1}\mathcal{N}_{l_1,l_2}\mathcal{G}_{k_2} \cdots \mathcal{N}_{l_{N-2},l_{N-1}}\mathcal{G}_{k_{N-1}}\mathcal{N}_{l_{N-1},0} \cdot 1, \tag{2.180}$$

where the partition λ is $\lambda' = ((l_1)^{k_1}, (l_2)^{k_2}, \cdots, (l_{N-1})^{k_{N-1}})$, namely corresponds to the following Young diagram,

$\lambda =$ 

.

For example, the Jack symmetric polynomial with a rectangular Young diagram (k^l) is

$$J_{(k^l)}(x) \propto \oint \prod_{j=1}^{l} \underline{dz}_j \cdot \prod_{j=1}^{l} \exp\Big(\beta \sum_{n>0} \frac{1}{n} z_j^n \sum_i x_i^n\Big) \cdot \prod_{i \neq j} \Big(1 - \frac{z_j}{z_i}\Big)^\beta \cdot \prod_{j=1}^{l} z_j^{-k}. \tag{2.181}$$

We will show the relation between the Jack symmetric polynomials and the Virasoro singular vectors [32]. States in the Fock space and symmetric polynomials have one-to-one correspondence,

$$\begin{array}{rcl} \mathcal{F}_\alpha & \longrightarrow & \{\text{symmetric function}\} \\ |f\rangle & \mapsto & f(x) = \langle\alpha| \exp\Big(\frac{1}{2}\sqrt{\beta}\sum_{n>0}\frac{1}{n} a_n p_n\Big)|f\rangle. \end{array} \tag{2.182}$$

Namely oscillator a_n and power sum p_n correspond as,

$$\frac{a_{-n}}{\sqrt{\beta}} \leftrightarrow p_n, \quad \frac{a_n}{\sqrt{\beta}} \leftrightarrow \frac{2n}{\beta}\frac{\partial}{\partial p_n}. \tag{2.183}$$

Using this correspondence, Hamiltonian H_β (2.162), which is a differential operator with respect to x_j (or p_n), is bosonized,

$$H_\beta \langle\alpha| \exp\Big(\frac{1}{2}\sqrt{\beta}\sum_{n>0}\frac{1}{n}a_n p_n\Big) = \langle\alpha| \exp\Big(\frac{1}{2}\sqrt{\beta}\sum_{n>0}\frac{1}{n}a_n p_n\Big)\hat{H}_\beta. \tag{2.184}$$

Bosonized Hamiltonian $\hat{H}_\beta$ is an operator on $\mathcal{F}_\alpha$. It is cubic in a_n and can be rewritten by using the Virasoro generator L_n (2.56),

$$\begin{aligned}\hat{H}_\beta &= \tfrac{1}{4}\sqrt{\beta}\sum_{n,m>0}(a_{-n-m}a_n a_m + 2a_{-n}a_{-m}a_{n+m}) + \tfrac{1}{2}\sum_{n>0}a_{-n}a_n\Big((1-\beta)n + N_0\beta\Big) \\ &= \sqrt{\beta}\sum_{n>0}a_{-n}L_n + \tfrac{1}{2}\sum_{n>0}a_{-n}a_n\Big(N_0\beta + \beta - 1 - \sqrt{\beta}a_0\Big).\end{aligned} \tag{2.185}$$

When $\hat{H}_\beta$ acts on the singular vector (2.80), the first term of $\hat{H}_\beta$ vanishes because of the property of singular vector, and the second term is already diagonal,

$$\hat{H}_\beta|\chi_{-l,-k}\rangle = lk\Big(N_0\beta + \beta - 1 - \sqrt{\beta}\alpha_{-l,-k}\Big)|\chi_{-l,-k}\rangle = \varepsilon_{\beta,(k^l)}|\chi_{-l,-k}\rangle. \tag{2.186}$$

Therefore $|\chi_{-l,-k}\rangle$ is an eigenstate of $\hat{H}_\beta$, i.e., it gives the Jack symmetric polynomial by the map (2.182). In fact the polynomial obtained from (2.80) by (2.182) agrees with the integral representation of the Jack symmetric polynomial with the partition (k^l), (2.181).

The Jack symmetric polynomial with general partition whose Young diagram is composed of $N-1$ rectangles is related to the singular vector of W_N algebra [32].

3 Deformed Virasoro Algebra ($A_1^{(1)}$ type)

3.1 Definition and consistency

Definition Deformed Virasoro algebra(DVA) ($A_1^{(1)}$ type) is an associative algebra over $\mathbb{C}$ generated by T_n ($n \in \mathbb{Z}$) with two parameters x and r, and their relation is [17] (see (3.128) for correspondence of parameters)

$$[T_n, T_m] = -\sum_{\ell=1}^{\infty} f_\ell (T_{n-\ell}T_{m+\ell} - T_{m-\ell}T_{n+\ell}) - (x-x^{-1})^2[r]_x[r-1]_x[2n]_x\delta_{n+m,0}, \tag{3.1}$$

where the structure constants f_ℓ is given by

$$\begin{aligned} f(z) &= \sum_{\ell=0}^{\infty} f_\ell z^\ell = \exp\Big(-\sum_{n>0}\frac{z^n}{n}\frac{(x^{rn}-x^{-rn})(x^{(r-1)n}-x^{-(r-1)n})}{x^n+x^{-n}}\Big) \\ &= \frac{1}{1-z}\frac{(x^{2r}z, x^{-2(r-1)}z; x^4)_\infty}{(x^{2+2r}z, x^{2-2(r-1)}z; x^4)_\infty}.\end{aligned} \tag{3.2}$$

Here we have used the notation (A.1), (A.3). By introducing DVA current $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$, the above relation can be written as a formal power series,

$$f(\tfrac{w}{z})T(z)T(w) - T(w)T(z)f(\tfrac{z}{w}) = -(x-x^{-1})[r]_x[r-1]_x\Big(\delta(x^2\tfrac{w}{z}) - \delta(x^{-2}\tfrac{w}{z})\Big). \quad (3.3)$$

For later use we add a grading operator d,

$$[d, T_n] = -nT_n. \quad (3.4)$$

The above relation (3.1) is invariant under

$$T_n \mapsto -T_n. \quad (3.5)$$

It is also invariant under the following two transformations:

$$\text{(i)} \quad \theta: \quad x \mapsto x^{-1}, \quad r \mapsto r, \quad (3.6)$$

$$\text{(ii)} \quad \omega: \quad x \mapsto x, \quad r \mapsto 1-r. \quad (3.7)$$

In the case of (i) $f(z)$ is understood as the first line of (3.2). Let us introduce β as

$$\beta = \frac{r}{r-1}, \quad (3.8)$$

then

$$\theta\cdot\beta = \beta, \quad \omega\cdot\beta = \beta^{-1}. \quad (3.9)$$

α_0 in (2.19) is

$$\alpha_0 = \frac{1}{\sqrt{r(r-1)}}. \quad (3.10)$$

Consistency In subsection 2.2.1 the central term of the Virasoro algebra is determined by the Jacobi identity. Here we will show that the structure function $f(z)$ is determined by associativity [41].

Let us consider the following relation,

$$f(\tfrac{w}{z})T(z)T(w) - T(w)T(z)f(\tfrac{z}{w}) = c_0\Big(\delta(x^2\tfrac{w}{z}) - \delta(x^{-2}\tfrac{w}{z})\Big), \quad (3.11)$$

where c_0 is a normalization constant and x is a parameter, and $f(z)$ is an unknown Taylor series $f(z) = \sum_{\ell=0}^{\infty} f_\ell z^\ell$. By using this relation, $f(\frac{z_2}{z_1})f(\frac{z_3}{z_1})f(\frac{z_3}{z_2})T(z_1)T(z_2)T(z_3)$ is related to $f(\frac{z_1}{z_2})f(\frac{z_1}{z_3})f(\frac{z_2}{z_3})T(z_3)T(z_2)T(z_1)$ in two ways,

$$\begin{array}{ccccc} (123) & \rightarrow & (132) & \rightarrow & (312) \\ \downarrow & & & & \downarrow \\ (213) & \rightarrow & (231) & \rightarrow & (321) \end{array}.$$

These two results should agree. So we obtain an equation containing delta functions,

$$c_0T(z_1)\Big(\delta(x^2\tfrac{z_3}{z_2})-\delta(x^{-2}\tfrac{z_3}{z_2})\Big)\Big(f(\tfrac{z_2}{z_1})f(\tfrac{z_3}{z_1})-f(\tfrac{z_1}{z_2})f(\tfrac{z_1}{z_3})\Big)+\text{cyclic}=0. \tag{3.12}$$

This is equivalent to

$$c_0T(z_1)\Big(\delta(x^2\tfrac{z_3}{z_2})g(\tfrac{z_2}{z_1})-\delta(x^{-2}\tfrac{z_3}{z_2})g(\tfrac{z_3}{z_1})\Big)+\text{cyclic}=0, \tag{3.13}$$

where $g(z)$ is

$$g(z)=f(z)f(x^{-2}z)-f(z^{-1})f(x^2z^{-1})=-g(x^2z^{-1}). \tag{3.14}$$

In mode expansion $g(z)=\sum_{n\in\mathbb{Z}}g_nz^n$, this equation becomes

$$(x^{2n}-x^{2m})g_{n+m}+(x^{2m}-x^{2l})g_{m+l}+(x^{2l}-x^{2n})g_{l+n}=0, \tag{3.15}$$

$$g_{-n}=-x^{2n}g_n. \tag{3.16}$$

From (3.16), we have $g_0=0$. From (3.15) with $(m,l)=(n-1,1-n)$, $(m,l)=(n-2,2-n)$, we have $g_{2n-1}=\frac{1-x^{-2(2n-1)}}{1-x^{-2}}g_1$ and $g_{2n-2}=\frac{1-x^{-2(2n-2)}}{1-x^{-4}}g_2$. And from (3.15) with $(m,l)=(n-1,n+1)$ we have $g_2=(1+x^{-2})g_1$. Combining these we get $g_n=(1-x^{-2n})g_1$ and this satisfies both (3.15) and (3.16). Therefore the solution is

$$g_n=c_0'(1-x^{-2n}),\quad g(z)=c_0'\Big(\delta(z)-\delta(x^{-2}z)\Big), \tag{3.17}$$

where c_0' is a constant. By setting $F(z)=f(z)f(x^2z)$, (3.14) and (3.17) become

$$F(z)-F(x^{-2}z^{-1})=-c_0'\sum_{n\in\mathbb{Z}}(1-x^{2n})z^n. \tag{3.18}$$

Since $F(z)$ is also a Taylor series, this equation implies

$$F(z)=-c_0'\Big(\alpha+\sum_{n>0}(1-x^{2n})z^n\Big), \tag{3.19}$$

where a new parameter α has appeared. If we express this α by a new parameter r as

$$\alpha=\frac{x-x^{-1}}{(x^r-x^{-r})(x^{r-1}-x^{-(r-1)})}, \tag{3.20}$$

then $F(z)$ becomes

$$F(z)=-c_0'\alpha\frac{(1-x^{2r}z)(1-x^{-2(r-1)})}{(1-z)(1-x^2z)}. \tag{3.21}$$

From this equation $f(z)$ is calculated as ($|x|<1$)

$$\begin{aligned} f(z) &= \frac{F(z)}{F(x^2z)}f(x^4z)=\frac{F(z)}{F(x^2z)}\frac{F(x^4z)}{F(x^6z)}\cdots\times f(0)\\ &= f(0)\exp\Big(\sum_{n>0}\frac{z^n}{n}\frac{(1-x^{2rn})(1-x^{-2(r-1)n})}{1+x^{2n}}\Big). \end{aligned} \tag{3.22}$$

We choose the normalization of $f(z)$ as $f(0)=1$ ($c_0'=-\alpha^{-1}$), then this $f(z)$ is just (3.2). We fix the normalization of $T(z)$ by $c_0=-\alpha^{-1}$.

3.2 Conformal limit

Quantum group(algebra) $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$ and it reduces to undeformed one in the $q \to 1$ limit,

$$U_q(\mathfrak{g}) \xrightarrow{q\to 1} U(\mathfrak{g}).$$

The deformed Virasoro algebra defined in the previous subsection is a deformation of the Virasoro algebra as expected from its name. Then in what limit it reduces to the usual Virasoro algebra?

$$\mathrm{DVA} \xrightarrow{??} \mathrm{Vir}.$$

Since DVA contains two parameters x and r (or β (3.8)), it admits various limits[41]. In this subsection we will show that DVA reduces to the Virasoro algebra in the conformal limit.

The conformal limit is

$$x \to 1, \quad r \text{ (or } \beta) : \text{ fixed.} \tag{3.23}$$

To study this limit we write x as

$$x = e^{-\frac{1}{2}\alpha_0 \hbar}, \tag{3.24}$$

where α_0 is given in (2.19) and $\hbar$ is a fictitious Plank constant, and take $\hbar \to 0$. $f(z)$ has the expansion,

$$f(z) = 1 + \hbar^2 f^{[2]}(z) + \hbar^4 f^{[4]}(z) + \cdots. \tag{3.25}$$

Taking into account the invariance under (3.6), we assume the expansion

$$T(z) = 2 + \hbar^2 T^{[2]}(z) + \hbar^4 T^{[4]}(z) + \hbar^6 T^{[6]}(z) + \cdots, \tag{3.26}$$

(Remark that $-T(z)$ is also a solution.) and we set

$$T^{[2]}(z) = z^2 L(z) + \tfrac{1}{4}\alpha_0^2. \tag{3.27}$$

Then LHS of (3.3) is

$$\begin{aligned}
&\hbar^2 4\Big(f^{[2]}(\zeta) - f^{[2]}(\zeta^{-1})\Big) + \hbar^4 \bigg(\Big[T^{[2]}(z), T^{[2]}(w)\Big] \\
&\quad + 2\Big(f^{[2]}(\zeta) - f^{[2]}(\zeta^{-1})\Big)\Big(T^{[2]}(z) + T^{[2]}(w)\Big) + 4\Big(f^{[4]}(\zeta) - f^{[4]}(\zeta^{-1})\Big)\bigg) + \cdots \\
= &\ \hbar^2\Big(-2\zeta\delta'(\zeta)\Big) + \hbar^4 \bigg(\Big[z^2 L(z), w^2 L(w)\Big] \\
&\quad -\zeta\delta'(\zeta)\Big(z^2 L(z) + w^2 L(w)\Big) - \zeta\delta'(\zeta)\frac{1+2\alpha_0^2}{6} - \zeta^2\delta'''(\zeta)\frac{1-2\alpha_0^2}{12}\bigg) + \cdots,
\end{aligned} \tag{3.28}$$

where $\zeta = \frac{w}{z}$. On the other hand RHS of (3.3) is

$$\hbar^2\Big(-2\zeta\delta'(\zeta)\Big) + \hbar^4\bigg(-\zeta\delta'(\zeta)\frac{1+2\alpha_0^2}{6} - \zeta^2\delta'''(\zeta)\frac{\alpha_0^2}{3}\bigg) + \cdots. \tag{3.29}$$

Comparing these equations and (2.5) shows that $L(z)$ in (3.27) is the Virasoro current with the central charge (2.17).

Therefore the Virasoro current $L(z)$ is found in $\hbar^2$ term of the DVA current $T(z)$. And also (conformal)spin 4 current $T^{[4]}(z)$, spin 6 current $T^{[6]}(z)$, $\cdots$ exist in the DVA current. In massless case only one Virasoro current $L(z)$ controls (chiral part of)CFT. In massive case, however, $L(z)$ and infinitely many higher spin currents $T^{[2n]}(z)$ are needed to control massive theory, and they gather and form the DVA current $T(z)$. DVA current is a 'dressed' Virasoro current.

3.3 Representation theory

Let us consider the highest weight representation. The highest weight state $|\lambda\rangle$ ($\lambda \in \mathbb{C}$) is characterized by

$$T_n|\lambda\rangle = 0 \quad (n > 0), \qquad T_0|\lambda\rangle = \lambda|\lambda\rangle, \tag{3.30}$$

and the Verma module is

$$M = \bigoplus_{l \geq 0} \bigoplus_{n_1 \geq \cdots \geq n_l > 0} \mathbb{C} T_{-n_1} \cdots T_{-n_l}|\lambda\rangle. \tag{3.31}$$

M is a graded module with grading d,

$$d|\lambda\rangle = d_\lambda|\lambda\rangle \ (d_\lambda \in \mathbb{C}), \quad d \cdot T_{-n_1} \cdots T_{-n_l}|\lambda\rangle = \Big(d_\lambda + \sum_{i=1}^{l} n_l\Big) T_{-n_1} \cdots T_{-n_l}|\lambda\rangle,$$

and we call $\sum_{i=1}^{l} n_l$ as a level.

To obtain the irreducible module from the Verma module we have to quotient out invariant submodules, which are generated by singular vectors. Singular vector at level N, $|\chi\rangle$, is

$$T_n|\chi\rangle = 0 \quad (n > 0), \quad T_0|\chi\rangle = \lambda_\chi|\chi\rangle, \quad d|\chi\rangle = (d_\lambda + N)|\chi\rangle, \tag{3.32}$$

where λ_χ is some eigenvalue. Existence of singular vectors is detected by zeros of the Kac determinant. To define the Kac determinant let us introduce the dual module M^* on which the DVA act as $T_n^\dagger = T_{-n}$. M^* is generated by $\langle\lambda|$ which satisfies $\langle\lambda|T_n = 0$ ($n < 0$), $\langle\lambda|T_0 = \lambda\langle\lambda|$ and $\langle\lambda|\lambda\rangle = 1$. At level N there are $p(N)$ states $|\lambda; N, 1\rangle = T_{-N}|\lambda\rangle$, $|\lambda; N, 2\rangle = T_{-N+1}T_{-1}|\lambda\rangle$, $\cdots$, $|\lambda; N, p(N)\rangle = T_{-1}^N|\lambda\rangle$ in M, and $\langle\lambda; N, 1| = \langle\lambda|T_N$, $\langle\lambda; N, 2| = \langle\lambda|T_1 T_{N-1}$, $\cdots$, $\langle\lambda; N, p(N)| = \langle\lambda|T_1^N$ in M^*.

The Kac determinant at level N is given by [17, 42]

$$\det\langle\lambda; N, i|\lambda; N, j\rangle = \prod_{\substack{l,k \geq 1 \\ lk \leq N}} \left(\frac{(x^{rl} - x^{-rl})(x^{(r-1)l} - x^{-(r-1)l})}{x^l + x^{-l}} (\lambda^2 - \lambda_{l,k}^2) \right)^{p(N-lk)}, \tag{3.33}$$

where $\lambda_{l,k}$ is

$$\lambda_{l,k}=x^{rl-(r-1)k}+x^{-rl+(r-1)k}. \tag{3.34}$$

λ dependence appears through λ^2 because of the symmetry (3.5). Conformal limit (3.23) of this determinant is

$$\begin{aligned}&\prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(\frac{\hbar^2 l^2}{2}\cdot\hbar^2(h-h_{l,k})\cdot 4\Big)^{p(N-lk)}+\cdots,\\ =\ &\hbar^A\prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(2lk(h-h_{l,k})\Big)^{p(N-lk)}+\cdots,\quad A=4\sum_{\substack{l,k\geq 1\\ lk\leq N}}p(N-lk).\end{aligned} \tag{3.35}$$

This is just the Kac determinant of the Virasoro algebra (2.16), and the order of $\hbar$ is consistent with (3.26) (use (A.30)).

For generic values of x and r, this Kac determinant (3.33) has essentially the same structure as the Virasoro one (2.16). Therefore embedding pattern is same as (2.27), and the character which counts the degeneracy at each level ($\mathrm{tr}\, q^d$) is also the same. For special value of x and r, for example x^r is a root of unity, the Kac determinant has more zeros. In this case we need special study, see [42].

3.4 Free field realization

3.4.1 free field realization

Let us introduce free boson oscillator h_n ($n\in\mathbb{Z}_{\neq 0}$),

$$[h_n,h_m]=(x-x^{-1})^2\frac{1}{n}\frac{[n]_x[rn]_x[(r-1)n]_x}{[2n]_x}\delta_{n+m,0}, \tag{3.36}$$

and use zero mode a_0 and Q defined in (2.49) (or a'_0 in (2.72)). The Fock space $\mathcal{F}_\alpha$ is defined by

$$\mathcal{F}_\alpha=\bigoplus_{l\geq 0}\bigoplus_{n_1\geq\cdots\geq n_l>0}\mathbb{C}h_{-n_1}\cdots h_{-n_l}|\alpha\rangle_B, \tag{3.37}$$

where $|\alpha\rangle_B$ is given by (2.51) with replacing a_n by h_n.

The DVA current $T(z)$ is realized as follows:

$$\begin{aligned}&T(z)=\Lambda_+(z)+\Lambda_-(z),\\ &\Lambda_\pm(z)=:\exp\Big(\pm\sum_{n\neq 0}h_n(x^{\pm 1}z)^{-n}\Big):\times x^{\pm\sqrt{r(r-1)}a'_0}.\end{aligned} \tag{3.38}$$

To prove this we need the OPE formula,

$$\begin{aligned}f(\tfrac{w}{z})\Lambda_\pm(z)\Lambda_\pm(w)\ &=\ :\Lambda_\pm(z)\Lambda_\pm(w):,\\ f(\tfrac{w}{z})\Lambda_\pm(z)\Lambda_\mp(w)\ &=\ :\Lambda_\pm(z)\Lambda_\mp(w):\gamma(x^{\mp 1}\tfrac{w}{z}),\end{aligned} \tag{3.39}$$

and the relation of $\Lambda_\pm$,

$$:\Lambda_+(x^{-1}z)\Lambda_-(xz):=1. \tag{3.40}$$

Here $\gamma(z)$ is

$$\gamma(z)=\frac{(1-x^{2r-1}z)(1-x^{-(2r-1)}z)}{(1-xz)(1-x^{-1}z)}, \tag{3.41}$$

and from (A.24) we have

$$\gamma(z)-\gamma(z^{-1})=-(x-x^{-1})[r]_x[r-1]_x\,\Big(\delta(xz)-\delta(x^{-1}z)\Big). \tag{3.42}$$

The grading operator d is realized by

$$\begin{aligned} d&=d^{\mathrm{osc}}+d^{\mathrm{zero}},\\ d^{\mathrm{osc}}&=\sum_{n>0}\frac{n^2[2n]_x}{(x-x^{-1})^2[n]_x[rn]_x[(r-1)n]_x}h_{-n}h_n,\quad d^{\mathrm{zero}}=\frac{1}{4}{a'_0}^2-\frac{1}{24}, \end{aligned} \tag{3.43}$$

which satisfies

$$[d,h_n]=-nh_n,\quad [d,Q]=a'_0,\quad d|\alpha_{l,k}\rangle_B=(h_{l,k}-\tfrac{c}{24})|\alpha_{l,k}\rangle_B, \tag{3.44}$$

where c and $h_{l,k}$ are given by (2.17) and (2.18) respectively.

$|\alpha\rangle_B$ is the highest weight state of DVA with $\lambda=\lambda(\alpha)$,

$$|\alpha\rangle_B=|\lambda(\alpha)\rangle, \tag{3.45}$$

$$\lambda(\alpha)=x^{\sqrt{r(r-1)}(\alpha-\alpha_0)}+x^{-\sqrt{r(r-1)}(\alpha-\alpha_0)}. \tag{3.46}$$

The dual space $\mathcal{F}^*_\alpha$ becomes a DVA module by (2.63) with

$${}^tT_n\;=\;T_{-n}, \tag{3.47}$$

$${}^th_n\;=\;-h_{-n},\quad \text{eq.(2.66)}, \tag{3.48}$$

and (2.67). By (3.48), $\mathcal{F}^*_\alpha$ is isomorphic to $\mathcal{F}_{2\alpha_0-\alpha}$ as DVA module,

$$\mathcal{F}^*_\alpha\;\cong\;\mathcal{F}_{2\alpha_0-\alpha}\quad \text{(DVA module)}. \tag{3.49}$$

$$|\alpha\rangle^*_B\;\leftrightarrow\;|2\alpha_0-\alpha\rangle_B. \tag{3.50}$$

Remark that $\lambda(\alpha)=\lambda(2\alpha_0-\alpha)$ by (3.46).

In the conformal limit (3.23), oscillator h_n is expressed by a_n in (2.49) as follows:

$$h_n=\frac{\hbar}{2}\sqrt{\frac{2}{x^n+x^{-n}}\frac{x^{rn}-x^{-rn}}{rn\alpha_0\hbar}\frac{x^{(r-1)n}-x^{-(r-1)n}}{(r-1)n\alpha_0\hbar}}\,a_n. \tag{3.51}$$

Substituting this expression into (3.38) and expanding in $\hbar$, we get (3.26) with $L(z)$ in (2.55) (or (2.73)).

3.4.2 singular vectors and Kac determinant

In the case of $\lambda = \lambda_{l,k}$ there are singular vectors. In the free boson realization they can be expressed by using the screening currents. For later convenience we denote screening currents as $x_\pm(z)$. $x_\pm(z)$ is defined by

$$x_+(z) = \,:\exp\Big(-\sum_{n\neq 0}\frac{\alpha_n}{[n]_x}z^{-n}\Big):\times e^{\sqrt{\frac{r}{r-1}}Q}z^{\sqrt{\frac{r}{r-1}}a'_0+\frac{r}{r-1}}, \tag{3.52}$$

$$x_-(z) = \,:\exp\Big(\sum_{n\neq 0}\frac{\alpha'_n}{[n]_x}z^{-n}\Big):\times e^{-\sqrt{\frac{r-1}{r}}Q}z^{-\sqrt{\frac{r-1}{r}}a'_0+\frac{r-1}{r}}, \tag{3.53}$$

where oscillators α_n, α'_n $(n\in\mathbb{Z}_{\neq 0})$ are related to h_n as

$$h_n = (x-x^{-1})(-1)^n\frac{[(r-1)n]_x}{[2n]_x}\alpha_n = (x-x^{-1})\frac{[rn]_x}{[2n]_x}\alpha'_n. \tag{3.54}$$

Conformal limit (3.23) of $x_-(z)$ is $zS_-(z)$ in (2.76) and that of $x_+(z)$ is $zS_+(-z)$ (up to phase) due to $(-1)^n$ factor in (3.54).

Commutation relation of DVA generator and screening currents is a total difference

$$[T_n, x_+(w)] = (x^r - x^{-r})\Big((-x^{r-1}w)^n A_+(x^{r-1}w) - (-x^{-(r-1)}w)^n A_+(x^{-(r-1)}w)\Big), \tag{3.55}$$

$$[T_n, x_-(w)] = (x^{r-1} - x^{-(r-1)})\Big((x^r w)^n A_-(x^r w) - (x^{-r}w)^n A_-(x^{-r}w)\Big), \tag{3.56}$$

where $A_\pm(w)$ is

$$A_+(w) = x^{\pm r} :\Lambda_\pm(-w)x_+(x^{\mp(r-1)}w):, \tag{3.57}$$

$$A_-(w) = x^{\pm(r-1)} :\Lambda_\mp(w)x_-(x^{\mp r}w): . \tag{3.58}$$

Hence their zero modes (screening charge) commute with DVA,

$$\Big[T_n, \oint_0 \frac{dz}{2\pi i z}x_\pm(z)\Big] = 0. \tag{3.59}$$

(Exactly speaking we have to specify the Fock space on which they act. See subsection 3.4.3.) We will use $x_+(z)$ in what follows. $x_-(z)$ can be treated similarly.

The representation with $\lambda = \lambda_{l,k}$ is realized on $\mathcal{F}_{l,k} = \mathcal{F}_{\alpha_{l,k}}$,

$$\lambda_{l,k} = \lambda(\alpha_{l,k}), \tag{3.60}$$

where $\alpha_{l,k}$ is given in (2.79). To contact with our previous paper [17, 18] ($\alpha_{r,s}$ there is $\propto \alpha_{-r,-s}$ here.), we consider dual space $\mathcal{F}^*_{l,k} = \mathcal{F}_{-l,-k}$. Like as the Virasoro case (2.80), singular vectors of DVA are expressed by product of screening charges. Moreover this product becomes more clear for the deformed case, if we include Lukyanov's zero mode factor (see subsection 3.4.3). In the Verma module with $\lambda_{-l,-k}$, the singular vector at level lk is expressed as

$$
\begin{aligned}
|\chi_{-l,-k}\rangle &= Q_l|\alpha_{l,-k}\rangle_B \qquad (3.61)\\
&= \int\prod_{j=1}^{l}\underline{dz}_j\cdot\prod_{\substack{i,j=1\\ i\neq j}}^{l}\frac{(\frac{z_i}{z_j};x^{2(r-1)})_\infty}{(x^{2r}\frac{z_i}{z_j};x^{2(r-1)})_\infty}\cdot C(z)\prod_{i=1}^{l}z_i^{-k}\cdot\prod_{j=1}^{l}e^{\sum_{n>0}\frac{1}{[n]_x}\alpha_{-n}z_j^n}|\alpha_{-l,-k}\rangle_B,
\end{aligned}
$$

where $\underline{dz}_j$ is given in (A.14), the BRST charge Q_l will be given in (3.77), and $C(z)$ is

$$
\begin{aligned}
C(z) &= \prod_{1\leq i<j\leq l}\frac{(x^{2r}\frac{z_i}{z_j};x^{2(r-1)})_\infty}{(\frac{z_i}{z_j};x^{2(r-1)})_\infty}\frac{(x^{-2}\frac{z_j}{z_i};x^{2(r-1)})_\infty}{(x^{2(r-1)}\frac{z_j}{z_i};x^{2(r-1)})_\infty}\cdot\prod_{i=1}^{l}z_i^{\frac{r}{r-1}(l+1-2i)}\\
&\quad\times\prod_{i=1}^{l}\frac{[-u_i+\frac{1}{2}-(2i-l)]^*}{[-u_i-\frac{1}{2}]^*}. \qquad (3.62)
\end{aligned}
$$

Here we have used the notation (A.6) with $r^*=r-1$ and $z_i=x^{2u_i}$. Since the BRST charge Q_l, which is a map from $\mathcal{F}_{l,-k}$ to $\mathcal{F}_{-l,-k}$, commutes with T_n, $|\chi_{-l,-k}\rangle$ is annihilated by T_n ($n>0$). This state $|\chi_{-l,-k}\rangle$ is non zero because its conformal limit gives non zero state (2.80).

Like as in subsection 2.3.2 let us introduce matrices $C(N,\alpha)$ and $C'(N,\alpha)$,

$$
T_{-I}|\alpha\rangle_B = \sum_J C(N,\alpha)_{I,J}h_{-J}|\alpha\rangle_B, \qquad (3.63)
$$

$$
({}^tT)_I|\alpha\rangle_B^* = \sum_J C'(N,\alpha)_{I,J}({}^th)_J|\alpha\rangle_B^*. \qquad (3.64)
$$

C' can be expressed by C as (2.92) with (2.91). Their determinants are given by

$$
\det C(N,\alpha)_{I,J} = \prod_{\substack{l,k\geq 1\\ lk\leq N}}\left(x^{\sqrt{r(r-1)}(\alpha-\alpha_{l,k})}-x^{-\sqrt{r(r-1)}(\alpha-\alpha_{l,k})}\right)^{p(N-lk)}, \qquad (3.65)
$$

$$
\begin{aligned}
\det C'(N,\alpha)_{I,J} &= \det C(N,2\alpha_0-\alpha)\cdot\det D\\
&= \prod_{\substack{l,k\geq 1\\ lk\leq N}}\left(x^{\sqrt{r(r-1)}(\alpha+\alpha_{l,k}-2\alpha_0)}-x^{-\sqrt{r(r-1)}(\alpha+\alpha_{l,k}-2\alpha_0)}\right)^{p(N-lk)}. \qquad (3.66)
\end{aligned}
$$

The inner product of two states in the Verma module becomes (see (2.94))

$$
\begin{aligned}
\langle\lambda|T_IT_{-J}|\lambda\rangle &= <({}^tT)_I|\alpha\rangle_B^*, T_{-J}|\alpha\rangle_B>\\
&= \sum_{K,L}C'(N,\alpha)_{I,K}G_{K,L}C(N,\alpha)_{J,L}. \qquad (3.67)
\end{aligned}
$$

Here $G_{K,L}$ is

$$
\begin{aligned}
G_{K,L} &= <({}^th)_K|\alpha\rangle_B^*, h_{-L}|\alpha\rangle_B>\\
&= \delta_{K,L}\prod_i\left(\frac{1}{i}\frac{(x^{ri}-x^{-ri})(x^{(r-1)i}-x^{-(r-1)i})}{x^i+x^{-i}}\right)^{k_i}k_i!, \qquad (3.68)
\end{aligned}
$$

and its determinant is (use (A.28) and (A.30))

$$\begin{aligned}
\det G_{K,L} &= \prod_{\substack{\{k_i\} \\ \Sigma_i ik_i=N}} \prod_i \Bigl(\frac{1}{i}\frac{(x^{ri}-x^{-ri})(x^{(r-1)i}-x^{-(r-1)i})}{x^i+x^{-i}}\Bigr)^{k_i} k_i! \\
&= \prod_{\substack{l,k\geq 1 \\ lk\leq N}} \Bigl(\frac{(x^{rl}-x^{-rl})(x^{(r-1)l}-x^{-(r-1)l})}{x^l+x^{-l}}\Bigr)^{p(N-lk)}. \qquad (3.69)
\end{aligned}$$

Therefore we obtain the Kac determinant (3.33) [17, 42],

$$\begin{aligned}
\det\langle\lambda|T_I T_{-J}|\lambda\rangle &= \det C'(N,\alpha)\cdot\det G\cdot\det {}^tC(N,\alpha) \\
&= \prod_{\substack{l,k\geq 1 \\ lk\leq N}} \Bigl(\frac{(x^{rl}-x^{-rl})(x^{(r-1)l}-x^{-(r-1)l})}{x^l+x^{-l}}(\lambda^2-\lambda_{l,k}^2)\Bigr)^{p(N-lk)}. \qquad (3.70)
\end{aligned}$$

3.4.3 Felder complex

We consider the representation of $\lambda=\lambda_{l,k}$ in (3.60) with (2.20) and (2.23), i.e.

$$r=\frac{p''}{p''-p'}. \qquad (3.71)$$

We set

$$r^*=r-1.$$

Let us define the screening operator $X(z)$ as the integral of $x_+(z')$ plus extra zero mode factor [43],

$$X(z)=\oint_C \underline{dz'}\, x_+(z')\frac{[u-u'+\frac{1}{2}-\hat{l}\,]^*}{[u-u'-\frac{1}{2}]^*}, \qquad (3.72)$$

where $z=x^{2u}$, $z'=x^{2u'}$, $\underline{dz'}=\frac{dz'}{2\pi i z'}$, and the integration contour C is a simple closed curve that encircles $z'=x^{-1+2(r-1)n}z$ but not $z'=x^{-1-2(r-1)(n+1)}z$ $(n=0,1,2,\cdots)$. $\hat{l}$ is $\hat{l}=l\times\mathrm{id}$ on $\mathcal{F}_{l,k}$ (see subsection 5.2). z is an arbitrary point, for example we take $z=1$.

This $X(z)$ is well defined on $\mathcal{F}_{l',k'}$ $(\forall l',k')$

$$X(z):\mathcal{F}_{l',k'}\mapsto\mathcal{F}_{l'-2,k'}. \qquad (3.73)$$

Product of $X(z)$'s is [43]

$$\begin{aligned}
&X(z)^m \;:\; \mathcal{F}_{l',k'}\mapsto\mathcal{F}_{l'-2m,k'} \\
&= \oint\prod_{j=1}^m \underline{dz}_j\cdot x_+(z_1)\cdots x_+(z_m)\prod_{i=1}^m\frac{[u-u_i+\frac{1}{2}-(\hat{l}-2(m-i))]^*}{[u-u_i-\frac{1}{2}]^*} \\
&= \oint\prod_{j=1}^m \underline{dz}_j\cdot x_+(z_1)\cdots x_+(z_m)
\end{aligned}$$

$$\times \frac{1}{m!}\sum_{\sigma\in S_m}\prod_{i=1}^{m}\frac{[u-u_{\sigma(i)}+\frac{1}{2}-(\hat{l}-2(m-i))]^*}{[u-u_{\sigma(i)}-\frac{1}{2}]^*}\cdot\prod_{\substack{1\le i<j\le m\\ \sigma(i)>\sigma(j)}} h^*(u_{\sigma(i)}-u_{\sigma(j)})$$

$$= \oint\prod_{j=1}^{m}\underline{dz}_j\cdot x_+(z_1)\cdots x_+(z_m)$$

$$\times\frac{1}{m!}\prod_{i=1}^{m}\frac{[i]^*}{[1]^*}\cdot\prod_{1\le i<j\le m}\frac{[u_i-u_j]^*}{[u_i-u_j+1]^*}\cdot\prod_{i=1}^{m}\frac{[u-u_i-\frac{1}{2}+m-\hat{l}\,]^*}{[u-u_i-\frac{1}{2}]^*}, \tag{3.74}$$

where $z=x^{2u}$, $z_i=x^{2u_i}$ and $\underline{dz}_j$ is given in (A.14). Here we have used $x_+(z_1)x_+(z_2)=h^*(u_1-u_2)x_+(z_2)x_+(z_1)$ (as a meromorphic function) with $h^*(u)=\frac{[u+1]^*}{[u-1]^*}$ (see subsection 5.2), and the formula (A.12) with $r\to r-1$. From the factor $[i]^*$ in the last line of (3.74), we have (use (A.10))

$$X(z)^{p'}=0. \tag{3.75}$$

Moreover $X(z)$ has the property,

$$[T_n, X(z)^l]=0 \quad \text{on } \mathcal{F}_{l',k} \quad l'\equiv l \pmod{p'}. \tag{3.76}$$

This can be proved by (3.55) and careful analysis of location of poles [43]. We define the BRST charge Q_m as

$$Q_m = X(1)^m. \tag{3.77}$$

Let us consider the Felder complex $C_{l,k}$,

$$\cdots \xrightarrow{X_{-3}} C_{-2}\xrightarrow{X_{-2}} C_{-1}\xrightarrow{X_{-1}} C_0\xrightarrow{X_0} C_1\xrightarrow{X_1} C_2\xrightarrow{X_2}\cdots, \tag{3.78}$$

where C_j and $X_j\ :\ C_j\to C_{j+1}$ $(j\in\mathbb{Z})$ are

$$C_{2j}=\mathcal{F}_{l-2p'j,k},\quad C_{2j+1}=\mathcal{F}_{-l-2p'j,k}, \tag{3.79}$$

$$X_{2j}=Q_l,\quad X_{2j+1}=Q_{p'-l}. \tag{3.80}$$

X satisfies the BRST property,

$$X_jX_{j-1}=0. \tag{3.81}$$

We assume that this Felder complex has the same structure as the Virasoro case because it formally tends to Virasoro one in the conformal limit ($x\to 1$ and r and $z=x^{2u}$ fixed kept). Then the cohomology groups of the complex $C_{l,k}$ are

$$H^j(C_{l,k})=\text{Ker } X_j/\text{Im } X_{j-1}=\begin{cases}0 & j\neq 0,\\ \mathcal{L}_{l,k} & j=0,\end{cases} \tag{3.82}$$

where $\mathcal{L}_{l,k}$ is the irreducible DVA module of $\lambda=\lambda_{l,k}$. The trace of operator $\mathcal{O}$ over $\mathcal{L}_{l,k}$ can be written as (see (2.122))

$$\mathrm{tr}_{\mathcal{L}_{l,k}}\mathcal{O}=\mathrm{tr}_{H^0(C_{l,k})}\mathcal{O}^{(0)}=\mathrm{tr}_{H^*(C_{l,k})}\mathcal{O}=\sum_{j\in\mathbb{Z}}(-1)^j\mathrm{tr}_{C_j}\mathcal{O}^{(j)}, \tag{3.83}$$

where $\mathcal{O}^{(j)}$ is an operator $\mathcal{O}$ realized on C_j.

If we use $x_-(z)$ instead of $x_+(z)$, another complex $C'_{l,k}$ in which k' of $\mathcal{F}_{l,k'}$ changes is obtained. The corresponding screening operator is

$$X'(z) = \oint_{C'} \underline{dz}' x_-(z') \frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]}, \tag{3.84}$$

where $[u]$ in (A.5), $z = x^{2u}$, $z' = x^{2u'}$, $\underline{dz}' = \frac{dz'}{2\pi i z'}$, and the integration contour C' is a simple closed curve that encircles $z' = x^{1+2rn}z$ but not $z' = x^{1-2r(n+1)}z$ $(n = 0, 1, 2, \cdots)$. $\hat{k}$ is $\hat{k} = k \times \mathrm{id}$ on $\mathcal{F}_{l,k}$ (see subsection 5.2). $X'(z)$ is well defined on $\mathcal{F}_{l',k'}$ $(\forall l', k')$, $X'(z) : \mathcal{F}_{l',k'} \mapsto \mathcal{F}_{l',k'-2}$, and $X'(z)$ satisfies

$$X'(z)^{p''} = 0, \tag{3.85}$$

$$[T_n, X'(z)^k] = 0 \quad \text{on } \mathcal{F}_{l,k'} \quad k' \equiv k \pmod{p''}. \tag{3.86}$$

By setting the BRST charge as $Q'_m = X'(1)^m$, another Felder complex $C'_{l,k} = \{C'_j, X'_j : C'_j \to C'_{j+1}\}$ is $C'_{2j} = \mathcal{F}_{l,k-2p''j}$, $C'_{2j+1} = \mathcal{F}_{l,-k-2p''j}$, $X'_{2j} = Q'_k$, $X'_{2j+1} = Q'_{p''-k}$, and the cohomology is $H^j(C'_{l,k}) = \mathrm{Ker}\ X'_j / \mathrm{Im}\ X'_{j-1} = \delta_{j,0}\mathcal{L}_{l,k}$.

3.4.4 Trigonometric Ruijsenaars-Schneider model and Macdonald symmetric polynomial

The trigonometric Ruijsenaars-Schneider model (tRSM) is a relativistic version of the Calogero-Sutherland model. The Ruijsenaars-Schneider model is a many body system with sufficiently enough conserved quantities $H_{\pm k}$ $(k = 1, \cdots, N_0)$ [44] (see also [45]),

$$H_{\pm k} = \sum_{\substack{I \subset \{1,\cdots,N_0\} \\ |I|=k}} \prod_{\substack{i\in I \\ j \notin I}} h(\pm(q_i - q_j))^{\frac{1}{2}} \cdot e^{\mp \sum_{i\in I}\theta_i} \cdot \prod_{\substack{i\in I \\ j \notin I}} h(\mp(q_i - q_j))^{\frac{1}{2}}, \tag{3.87}$$

where θ_i $(i = 1, \cdots, N_0)$ is a rapidity and $\bar{q}_i$ is its conjugate variable and $q_i = \dfrac{\bar{q}_i}{mc}$ is a coordinate. Their dimensions are $[\theta_i] = 1$, $[\bar{q}_i] = \hbar$ and $[q_i] = L$. (Exactly speaking θ_i and $\bar{q}_i$ are related to the coordinate q_i and momentum p_i by $\bar{q}_i = mcq_i \cosh\theta_i$ and $p_i = mc\sinh\theta_i$. But we skip this procedure. See [46].) θ_j is

$$\theta_j = \frac{\hbar}{i}\frac{\partial}{\partial \bar{q}_j} = \frac{1}{i}\frac{\hbar}{mc}\frac{\partial}{\partial q_j}, \tag{3.88}$$

and $h(q)$ is

$$h(q) = \frac{\sigma(\bar{q} + i\bar{\beta})}{\sigma(\bar{q})}, \tag{3.89}$$

where $\bar{\beta}$ is a coupling constant and $\sigma(z)$ is the Weierstrass σ function. These conserved quantities commute mutually

$$[H_k, H_l] = 0 \quad (k, l = -N_0, \cdots, N_0). \tag{3.90}$$

This model is a 'relativistic' model because it has Poincaré invariance,

$$\begin{array}{ll} H = mc^2\frac{1}{2}(H_{-1}+H_1), & [H,P]=0, \\ P = mc(H_{-1}-H_1), & [H,B]=i\hbar P, \\ B = -\frac{1}{c}\sum_{i=1}^{N_0}\bar{q}_i, & [P,B]=i\hbar\frac{1}{c^2}H. \end{array} \tag{3.91}$$

In the 'non-relativistic' limit (the speed of light $c\to\infty$), the Hamiltonian H becomes

$$\lim_{c\to\infty}(H-N_0mc^2)=\sum_{j=1}^{N_0}\frac{1}{2m}\Big(\frac{\hbar}{i}\frac{\partial}{\partial q_j}\Big)^2+\frac{1}{m}\bar{\beta}(\bar{\beta}-\hbar)\sum_{1\leq i<j\leq N_0}\wp(q_i-q_j), \tag{3.92}$$

where $\wp(z)$ is the Weierstrass $\wp$ function and we have rescaled the periods of $\sigma(z)$.

tRSM is a trigonometric case of RSM,

$$h(q)=\frac{\sin\frac{\pi}{mcL}(\bar{q}+i\bar{\beta})}{\sin\frac{\pi}{mcL}\bar{q}}. \tag{3.93}$$

Its 'non-relativistic' limit is

$$\lim_{c\to\infty}(H_{tRS}-N_0mc^2)=H_{CS}, \tag{3.94}$$

where H_{CS} is given in (2.154). Let us introducing dimensionless quantities x_j and β in (2.159) and

$$q=e^{-2\pi\frac{\hbar}{mcL}},\quad t=q^{\beta}. \tag{3.95}$$

Removing the contribution from the ground state, we obtain

$$\Delta^{-\frac{1}{2}}H_{\pm k}\Delta^{\frac{1}{2}}=t^{\mp\frac{1}{2}k(N_0-1)}D_k(q^{\pm1},t^{\pm1}). \tag{3.96}$$

Here Δ is

$$\Delta=\Delta(x)=\Delta(x;q,t)=\prod_{\substack{i,j=1\\ i\neq j}}^{N_0}\frac{(\frac{x_i}{x_j};q)_\infty}{(t\frac{x_i}{x_j};q)_\infty}, \tag{3.97}$$

and $D_k(q,t)$ $(k=1,\cdots,N_0)$ is the Macdonald operator

$$D_k(q,t)=t^{\frac{1}{2}k(k-1)}\sum_{\substack{I\subset\{1,\cdots,N_0\}\\ |I|=k}}\prod_{\substack{i\in I\\ j\notin I}}\frac{tx_i-x_j}{x_i-x_j}\cdot\prod_{i\in I}q^{D_i}, \tag{3.98}$$

where $D_i=x_i\frac{\partial}{\partial x_i}$. q^{D_i} is a q-shift, $q^{D_i}f(x_1,\cdots,x_i,\cdots,x_{N_0})=f(x_1,\cdots,qx_i,\cdots,x_{N_0})$.

The Macdonald symmetric polynomial $P_\lambda=P_\lambda(x;q,t)$ is a multivariable orthogonal polynomial with two parameters q and t, which is determined uniquely by the following conditions [38],

$$\text{(i)}\quad P_\lambda=\sum_{\mu\leq\lambda}u_{\lambda,\mu}m_\mu(x),\quad u_{\lambda,\lambda}=1, \tag{3.99}$$

$$\text{(ii)}\quad \langle P_\lambda,P_\mu\rangle_{q,t}=0\quad \text{if } \lambda\neq\mu, \tag{3.100}$$

where the inner product is

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} \prod_i i^{m_i} m_i! \cdot \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}. \tag{3.101}$$

The condition (ii) can be replaced by (ii)',

$$\text{(ii)'} \quad D_1(q,t)P_\lambda = \sum_{i=1}^{N_0} t^{N_0-i} q^{\lambda_i} \cdot P_\lambda. \tag{3.102}$$

Moreover the Macdonald symmetric polynomial is the simultaneous eigenfunction of the Macdonald operators,

$$\sum_{k=0}^{N_0} (-u)^k D_k(q,t) P_\lambda(x;q,t) = \prod_{i=1}^{N_0} (1 - u t^{N_0-i} q^{\lambda_i}) \cdot P_\lambda(x;q,t), \tag{3.103}$$

or

$$D_k(q^{\pm 1}, t^{\pm 1}) P_\lambda(x;q,t) = \sum_{1 \le i_1 < \cdots < i_k \le N_0} \prod_{l=1}^{k} t^{N_0 - i_l} q^{\lambda_{i_l}} \cdot P_\lambda(x;q,t). \tag{3.104}$$

Remark that $P_\lambda(x;q,t) = P_\lambda(x;q^{-1},t^{-1})$ and $\langle f,g \rangle_{q^{-1},t^{-1}} = (qt^{-1})^{N_0} \langle f,g \rangle_{q,t}$. In the 'conformal limit',

$$t = q^\beta, \quad q \to 1, \quad \beta : \text{ fixed}, \tag{3.105}$$

which corresponds to the non-relativistic limit ($c \to \infty$) (see (3.95)), the Macdonald polynomial reduces to the Jack polynomial

$$\lim_{\substack{q \to 1 \\ t = q^\beta}} P_\lambda(x;q,t) = J_\lambda(x;\beta). \tag{3.106}$$

$\beta \to 1$ limit of the Jack polynomial is the Schur polynomial $s_\lambda(x) = \lim_{\beta \to 1} J_\lambda(x;\beta)$. $t \to q$ limit of the Macdonald polynomial is also $s_\lambda(x) = \lim_{t \to q} P_\lambda(x;q,t)$.

There exists another inner product,

$$\langle f,g \rangle'_{l;q,t} = \frac{1}{l!} \oint \prod_{j=1}^{l} \underline{dx}_j \cdot \Delta(x;q,t) f(\bar{x}) g(x), \quad \underline{dx}_j = \frac{dx_j}{2\pi i x_j}, \tag{3.107}$$

which satisfies

$$\langle\ ,\ \rangle'_{l;q,t} \propto \langle\ ,\ \rangle_{q,t}. \tag{3.108}$$

Like as the Jack symmetric polynomial, the Macdonald symmetric polynomial has an integral representation obtained by the following two transformations,

$$\mathcal{G}_k \quad : \quad \Big(\mathcal{G}_k f\Big)(x_1, \cdots, x_l) = \prod_{i=1}^{l} x^k \cdot f(x_1, \cdots, x_l), \tag{3.109}$$

$$\mathcal{N}_{l',l} \;:\; \Big(\mathcal{N}_{l',l}f\Big)(x'_1,\cdots,x'_{l'}) = \oint\prod_{j=1}^{l}\underline{dx}_j\cdot\Pi(x',\bar{x})\Delta(x)f(x_1,\cdots,x_l), \qquad (3.110)$$

$$\Pi(x,y) = \Pi(x,y;q,t) = \prod_i\prod_j\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}. \qquad (3.111)$$

The Macdonald symmetric polynomial with λ ($\lambda' = ((l_1)^{k_1},(l_2)^{k_2},\cdots,(l_{N-1})^{k_{N-1}})$) is expressed as [47]

$$P_\lambda(x)\propto\mathcal{N}_{l,l_1}\mathcal{G}_{k_1}\mathcal{N}_{l_1,l_2}\mathcal{G}_{k_2}\cdots\mathcal{N}_{l_{N-2},l_{N-1}}\mathcal{G}_{k_{N-1}}\mathcal{N}_{l_{N-1},0}\cdot 1, \qquad (3.112)$$

because the Macdonald symmetric polynomial has two properties

$$P_{(k^l)+\lambda} = \mathcal{G}_kP_\lambda \quad (l \text{ variables}), \qquad (3.113)$$

$$P_\lambda = \frac{\langle P_\lambda,P_\lambda\rangle_{q,t}}{l!\langle P_\lambda,P_\lambda\rangle'_{l;q,t}}\mathcal{N}_{l',l}P_\lambda \quad \begin{array}{l}(P_\lambda \text{ in LHS : } l' \text{ variables})\\ (P_\lambda \text{ in RHS : } l \text{ variables})\end{array}. \qquad (3.114)$$

This second property is valid if $\mathcal{N}_{l',l}$ is modified in the following way [18],

$$P_\lambda\propto\mathcal{N}'_{l',l}P_\lambda,$$
$$\Big(\mathcal{N}'_{l',l}f\Big)(x'_1,\cdots,x'_{l'}) = \oint\prod_{j=1}^{l}\underline{dx}_j\cdot\Pi(x',\bar{x})\Delta(x)C(x)f(x_1,\cdots,x_l), \qquad (3.115)$$

where $C(x) = C(x_1,\cdots,x_l)$ is an arbitrary pseudoconstant with respect to the q-shift, $q^{D_i}C(x) = C(x)$ $(\forall i)$. (Remark that $\langle D_1(q,t)f,g\rangle'_{l;q,t} = \langle f,D_1(q,t)g\rangle'_{l;q,t}$.)
For example, the Macdonald symmetric polynomial with a rectangular Young diagram (k^l) is

$$P_{(k^l)}(x)\propto\oint\prod_{j=1}^{l}\underline{dz}_j\cdot\prod_{j=1}^{l}\exp\Big(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}z_j^n\sum_i x_i^n\Big)\cdot\prod_{\substack{i,j=1\\i\neq j}}^{l}\frac{(\frac{z_i}{z_j};q)_\infty}{(t\frac{z_i}{z_j};q)_\infty}\cdot\prod_{j=1}^{l}z_j^{-k}. \qquad (3.116)$$

Next let us consider bosonization of the Macdonald operator $\mathcal{D} = D_1(q,t)$ [17, 18]. $\mathcal{D}$ can be expressed by power sum polynomials,

$$\begin{aligned}\mathcal{D} &= \sum_{i=1}^{N_0}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\cdot q^{D_i} &(3.117)\\ &= \frac{1-t^{N_0}}{1-t}+\frac{t^{N_0}}{t-1}\bigg(\oint_0\underline{dz}\exp\Big(\sum_{n>0}\frac{1}{n}(1-t^{-n})p_nz^n\Big)\exp\Big(\sum_{n>0}(q^n-1)\frac{\partial}{\partial p_n}z^{-n}\Big)-1\bigg).\end{aligned}$$

Proof. Since $q^{D_i}p_n = p_n + x_i^n(q^n-1)$, q^{D_i} can be written as

$$q^{D_i} = \exp\Big(\sum_{n>0}x_i^n(q^n-1)\frac{\partial}{\partial p_n}\Big) = \sum_{n\geq 0}x_i^n\tilde{q}_n(-m\frac{\partial}{\partial p_n};q), \qquad (3.118)$$

where $\tilde{q}_n(p_m;t)=q_n(x_m;t)$ is (see [38] p.209)

$$q_n(x;t)=(1-t)\sum_{i=1}^{N_0}x_i^n\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j},\quad \sum_{n\geq 0}q_n(x;t)z^n=\exp\Big(\sum_{n>0}\frac{1}{n}(1-t^n)p_nz^n\Big). \tag{3.119}$$

Then we have

$$\begin{aligned}\mathcal{D} &= \sum_{i=1}^{N_0}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\cdot\sum_{n\geq 0}x_i^n\tilde{q}_n(-m\frac{\partial}{\partial p_n};q)\\ &= \frac{1-t^{N_0}}{1-t}+\frac{t^{N_0}}{t-1}\sum_{n>0}q_n(x_m;t^{-1})\tilde{q}_n(-m\frac{\partial}{\partial p_m};q)\end{aligned} \tag{3.120}$$

and get the result by (3.119). □

To bosonize this operator we introduce boson oscillator $\tilde{a}_n$ ($n\in\mathbb{Z}_{\neq 0}$),

$$[\tilde{a}_n,\tilde{a}_m]=n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}. \tag{3.121}$$

A state in the Fock space is mapped to a symmetric function by

$$|f\rangle\mapsto f(x)=\langle\alpha|\exp\Big(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\tilde{a}_np_n\Big)|f\rangle, \tag{3.122}$$

namely

$$\tilde{a}_{-n}\leftrightarrow p_n,\quad \tilde{a}_n\leftrightarrow n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n}. \tag{3.123}$$

This normalization for $\tilde{a}_n$ is chosen so that it produces the inner product (3.101). Then $\mathcal{D}$ is realized on the Fock space :

$$\mathcal{D}\langle\alpha|\exp\Big(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\tilde{a}_np_n\Big)=\langle\alpha|\exp\Big(\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}\tilde{a}_np_n\Big)\hat{\mathcal{D}}. \tag{3.124}$$

The bosonized operator $\hat{\mathcal{D}}$ is

$$\begin{aligned}\hat{\mathcal{D}} &= \frac{1-t^{N_0}}{1-t}+\frac{t^{N_0}}{t-1}\Big(\oint_0\underline{dz}\exp\Big(\sum_{n>0}\frac{1}{n}(1-t^{-n})\tilde{a}_{-n}z^n\Big)\exp\Big(-\sum_{n>0}(1-t^n)\tilde{a}_nz^{-n}\Big)-1\Big)\\ &= \frac{1-t^{N_0}}{1-t}+\frac{t^{N_0}}{t-1}\Big(\oint_0\underline{dz}\Big(\psi(z)T(z)-\exp\Big(-\sum_{n>0}h_nx^nz^{-n}\Big)x^{-2\sqrt{r(r-1)}a_0'}\Big)-1\Big)\\ &= \frac{1-t^{N_0}}{1-t}+\frac{t^{N_0}}{t-1}\Big(\sum_{n\geq 0}\psi_{-n}T_n-x^{-2\sqrt{r(r-1)}a_0'}-1\Big),\end{aligned} \tag{3.125}$$

where $T(z)$ is the DVA current (3.38) and $\psi(z)$ is

$$\psi(z)=\sum_{n\geq 0}\psi_{-n}z^n=\exp\Big(\sum_{n>0}h_{-n}x^{-n}z^n\Big)x^{-\sqrt{r(r-1)}a_0'}. \tag{3.126}$$

Here $\tilde{a}_n$ is expressed by h_n in (3.36)

$$\tilde{a}_n = n\frac{1}{x^{rn} - x^{-rn}} h_n, \quad \tilde{a}_{-n} = nx^{-n}\frac{x^n + x^{-n}}{x^{rn} - x^{-rn}} h_{-n}, \quad (n > 0), \tag{3.127}$$

and parameters are identified as

$$q = x^{2(r-1)}, \quad t = q^\beta = x^{2r}, \quad p = qt^{-1} = x^{-2}, \quad \beta = \frac{r}{r-1}. \tag{3.128}$$

When $\hat{\mathcal{D}}$ acts on the singular vector (3.61), $\sum_{n>0} \psi_{-n} T_n$ term vanishes because of the property of singular vector and the remaining terms are already diagonal,

$$\hat{\mathcal{D}}|\chi_{-l,-k}\rangle = \sum_{i=1}^{N_0} t^{N_0 - i} q^{\lambda_i} |\chi_{-l,-k}\rangle, \tag{3.129}$$

where the partition λ is $\lambda = (k^l, 0^{N_0 - l})$. Therefore $|\chi_{-l,-k}\rangle$ is an eigenstate of $\hat{\mathcal{D}}$, i.e., it gives the Macdonald symmetric polynomial by the map (3.122). In fact the polynomial obtained from (3.61) by (3.122) agrees with the integral representation of the Macdonald symmetric polynomial with partition (k^l) in (3.116) up to $x_i \to -x_i$ and $C(z)$ in (3.62), which is irrelevant because $C(z)$ is a pseudoconstant with respect to the q-shift, $q^{D_{z_i}} C(z) = C(z)$ $(\forall i)$.

The Macdonald symmetric polynomial with general partition whose Young diagram is composed of $N-1$ rectangles is related to the singular vector of the deformed W_N algebra [18].

We have established the relation between the Macdonald symmetric polynomials and the singular vectors of DVA in free field realization. When the DVA was firstly formulated in [17], this relation was not the derived property but the guiding principle to find the DVA. At that time we knew the two facts:

(i) In the free field realization, the singular vectors of the Virasoro and W_N algebras realize the Jack symmetric polynomials [37, 32].

(ii) The Jack symmetric polynomials have the good q-deformation, the Macdonald symmetric polynomials [38].

Based on these, we set up the following ‘natural’ question:

- Construct the algebras whose singular vectors in the free field realization realize the Macdonald symmetric polynomials.

The resultant algebra are worth being called quantum deformation (q-deformation) of the Virasoro and W_N algebras in this sense. This scenario is illustrated in the following figure,

$$
\begin{tikzcd}
\boxed{\begin{array}{c} Vir \\ W_N \end{array}} \arrow[r, "\text{free field realization}", "\text{singular vector}"'] \arrow[d, Rightarrow, "q\text{-deformation!}"] & \boxed{\begin{array}{c} \text{Jack} \\ \text{polynomial} \end{array}} \arrow[d, "q\text{-deformation}"] & \leftrightarrow \text{CSM} \\
\boxed{\begin{array}{c} q\text{-}Vir \\ q\text{-}W_N \end{array}} \arrow[r, "\text{free field realization}", "\text{singular vector}"'] & \boxed{\begin{array}{c} \text{Macdonald} \\ \text{polynomial} \end{array}} & \leftrightarrow \text{tRSM}
\end{tikzcd}
.
$$

See [17] how the DVA was found.

3.5 Higher DVA currents

In this subsection we present higher DVA currents. We define higher DVA currents $T_{(n)}(z)$ $(n = 1, 2, \cdots)$ by 'fusion',

$$
\begin{aligned}
T_{(1)}(z) &= T(z), \quad T_{(0)}(z) = 1, \\
T_{(n)}(z) &= f_{(1),(n-1)}(x^{rn}\tfrac{z}{z'})T_{(1)}(x^{-r(n-1)}z')T_{(n-1)}(x^r z)\Big|_{z'\to z},
\end{aligned}
\tag{3.130}
$$

where $f_{(n),(m)}(z)$ is given by

$$
f_{(n),(m)}(z) = \prod_{i=0}^{n-1}\prod_{j=0}^{m-1} f(x^{r(2j-m+1)-r(2i-n+1)}z) = f_{(m),(n)}(z), \tag{3.131}
$$

with $f(z)$ in (3.2). In free boson realization (3.38), we have

$$
T_{(n)}(z) = \sum_{i=0}^{n} {}_nB_i\, \Lambda_{n,i}(z), \tag{3.132}
$$

where ${}_nB_i$ and $\Lambda_{n,i}(z)$ are

$$
{}_nB_i = \frac{b_i\, b_{n-i}}{b_n}, \quad b_i = \prod_{i'=0}^{i-1} \frac{[ri'+1]_x}{[r(i'+1)]_x}, \tag{3.133}
$$

$$
\Lambda_{n,i}(z) = :\prod_{i'=0}^{i-1}\Lambda_-(x^{r(2i'-n+1)}z)\cdot\prod_{i'=i}^{n-1}\Lambda_+(x^{r(2i'-n+1)}z): . \tag{3.134}
$$

$T_{(n)}$ consists of $n+1$ terms, which corresponds to the spin $\frac{n}{2}$ representation of A_1. $T_{(n)}(z)$ satisfies the relation

$$
f_{(n),(m)}(\tfrac{z_2}{z_1})T_{(n)}(z_1)T_{(m)}(z_2) - T_{(m)}(z_2)T_{(n)}(z_1)f_{(m),(n)}(\tfrac{z_1}{z_2}) \qquad (n \le m)
$$

$$
\begin{aligned}
= \quad & -(x-x^{-1})\sum_{a=1}^{n} {}_nB_a\, {}_mB_a\, [ra]_x[ra+1]_x \prod_{b=1}^{a}\frac{[rb-1]_x}{[rb+1]_x} \\
& \times\Big(\delta(x^{r(n+m-2a)+2}\tfrac{z_2}{z_1})\,{}^{\circ}_{\circ}T_{(n-a)}(x^{ra}z_1)T_{(m-a)}(x^{-ra}z_2){}^{\circ}_{\circ} \\
& \qquad -\delta(x^{-r(n+m-2a)-2}\tfrac{z_2}{z_1})\,{}^{\circ}_{\circ}T_{(n-a)}(x^{-ra}z_1)T_{(m-a)}(x^{ra}z_2){}^{\circ}_{\circ}\Big).
\end{aligned} \tag{3.135}
$$

Here normal ordering ${}^{\circ}_{\circ}T_{(a)}(\alpha z)T_{(b)}(z){}^{\circ}_{\circ}$ is

$$
\begin{aligned}
& {}^{\circ}_{\circ}T_{(a)}(\alpha z)T_{(b)}(z){}^{\circ}_{\circ} \\
\stackrel{\mathrm{def}}{=} \; & \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{m} f_{(a),(b),\ell}\Big(\alpha^{m-\ell}T_{(a),-m}T_{(b),n+m}+\alpha^{\ell-m-1}T_{(b),n-m-1}T_{(a),m+1}\Big)\cdot z^{-n} \\
= \; & \oint\frac{dy}{2\pi i y}\Big(\sum_{j=0}^{\infty}(\tfrac{\alpha z}{y})^{j}\cdot f_{(a),(b)}(\tfrac{z}{y})T_{(a)}(y)T_{(b)}(z)+\sum_{j=0}^{\infty}(\tfrac{y}{\alpha z})^{j+1}\cdot T_{(b)}(z)T_{(a)}(y)f_{(a),(b)}(\tfrac{y}{z})\Big) \\
= \; & \oint_{|y|>|\alpha z|}\frac{dy}{2\pi i y}\frac{1}{1-\frac{\alpha z}{y}}\cdot f_{(a),(b)}(\tfrac{z}{y})T_{(a)}(y)T_{(b)}(z) \\
& \quad +\oint_{|y|<|\alpha z|}\frac{dy}{2\pi i y}\frac{\frac{y}{\alpha z}}{1-\frac{y}{\alpha z}}\cdot T_{(b)}(z)T_{(a)}(y)f_{(a),(b)}(\tfrac{y}{z}) \\
= \; & \oint_{\alpha z}\frac{dy}{2\pi i}\frac{1}{y-\alpha z}\cdot f_{(a),(b)}(\tfrac{z}{y})T_{(a)}(y)T_{(b)}(z),
\end{aligned} \tag{3.136}
$$

where mode expansions are

$$
f_{(a),(b)}(z)=\sum_{\ell=0}^{\infty}f_{(a),(b),\ell}z^{\ell},\quad T_{(a)}(z)=\sum_{n\in\mathbb{Z}}T_{(a),n}z^{-n}. \tag{3.137}
$$

This is a generalization of the normal ordering for currents used in the CFT,

$$
(AB)(z)=\oint_{z}\frac{dy}{2\pi i}\frac{1}{y-z}\cdot A(y)B(z). \tag{3.138}
$$

We remark that $T_{(n)}(z)$ is a composite field of $T(z)$,

$$
{}^{\circ}_{\circ}T_{(n)}(x^{-rm}z)T_{(m)}(x^{rn}z){}^{\circ}_{\circ}=T_{(n+m)}(z). \tag{3.139}
$$

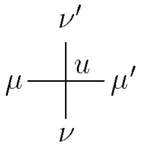
We remark also that another higher currents are obtained by replacing r with $1-r$.

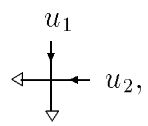
Concerning the higher currents for the deformed W_N algebra, see appendix C in [48] where delta function terms are neglected. (The arguments of $W_{(n)}(u)$ and $f_{(n),(m)}(u,v)$ in [48] should be shifted in order to compare them with $T_{(n)}(z)$ and $f_{(n),(m)}(z)$ here.)

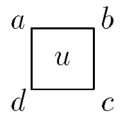
4 Solvable Lattice Models and Elliptic Algebras

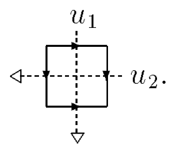
4.1 Solvable lattice models and Yang-Baxter equation

A statistical lattice model is a statistical mechanical system on a lattice [6]. In this lecture we consider classical systems on two dimensional space square lattice. There are two types of lattice models, vertex models and face models. For vertex models dynamical variables 'spin' are located on edges and the Boltzmann weights are assigned to each vertex. On the other hand, for face models, dynamical variables 'height' are located on vertices and the Boltzmann weights are assigned to each face.

Vertex model : 

$R_{\mu'\nu',\mu\nu}(u), \quad u = u_1 - u_2$ 

Face model : 

$W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\middle| u\right), \quad u = u_1 - u_2$ 

(4.1)

Here μ, ν, μ', ν' are spins, a, b, c, d are heights, and $R_{\mu'\nu',\mu\nu}(u)$ and $W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\middle| u\right)$ are Boltzmann weights. We have assumed that the spectral parameters enter in the Boltzmann weight only through their difference u. Roughly speaking, a vertex model on a lattice is a face model on its dual lattice. A face model is also called an interaction round a face (IRF) model or a solid on solid (SOS) model. If the height variable takes a finite number of states, a SOS model is called a restricted SOS (RSOS) model.

So-called solvable lattice model is a statistical lattice model whose Boltzmann weight satisfies the Yang-Baxter equation (YBE) [5, 6, 7],

$$R^{(12)}(u-v)R^{(13)}(u)R^{(23)}(v) = R^{(23)}(v)R^{(13)}(u)R^{(12)}(u-v), \tag{4.2}$$

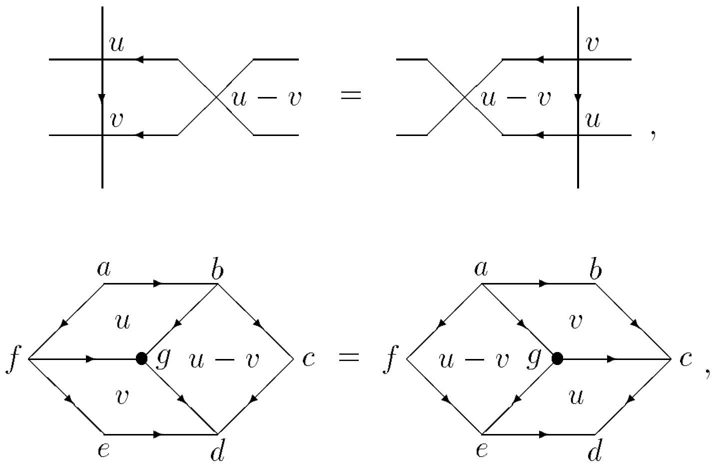
for a vertex model and

$$\sum_g W\left(\begin{smallmatrix} b & c \\ g & d \end{smallmatrix}\middle| u-v\right) W\left(\begin{smallmatrix} a & b \\ f & g \end{smallmatrix}\middle| u\right) W\left(\begin{smallmatrix} f & g \\ e & d \end{smallmatrix}\middle| v\right) = \sum_g W\left(\begin{smallmatrix} a & b \\ g & c \end{smallmatrix}\middle| v\right) W\left(\begin{smallmatrix} g & c \\ e & d \end{smallmatrix}\middle| u\right) W\left(\begin{smallmatrix} a & g \\ f & e \end{smallmatrix}\middle| u-v\right), \tag{4.3}$$

for a face model. Here $R(u) = (R_{\mu'\nu',\mu\nu}(u))$ is a matrix on $V \otimes V$ and (4.2) is an equation on $V \otimes V \otimes V$ and the superscript $(12), (13)$ etc. refer to the tensor components, e.g., $R^{(12)}(u) = R(u) \otimes \mathrm{id}$. YBE (4.3) is also satisfied by the new Boltzmann weight obtained by the following transformation (gauge transformation),

$$W'\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\middle| u\right) = \frac{F(a,b)F(b,d)}{F(a,c)F(c,d)} W\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\middle| u\right), \tag{4.4}$$

where $F(a,b)$ is an arbitrary u-independent function of a and b. The following graphical expression of the Yang-Baxter equation will help us.



(4.5)

where summation is taken over the height g at site $\bullet$.

Solutions of the Yang-Baxter equation were studied by many people and the relation to the Lie algebras was clarified [5, 8, 9, 10]. Three types of solutions are known; rational, trigonometric and ellipic. Associated for each type of solution (R matrix), algebras are defined [11],

$$\begin{array}{lcl} \text{rational} & \to & \text{Yangian}, \\ \text{trigonometric} & \to & \text{quantum group (quantum algebra)}, \\ \text{elliptic} & \to & \text{elliptic quantum group (elliptic algebra)}. \end{array} \tag{4.6}$$

Elliptic quntum groups will be introduced in subsection 4.4.

Example: Vector representation of A_n algebra [10].

$$\begin{aligned} &W\Big(\begin{matrix} a & a+\hat{\mu} \\ a+\hat{\mu} & a+2\hat{\mu} \end{matrix}\Big| u\Big) = \frac{[1+u]}{[1]}, \\ &W\Big(\begin{matrix} a & a+\hat{\mu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\Big| u\Big) = \frac{[a_\mu - a_\nu - u]}{[a_\mu - a_\nu]} \quad (\mu \neq \nu), \\ &W\Big(\begin{matrix} a & a+\hat{\nu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\Big| u\Big) = \frac{[u]}{[1]} \left(\frac{[a_\mu - a_\nu + 1][a_\mu - a_\nu - 1]}{[a_\mu - a_\nu]^2}\right)^{\frac{1}{2}} \quad (\mu \neq \nu). \end{aligned} \tag{4.7}$$

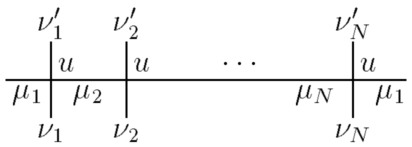
Notation is as follows. a is an element of the weight space of $A_n^{(1)}$. $\hat{\mu}$ is $\hat{\mu} = \varepsilon_\mu - \frac{1}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$ $(\mu = 1, 2, \cdots, n+1)$, where ε_μ is an orothonomal basis of $\mathbb{C}^{n+1}$. a_μ is $\langle a+\rho, \hat{\mu}\rangle$, where ρ is a sum of fundamental weight of $A_1^{(1)}$. So $\bar{a} + \bar{\rho} = \sum_{\mu=1}^{n+1} a_\mu \varepsilon_\mu$, $\sum_{\mu=1}^{n+1} a_\mu = 0$.

The partition function of a lattice model is

$$Z = \sum_{\text{config.}} \prod (\text{Boltzmann weight}), \tag{4.8}$$

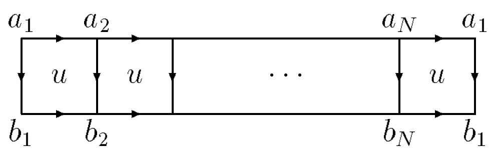
where the summation is taken over all configurations and the product is taken over all verteces or faces. This partition function can be calculated by using the transfer matrix. The row-to-row transfer matrix $\mathcal{T}(u)$ is

$$\mathcal{T}(u) = \Big(\mathcal{T}(u)_{\underline{\nu}',\underline{\nu}}\Big) = \sum_{\mu_1,\cdots,\mu_N}$$



, (4.9)

$$\mathcal{T}(u) = \Big(\mathcal{T}(u)_{\underline{a},\underline{b}}\Big) =$$



,

where $\underline{\nu} = (\nu_1, \cdots, \nu_N)$, $\underline{\nu}' = (\nu'_1, \cdots, \nu'_N)$, $\underline{a} = (a_1, \cdots, a_N)$ and $\underline{b} = (b_1, \cdots, b_N)$. Here lattice size is N (horizontal) by M (vertical), and we assume the periodic boundary condition. Then the partition function is

$$Z = \mathrm{tr}\Big(\mathcal{T}(u)^M\Big). \tag{4.10}$$

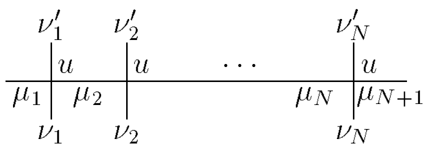
In the thermodynamic limit ($N, M \to \infty$), only the maximal eigenvalue of the transfer matrix contributes to the partition function. Once we know information about the eigenvalues and eigenvectors of the transfer matrix, we can calculate various phyical quantities.

If the Boltzmann weight satisfies the Yang-Baxter equation, the transfer matrix has good properties. For example the trasfer matrices with different spectral parameters commute each other,

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0. \tag{4.11}$$

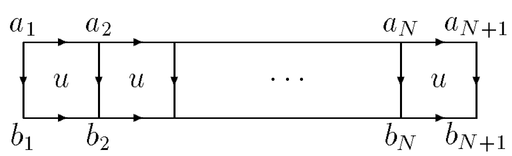
To show this let us introduce the monodromy matrix $\hat{\mathcal{T}}(u)$

$$\hat{\mathcal{T}}(u) = \sum_{\mu_2,\cdots,\mu_N}$$



, (4.12)

$$\hat{\mathcal{T}}(u) =$$

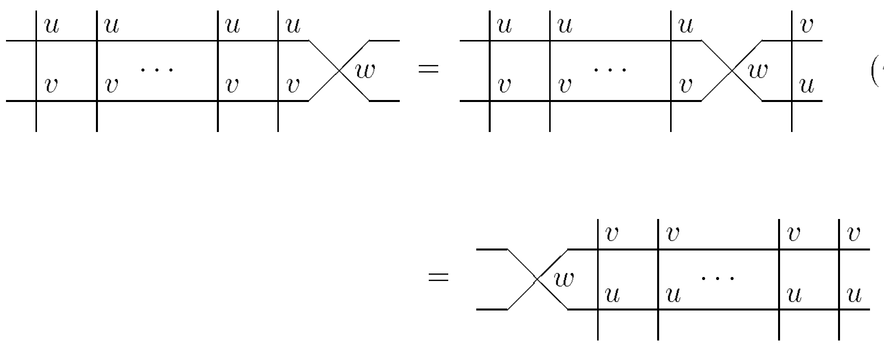


.

Then the transfer matrix is expressed as a trace of the monodromy matrix

$$\mathcal{T}(u) = \mathrm{tr}\,\hat{\mathcal{T}}(u) \qquad \Big(\mathrm{tr} = \sum_{\mu_1,\mu_{N+1}} \delta_{\mu_1,\mu_{N+1}} \text{ or } \sum_{a_{N+1}} \delta_{a_1,a_{N+1}}\Big). \tag{4.13}$$

The following figure (for face model see similar figure in (4.50))



$(w = u - v)$

(4.14)

shows

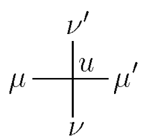
$$R(u-v)\hat{\mathcal{T}}(u)\hat{\mathcal{T}}(v) = \hat{\mathcal{T}}(v)\hat{\mathcal{T}}(u)R(u-v) \tag{4.15}$$

$$\text{or} \qquad W(u-v)\hat{\mathcal{T}}(u)\hat{\mathcal{T}}(v) = \hat{\mathcal{T}}(v)\hat{\mathcal{T}}(u)W(u-v). \tag{4.16}$$

(Here we do not write tensor product explicitly.) Therefore we obtain (4.11) by taking a trace of $R(u-v)\hat{\mathcal{T}}(u)\hat{\mathcal{T}}(v)R(u-v)^{-1} = \hat{\mathcal{T}}(v)\hat{\mathcal{T}}(u)$.

If we interpret a R matrix as a L matrix (L operator), whose entries are operators acting on vertical vector space,

$$L(u) = \Big(L(u)_{\mu',\mu}\Big), \quad \Big(L(u)_{\mu',\mu}\Big)_{\nu',\nu} =$$



(4.17)

then (4.2) is rewritten as

$$R^{(12)}(u-v)L^{(1)}(u)L^{(2)}(v) = L^{(2)}(v)L^{(1)}(u)R^{(12)}(u-v). \tag{4.18}$$

The monodoromy matrix is

$$\hat{\mathcal{T}}(u) = L_1(u)L_2(u)\cdots L_N(u), \tag{4.19}$$

where $L_i(u)$ acts nontrivially on i-th component, $L_i(u) = 1 \otimes \cdots \otimes \overset{i}{\smile}{L(u)} \otimes \cdots \otimes 1$.

The R matrix obtained from the Boltzmann weight of the face model satisfies the dynamical Yang-Baxter equation [24],

$$\begin{aligned} &R^{(12)}(u-v;\lambda+h^{(3)})R^{(13)}(u;\lambda)R^{(23)}(v;\lambda+h^{(1)}) \\ &\quad = R^{(23)}(v;\lambda)R^{(13)}(u;\lambda+h^{(2)})R^{(12)}(u-v;\lambda). \end{aligned} \tag{4.20}$$

Here h is an element of the Cartan subalgebra (see subsections 4.3, 4.4). For simpliclity we consider A_n vector representation case. Associated to the face Boltzmann weight (4.7), R matrix is introduced by

$$R(u,\lambda)=\sum_{\substack{\mu,\nu,\mu',\nu'\\ \mu+\nu=\mu'+\nu'}} W\left(\begin{matrix}\lambda & \lambda+\hat{\nu}' \\ \lambda+\hat{\mu} & \lambda+\hat{\mu}+\hat{\nu}\end{matrix}\middle| u\right) E_{\mu',\mu}\otimes E_{\nu',\nu}, \tag{4.21}$$

where $E_{\mu',\mu}$ is a matrix unit, $(E_{\mu',\mu})_{\nu',\nu}=\delta_{\mu',\mu}\delta_{\nu',\nu}$. h acts on $E_{\mu',\mu}$ as $hE_{\mu',\mu}=\hat{\mu}'E_{\mu',\mu}$, $E_{\mu',\mu}h=\hat{\mu}E_{\mu',\mu}$. Using this and (4.3) we can show (4.20). Similariy $L(u,\lambda)$

$$\begin{aligned} L(u,\lambda) &= \Big(L(u,\lambda)_{\mu',\mu}\Big)=\sum_{\mu,\mu'} L(u,\lambda)_{\mu',\mu}E_{\mu',\mu}, \\ L(u,\lambda)_{\mu',\mu} &= \sum_{\substack{\nu,\nu'\\ \mu+\nu=\mu'+\nu'}} W\left(\begin{matrix}\lambda & \lambda+\hat{\nu}' \\ \lambda+\hat{\mu} & \lambda+\hat{\mu}+\hat{\nu}\end{matrix}\middle| u\right) E_{\nu',\nu}, \end{aligned} \tag{4.22}$$

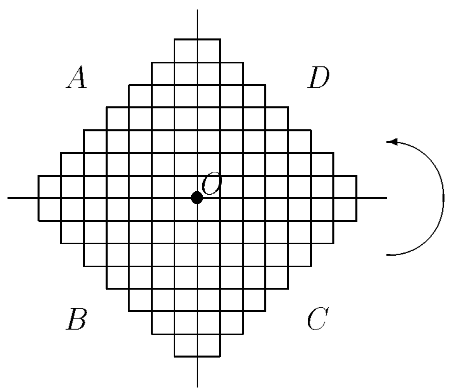
satisfies the dynamical RLL-relation

$$\begin{aligned} &R^{(12)}(u-v;\lambda+h)L^{(1)}(u;\lambda)L^{(2)}(v;\lambda+h^{(1)}) \\ &\quad = L^{(2)}(v;\lambda)L^{(1)}(u;\lambda+h^{(2)})R^{(12)}(u-v;\lambda). \end{aligned} \tag{4.23}$$

4.2 Corner transfer matrices and vertex operators

In the previous subsection the row-to-row transfer matrix is explained. There is another powerful method, corner trasfer matrix (CTM) method, which was developed by Baxter [6]. In the following we consider face models in which heights take values in a set I. Vertex models are also treated similarly.

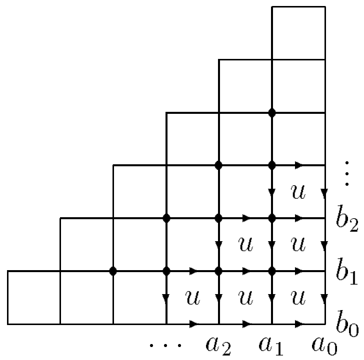
Let us consider the square lattice. We take the central site as the reference site O and devide a lattice into four quadrants.



(4.24)

For each quadrant we assign a matrix A, B, C, D, which are called corner transfer matrices. We take a multiplication of matrices counterclockwise. Sites on each column (row) are numbered increasingly from south to north (from east to west), $\cdots, -2, -1, 0, 1, 2, \cdots$, where the zeroth site is the reference site O. For example $A(u) = (A_{\underline{a},\underline{b}}(u))$ is

$$A_{\underline{a},\underline{b}}(u) = \delta_{a_0,b_0}$$



, (4.25)

where $\underline{a} = (a_0, a_1, a_2, \cdots)$, $\underline{b} = (b_0, b_1, b_2, \cdots)$, and summation is taken over inner sites $\bullet$. Using these CTM's the partition function is expressed as

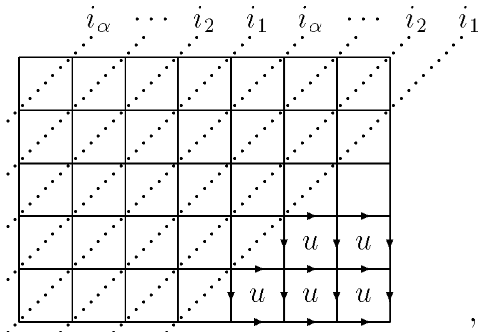
$$Z = \mathrm{tr}\,(DCBA), \tag{4.26}$$

and one-point local height probability (LHP), which is a probability of finding local height at O to be k, is

$$P_k = Z^{-1}\mathrm{tr}_{a_0=k}(DCBA). \tag{4.27}$$

To discuss these quantities, however, we have to specify the ground state.

We restrict ourselves to consider the following ground state configuration



, (4.28)

which is invariant under the shift in the NE-SW direction. Many interesting models have this type of ground states. We label this ground state configuration (i.e. an ordered set $(i_1, i_2, \cdots, i_\alpha)$ whose cyclic permutations are identified) as an integer l. On each column or row the height sequence is $\cdots, i_1, i_2, \cdots, i_\alpha, i_1, i_2, \cdots, i_\alpha, \cdots$, but there are α configurations because the height on the zeroth site takes one of α values i_{m+1}. We distinguish these configurations by an integer $m \in \mathbb{Z}/\alpha\mathbb{Z}$. Namely the ground state configuration (l, m), which we denote $\bar{a}(l,m)$, is

$$\bar{a}_{j-m}(l,m) = i_{j+1} \quad (0 \le j \le \alpha - 1),$$
$$\bar{a}_{j+\alpha}(l,m) = \bar{a}_j(l,m) \quad (j \in \mathbb{Z}). \tag{4.29}$$

When we discuss this face model, we impose the condition that heights at the boundary (or sites far away from O) take one of the ground state configurations.

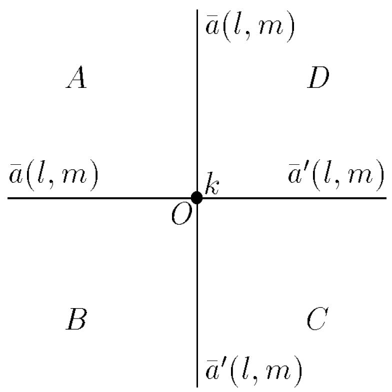
In the north and west directions the ground state configurations are $\cdots, i_1, i_2, \cdots, i_\alpha, \cdots$. Let $\mathcal{H}^{(k)}_{l,m}$ denote the space of states of the half-infinite lattice where the central height (i.e. the one on the zeroth site) is fixed to k and the boundary heights are in the ground state (l,m). Formally it is

$$\mathcal{H}^{(k)}_{l,m} = \mathrm{Span}\Big\{ (a_j)_{j=0}^{\infty} \;\Big|\; a_j \in I,\, a_0 = k,\, a_j = \bar{a}_j(l,m)\; (j \gg 1) \Big\}. \tag{4.30}$$

In the south and east directions we relabel the numbering of sites : $0, -1, -2, \cdots \mapsto 0, 1, 2, \cdots$. Then the ground state configurations are in the south and east directions are $\cdots, i_\alpha, \cdots, i_2, i_1, \cdots$. So we introduce

$$\mathcal{H}'^{(k)}_{l,m} = \mathrm{Span}\Big\{ (a_j)_{j=0}^{\infty} \;\Big|\; a_j \in I,\, a_0 = k,\, a_j = \bar{a}'_j(l,m)\; (j \gg 1) \Big\}, \tag{4.31}$$

where $\bar{a}'_j(l,m) = \bar{a}_{-j}(l,m)$. We have



(4.32)

For later convenience we define

$$\mathcal{H}^{(k)}_l = \bigoplus_m \mathcal{H}^{(k)}_{l,m}, \quad \mathcal{H}_l = \bigoplus_k \mathcal{H}^{(k)}_l, \quad \mathcal{H}'^{(k)}_l = \bigoplus_m \mathcal{H}'^{(k)}_{l,m}, \quad \mathcal{H}'_l = \bigoplus_k \mathcal{H}'^{(k)}_l. \tag{4.33}$$

Since the factor δ_{a_0,b_0} in (4.25), $A(u)$ has a block structure

$$A = \bigoplus_k A^{(k)}, \tag{4.34}$$

where $A^{(k)}$ is a matrix in $a_0 = k$ sector. B, C and D have also this block structure. CTM's are linear maps:

$$\begin{aligned} A^{(k)} &: \mathcal{H}^{(k)}_{l,m} \to \mathcal{H}^{(k)}_{l,m}, \\ B^{(k)} &: \mathcal{H}^{(k)}_{l,m} \to \mathcal{H}'^{(k)}_{l,m}, \\ C^{(k)} &: \mathcal{H}'^{(k)}_{l,m} \to \mathcal{H}'^{(k)}_{l,m}, \\ D^{(k)} &: \mathcal{H}'^{(k)}_{l,m} \to \mathcal{H}^{(k)}_{l,m}. \end{aligned} \tag{4.35}$$

One-point LHP $P_k(l,m)$, which is a probability of finding local height at O to be k and the boundary heights in the north direction on the column through O is in the ground state (l,m), is

$$P_k(l,m) = Z^{-1}_{l,m} \mathrm{tr}_{\mathcal{H}^{(k)}_{l,m}} (D^{(k)} C^{(k)} B^{(k)} A^{(k)}), \tag{4.36}$$

where $Z_{l,m}$ is the partition function for the ground state (l,m)

$$Z_{l,m} = \sum_k \mathrm{tr}_{\mathcal{H}^{(k)}_{l,m}} (D^{(k)} C^{(k)} B^{(k)} A^{(k)}). \tag{4.37}$$

Baxter's important observation is the following. Let us assume that the Boltzmann weight W satisfies the YBE (4.3),
initial condition

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| 0\right) = \delta_{b,c}, \tag{4.38}$$

unitarity

$$\sum_g W\left(\begin{matrix} a & b \\ g & d \end{matrix}\middle| u\right) W\left(\begin{matrix} a & g \\ c & d \end{matrix}\middle| -u\right) = \delta_{b,c}, \tag{4.39}$$

and the second inversion relation

$$\sum_g G_g W\left(\begin{matrix} a & b \\ c & g \end{matrix}\middle| \lambda - u\right) W\left(\begin{matrix} d & c \\ b & g \end{matrix}\middle| \lambda + u\right) = \frac{G_b G_c}{G_a} \delta_{a,d}, \tag{4.40}$$

where λ and G_a are some constants, and W is a double periodic function in u. Then in the thermodynamic limit (infinite lattice size limit), u-dependence of CTM becomes

$$\begin{aligned} A^{(a)}(u) &= x^{-2uH^{(a)}_C} &&: \mathcal{H}^{(a)}_{l,m} \to \mathcal{H}^{(a)}_{l,m}, \\ B^{(a)}(u) &= \sqrt{G_a} P^{(a)^{-1}} A^{(a)}(\lambda - u) &&: \mathcal{H}^{(a)}_{l,m} \to \mathcal{H}'^{(a)}_{l,m}, \\ C^{(a)}(u) &= P^{(a)^{-1}} A^{(a)}(u) P^{(a)} &&: \mathcal{H}'^{(a)}_{l,m} \to \mathcal{H}'^{(a)}_{l,m}, \\ D^{(a)}(u) &= \sqrt{G_a} A^{(a)}(\lambda - u) P^{(a)} &&: \mathcal{H}'^{(a)}_{l,m} \to \mathcal{H}^{(a)}_{l,m}, \end{aligned} \tag{4.41}$$

where $H^{(a)}_C$ and $P^{(a)}$ are u-independent matrices and x is a parameter of the model. We have neglected multiplicative constant factors (normalization of CTM's) because they do not contribute to LHP. $P^{(a)}$ gives an isomorphism of vector spaces

$$P^{(a)} : \mathcal{H}'^{(a)}_{l,m} \to \mathcal{H}^{(a)}_{l,m}. \tag{4.42}$$

The spectrum of the corner Hamiltonian H_C has properties

bounded below, discrete, equidistance.

The multiplicity of the spectrum of H_C depends on the model and it is summarized in a character

$$\chi_{l,m,k}(q) = \mathrm{tr}_{\mathcal{H}^{(k)}_{l,m}}\, q^{H^{(k)}_C}. \tag{4.43}$$

By using these, one-point LHP (4.36) becomes

$$P_k(l,m) = Z^{-1}_{l,m} G_k \,\mathrm{tr}_{\mathcal{H}^{(k)}_{l,m}}\, x^{-4\lambda H^{(k)}_C} = Z^{-1}_{l,m} G_k \,\chi_{l,m,k}(x^{-4\lambda}), \tag{4.44}$$

where the partition function (4.37) is

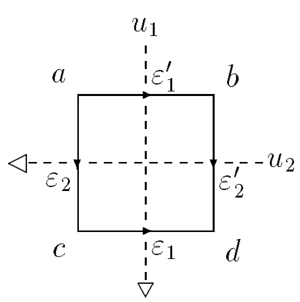
$$Z_{l,m} = \sum_k G_k \,\chi_{l,m,k}(x^{-4\lambda}). \tag{4.45}$$

We remark that this $Z_{l,m}$ is independent on m because we are considering an infinitely large lattice.

To formulate multi-point LHP let us introduce the vertex operator (VO) of type I [49, 50]. Foda et al. [49] presented a formulation of VO's in solvable lattice models. We explain a face model case following refs. [49, 50]. Here we use intuitive graphical argument. For precise representation theoretical argument, see subsection 4.4.

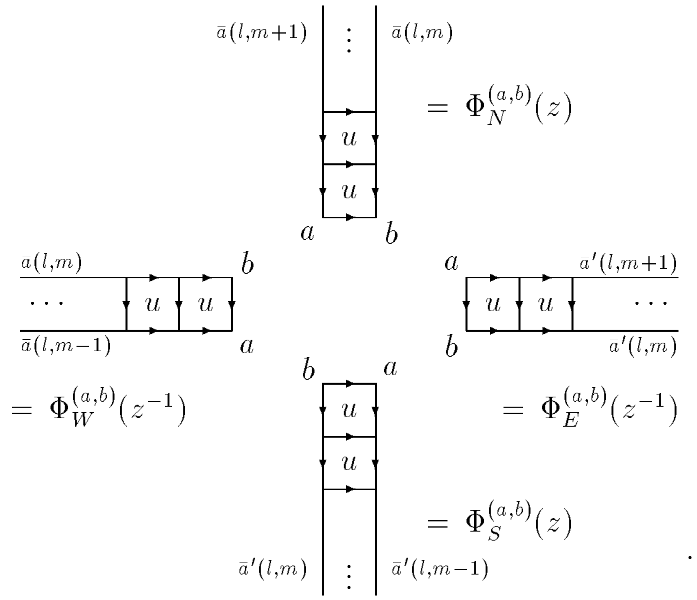
Graphically the Boltzmann weight is

$$W\begin{pmatrix} a & b \\ c & d \end{pmatrix}\!\Big|\, u_1 - u_2 \Big) \;=$$



$$, \qquad \begin{aligned} b &= a + \varepsilon'_1, \\ c &= a + \varepsilon_2, \\ d &= b + \varepsilon'_2 = c + \varepsilon_1. \end{aligned} \tag{4.46}$$

Let $\Phi^{(a,b)}_N(z)$, $\Phi^{(a,b)}_W(z^{-1})$, $\Phi^{(a,b)}_S(z)$ and $\Phi^{(a,b)}_E(z^{-1})$ be the half-infinite transfer matrices extending to infinity in the north, west, south and east direction respectively $(z = x^{2u})$:



(4.47)

They are linear maps:

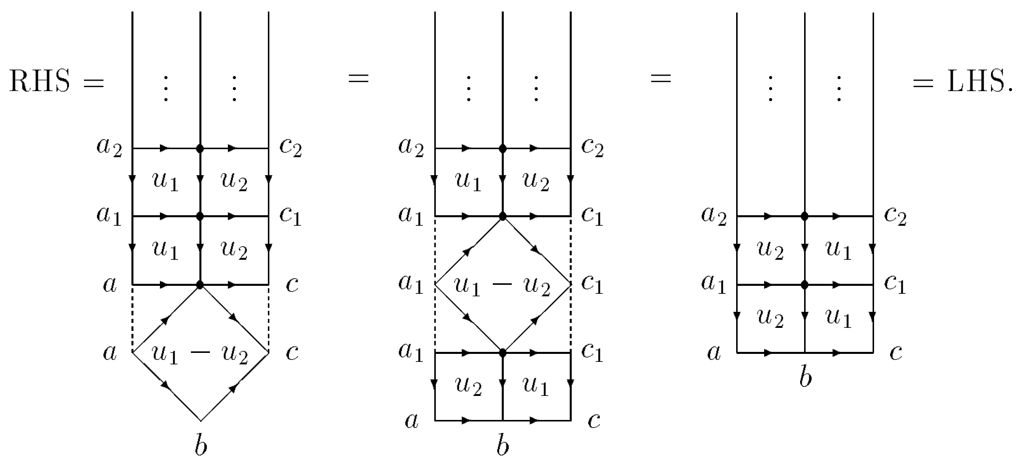
$$
\begin{aligned}
\Phi_N^{(a,b)}(z) &: \mathcal{H}_{l,m}^{(b)} \to \mathcal{H}_{l,m+1}^{(a)}\,, \\
\Phi_S^{(a,b)}(z) &: \mathcal{H}_{l,m}'^{(b)} \to \mathcal{H}_{l,m-1}'^{(a)}\,, \\
\Phi_W^{(a,b)}(z) &: \mathcal{H}_{l,m}^{(b)} \to \mathcal{H}_{l,m-1}^{(a)}\,, \\
\Phi_E^{(a,b)}(z) &: \mathcal{H}_{l,m}'^{(b)} \to \mathcal{H}_{l,m+1}'^{(a)}\,.
\end{aligned}
\tag{4.48}
$$

For CTM's and type I VO's we suppress l dependence.

$\Phi_N^{(a,b)}(z)$ satisfies the commutation relation

$$
\Phi_N^{(a,b)}(z_2)\Phi_N^{(b,c)}(z_1) = \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| u_1 - u_2\right)\Phi_N^{(a,g)}(z_1)\Phi_N^{(g,c)}(z_2) \quad (z_j = x^{2u_j}),
\tag{4.49}
$$

which can be shown by repeated use of (4.3),



(4.50)

Similarly we have ($z_j = x^{2u_j}$)

$$\Phi_W^{(a,b)}(z_2)\Phi_W^{(b,c)}(z_1) = \sum_g W\begin{pmatrix} c & b \\ g & a \end{pmatrix}\!\!\Big| u_1 - u_2 \Big) \Phi_W^{(a,g)}(z_1)\Phi_W^{(g,c)}(z_2), \tag{4.51}$$

$$\Phi_N^{(a,b)}(z_1)\Phi_W^{(b,c)}(z_2) = \sum_g W\begin{pmatrix} g & c \\ a & b \end{pmatrix}\!\!\Big| u_1 - u_2 \Big) \Phi_W^{(a,g)}(z_2)\Phi_N^{(g,c)}(z_1), \tag{4.52}$$

and

$$\Phi_S^{(a,b)}(z_2)\Phi_S^{(b,c)}(z_1) = \sum_g W\begin{pmatrix} c & b \\ g & a \end{pmatrix}\!\!\Big| u_1 - u_2 \Big) \Phi_S^{(a,g)}(z_1)\Phi_S^{(g,c)}(z_2), \quad \text{etc.} \tag{4.53}$$

The initial condition (4.38) implies

$$\begin{aligned}
A^{(a)}(u)\Phi_N^{(a,b)}(z) &= \Phi_N^{(a,b)}(1)A^{(b)}(u), \quad \Phi_W^{(a,b)}(z^{-1})A^{(b)}(u) = A^{(a)}(u)\Phi_W^{(a,b)}(1), \\
B^{(a)}(u)\Phi_W^{(a,b)}(1) &= \Phi_S^{(a,b)}(z)B^{(b)}(u), \quad \Phi_S^{(a,b)}(1)B^{(b)}(u) = B^{(a)}(u)\Phi_W^{(a,b)}(z^{-1}), \\
C^{(a)}(u)\Phi_S^{(a,b)}(z) &= \Phi_S^{(a,b)}(1)C^{(b)}(u), \quad \Phi_E^{(a,b)}(z^{-1})C^{(b)}(u) = C^{(a)}(u)\Phi_E^{(a,b)}(1), \\
D^{(a)}(u)\Phi_E^{(a,b)}(1) &= \Phi_N^{(a,b)}(z)D^{(b)}(u), \quad \Phi_N^{(a,b)}(1)D^{(b)}(u) = D^{(a)}(u)\Phi_E^{(a,b)}(z^{-1}),
\end{aligned} \tag{4.54}$$

and therefore by combining (4.41) we have

$$\Phi_N^{(a,b)}(z) = x^{2uH_C^{(a)}}\Phi_N^{(a,b)}(1)x^{-2uH_C^{(b)}}, \tag{4.55}$$

$$\Phi_W^{(a,b)}(z) = x^{2uH_C^{(a)}}\Phi_W^{(a,b)}(1)x^{-2uH_C^{(b)}}, \tag{4.56}$$

$$P^{(a)}\Phi_S^{(a,b)}(z)P^{(b)^{-1}} = x^{2uH_C^{(a)}}P^{(a)}\Phi_S^{(a,b)}(1)P^{(b)^{-1}}x^{-2uH_C^{(b)}}, \tag{4.57}$$

$$P^{(a)}\Phi_E^{(a,b)}(z)P^{(b)^{-1}} = x^{2uH_C^{(a)}}P^{(a)}\Phi_E^{(a,b)}(1)P^{(b)^{-1}}x^{-2uH_C^{(b)}}, \tag{4.58}$$

$$P^{(a)}\Phi_S^{(a,b)}(z)P^{(b)^{-1}} = \sqrt{\frac{G_a}{G_b}}\Phi_W^{(a,b)}(x^{-2\lambda}z), \tag{4.59}$$

$$P^{(a)}\Phi_E^{(a,b)}(z)P^{(b)^{-1}} = \sqrt{\frac{G_b}{G_a}}\Phi_N^{(a,b)}(x^{2\lambda}z). \tag{4.60}$$

The unitarity (4.39) gives

$$\sum_g \Phi_W^{(a,g)}(z)\Phi_N^{(g,a)}(z) = \mathrm{id}, \tag{4.61}$$

$$\sum_g \Phi_E^{(a,g)}(z)\Phi_S^{(g,a)}(z) = \mathrm{id}. \tag{4.62}$$

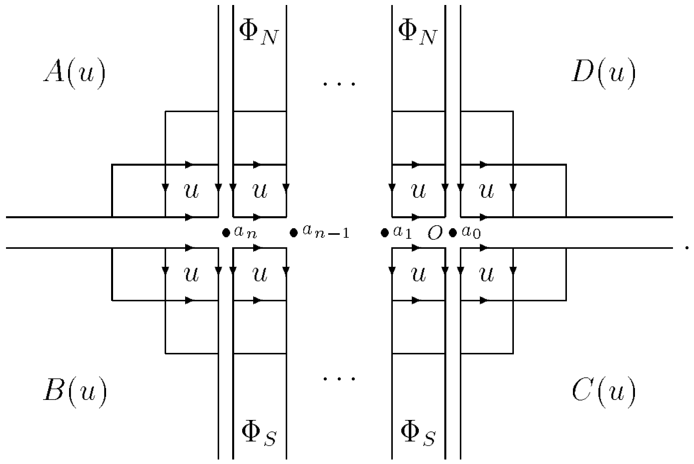
So (4.49),(4.38) and (4.61) imply

$$\Phi_N^{(a,b)}(z)\Phi_W^{(b,a)}(z) = \mathrm{id}. \tag{4.63}$$

The second inversion relation (4.40) entails

$$G_a^{-1}\sum_g G_g \Phi_N^{(a,g)}(x^{2\lambda}z)\Phi_W^{(g,a)}(x^{-2\lambda}z) = \mathrm{id}. \tag{4.64}$$

Consider neighboring $n+1$ sites in a row, whose most east site is the reference site O. We divide a lattice into $2n+4$ parts,



(4.65)

Let $P_{a_n,\cdots,a_1,a_0}(l,m)$ denote the probability of finding these local variables to be $(a_n,\cdots,a_0)$ under the condition that the boundary heights in the north direction on the column through O are in the ground state (l,m). $P_{a_n,\cdots,a_0}(l,m)$ satisfies obvious recursion relations and normalization condition,

$$\sum_{a_n} P_{a_n,\cdots,a_0}(l,m) = P_{a_{n-1},\cdots,a_0}(l,m), \quad \sum_{a_0} P_{a_n,\cdots,a_0}(l,m) = P_{a_n,\cdots,a_1}(l,m+1), \tag{4.66}$$

$$\sum_{a_0,\cdots,a_n} P_{a_n,\cdots,a_0}(l,m) = 1. \tag{4.67}$$

From (4.65) we have

$$\begin{aligned} P_{a_n,\cdots,a_0}(l,m) = Z_{l,m}^{-1} \operatorname{tr}_{\mathcal{H}_{l,m}^{(a_0)}} \Big(& D^{(a_0)}(u) C^{(a_0)}(u) \Phi_S^{(a_0,a_1)}(z) \cdots \Phi_S^{(a_{n-1},a_n)}(z) \\ & \times B^{(a_n)}(u) A^{(a_n)}(u) \Phi_N^{(a_n,a_{n-1})}(z) \cdots \Phi_N^{(a_1,a_0)}(z) \Big). \end{aligned} \tag{4.68}$$

By using

$$\Phi_S^{(a,b)}(w) B^{(b)}(u) A^{(b)}(u) = B^{(a)}(u) A^{(a)}(u) \Phi_W^{(a,b)}(w), \tag{4.69}$$

which is derived from (4.41) and (4.59), it can be written as

$$\begin{aligned} & P_{a_n,\cdots,a_0}(l,m) \\ = \; & Z_{l,m}^{-1} \operatorname{tr}_{\mathcal{H}_{l,m}^{(a_0)}} \Big(D^{(a_0)}(u) C^{(a_0)}(u) B^{(a_0)}(u) A^{(a_0)}(u) \\ & \qquad \times \Phi_W^{(a_0,a_1)}(z) \cdots \Phi_W^{(a_{n-1},a_n)}(z) \Phi_N^{(a_n,a_{n-1})}(z) \cdots \Phi_N^{(a_1,a_0)}(z) \Big) \\ = \; & Z_{l,m}^{-1} G_{a_0} \operatorname{tr}_{\mathcal{H}_{l,m}^{(a_0)}} \Big(x^{-4\lambda H_C^{(a_0)}} \Phi_W^{(a_0,a_1)}(z) \cdots \Phi_W^{(a_{n-1},a_n)}(z) \Phi_N^{(a_n,a_{n-1})}(z) \cdots \Phi_N^{(a_1,a_0)}(z) \Big), \end{aligned} \tag{4.70}$$

where the partition function $Z_{l,m}$ is given in (4.45). Using this expression and (4.61), the cyclic property of trace, (4.55) and (4.64), we can reproduce the recursion relations (4.66). By changing spectral parameters of VO's we can formulate more general quantities

$$\begin{aligned} &P_{a_n,\cdots,a_0}(l,m\,;{}^{z_n,\cdots,z_1}_{z'_n,\cdots,z'_1}) \\ &= Z_{l,m}^{-1}G_{a_0}\,\mathrm{tr}_{\mathcal{H}_{l,m}^{(a_0)}}\Big(x^{-4\lambda H_C^{(a_0)}}\,\Phi_W^{(a_0,a_1)}(z'_1)\cdots\Phi_W^{(a_{n-1},a_n)}(z'_n)\Phi_N^{(a_n,a_{n-1})}(z_n)\cdots\Phi_N^{(a_1,a_0)}(z_1)\Big). \end{aligned} \tag{4.71}$$

We remark that these LHP's are independent of $z=x^{2u}$ because

$$\begin{aligned} &\mathrm{tr}\Big(x^{-4\lambda H_C}\mathcal{O}(w_i)\Big) = \mathrm{tr}\Big(w^{H_C}w^{-H_C}x^{-4\lambda H_C}\mathcal{O}(w_i)\Big) \\ &= \mathrm{tr}\Big(x^{-4\lambda H_C}w^{-H_C}\mathcal{O}(w_i)w^{H_C}\Big) = \mathrm{tr}\Big(x^{-4\lambda H_C}\mathcal{O}(\tfrac{w_i}{w})\Big). \end{aligned} \tag{4.72}$$

By averaging $P_{a_n,\cdots,a_0}(l,m)$ over m we obtain

$$\begin{aligned} &P_{a_n,\cdots,a_0}(l) = \sum_m \frac{Z_{l,m}}{Z_l}P_{a_n,\cdots,a_0}(l,m) \\ &= Z_l^{-1}G_{a_0}\,\mathrm{tr}_{\mathcal{H}_l^{(a_0)}}\Big(x^{-4\lambda H_C^{(a_0)}}\,\Phi_W^{(a_0,a_1)}(z)\cdots\Phi_W^{(a_{n-1},a_n)}(z)\Phi_N^{(a_n,a_{n-1})}(z)\cdots\Phi_N^{(a_1,a_0)}(z)\Big), \end{aligned} \tag{4.73}$$

where Z_l is

$$Z_l = \sum_m Z_{l,m} = \sum_a G_a \mathrm{tr}_{\mathcal{H}_l^{(a)}}x^{-4\lambda H_C^{(a)}} = \sum_{a,m} G_a\chi_{l,m,a}(x^{-4\lambda}). \tag{4.74}$$

For later use we define the following quantities

$$\begin{aligned} &Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O}) = Z_{l,m}^{-1}G_{a_0}\,\mathrm{tr}_{\mathcal{H}_{l,m}^{(a_0)}}\Big(x^{-4\lambda H_C^{(a_0)}}\mathcal{O}_\Phi(z)\mathcal{O}\Big), \\ &Q_{a_n,\cdots,a_0}(l\,;z|\mathcal{O}) = \sum_m \frac{Z_{l,m}}{Z_l}Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O}) = Z_l^{-1}G_{a_0}\,\mathrm{tr}_{\mathcal{H}_l^{(a_0)}}\Big(x^{-4\lambda H_C^{(a_0)}}\mathcal{O}_\Phi(z)\mathcal{O}\Big), \end{aligned} \tag{4.75}$$

where $\mathcal{O}_\Phi(z) = \Phi_W^{(a_0,a_1)}(z)\cdots\Phi_W^{(a_{n-1},a_n)}(z)\Phi_N^{(a_n,a_{n-1})}(z)\cdots\Phi_N^{(a_1,a_0)}(z)$ and $\mathcal{O}$ is a linear map $\mathcal{O}:\mathcal{H}_{l,m}^{(a)}\to\mathcal{H}_{l,m}^{(a)}$. Note that $Q_{a_n,\cdots,a_0}(l,m\,;z|1) = P_{a_n,\cdots,a_0}(l,m)$. Like as $P_{a_n,\cdots,a_0}(l,m)$, $Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O})$ satisfies recursion relations. From (4.61) we have

$$\sum_{a_n} Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O}) = Q_{a_{n-1},\cdots,a_0}(l,m\,;z|\mathcal{O}). \tag{4.76}$$

If $\mathcal{O}$ satisfies $\Phi_N^{(a,b)}(z)\mathcal{O} = f(z)\mathcal{O}\Phi_N^{(a,b)}(z)$ where $f(z)$ is a function independent on a and b, then we have another recursion relation from (4.64)

$$\sum_{a_0} Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O}) = f(z)Q_{a_{n-1},\cdots,a_1}(l,m+1\,;z|\mathcal{O}). \tag{4.77}$$

Type I VO's are half-infinite transfer matrices but a 'physical' transfer matrix is the row-to-row (or column-to-column) transfer matrix discussed in the previous subsection.

The column-to-column transfer matrix $\mathcal{T}_{\rm col}$, which adds one column from west to east, is a linear map

$$\mathcal{T}_{\rm col}(u) : \widetilde{\mathcal{H}}_l \to \widetilde{\mathcal{H}}_l, \tag{4.78}$$

where $\widetilde{\mathcal{H}}_l$ is defined by

$$\widetilde{\mathcal{H}}_l = \bigoplus_k \widetilde{\mathcal{H}}_l^{(k)}, \quad \widetilde{\mathcal{H}}_l^{(k)} = \bigoplus_m \mathcal{H}'^{(k)}_{l,m} \otimes \mathcal{H}^{(k)}_{l,m}. \tag{4.79}$$

Here the numbering of sites in the south direction is the original one (i.e. increasingly from south to north $\cdots, -2, -1, 0, 1, 2, \cdots$) and the zeroth site is identified. Its matrix element is

$$\mathcal{T}_{\rm col}(u)_{\cdots a_{-1}a_0a_1\cdots,\cdots b_{-1}b_0b_1\cdots} = \Phi_S^{(a_0,b_0)}(z)_{a_0a_{-1}\cdots,b_0b_{-1}\cdots}\Phi_N^{(b_0,a_0)}(z)_{b_0b_1\cdots,a_0a_1\cdots}, \tag{4.80}$$

and its matrix form is

$$\Big(\mathcal{T}_{\rm col}(u)\Big)^{(a,b)} = \Phi_S^{(a,b)}(z) \otimes {}^t\Phi_N^{(b,a)}(z), \tag{4.81}$$

where a, b are heights on the zeroth site. $\mathcal{T}_{\rm col}$ with different spectral parameters commute each other

$$\Big[\mathcal{T}_{\rm col}(u), \mathcal{T}_{\rm col}(v)\Big] = 0, \tag{4.82}$$

because

$$\begin{aligned} &\Big(\mathcal{T}_{\rm col}(u_2)\mathcal{T}_{\rm col}(u_1)\Big)^{(a,c)} = \sum_b \Big(\mathcal{T}_{\rm col}(u_2)\Big)^{(a,b)}\Big(\mathcal{T}_{\rm col}(u_1)\Big)^{(b,c)} \\ &= \sum_b \Phi_S^{(a,b)}(z_2)\Phi_S^{(b,c)}(z_1) \otimes {}^t(\Phi_N^{(c,b)}(z_1)\Phi_N^{(b,a)}(z_2)) \\ &= \sum_g \Phi_S^{(a,g)}(z_1)\Phi_S^{(g,c)}(z_2) \otimes {}^t(\Phi_N^{(c,g)}(z_2)\Phi_N^{(g,a)}(z_1)) = \Big(\mathcal{T}_{\rm col}(u_1)\mathcal{T}_{\rm col}(u_2)\Big)^{(a,c)}, \end{aligned} \tag{4.83}$$

where we have used (4.49), (4.53) and (4.39).

When we consider the action of $\mathcal{T}_{\rm col}$ in this vertex operator approach, it is convenient to regard a state of $\widetilde{\mathcal{H}}_l \subset \mathcal{H}'_l \otimes \mathcal{H}_l$ as a linear map on $\mathcal{H}_l$. In general for two vector spaces V_i $(i = 1, 2)$ a vector of $V_1 \otimes V_2$ can be regarded as a linear map from V_2 to V_1 by $f_1 \otimes f_2 \in V_1 \otimes V_2 \mapsto f_1{}^tf_2$ where f_1 and f_2 are understood as column vectors. When there is an isomorphism $P : V_1 \to V_2$, a vector of $V_1 \otimes V_2$ can be regarded as a linear map on V_2,

$$|f\rangle = f_1 \otimes f_2 \in V_1 \otimes V_2 \mapsto Pf_1{}^tf_2 \in {\rm End}(V_2). \tag{4.84}$$

Similarly

$$\langle f| = {}^tf_1 \otimes {}^tf_2 \in (V_1 \otimes V_2)^* \mapsto f_2{}^tf_1P^{-1} \in {\rm End}(V_2). \tag{4.85}$$

Inner product (pairing) of two vectors $|f\rangle = f_1 \otimes f_2, |g\rangle = g_1 \otimes g_2 \in V_1 \otimes V_2$ can be expressed as a trace over V_2,

$$\langle f|g\rangle = {}^t f_1 g_1 \cdot {}^t f_2 g_2 = \mathrm{tr}_{V_2}\Big(f_2 {}^t f_1 P^{-1} \cdot P g_1 {}^t g_2\Big), \tag{4.86}$$

For $A = A_1 \otimes A_2$ ($A_i \in \mathrm{End}(V_i)$ $(i = 1, 2)$), $|f\rangle = f_1 \otimes f_2$ and $\langle f| = {}^t f_1 \otimes {}^t f_2$, we have the following correspondence

$$\begin{aligned} A|f\rangle = A_1 f_1 \otimes A_2 f_2 &\mapsto P(A_1 f_1)^t(A_2 f_2) = P A_1 P^{-1} \cdot P f_1 {}^t f_2 \cdot {}^t A_2, \\ \langle f|A = {}^t f_1 A_1 \otimes {}^t f_2 A_2 &\mapsto {}^t({}^t f_2 A_2)({}^t f_1 A_1) P^{-1} = {}^t A_2 \cdot f_2 {}^t f_1 P^{-1} \cdot P A_1 P^{-1}, \end{aligned} \tag{4.87}$$

and

$$\langle f|A|g\rangle = \langle f|Ag\rangle = \mathrm{tr}_{V_2}\Big(f_2 {}^t f_1 P^{-1} \cdot (P A_1 P^{-1} \cdot P g_1 {}^t g_2 \cdot {}^t A_2)\Big). \tag{4.88}$$

Using this correspondence $\mathcal{T}_{\mathrm{col}}$ acts on a vector $|f\rangle \in \widetilde{\mathcal{H}}_l$ as

$$\Big(\mathcal{T}_{\mathrm{col}}(u)|f\rangle\Big)^{(a)} = \sum_b P^{(a)} \Phi_S^{(a,b)}(z) P^{(b)-1} \cdot f \cdot \Phi_N^{(b,a)}(z), \tag{4.89}$$

where in the RHS f is understood as a linear map on $\mathcal{H}_l$, which does not change the central height ($f : \mathcal{H}_l^{(a)} \to \mathcal{H}_l^{(a)}$). The translation invariant vacuum state $|\mathrm{vac}\rangle$ is given by

$$|\mathrm{vac}\rangle^{(a)} = \sqrt{G_a} x^{-2\lambda H_C^{(a)}}. \tag{4.90}$$

In fact it is translationally invariant

$$\begin{aligned} \Big(\mathcal{T}_{\mathrm{col}}(u)|\mathrm{vac}\rangle\Big)^{(a)} &= \sum_b P^{(a)} \Phi_S^{(a,b)}(z) P^{(b)-1} \cdot \sqrt{G_b} x^{-2\lambda H_C^{(b)}} \cdot \Phi_N^{(b,a)}(z) \\ &= \sum_b \sqrt{G_b} x^{-2\lambda H_C^{(a)}} P^{(a)} \Phi_S^{(a,b)}(x^{2\lambda} z) P^{(b)-1} \Phi_N^{(b,a)}(z) \\ &= \sqrt{G_a} x^{-2\lambda H_C^{(a)}} \sum_b \Phi_W^{(a,b)}(z) \Phi_N^{(b,a)}(z) \\ &= \sqrt{G_a} x^{-2\lambda H_C^{(a)}} = |\mathrm{vac}\rangle^{(a)}. \end{aligned} \tag{4.91}$$

For $\widetilde{\mathcal{O}} = 1 \otimes {}^t\mathcal{O}$ where $\mathcal{O}$ is a linear map $\mathcal{O} : \mathcal{H}_l^{(a)} \to \mathcal{H}_l^{(b)}$, we have

$$\langle \mathrm{vac}|\widetilde{\mathcal{O}}|\mathrm{vac}\rangle = \sum_{a,b} \mathrm{tr}_{\mathcal{H}_l^a}\Big(\sqrt{G_a} x^{-2\lambda H_C^{(a)}} \cdot P^{(a)} P^{(b)-1} \cdot \sqrt{G_b} x^{-2\lambda H_C^{(b)}} \cdot \mathcal{O}^{(b,a)}\Big). \tag{4.92}$$

In particular when $\mathcal{O}$ is a linear map $\mathcal{O} : \mathcal{H}_l^{(a)} \to \mathcal{H}_l^{(a)}$, we have

$$\langle \mathrm{vac}|\widetilde{\mathcal{O}}|\mathrm{vac}\rangle = \sum_a {}^{(a)}\langle \mathrm{vac}|\widetilde{\mathcal{O}}|\mathrm{vac}\rangle^{(a)}, \quad {}^{(a)}\langle \mathrm{vac}|\widetilde{\mathcal{O}}|\mathrm{vac}\rangle^{(a)} = G_a \mathrm{tr}_{\mathcal{H}_l^{(a)}}\Big(x^{-4\lambda H_C^{(a)}} \mathcal{O}\Big), \tag{4.93}$$

$$\langle \mathrm{vac}|\mathrm{vac}\rangle = \sum_a G_a \mathrm{tr}_{\mathcal{H}_l^{(a)}}\Big(x^{-4\lambda H_C^{(a)}}\Big) = Z_l\,. \tag{4.94}$$

Therefore LHP (4.73) can be written as a vacuum expectation value with the fixed central value

$$P_{a_n,\cdots,a_0}(l) = \frac{{}^{(a_0)}\langle \mathrm{vac}|\widetilde{\mathcal{O}}|\mathrm{vac}\rangle^{(a_0)}}{\langle \mathrm{vac}|\mathrm{vac}\rangle}, \tag{4.95}$$

where $\widetilde{\mathcal{O}} = 1 \otimes {}^t\mathcal{O}$ with $\mathcal{O} = \Phi_W^{(a_0,a_1)}(z)\cdots\Phi_W^{(a_{n-1},a_n)}(z)\Phi_N^{(a_n,a_{n-1})}(z)\cdots\Phi_N^{(a_1,a_0)}(z)$.

Similarly the row-to-row transfer matrix, which adds one row from south to north, can be written as

$$\Big(\mathcal{T}_{\mathrm{row}}(u)\Big)^{(a,b)} = \Phi_E^{(a,b)}(z^{-1}) \otimes {}^t\Phi_W^{(b,a)}(z^{-1}) \ :\ \widetilde{\mathcal{H}}_l \to \widetilde{\mathcal{H}}_l, \tag{4.96}$$

and it acts on a state $|f\rangle \in \widetilde{\mathcal{H}}_l$ as

$$\Big(\mathcal{T}_{\mathrm{row}}(u)|f\rangle\Big)^{(a)} = \sum_b P^{(a)}\Phi_E^{(a,b)}(z^{-1})P^{(b)^{-1}} \cdot f \cdot \Phi_W^{(b,a)}(z^{-1}). \tag{4.97}$$

The state (4.90) is translationally invariant

$$\begin{aligned}
\Big(\mathcal{T}_{\mathrm{row}}(u)|\mathrm{vac}\rangle\Big)^{(a)} &= \sum_b P^{(a)}\Phi_E^{(a,b)}(z^{-1})P^{(b)^{-1}} \cdot \sqrt{G_b}\,x^{-2\lambda H_C^{(b)}} \cdot \Phi_W^{(b,a)}(z^{-1}) \\
&= \sum_b \sqrt{G_b}\,x^{-2\lambda H_C^{(a)}} P^{(a)}\Phi_E^{(a,b)}(x^{2\lambda}z^{-1})P^{(b)^{-1}}\Phi_W^{(b,a)}(z^{-1}) \\
&= \sqrt{G_a}\,x^{-2\lambda H_C^{(a)}} \sum_b \frac{G_b}{G_a}\Phi_N^{(a,b)}(x^{4\lambda}z^{-1})\Phi_W^{(b,a)}(z^{-1}) \\
&= \sqrt{G_a}\,x^{-2\lambda H_C^{(a)}} = |\mathrm{vac}\rangle^{(a)}.
\end{aligned} \tag{4.98}$$

We remark that these $\mathcal{T}_{\mathrm{col}}$ and $\mathcal{T}_{\mathrm{row}}$ are defined on an infinitely large lattice from the beginning in contrast to those in subsection 4.1. Their excited states are created by another type of vertex operators, type II VO's, from the vacuum state (Note that we need also the translation non-invariant 'vacuum' states to obtain a complete set of excited states) [50]. We will give an example in subsection 5.4. Commutation relations for type II VO will be given in subsection 4.4, section 5 and section 6.

In the rest of this subsection we assume that W has the crossing symmetry,

$$W\Big(\begin{matrix} b & d \\ a & c \end{matrix}\Big|\lambda - u\Big) = \sqrt{\frac{G_aG_d}{G_bG_c}}\,W\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big|u\Big). \tag{4.99}$$

Remark that unitarity (4.39) and crossing symmetry (4.99) imply the second inversion relation (4.40). We also remark that there are models which enjoy the second inversion relation but do not have crossing symmetry. Here we assume the crossing symmetry. Then graphical argument shows that CTM's in the thermodynamic limit are ((4.41) with $P^{(a)} = 1$)

$$\left.\begin{aligned} A^{(a)}(u) = C^{(a)}(u) &= x^{-2uH_C^{(a)}} \\ B^{(a)}(u) = D^{(a)}(u) &= \sqrt{G_a}\,x^{-2(\lambda-u)H_C^{(a)}} \end{aligned}\right. \ :\ \mathcal{H}_{l,m}^{(a)} \to \mathcal{H}_{l,m}^{(a)}, \tag{4.100}$$

and type I VO's are related each other,

$$\Phi_W^{(a,b)}(z) = \sqrt{\tfrac{G_b}{G_a}}\Phi_N^{(a,b)}(x^{2\lambda}z), \tag{4.101}$$
$$\Phi_S^{(a,b)}(z) = \Phi_N^{(a,b)}(z), \tag{4.102}$$
$$\Phi_E^{(a,b)}(z) = \sqrt{\tfrac{G_b}{G_a}}\Phi_N^{(a,b)}(x^{2\lambda}z) = \Phi_W^{(a,b)}(z). \tag{4.103}$$

We write

$$\Phi^{(a,b)}(z) = \Phi_N^{(a,b)}(z) \quad : \quad \mathcal{H}_{l,m}^{(b)} \to \mathcal{H}_{l,m+1}^{(a)}, \tag{4.104}$$
$$\Phi^{*(a,b)}(z) = \Phi_W^{(a,b)}(z) \quad : \quad \mathcal{H}_{l,m}^{(b)} \to \mathcal{H}_{l,m-1}^{(a)}. \tag{4.105}$$

Then $\Phi^{(a,b)}(z)$ and $\Phi^{*(a,b)}(z)$ enjoy the following properties,

$$\Phi^{*(a,b)}(z) = \sqrt{\tfrac{G_b}{G_a}}\Phi^{(a,b)}(x^{2\lambda}z), \tag{4.106}$$
$$\Phi^{(a,b)}(z_2)\Phi^{(b,c)}(z_1) = \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| u_1 - u_2\right)\Phi^{(a,g)}(z_1)\Phi^{(g,c)}(z_2), \tag{4.107}$$
$$w^{H_C}\Phi^{(a,b)}(z)w^{-H_C} = \Phi^{(a,b)}(wz), \tag{4.108}$$
$$\sum_g \Phi^{*(a,g)}(z)\Phi^{(g,a)}(z) = 1, \tag{4.109}$$
$$\Phi^{(a,b)}(z)\Phi^{*(b,c)}(z) = \delta_{a,c}. \tag{4.110}$$

The first equation is (4.101), the second one is (4.49), the third one is obtained by (4.55), the fourth one is (4.61), and the fifth one is derived by using (4.106), (4.107), (4.99), (4.38) and (4.109). Multi-point LHP (4.70) becomes

$$\begin{aligned} & P_{a_n,\cdots,a_0}(l,m) \\ = \; & Z_{l,m}^{-1} G_{a_0}\, \mathrm{tr}_{\mathcal{H}_{l,m}^{(a_0)}}\left(x^{-4\lambda H_C^{(a_0)}}\Phi^{*(a_0,a_1)}(z)\cdots\Phi^{*(a_{n-1},a_n)}(z)\Phi^{(a_n,a_{n-1})}(z)\cdots\Phi^{(a_1,a_0)}(z)\right). \end{aligned} \tag{4.111}$$

Since the Boltzmann weight is invariant under 180° rotation $W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = W\left(\begin{matrix} d & c \\ b & a \end{matrix}\middle| u\right)$, $P_{a_n,\cdots,a_0}(l)$ satisfies

$$P_{a_n,\cdots,a_0}(l) = P_{a_0,\cdots,a_n}(l). \tag{4.112}$$

Eq. (4.89) becomes

$$\Big(\mathcal{T}_{\mathrm{col}}(u)|f\rangle\Big)^{(a)} = \sum_b \Phi^{(a,b)}(z)\cdot f\cdot \Phi^{(b,a)}(z). \tag{4.113}$$

In section 5 and subsection 6.4 we will calculate LHP's by using bosonization technique for the VO's.

4.3 Introduction to quasi-Hopf algebra

In this subsection we illustrate an outline of a quasi-Hopf algebra. For more details we refer the readers to refs.[11, 26]

In quantum mechanics, we know an addition of angular momentums very well. For two particles system the total angular momentum $\vec{J}$ is obtained simply by an addition of each angular momentum $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$,

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}. \tag{4.114}$$

In mathematics, this formula is written in the following way; $\vec{J}^{(1)}$ acts on the representation space V_1, $\vec{J}^{(2)}$ acts on V_2, and the total angular momentum $\vec{J}$ acts on $V_1 \otimes V_2$ by,

$$\vec{J} = \vec{J} \otimes 1 + 1 \otimes \vec{J}. \tag{4.115}$$

This is called the tensor product representation of Lie algebra $so(3)$. If a system has rotational symmetry, for example the Heisenberg spin chain (XXX spin chain), one can apply the representation theory of the rotational group $SO(3)$ (or its Lie algebra $so(3)$) to it. But if the system is perturbed and looses the rotational symmetry, then one can not apply $so(3)$ to it. Some models, however, have a good property. For example the XXZ spin chain has the same degeneracy of energy as the XXX spin chain. To treat such models we need some deformation of the Lie algebra or some deformation of the tensor product representation.

1. algebra and coalgebra

Let us begin with the definitions of an algebra and a coalgebra. For simplicity we take the complex field $\mathbb{C}$ as a base field. An algebra A is a vector space with two operations, product (multiplication) m and unit u, which satisfy

$$\begin{array}{ccccl}
A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{id}} & A\otimes A & \text{product} & m : A\otimes A \to A \\
{\scriptstyle \mathrm{id}\otimes m}\downarrow & & \downarrow{\scriptstyle m} & \text{unit} & u : \mathbb{C} \to A \\
A\otimes A & \xrightarrow[m]{} & A & \text{associativity} & m\circ(m\otimes \mathrm{id}) = m\circ(\mathrm{id}\otimes m),
\end{array} \tag{4.116}$$

and $m\circ(\mathrm{id}\otimes u) = \mathrm{id} = m\circ(u\otimes \mathrm{id})$ ($A\otimes\mathbb{C}$, A and $\mathbb{C}\otimes A$ are identified). If we write $m(a\otimes b) = ab$, the associativity becomes a usual form $(ab)c = a(bc)$.

A coalgebra is defined by reversing the arrows. A coalgebra A is a vector space with two operations, coproduct Δ and counit ε, which satisfy

$$\begin{array}{ccccl}
A\otimes A\otimes A & \xleftarrow{\Delta\otimes \mathrm{id}} & A\otimes A & \text{coproduct} & \Delta : A \to A\otimes A \\
{\scriptstyle \mathrm{id}\otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} & \text{counit} & \varepsilon : A \to \mathbb{C} \\
A\otimes A & \xleftarrow[\Delta]{} & A & \text{coassociativity} & (\Delta\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta,
\end{array} \tag{4.117}$$

and $(\mathrm{id}\otimes\varepsilon)\circ\Delta=\mathrm{id}=(\varepsilon\otimes\mathrm{id})\circ\Delta$ ($A\otimes\mathbb{C}$, A and $\mathbb{C}\otimes A$ are identified).

Let us introduce σ ($\sigma : A\otimes A\to A\otimes A$, $\sigma(a\otimes b)=b\otimes a$) and define $m'=m\circ\sigma$ and $\Delta'=\sigma\circ\Delta$. An algebra is called commutative if $m'=m$, and a coalgebra is called cocommutative if $\Delta'=\Delta$.

2. Hopf algebra

A Hopf algebra is a set $(A,m,u,\Delta,\varepsilon,S)$ satisfying the following conditions: A is an algebra and a coalgebra; m, u, Δ, ε are homomorphism; antipode $S:A\to A$ satisfies $m\circ(S\otimes\mathrm{id})\circ\Delta=u\circ\varepsilon=m\circ(\mathrm{id}\otimes S)\circ\Delta$. S is an anti-homomorphism.

We give two examples of a Hopf algebra, a group G and a Lie algebra $\mathfrak{g}$, exactly speaking a function algebra of group $\mathrm{Fun}(G)$ and an enveloping algebra of Lie algebra $U(\mathfrak{g})$ respectively. Their Hopf algebra structures are

$$\begin{array}{lll} & \mathrm{Fun}(G)=\mathrm{Map}(G,\mathbb{C}) & U(\mathfrak{g}) \\ \text{product} & (f_1f_2)(x)=f_1(x)f_2(x) & XY \\ \text{unit} & (u(a))(x)=a & u(a)=a1 \\ \text{coproduct} & (\Delta(f))(x_1,x_2)=f(x_1x_2) & \Delta(X)=X\otimes 1+1\otimes X \\ \text{counit} & (\varepsilon(f))(x)=f(e) & \varepsilon(X)=0 \\ \text{antipode} & (S(f))(x)=f(x^{-1}) & S(X)=-X, \end{array} \tag{4.118}$$

and $(m(g))(x)=g(x,x)$ for $g(x_1,x_2)\in\mathrm{Map}(G\times G,\mathbb{C})$, and $\Delta(1)=1\otimes 1$, $\varepsilon(1)=1$, $S(1)=1$ for $U(\mathfrak{g})$. Roughly speaking Δ, ε and S correspond to

$$\begin{array}{llll} & & \mathrm{Fun}(G) & U(\mathfrak{g}) \\ \Delta & \leftrightarrow & \text{product of } G & \text{tensor product rep. of } U(\mathfrak{g}) \\ \varepsilon & \leftrightarrow & \text{unit element of } G & \text{trivial rep. of } U(\mathfrak{g}) \\ S & \leftrightarrow & \text{inverse element of } G & \text{contragredient rep. of } U(\mathfrak{g}). \end{array} \tag{4.119}$$

$\mathrm{Fun}(G)$ is a commutative (and non-cocommutative) Hopf algebra and $U(\mathfrak{g})$ is a cocommutative (and non-commutative) Hopf algebra. Non-commutative and non-cocommutative Hopf algebra may be regarded as an extension (deformation) of group or Lie algebra in this sense. This is the idea of quantum group (quantum algebra).

The quantum group (quantum algebra) is a Hopf algebra. We give an example of the quantum group, $U_q(\mathfrak{sl}_2)$, which is a deformation of $U(\mathfrak{sl}_2)$. $U_q(\mathfrak{sl}_2)$ is generated by $t=q^h$, e and f, which satisfy

$$\begin{array}{llll} [h,e]=2e & \Delta(h)=h\otimes 1+1\otimes h & \varepsilon(h)=0 & S(h)=-h \\ [h,f]=-2f & \Delta(e)=e\otimes 1+t\otimes e & \varepsilon(e)=0 & S(e)=-t^{-1}e \\ [e,f]=\frac{t-t^{-1}}{q-q^{-1}} & \Delta(f)=f\otimes t^{-1}+1\otimes f & \varepsilon(f)=0 & S(f)=-ft. \end{array} \tag{4.120}$$

This quantum algebra appears in the XXZ spin chain as a symmetry,

$$
\begin{aligned}
&[H_{\mathrm{XXZ}}, U_q(\mathfrak{sl}_2)] = 0,\\
&H_{\mathrm{XXZ}} = J\sum_{i=1}^{N-1}\Big(s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + \frac{q+q^{-1}}{2}s_i^z s_{i+1}^z\Big) + J\frac{q-q^{-1}}{4}(s_1^z - s_N^z),\\
&h = \sum_i 2s_i^z, \quad e = \sum_i q^{\sum_{j<i}2s_j^z}s_i^+, \quad f = \sum_i s_i^- q^{-\sum_{j>i}2s_j^z},
\end{aligned}
\tag{4.121}
$$

where $\vec{s} = \frac{1}{2}\vec{\sigma}$ and $s^{\pm} = s^1 \pm is^2$. In the $q \to 1$ limit, $U_q(\mathfrak{sl}_2)$ reduces to $U(\mathfrak{sl}_2)$.

3. quasi-triangular Hopf algebra

Using the coproduct, a tensor product representation of two representations (π_i, V_i) $(i = 1,2)$ of the Hopf algebra can be defined in the following way,

$$\Big((\pi_1\otimes\pi_2)\circ\Delta, V_1\otimes V_2\Big). \tag{4.122}$$

Coassociativity implies the isomorphism,

$$(V_1\otimes V_2)\otimes V_3 \cong V_1\otimes(V_2\otimes V_3) \quad (\text{as } A \text{ module}). \tag{4.123}$$

But cocommutativity does not hold in general, so the following isomorphism depends on the detail of the Hopf algebra:

$$V_1\otimes V_2 \overset{?}{\cong} V_2\otimes V_1 \quad (\text{as } A \text{ module}). \tag{4.124}$$

Of course we have $V_1\otimes V_2 \cong V_2\otimes V_1$ as vector space, by $P_{V_1V_2} : V_1\otimes V_2 \overset{\cong}{\to} V_2\otimes V_1$, $P_{V_1V_2}(v_1\otimes v_2) = v_2\otimes v_1$. But the problem is the commutativity of the action of A and $P_{V_1V_2}$.

Drinfeld considered the situation that the isomorphism (4.124) does hold. A quasi-triangular Hopf algebra $(A, m, u, \Delta, \varepsilon, S, \mathcal{R})$ (we abbreviate it as $(A, \Delta, \mathcal{R})$) is a Hopf algebra with a universal R matrix $\mathcal{R}$, which satisfies

$$
\begin{aligned}
\mathcal{R} \;&\in A\otimes A : \underline{\text{universal } R \text{ matrix}}\\
&\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad (\forall a\in A),\\
&(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(23)}, \quad (\varepsilon\otimes\mathrm{id})\mathcal{R} = 1,\\
&(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(12)}, \quad (\mathrm{id}\otimes\varepsilon)\mathcal{R} = 1.
\end{aligned}
\tag{4.125}
$$

Then we have an intertwiner

$$R_{V_1V_2} = P_{V_1V_2}\circ(\pi_1\otimes\pi_2)(\mathcal{R}) : V_1\otimes V_2 \xrightarrow{\cong} V_2\otimes V_1 \quad (\text{as } A \text{ module}), \tag{4.126}$$

and $\mathcal{R}$ satisfies the Yang-Baxter equation,

$$\mathcal{R}^{(12)}\mathcal{R}^{(13)}\mathcal{R}^{(23)} = \mathcal{R}^{(23)}\mathcal{R}^{(13)}\mathcal{R}^{(12)}. \tag{4.127}$$

4. quasi-triangular quasi-Hopf algebra

As presented in **2** the quantum group is obtained from the Lie algebra by relaxing one condition, cocommutativity. Here we relax one more condition, coassociativity. Coassociativity (4.117) is modified by a coassociator Φ in the following way,

$$\begin{aligned}
\Phi\ \ &\in A\otimes A\otimes A : \underline{\text{coassociator}}\\
&(\mathrm{id}\otimes\Delta)\Delta(a)=\Phi(\Delta\otimes\mathrm{id})\Delta(a)\Phi^{-1}\quad(\forall a\in A),\\
&(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Phi\cdot(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Phi=(1\otimes\Phi)\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Phi\cdot(\Phi\otimes 1),\\
&(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\Phi=1.
\end{aligned}\tag{4.128}$$

A $\underline{\text{quasi-triangular quasi-Hopf algebra}}$ $(A,m,u,\Delta,\varepsilon,\Phi,S,\alpha,\beta,\mathcal{R})$ (we abbreviate it as $(A,\Delta,\Phi,\mathcal{R})$) satisfies (4.128) and

$$\begin{aligned}
\mathcal{R}\ \ &\in A\otimes A\\
&\Delta'(a)=\mathcal{R}\Delta(a)\mathcal{R}^{-1}\quad(\forall a\in A),\\
&(\Delta\otimes\mathrm{id})\mathcal{R}=\Phi^{(312)}\mathcal{R}^{(13)}\Phi^{(132)^{-1}}\mathcal{R}^{(23)}\Phi^{(123)},\\
&(\mathrm{id}\otimes\Delta)\mathcal{R}=\Phi^{(231)^{-1}}\mathcal{R}^{(13)}\Phi^{(213)}\mathcal{R}^{(12)}\Phi^{(123)^{-1}},
\end{aligned}\tag{4.129}$$

where $\Phi^{(312)}$ means $\Phi^{(312)}=\sum_i Z_i\otimes X_i\otimes Y_i$ for $\Phi=\sum_i X_i\otimes Y_i\otimes Z_i$. $\alpha,\beta\in A$ satisfy

$$\sum_i X_i\beta S(Y_i)\alpha Z_i=1,\tag{4.130}$$

$$\sum_i S(b_i)\alpha c_i=\varepsilon(a)\alpha,\quad \sum_i b_i\beta S(c_i)=\varepsilon(a)\beta\quad(\forall a\in A),\tag{4.131}$$

where $\Delta(a)=\sum_i b_i\otimes c_i$. Then $\mathcal{R}$ enjoys the Yang-Baxter type equation,

$$\mathcal{R}^{(12)}\Phi^{(312)}\mathcal{R}^{(13)}\Phi^{(132)^{-1}}\mathcal{R}^{(23)}\Phi^{(123)}=\Phi^{(321)}\mathcal{R}^{(23)}\Phi^{(231)^{-1}}\mathcal{R}^{(13)}\Phi^{(213)}\mathcal{R}^{(12)}.\tag{4.132}$$

A quasi-Hopf algebra with $\Phi=1$ is nothing but a Hopf algebra.

5. twist

Quasi-Hopf algebras admit an important operation, <u>twist</u>. For any invertible element $F\in A\otimes A$ ($(\varepsilon\otimes\mathrm{id})F=(\mathrm{id}\otimes\varepsilon)F=1$), which is called as a twistor, there is a map from quasi-Hopf algebras to quasi-Hopf algebras:

$$\begin{aligned}
\text{quasi-Hopf algebra}\ &\longrightarrow\ \text{quasi-Hopf algebra}\\
(A,\Delta,\Phi,\mathcal{R})\ &\xrightarrow{F}\ (A,\widetilde{\Delta},\widetilde{\Phi},\widetilde{\mathcal{R}}).
\end{aligned}\tag{4.133}$$

New coproduct, coassociator, R matrix etc. are given by

$$\begin{aligned}
F \quad & \in A \otimes A : \underline{\text{twistor}} \\
& \widetilde{\Delta} = F\Delta(a)F^{-1} \quad (\forall a \in A), \\
& \widetilde{\Phi} = \Big(F^{(23)}(\mathrm{id} \otimes \Delta)F\Big)\Phi\Big(F^{(12)}(\Delta \otimes \mathrm{id})F\Big)^{-1}, \\
& \widetilde{\mathcal{R}} = F^{(21)}\mathcal{R}F^{-1}, \\
& \widetilde{\varepsilon} = \varepsilon, \quad \widetilde{S} = S, \quad \widetilde{\alpha} = \sum_i S(d_i)\alpha e_i, \quad \tilde{\beta} = \sum_i f_i \beta S(g_i),
\end{aligned} \tag{4.134}$$

where $\sum_i d_i \otimes e_i = F^{-1}$ and $\sum_i f_i \otimes g_i = F$. We remark that an algebra A itself is unchanged. If a twistor F satisfies the cocycle condition, this twist operation maps a Hopf algebra to a Hopf algebra. For a general twistor F, however, a Hopf algebra is mapped to a quasi-Hopf algebra:

$$\begin{aligned}
\text{Hopf algebra} \quad & \longrightarrow \quad \text{quasi-Hopf algebra} \\
(A, \Delta, \mathcal{R}) \quad & \stackrel{F}{\longrightarrow} \quad (A, \widetilde{\Delta}, \widetilde{\Phi}, \widetilde{\mathcal{R}}).
\end{aligned} \tag{4.135}$$

Let H be an Abelian subalgebra of A, with the product written additively. A twistor $F(\lambda) \in A \otimes A$ depending on $\lambda \in H$ is a shifted cocycle if it satisfies the relation (shifted cocycle condition),

$$\begin{aligned}
& F(\lambda) : \underline{\text{shifted cocycle}} \\
\Leftrightarrow \quad & F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda),
\end{aligned} \tag{4.136}$$

for some $h \in H$.

When a twistor $F(\lambda)$ satisfies the shifted cocycle condition, we obtain a quasi-triangular quasi-Hopf algebra from a quasi-triangular Hopf algebra by twisting,

$$\begin{aligned}
\text{Hopf algebra} \quad & \longrightarrow \quad \text{quasi-Hopf algebra} \\
(A, \Delta, \mathcal{R}) \quad & \xrightarrow{F(\lambda)} \quad (A, \Delta_\lambda, \Phi(\lambda), \mathcal{R}(\lambda)),
\end{aligned} \tag{4.137}$$

and we have

$$\begin{aligned}
& \Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}, \\
& (\Delta_\lambda \otimes \mathrm{id})\mathcal{R}(\lambda) = \Phi^{(312)}(\lambda)\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}), \\
& (\mathrm{id} \otimes \Delta_\lambda)\mathcal{R}(\lambda) = \mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda)\Phi^{(123)}(\lambda)^{-1},
\end{aligned} \tag{4.138}$$

and R matrix satisfies the dynamical Yang-Baxter equation,

$$\mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{4.139}$$

4.4 Elliptic quantum groups

The solutions of YBE (R-matrices) are classified into two types, vertex-type and face-type. Corresponding to these two there are two types of elliptic quantum groups (algebras).

The vertex-type elliptic algebras are associated with the R-matrix $R(u)$ of Baxter [51] and Belavin [52]. The first example of this sort is the Sklyanin algebra [53], designed as an elliptic deformation of the Lie algebra $\mathfrak{sl}_2$. It is presented by the 'RLL'-relation

$$R^{(12)}(u_1-u_2)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2), \tag{4.140}$$

together with a specific choice of the form for $L(u)$. R and L depend on an elliptic modulus r. Foda et al.[23] proposed its affine version, $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$,

$$R^{(12)}(u_1-u_2,r)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2,r-c), \tag{4.141}$$

whose main point of is the shift of r by a central element c.

The face-type algebras are based on R-matrices of Andrews, Baxter, Forrester [54] and generalizations [55, 8, 10]. In this case, besides the elliptic modulus, R and L depend also on extra parameter(s) λ. As Felder has shown [24], the RLL relation undergoes a 'dynamical' shift by elements h of the Cartan subalgebra,

$$\begin{aligned}&R^{(12)}(u_1-u_2,\lambda+h)L^{(1)}(u_1,\lambda)L^{(2)}(u_2,\lambda+h^{(1)})\\&\quad=L^{(2)}(u_2,\lambda)L^{(1)}(u_1,\lambda+h^{(2)})R^{(12)}(u_1-u_2,\lambda),\end{aligned} \tag{4.142}$$

and the YBE itself is modified to a dynamical one (4.139), (4.20). As we shall see, a central extension of this algebra is obtained by introducing further a shift of the elliptic modulus analogous to (4.141) (see (4.196)-(4.197) and the remark following them).

These two algebras, RLL relations (4.141) and (4.142), seemed to be different but Frønsdal [25] pointed out that they have a common structure; they are quasi-Hopf algebras obtained by twisting quantum affine algebras. Namely, there exist two types of twistors which give rise to different comultiplications on the quantum affine algebras $U_q(\mathfrak{g})$, and the resultant quasi-Hopf algebras are nothing but the two types of elliptic quantum groups. In [26] explicit formulas for the twistors satisfying the shifted cocycle condition were presented and two types of elliptic quantum groups, $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{B}_{q,\lambda}(\mathfrak{g})$, were defined:

$$\begin{array}{ccccc}\text{type} & \text{twistor} & \text{Hopf algebra} & & \text{quasi-Hopf algebra}\\ \text{face} & F(\lambda) & U_q(\mathfrak{g}) & \xrightarrow{F(\lambda)} & \mathcal{B}_{q,\lambda}(\mathfrak{g})\\ \text{vertex} & E(r) & U_q(\widehat{\mathfrak{sl}}_n) & \xrightarrow{E(r)} & \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n).\end{array} \tag{4.143}$$

The face-type algebra has also been given an alternative formulation in terms of the Drinfeld currents. This is the approach adopted by Enriquez and Felder [56] and Konno [57]. The Drinfeld currents are suited to deal with infinite dimensional representations.

1. quantum group

First let us fix the notation. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ [58]. We fix an invariant inner product $(\ ,\ )$ on the Cartan subalgebra $\mathfrak{h}$ and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via $(\ ,\ )$. If $\{\alpha_i\}_{i\in I}$ denotes the set of simple roots, then $(\alpha_i, \alpha_j) = d_i a_{ij}$, where $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$.

Consider the corresponding quantum group $U = U_q(\mathfrak{g})$. Hereafter we fix a complex number $q \neq 0$, $|q| < 1$. The algebra U has generators e_i, f_i $(i \in I)$ and h $(h \in \mathfrak{h})$, satisfying the standard relations

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h}), \tag{4.144}$$

$$[h, e_i] = (h, \alpha_i)e_i, \qquad [h, f_i] = -(h, \alpha_i)f_i \qquad (i \in I, h \in \mathfrak{h}), \tag{4.145}$$

$$[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \qquad (i, j \in I), \tag{4.146}$$

and the Serre relations which we omit. In (4.146) we have set $q_i = q^{d_i}$, $t_i = q^{\alpha_i}$. We adopt the Hopf algebra structure given as follows.

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \tag{4.147}$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \tag{4.148}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(h) = 0, \tag{4.149}$$

$$S(e_i) = -t_i^{-1}e_i, \quad S(f_i) = -f_i t_i, \quad S(h) = -h, \tag{4.150}$$

where $i \in I$ and $h \in \mathfrak{h}$.

Let $\mathcal{R} \in U^{\otimes 2}$ denote the universal R matrix of U. It has the form

$$\mathcal{R} = q^{-T}\mathcal{C}, \tag{4.151}$$

$$\mathcal{C} = \sum_{\beta\in Q^+} q^{(\beta,\beta)}\Big(q^{-\beta} \otimes q^{\beta}\Big)\mathcal{C}_\beta = 1 - \sum_{i\in I}(q_i - q_i^{-1})e_i t_i^{-1} \otimes t_i f_i + \cdots. \tag{4.152}$$

Here the notation is as follows. Take a basis $\{h_l\}$ of $\mathfrak{h}$, and its dual basis $\{h^l\}$. Then

$$T = \sum_l h_l \otimes h^l \tag{4.153}$$

denotes the canonical element of $\mathfrak{h} \otimes \mathfrak{h}$. The element $\mathcal{C}_\beta = \sum_j u_{\beta,j} \otimes u^j_{-\beta}$ is the canonical element of $U^+_\beta \otimes U^-_{-\beta}$ with respect to a certain Hopf pairing, where U^+ (resp. U^-) denotes the subalgebra of U generated by the e_i (resp. f_i), and $U^\pm_{\pm\beta}$ $(\beta \in Q^+)$ signifies the homogeneous components with respect to the natural gradation by $Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. (For the details the reader is referred e.g. to [11, 59].) Basic properties of the universal R matrix is given in (4.125).

2. face type algebra

Let $\rho \in \mathfrak{h}$ be an element such that $(\rho, \alpha_i) = d_i$ for all $i \in I$. Let ϕ be an automorphism of U given by

$$\phi = \mathrm{Ad}(q^{\frac{1}{2}\sum_l h_l h^l - \rho}), \tag{4.154}$$

where $\{h_l\}$, $\{h^l\}$ are as in (4.153). In other words,

$$\phi(e_i) = e_i t_i, \quad \phi(f_i) = t_i^{-1} f_i, \quad \phi(q^h) = q^h. \tag{4.155}$$

Since

$$\mathrm{Ad}(q^T) \circ (\phi \otimes \phi) = \mathrm{Ad}(q^{\frac{1}{2}\sum_l \Delta(h_l h^l) - \Delta(\rho)}), \tag{4.156}$$

we have

$$\mathrm{Ad}(q^T) \circ (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi. \tag{4.157}$$

For $\lambda \in \mathfrak{h}$, introduce an automorphism

$$\varphi_\lambda = \mathrm{Ad}(q^{\sum_l h_l h^l + 2(\lambda - \rho)}) = \phi^2 \circ \mathrm{Ad}(q^{2\lambda}). \tag{4.158}$$

Then the expression $(\varphi_\lambda \otimes \mathrm{id})(q^T \mathcal{R})$ is a formal power series in the variables $x_i = q^{2(\lambda, \alpha_i)}$ $(i \in I)$ of the form $1 - \displaystyle\sum_i (q_i - q_i^{-1}) x_i e_i t_i \otimes t_i f_i + \cdots$. We define the twistor $F(\lambda)$ as follows.

[Face type twistor]

$$F(\lambda) = \prod_{k \geq 1}^{\curvearrowleft} \Big(\varphi_\lambda^k \otimes \mathrm{id}\Big) \Big(q^T \mathcal{R}\Big)^{-1}. \tag{4.159}$$

Here and after, we use the ordered product symbol $\displaystyle\prod_{k \geq 1}^{\curvearrowleft} A_k = \cdots A_3 A_2 A_1$. Note that the k-th factor in the product (4.159) is a formal power series in the x_i^k with leading term 1, and hence the infinite product makes sense. Then the face type twistor (4.159) satisfies the shifted cocycle condition

$$F^{(12)}(\lambda)(\Delta \otimes \mathrm{id}) F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta) F(\lambda), \tag{4.160}$$

and

$$(\varepsilon \otimes \mathrm{id})\, F(\lambda) = (\mathrm{id} \otimes \varepsilon)\, F(\lambda) = 1. \tag{4.161}$$

A proof of this property and examples are given in [26]. In (4.160), if $\lambda = \sum_l \lambda_l h^l$, then $\lambda + h^{(1)}$ means $\sum_l (\lambda_l + h_l^{(1)}) h^l$. Hence we have, for example,

$$\begin{aligned}
\mathrm{Ad}(q^{2lT^{(12)}}) F_k^{(23)}(\lambda) &= F_k^{(23)}(\lambda + l h^{(1)}), \\
\mathrm{Ad}(q^{2lT^{(13)}}) F_k^{(23)}(\lambda) &= F_k^{(23)}(\lambda - l h^{(1)}).
\end{aligned}$$

For convenience, let us give a name to the quasi-Hopf algebra associated with the twistor (4.159).

[Face type algebra] *We define the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ of face type to be the set $(U_q(\mathfrak{g}), \Delta_\lambda, \Phi(\lambda), \mathcal{R}(\lambda))$ together with $\alpha_\lambda = \sum_i S(d_i)e_i$, $\beta_\lambda = \sum_i f_i S(g_i)$, the antiautomorphism S defined by (4.150) and ε defined by (4.149),*

$$\Delta_\lambda(a) = F^{(12)}(\lambda)\,\Delta(a)\,F^{(12)}(\lambda)^{-1}, \tag{4.162}$$

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\,\mathcal{R}\,F^{(12)}(\lambda)^{-1}, \tag{4.163}$$

$$\Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}, \tag{4.164}$$

and $\sum_i d_i \otimes e_i = F(\lambda)^{-1}$, $\sum_i f_i \otimes g_i = F(\lambda)$.

Let us consider the case where $\mathfrak{g}$ is of affine type, in which we are mainly interested. Let c be the canonical central element and d the scaling element. We set

$$\lambda - \rho = rd + s'c + \bar{\lambda} - \bar{\rho} \quad (r, s' \in \mathbb{C}), \tag{4.165}$$

where $\bar{\lambda}$ stands for the classical part of $\lambda \in \mathfrak{h}$. Denote by $\{\bar{h}_j\}$, $\{\bar{h}^j\}$ the classical part of the dual basis of $\mathfrak{h}$. Since c is central, φ_λ is independent of s'. Writing $p = q^{2r}$, we have

$$\varphi_\lambda = \mathrm{Ad}(p^d q^{2cd}) \circ \bar{\varphi}_\lambda, \quad \bar{\varphi}_\lambda = \mathrm{Ad}(q^{\sum \bar{h}_j \bar{h}^j + 2(\bar{\lambda} - \bar{\rho})}). \tag{4.166}$$

Set further

$$\mathcal{R}(z) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}), \tag{4.167}$$

$$F(z, \lambda) = \mathrm{Ad}(z^d \otimes 1)(F(\lambda)), \tag{4.168}$$

$$\mathcal{R}(z, \lambda) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}(\lambda)) = \sigma\Big(F(z^{-1}, \lambda)\Big)\mathcal{R}(z)F(z, \lambda)^{-1}. \tag{4.169}$$

(4.167) and (4.168) are formal power series in z, whereas (4.169) contains both positive and negative powers of z. Note that $q^{c\otimes d + d\otimes c}\mathcal{R}(z)|_{z=0}$ reduces to the universal R matrix of $U_q(\bar{\mathfrak{g}})$ corresponding to the underlying finite dimensional Lie algebra $\bar{\mathfrak{g}}$. From the definition (4.159) of $F(\lambda)$ we have the difference equation

$$F(pq^{2c^{(1)}} z, \lambda) = (\bar{\varphi}_\lambda \otimes \mathrm{id})^{-1}\Big(F(z, \lambda)\Big) \cdot q^T \mathcal{R}(pq^{2c^{(1)}} z), \tag{4.170}$$

with the initial condition $F(0, \lambda) = F_{\bar{\mathfrak{g}}}(\bar{\lambda})$, where $F_{\bar{\mathfrak{g}}}(\bar{\lambda})$ signifies the twistor corresponding to $\bar{\mathfrak{g}}$.

3. vertex type algebra

When $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$, it is possible to construct a different type of twistor. We call it *vertex type*.

Let us write $h_i = \alpha_i$ $(i = 0, \dots, n-1)$. A basis of $\mathfrak{h}$ is $\{h_0, \dots, h_{n-1}, d\}$. The element d gives the homogeneous grading,

$$[d, e_i] = \delta_{i0} e_i, \quad [d, f_i] = -\delta_{i0} f_i, \tag{4.171}$$

for all $i = 0, \dots, n-1$. Let the dual basis be $\{\Lambda_0, \dots, \Lambda_{n-1}, c\}$. The Λ_i are the fundamental weights and c is the canonical central element. Let τ be the automorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ such that

$$\tau(e_i) = e_{i+1 \bmod n}, \quad \tau(f_i) = f_{i+1 \bmod n}, \quad \tau(h_i) = h_{i+1 \bmod n} \tag{4.172}$$

and $\tau^n = \mathrm{id}$. Then we have

$$\tau(\Lambda_i) = \Lambda_{i+1 \bmod n} - \frac{n-1-2i}{2n} c. \tag{4.173}$$

The element $\rho = \sum_{i=0}^{n-1} \Lambda_i$ is invariant under τ. It gives the principal grading

$$[\rho, e_i] = e_i, \quad [\rho, f_i] = -f_i, \tag{4.174}$$

for all $i = 0, \dots, n-1$. Note also that

$$(\tau \otimes \tau) \circ \Delta = \Delta \circ \tau, \quad (\tau \otimes \tau)(\mathcal{C}_\beta) = \mathcal{C}_{\tau(\beta)}. \tag{4.175}$$

For $r \in \mathbb{C}$, we introduce an automorphism

$$\widetilde{\varphi}_r = \tau \circ \mathrm{Ad}\left(q^{\frac{2(r+c)}{n}\rho}\right), \tag{4.176}$$

and set

$$\widetilde{T} = \frac{1}{n}\left(\rho \otimes c + c \otimes \rho - \frac{n^2-1}{12} c \otimes c\right). \tag{4.177}$$

Then $\left(\widetilde{\varphi}_r \otimes \mathrm{id}\right)\left(q^{\widetilde{T}}\mathcal{R}\right)^{-1}$ is a formal power series in $p^{\frac{1}{n}}$ where $p = q^{2r}$. Unlike the face case, this is a formal series with a non-trivial leading term $q^{T-\widetilde{T}}(1 + \cdots)$. Nevertheless, the n-fold product

$$\prod_{n \geq k \geq 1}^{\curvearrowleft} \left(\widetilde{\varphi}_r^k \otimes \mathrm{id}\right)\left(q^{\widetilde{T}}\mathcal{R}\right)^{-1} \tag{4.178}$$

takes the form $1 + \cdots$, because of the relation

$$\sum_{k=1}^{n} \left(\tau^k \otimes \mathrm{id}\right)\left(T - \widetilde{T}\right) = 0. \tag{4.179}$$

We now define the vertex type twistor $E(r)$ as follows.

[Vertex type twistor]

$$E(r) = \prod_{k \geq 1}^{\curvearrowleft} \left(\widetilde{\varphi}_r^k \otimes \mathrm{id}\right)\left(q^{\widetilde{T}}\mathcal{R}\right)^{-1}. \tag{4.180}$$

The infinite product $\widehat{\prod_{k\geq 1}}$ is to be understood as $\lim_{N\to\infty}\widehat{\prod_{nN\geq k\geq 1}}$. In view of the remark made above, $E(r)$ is a well defined formal series in $p^{\frac{1}{n}}$. Then the vertex type twistor (4.180) satisfies the shifted cocycle condition

$$E^{(12)}(r)(\Delta\otimes \mathrm{id})E(r)=E^{(23)}(r+c^{(1)})(\mathrm{id}\otimes\Delta)E(r), \tag{4.181}$$

and

$$(\varepsilon\otimes \mathrm{id})\, E(r)=(\mathrm{id}\otimes\varepsilon)\, E(r)=1. \tag{4.182}$$

A proof of this property and an example are given in [26].

[Vertex type algebra] *We define the quasi-Hopf algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ ($p=q^{2r}$) of vertex type to be the set $(U_q(\widehat{\mathfrak{sl}}_n),\Delta_r,\Phi(r),\mathcal{R}(r))$ together with $\alpha_r=\sum_i S(d_i)e_i$, $\beta_r=\sum_i f_iS(g_i)$, the antiautomorphism S defined by (4.150) and ε defined by (4.149),*

$$\begin{aligned}\Delta_r(a) &= E^{(12)}(r)\,\Delta(a)\,E^{(12)}(r)^{-1}, &(4.183)\\ \mathcal{R}(r) &= E^{(21)}(r)\,\mathcal{R}\,E^{(12)}(r)^{-1}, &(4.184)\\ \Phi(r) &= E^{(23)}(r)E^{(23)}(r+c^{(1)})^{-1}, &(4.185)\end{aligned}$$

and $\sum_i d_i\otimes e_i=E(r)^{-1}$, $\sum_i f_i\otimes g_i=E(r)$.

Let us set

$$\begin{aligned}\widetilde{\mathcal{R}}'(\zeta) &= \Big(\mathrm{Ad}(\zeta^{\rho})\otimes\mathrm{id}\Big)(q^{\widetilde{T}}\mathcal{R}), &(4.186)\\ E(\zeta,r) &= \Big(\mathrm{Ad}(\zeta^{\rho})\otimes\mathrm{id}\Big)E(r). &(4.187)\end{aligned}$$

In just the same way as in the face type case, the definition (4.180) can be alternatively described as the unique solution of the difference equation

$$E(p^{\frac{1}{n}}q^{\frac{2}{n}c^{(1)}}\zeta,r)=(\tau\otimes\mathrm{id})^{-1}(E(\zeta,r))\cdot\widetilde{\mathcal{R}}'(p^{\frac{1}{n}}q^{\frac{2}{n}c^{(1)}}\zeta), \tag{4.188}$$

with the initial condition $E(0,r)=1$, where $p=q^{2r}$.

4. dynamical RLL-relations and vertex operators

The L-operators and vertex operators for the elliptic algebras can be constructed from those of $U_q(\mathfrak{g})$ by 'dressing' the latter with the twistors. In this subsection, we examine various commutation relations among these operators. We shall mainly discuss the case of the face type algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ where $\mathfrak{g}$ is of affine type.

Hereafter we write $U=U_q(\mathfrak{g})$, $\mathcal{B}=\mathcal{B}_{q,\lambda}(\mathfrak{g})$. By a representation of the quasi-Hopf algebra $\mathcal{B}$ we mean that of the underlying associative algebra U. Let (π_V,V) be a finite dimensional module over U, and $(\pi_{V,z}, V_z)$ be the evaluation representation associated with it where $\pi_{V,z} = \pi_V \circ \mathrm{Ad}(z^d)$.

We define L-operators for $\mathcal{B}$ by

$$\begin{aligned} L_V^{\pm}(z,\lambda) &= (\pi_{V,z}\otimes \mathrm{id})\,\mathcal{R}'^{\pm}(\lambda), & (4.189)\\ \mathcal{R}'^{+}(\lambda) &= q^{c\otimes d+d\otimes c}\mathcal{R}(\lambda), & (4.190)\\ \mathcal{R}'^{-}(\lambda) &= \mathcal{R}^{(21)}(\lambda)^{-1}q^{-c\otimes d-d\otimes c}. & (4.191)\end{aligned}$$

Likewise we set

$$R^{\pm}_{VW}(\tfrac{z_1}{z_2},\lambda) = (\pi_{V,z_1}\otimes\pi_{W,z_2})\mathcal{R}'^{\pm}(\lambda). \qquad (4.192)$$

Setting further

$$\mathcal{R}'^{\pm}(z,\lambda) = \mathrm{Ad}(z^d\otimes 1)\mathcal{R}'^{\pm}(\lambda), \qquad (4.193)$$

we find from the dynamical YBE (4.139) that

$$\begin{aligned} &\mathcal{R}'^{\pm(12)}(\tfrac{z_1}{z_2},\lambda+h^{(3)})\mathcal{R}'^{\pm(13)}(q^{\mp c^{(2)}}\tfrac{z_1}{z_3},\lambda)\mathcal{R}'^{\pm(23)}(\tfrac{z_2}{z_3},\lambda+h^{(1)})\\ &\quad= \mathcal{R}'^{\pm(23)}(\tfrac{z_2}{z_3},\lambda)\mathcal{R}'^{\pm(13)}(q^{\pm c^{(2)}}\tfrac{z_1}{z_3},\lambda+h^{(2)})\mathcal{R}'^{\pm(12)}(\tfrac{z_1}{z_2},\lambda), \qquad (4.194)\\ &\mathcal{R}'^{+(12)}(q^{c^{(3)}}\tfrac{z_1}{z_2},\lambda+h^{(3)})\mathcal{R}'^{+(13)}(\tfrac{z_1}{z_3},\lambda)\mathcal{R}'^{-(23)}(\tfrac{z_2}{z_3},\lambda+h^{(1)})\\ &\quad= \mathcal{R}'^{-(23)}(\tfrac{z_2}{z_3},\lambda)\mathcal{R}'^{+(13)}(\tfrac{z_1}{z_3},\lambda+h^{(2)})\mathcal{R}'^{+(12)}(q^{-c^{(3)}}\tfrac{z_1}{z_2},\lambda). \qquad (4.195)\end{aligned}$$

Applying $\pi_V\otimes\pi_W\otimes\mathrm{id}$, we obtain the dynamical RLL relation,

$$\begin{aligned} &R^{\pm(12)}_{VW}(\tfrac{z_1}{z_2},\lambda+h)L_V^{\pm(1)}(z_1,\lambda)L_W^{\pm(2)}(z_2,\lambda+h^{(1)})\\ &\quad= L_W^{\pm(2)}(z_2,\lambda)L_V^{\pm(1)}(z_1,\lambda+h^{(2)})R^{\pm(12)}_{VW}(\tfrac{z_1}{z_2},\lambda), \qquad (4.196)\\ &R^{+(12)}_{VW}(q^c\tfrac{z_1}{z_2},\lambda+h)L_V^{+(1)}(z_1,\lambda)L_W^{-(2)}(z_2,\lambda+h^{(1)})\\ &\quad= L_W^{-(2)}(z_2,\lambda)L_V^{+(1)}(z_1,\lambda+h^{(2)})R^{+(12)}_{VW}(q^{-c}\tfrac{z_1}{z_2},\lambda). \qquad (4.197)\end{aligned}$$

Here the index (1) (resp. (2)) refers to V (resp. W), and h, c (without superfix) are elements of $\mathfrak{h}\subset\mathcal{B}$. If we write

$$\lambda-\rho = rd+s'c+\bar\lambda-\bar\rho \quad (r,s'\in\mathbb{C},\bar\lambda\in\bar{\mathfrak{h}}), \qquad (4.198)$$

then

$$\lambda+h^{(1)} = (r+h^\vee+c^{(1)})d+(s'+d^{(1)})c+(\bar\lambda+\bar{h}^{(1)}), \qquad (4.199)$$

where $h^\vee$ is the dual Coxeter number. The parameter r plays the role of the elliptic modulus. Note that, in (4.196)-(4.197), r also undergoes a shift depending on the central element c.

Actually the two L-operators (4.189) are not independent. We have

$$L_V^{+}(pq^cz,\lambda) = q^{-2\bar{T}_{V,\bullet}}(\mathrm{Ad}(\bar X_\lambda)\otimes\mathrm{id})^{-1}L_V^{-}(z,\lambda), \qquad (4.200)$$

where

$$\bar{T}_{V,\bullet} = \sum_j \pi(\bar{h}_j) \otimes \bar{h}^j, \quad \bar{X}_\lambda = \pi(q^{\sum \bar{h}_j \bar{h}^j + 2(\bar{\lambda}-\bar{\rho})}). \tag{4.201}$$

Next let us consider vertex operators. Let $(\pi_{V,z}, V_z)$ be as before, and let $V(\mu)$ be a highest weight module with highest weight μ. Consider intertwiners of U-modules of the form

$$\Phi_V^{(\nu,\mu)}(z) \;:\; V(\mu) \longrightarrow V(\nu) \otimes V_z, \tag{4.202}$$
$$\Psi_V^{*(\nu,\mu)}(z) \;:\; V_z \otimes V(\mu) \longrightarrow V(\nu), \tag{4.203}$$

which are called vertex operators of type I and type II respectively. Define the corresponding VO's for $\mathcal{B}$ as follows [60]:

$$\Phi_V^{(\nu,\mu)}(z,\lambda) = (\mathrm{id} \otimes \pi_z) F(\lambda) \circ \Phi_V^{(\nu,\mu)}(z), \tag{4.204}$$
$$\Psi_V^{*(\nu,\mu)}(z,\lambda) = \Psi_V^{*(\nu,\mu)}(z) \circ (\pi_z \otimes \mathrm{id}) F(\lambda)^{-1}. \tag{4.205}$$

When there is no fear of confusion, we often drop the sub(super)scripts V or (ν,μ). It is clear that (4.204) and (4.205) satisfy the intertwining relations relative to the coproduct Δ_λ (4.162),

$$\Delta_\lambda(a)\Phi(z,\lambda) = \Phi(z,\lambda)a \qquad (\forall a \in \mathcal{B}), \tag{4.206}$$
$$a\Psi^*(z,\lambda) = \Psi^*(z,\lambda)\Delta_\lambda(a) \qquad (\forall a \in \mathcal{B}). \tag{4.207}$$

These intertwining relations can be encapsulated to commutation relations with the L-operators.

The 'dressed' VO's (4.204), (4.205) satisfy the following dynamical intertwining relations (see the diagram below):

$$\Phi_W(z_2,\lambda)L_V^+(z_1,\lambda) = R_{VW}^+(q^c\tfrac{z_1}{z_2}, \lambda+h)L_V^+(z_1,\lambda)\Phi_W(z_2,\lambda+h^{(1)}), \tag{4.208}$$
$$\Phi_W(z_2,\lambda)L_V^-(z_1,\lambda) = R_{VW}^-(\tfrac{z_1}{z_2}, \lambda+h)L_V^-(z_1,\lambda)\Phi_W(z_2,\lambda+h^{(1)}), \tag{4.209}$$
$$L_V^+(z_1,\lambda)\Psi_W^*(z_2,\lambda+h^{(1)}) = \Psi_W^*(z_2,\lambda)L_V^+(z_1,\lambda+h^{(2)})R_{VW}^+(\tfrac{z_1}{z_2},\lambda), \tag{4.210}$$
$$L_V^-(z_1,\lambda)\Psi_W^*(z_2,\lambda+h^{(1)}) = \Psi_W^*(z_2,\lambda)L_V^-(z_1,\lambda+h^{(2)})R_{VW}^-(q^c\tfrac{z_1}{z_2},\lambda). \tag{4.211}$$

$$\begin{tikzcd}
V_{z_1} \otimes V(\mu) \arrow[r, "\Phi_W"] \arrow[d, "L_V^\pm"'] & V_{z_1} \otimes V(\nu) \otimes W_{z_2} \arrow[r, "L_V^\pm"] & V_{z_1} \otimes V(\nu) \otimes W_{z_2} \arrow[d, "R_{VW}^\pm"] \\
V_{z_1} \otimes V(\mu) \arrow[rr, "\Phi_W"'] & & V_{z_1} \otimes V(\nu) \otimes W_{z_2}
\end{tikzcd} \tag{4.212}$$

$$\begin{tikzcd}
V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[r, "R^{\pm}_{VW}"] \arrow[d, "\Psi^*_W"'] & V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[r, "L^{\pm}_V"] & V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[d, "\Psi^*_W"] \\
V_{z_1} \otimes V(\nu) \arrow[rr, "L^{\pm}_V"'] & & V_{z_1} \otimes V(\nu)
\end{tikzcd} \tag{4.213}$$

From the theory of q-KZ-equation[61], we know the VO's for U satisfy the commutation relations of the form

$$\check{R}_{VV}(\tfrac{z_1}{z_2})\Phi_V^{(\nu,\mu)}(z_1)\Phi_V^{(\mu,\kappa)}(z_2) = \sum_{\mu'} \Phi_V^{(\nu,\mu')}(z_2)\Phi_V^{(\mu',\kappa)}(z_1) W_I\left(\begin{matrix}\kappa & \mu \\ \mu' & \nu\end{matrix}\middle| \tfrac{z_1}{z_2}\right), \tag{4.214}$$

$$\Psi_V^{*(\nu,\mu)}(z_1)\Psi_V^{*(\mu,\kappa)}(z_2)\check{R}_{VV}(\tfrac{z_1}{z_2})^{-1} = \sum_{\mu'} W_{II}\left(\begin{matrix}\kappa & \mu \\ \mu' & \nu\end{matrix}\middle| \tfrac{z_1}{z_2}\right)\Psi_V^{*(\nu,\mu')}(z_2)\Psi_V^{*(\mu',\kappa)}(z_1), \tag{4.215}$$

$$\Phi_V^{(\nu,\mu)}(z_1)\Psi_V^{*(\mu,\kappa)}(z_2) = \sum_{\mu'} W_{I,II}\left(\begin{matrix}\kappa & \mu \\ \mu' & \nu\end{matrix}\middle| \tfrac{z_1}{z_2}\right)\Psi_V^{*(\nu,\mu')}(z_2)\Phi_V^{(\mu',\kappa)}(z_1). \tag{4.216}$$

Here

$$\check{R}_{VV}(z) = PR_{VV}(z), \qquad P(v\otimes v') = v'\otimes v, \tag{4.217}$$

$$R_{VV}(\tfrac{z_1}{z_2}) = (\pi_{V,z_1}\otimes\pi_{V,z_2})\mathcal{R} \tag{4.218}$$

is the 'trigonometric' R matrix. In (4.214)-(4.216) we used a slightly abbreviated notation. For example, the left hand side of (4.214) means the composition

$$V(\kappa)\xrightarrow{\Phi(z_2)}V(\mu)\otimes V_{z_2}\xrightarrow{\Phi(z_1)\otimes \mathrm{id}}V(\nu)\otimes V_{z_1}\otimes V_{z_2}\xrightarrow{\mathrm{id}\otimes\check{R}(\frac{z_1}{z_2})}V(\nu)\otimes V_{z_2}\otimes V_{z_1}.$$

Similarly (4.215), (4.216) are maps

$$V_{z_2}\otimes V_{z_1}\otimes V(\kappa)\longrightarrow V(\nu), \quad V_{z_2}\otimes V(\kappa)\longrightarrow V(\nu)\otimes V_{z_1}, \tag{4.219}$$

respectively. For $U_q(\widehat{\mathfrak{sl}}_2)$, the formulas for the W-factors in the simplest case can be found e.g. in [62].

The 'dressed' VO's satisfy similar relations with appropriate dynamical shift. Setting $\check{R}_{VV}(z,\lambda) = PR_{VV}(z,\lambda)$, we have

$$\begin{aligned}
&\check{R}_{VV}(\tfrac{z_1}{z_2},\lambda+h^{(1)})\Phi_V^{(\nu,\mu)}(z_1,\lambda)\Phi_V^{(\mu,\kappa)}(z_2,\lambda) \\
&\qquad = \sum_{\mu'} \Phi_V^{(\nu,\mu')}(z_2,\lambda)\Phi_V^{(\mu',\kappa)}(z_1,\lambda) W_I\left(\begin{matrix}\kappa & \mu \\ \mu' & \nu\end{matrix}\middle| \tfrac{z_1}{z_2}\right),
\end{aligned} \tag{4.220}$$

$$\Psi_V^{*(\nu,\mu)}(z_1,\lambda)\Psi_V^{*(\mu,\kappa)}(z_2,\lambda+h^{(1)})\check{R}_{VV}(\tfrac{z_1}{z_2},\lambda)^{-1}$$

$$= \sum_{\mu'} W_{II}\begin{pmatrix} \kappa & \mu \\ \mu' & \nu \end{pmatrix}\Big|\tfrac{z_1}{z_2}\Big) \Psi_V^{*(\nu,\mu')}(z_2,\lambda)\Psi_V^{*(\mu',\kappa)}(z_1,\lambda+h^{(1)}), \qquad (4.221)$$

$$\Phi_V^{(\nu,\mu)}(z_1,\lambda)\Psi_V^{*(\mu,\kappa)}(z_2,\lambda)$$
$$= \sum_{\mu'} W_{I,II}\begin{pmatrix} \kappa & \mu \\ \mu' & \nu \end{pmatrix}\Big|\tfrac{z_1}{z_2}\Big) \Psi_V^{*(\nu,\mu')}(z_2,\lambda)\Phi_V^{(\mu',\kappa)}(z_1,\lambda+h^{(1)}). \qquad (4.222)$$

Notice that the W-factors stay the same with the trigonometric case, and are not affected by a dynamical shift.

The case of vertex type algebras can be treated in a parallel way. See [26].

5. Drinfeld currents

The Drinfeld currents are suited to deal with infinite dimensional representations. We explain it by taking $U_q(\widehat{\mathfrak{sl}}_2)$ as an example. (See [63] for general case.)

First let us recall the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_2)$ [64]. Let $x_n^{\pm}$ $(n \in \mathbb{Z})$, a_n $(n \in \mathbb{Z}_{\neq 0})$, h, c, d denote the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_2)$. In terms of the generating functions

$$x^{\pm}(z) \;=\; \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}, \qquad (4.223)$$

$$\psi(q^{\frac{c}{2}}z) \;=\; q^h \exp\Big((q-q^{-1})\sum_{n>0} a_n z^{-n}\Big), \qquad (4.224)$$

$$\varphi(q^{-\frac{c}{2}}z) \;=\; q^{-h} \exp\Big(-(q-q^{-1})\sum_{n>0} a_{-n} z^{n}\Big), \qquad (4.225)$$

the defining relations read as follows:

$$c: \ \text{central}\ , \qquad (4.226)$$

$$[h,d]=0, \quad [d,a_n]=na_n, \quad [d,x_n^{\pm}]=nx_n^{\pm}, \qquad (4.227)$$

$$[h,a_n]=0, \quad [h,x^{\pm}(z)]=\pm 2x^{\pm}(z), \qquad (4.228)$$

$$[a_n,a_m]=\frac{[2n]_q[cn]_q}{n}q^{-c|n|}\delta_{n+m,0}, \qquad (4.229)$$

$$[a_n,x^+(z)]=\frac{[2n]_q}{n}q^{-c|n|}z^n x^+(z), \quad [a_n,x^-(z)]=-\frac{[2n]_q}{n}z^n x^-(z), \qquad (4.230)$$

$$(z-q^{\pm 2}w)x^{\pm}(z)x^{\pm}(w)=(q^{\pm 2}z-w)x^{\pm}(w)x^{\pm}(z), \qquad (4.231)$$

$$[x^+(z),x^-(w)]=\frac{1}{q-q^{-1}}\Big(\delta(q^c\tfrac{w}{z})\psi(q^{\frac{c}{2}}w)-\delta(q^{-c}\tfrac{w}{z})\varphi(q^{-\frac{c}{2}}w)\Big), \qquad (4.232)$$

and the Serre relation which we omit.

For the level 1 case $(c=1)$, $x^{\pm}(z)$ has the following free boson realization [65],

$$x^+(z) \;=\; :\exp\Big(-\sum_{n\neq 0}\frac{a_n}{[n]_q}z^{-n}\Big): e^Q z^{a_0+1}, \qquad (4.233)$$

$$x^-(z) \;=\; :\exp\Big(\sum_{n\neq 0}\frac{a_n}{[n]_q}z^{-n}q^{|n|}\Big): e^{-Q} z^{-a_0+1}, \qquad (4.234)$$

where a_n ($n \in \mathbb{Z}_{\neq 0}$) is given in (4.229), a_0 and Q are given in (2.49), and $h = a_0$. For general level c, $U_q(\widehat{\mathfrak{sl}}_2)$ has the Wakimoto realization (see e.g. [66] and references therein).

We now introduce a new parameter p and modify (4.223)-(4.225) to define another set of currents. For notational convenience, we will frequently write

$$p = q^{2r}, \quad p^* = pq^{-2c} = q^{2r^*}, \tag{4.235}$$

where r^* is

$$r^* = r - c. \tag{4.236}$$

Let us introduce two currents $u^{\pm}(z,p) \in U_q(\widehat{\mathfrak{sl}}_2)$ depending on p by

$$u^+(z,p) \;=\; \exp\Big(\sum_{n>0} \frac{1}{[r^* n]_q} a_{-n} (q^r z)^n\Big), \tag{4.237}$$

$$u^-(z,p) \;=\; \exp\Big(-\sum_{n>0} \frac{1}{[rn]_q} a_n (q^{-r} z)^{-n}\Big). \tag{4.238}$$

We define the 'dressed' currents $x_+(z,p)$, $x_-(z,p)$, $\psi^{\pm}(z,p)$ by

$$x_+(z,p) \;=\; u^+(z,p) x^+(z), \tag{4.239}$$

$$x_-(z,p) \;=\; x^-(z) u^-(z,p), \tag{4.240}$$

$$\psi^+(z,p) \;=\; u^+(q^{\frac{c}{2}} z, p)\psi(z) u^-(q^{-\frac{c}{2}} z, p), \tag{4.241}$$

$$\psi^-(z,p) \;=\; u^+(q^{-\frac{c}{2}} z, p)\varphi(z) u^-(q^{\frac{c}{2}} z, p). \tag{4.242}$$

We call these as elliptic currents. They are 'Drinfeld currents' of $U_q(\widehat{\mathfrak{sl}}_2)$ or $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ which is nothing but $U_q(\widehat{\mathfrak{sl}}_2)$ equipped with a different coproduct. We will often drop p, and write $x_-(z,p)$ as $x_-(z)$ and so forth.

The merit of these currents is that they obey the following 'elliptic' commutation relations:

$$\psi^{\pm}(z)\psi^{\pm}(w) = \frac{\Theta_p(q^{-2}\frac{z}{w})}{\Theta_p(q^2\frac{z}{w})} \frac{\Theta_{p^*}(q^2\frac{z}{w})}{\Theta_{p^*}(q^{-2}\frac{z}{w})} \psi^{\pm}(w)\psi^{\pm}(z), \tag{4.243}$$

$$\psi^+(z)\psi^-(w) = \frac{\Theta_p(pq^{-c-2}\frac{z}{w})}{\Theta_p(pq^{-c+2}\frac{z}{w})} \frac{\Theta_{p^*}(p^* q^{c+2}\frac{z}{w})}{\Theta_{p^*}(p^* q^{c-2}\frac{z}{w})} \psi^-(w)\psi^+(z), \tag{4.244}$$

$$\psi^{\pm}(z) x_+(w) \psi^{\pm}(z)^{-1} = q^{-2} \frac{\Theta_{p^*}(q^{\pm\frac{c}{2}+2}\frac{z}{w})}{\Theta_{p^*}(q^{\pm\frac{c}{2}-2}\frac{z}{w})} x_+(w), \tag{4.245}$$

$$\psi^{\pm}(z) x_-(w) \psi^{\pm}(z)^{-1} = q^{2} \frac{\Theta_{p}(q^{\mp\frac{c}{2}-2}\frac{z}{w})}{\Theta_{p}(q^{\mp\frac{c}{2}+2}\frac{z}{w})} x_-(w), \tag{4.246}$$

$$[x_+(z), x_-(w)] = \frac{1}{q-q^{-1}} \Big(\delta(q^c \tfrac{w}{z}) \psi^+(q^{\frac{c}{2}} w) - \delta(q^{-c}\tfrac{w}{z}) \psi^-(q^{-\frac{c}{2}} w)\Big), \tag{4.247}$$

where $\Theta_p(z)$ is given in (A.4). It is convenient to consider also the current

$$k(z)=\exp\Big(\sum_{n>0}\frac{[n]_q}{[2n]_q[r^*n]_q}a_{-n}(q^cz)^n\Big)\exp\Big(-\sum_{n>0}\frac{[n]_q}{[2n]_q[rn]_q}a_nz^{-n}\Big). \tag{4.248}$$

The $\psi^\pm(z)$ are related to $k(z)$ by the formula

$$\psi^\pm(p^{\mp(r-\frac{c}{2})}z)=\kappa q^{\pm h}k(qz)k(q^{-1}z), \tag{4.249}$$

$$\kappa=\frac{\xi(z;p^*,q)}{\xi(z;p,q)}\bigg|_{z=q^{-2}}, \tag{4.250}$$

where the function

$$\xi(z;p,q)=\frac{(q^2z,pq^2z;p,q^4)_\infty}{(q^4z,pz;p,q^4)_\infty} \tag{4.251}$$

is a solution of the difference equation

$$\xi(z;p,q)\xi(q^2z;p,q)=\frac{(q^2z;p)_\infty}{(pz;p)_\infty}. \tag{4.252}$$

We have the commutation relations supplementing (4.243)-(4.247),

$$k(z)k(w)=\frac{\xi(\frac{w}{z};p,q)}{\xi(\frac{w}{z};p^*,q)}\frac{\xi(\frac{z}{w};p^*,q)}{\xi(\frac{z}{w};p,q)}k(w)k(z), \tag{4.253}$$

$$k(z)x_+(w)k(z)^{-1}=\frac{\Theta_{p^*}(p^{*\frac{1}{2}}q\frac{z}{w})}{\Theta_{p^*}(p^{*\frac{1}{2}}q^{-1}\frac{z}{w})}x_+(w), \tag{4.254}$$

$$k(z)x_-(w)k(z)^{-1}=\frac{\Theta_p(p^{\frac{1}{2}}q^{-1}\frac{z}{w})}{\Theta_p(p^{\frac{1}{2}}q\frac{z}{w})}x_-(w). \tag{4.255}$$

Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is obtained as a tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ ($\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$) and a Heisenberg algebra $[Q,P]=1$. For details see [57, 63]. (See also [67].)

5 Free Field Approach to ABF Model

Lukyanov and Pugai studied the Andrews-Baxter-Forrester (ABF) model and calculated LHS's by bosonizing VO's in subsection 4.2 [21]. Here we explain their work.

r, β, $x_\pm(z)$, $T(z)$, α_n, α'_n, h_n, $C_{l,k}$, etc. in this section are same as those in section 3. In this section r is a positive integer,

$$r\in\mathbb{Z},\quad r\geq 4, \tag{5.1}$$

and r^* is

$$r^*=r-1. \tag{5.2}$$

So we have

$$\beta=\frac{r}{r-1},\quad \alpha_0=\frac{1}{\sqrt{r(r-1)}},\quad p'=r-1,\quad p''=r. \tag{5.3}$$

5.1 ABF model

The ABF model [54] is a RSOS model associated to the vector representation of $A_1^{(1)}$ algebra [10]. The height variables $a, b, \cdots$ take one of the $r-1$ states $1, 2, \cdots r-1$ and those on neighboring sites are subject to the condition $|a-b| = 1$. We write the Boltzmann weight in the form

$$W\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big| u\Big) = \rho(u)\overline{W}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big| u\Big), \tag{5.4}$$

where an overall scalar factor $\rho(u)$ is given in (5.134) and chosen so that the partition function per site equals to 1. Non-zero components of $\overline{W}$ are given by

$$\begin{aligned}
\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{matrix}\middle| u\right) &= 1, \\
\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\middle| u\right) &= \frac{[a\mp u]}{[a]}\frac{[1]}{[1+u]}, \\
\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\middle| u\right) &= \frac{\sqrt{[a+1][a-1]}}{[a]}\frac{[-u]}{[1+u]}.
\end{aligned} \tag{5.5}$$

Here $[u]$ is given in (A.5) and we use a parameter x which is obtained from the original nome p by the modular transformation ($p = e^{2\pi i\tau} \mapsto x^{2r} = e^{2\pi i\frac{-1}{\tau}}$). Low temperature limit corresponds to $x \sim 0$ ($p \sim 1$).

This Boltzmann weight W enjoys YBE (4.3), initial condition (4.38), unitarity (4.39) and crossing symmetry (4.99) ($\lambda = -1$, $G_a = [a]$),

$$W\left(\begin{matrix} b & d \\ a & c \end{matrix}\middle| -1-u\right) = \sqrt{\frac{[a][d]}{[b][c]}}\,W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right). \tag{5.6}$$

In this gauge W enjoys also a reflection symmetry

$$W\left(\begin{matrix} a & c \\ b & d \end{matrix}\middle| u\right) = W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right). \tag{5.7}$$

Along the additive variable u, we often use the multiplicative variable $z = x^{2u}$.

We will restrict ourselves to the 'regime III' region defined by

$$0 < x < 1, \quad -1 < u < 0. \tag{5.8}$$

In this region W is positive. On the critical point ($x \to 1$ ($p \to 0$)), a correlation length becomes infinite and this model becomes scale invariant, i.e. it has conformal symmetry. So it is described by CFT, i.e. Virasoro algebra. In this case it corresponds to the minimal unitary series with $\beta = \frac{r}{r-1}$. On the off-critical point ($x < 1$), which corresponds to the $(1,3)$-perturbation of the minimal unitary CFT, the Virasoro symmetry is lost but the

DVA symmetry remains. In the low temperature limit ($x \to 0$ with $z = x^{2u}(\geq 1)$ fixed kept), W behaves as

$$
\begin{aligned}
&\rho(u) \to z^{-\frac{r^*}{2r}}, \\
&\overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{matrix}\middle| u\right) \to 1, \quad \overline{W}\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\middle| u\right) \to 0, \\
&\overline{W}\left(\begin{matrix} a & a+1 \\ a+1 & a \end{matrix}\middle| u\right) \to z^{\frac{r-a-1}{r}}(\geq 1), \quad \overline{W}\left(\begin{matrix} a & a-1 \\ a-1 & a \end{matrix}\middle| u\right) \to z^{\frac{a-1}{r}}(\geq 1).
\end{aligned} \tag{5.9}
$$

Therefore ground state configuration is

$$
\begin{array}{cccc}
l & l+1 & l & l+1 \\
l+1 & l & l+1 & l \\
l & l+1 & l & l+1 \\
l+1 & l & l+1 & l
\end{array} \quad . \tag{5.10}
$$

In the notation of subsection 4.2 this corresponds to $(i_1, i_2) = (l, l+1)$. We label this ground state by an integer l $(l = 1, 2, \cdots, r-2)$ and m takes two values $0, 1 \in \mathbb{Z}/2\mathbb{Z}$. Since heights on neighboring sites differ ± 1, we have

$$
\mathcal{H}^{(k)}_{l,m} = 0 \quad (m \not\equiv l-k \pmod 2). \tag{5.11}
$$

Therefore we can identify

$$
\mathcal{H}^{(k)}_{l} = \bigoplus_{m=0}^{1} \mathcal{H}^{(k)}_{l,m} \cong \mathcal{H}^{(k)}_{l,l-k}. \tag{5.12}
$$

This space of states $\mathcal{H}^{(k)}_{l}$, on which the corner Hamiltonian $H_C^{(k)}$ acts, has two labels l and k with range $1 \leq l \leq r-2$, $1 \leq k \leq r-1$. This range is same as (2.23).

In the thermodynamic limit CTM's (4.100) become

$$
A^{(k)}(u) = C^{(k)}(u) = x^{-2uH_C^{(k)}}, \quad B^{(k)}(u) = D^{(k)}(u) = \sqrt{[k]}x^{2(u+1)H_C^{(k)}}. \tag{5.13}
$$

Careful study of the corner Hamiltonian shows that the character (4.43) agrees with the Virasoro minimal unitary character [54, 9],

$$
\chi_{l,l-k,k}(q) = \chi^{\mathrm{Vir}}_{l,k}(q), \tag{5.14}
$$

where $\chi^{\mathrm{Vir}}_{l,k}(q)$ is given in (2.37). This character $\chi^{\mathrm{Vir}}_{l,k}(q)$, which agrees with DVA one, should be understood as a character of the representation of DVA. Comparing the free filed realization given in subsection 3.4, we make an identification [21]

$$\mathcal{H}_l^{(k)} = \mathcal{L}_{l,k}\,, \tag{5.15}$$

$$H_C^{(k)} = d\Big|_{\mathcal{L}_{l,k}}, \tag{5.16}$$

where $\mathcal{L}_{l,k}$ is an irreducible DVA module given in (3.82) as a cohomology of the Felder complex and d is realized in (3.43). We remark that $Z_{l,m}$ (4.45) is in fact m-independent, $Z_{l,0} = Z_{l,1}$.

For later use we define W^*,

$$W^*\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big) = \overline{W}\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big)\Big|_{r\to r^*} \times \rho^*(u), \tag{5.17}$$

where $\rho^*(u)$ is given in (5.135).

5.2 Vertex operators

Although bosons are already introduced in subsection 3.4, we present their definitions again.

Let us introduce free boson oscillator α_n $(n \in \mathbb{Z}_{\neq 0})$,

$$[\alpha_n, \alpha_m] = \frac{[n]_x[2n]_x}{n}\frac{[rn]_x}{[r^*n]_x}\delta_{n+m,0}, \tag{5.18}$$

and use zero mode a_0 and Q defined in (2.49) (or a_0' in (2.72)). The Fock space $\mathcal{F}_{l,k}$ is defined by

$$\mathcal{F}_{l,k} = \bigoplus_{m\geq 0}\bigoplus_{n_1\geq\cdots\geq n_m>0} \mathbb{C}\alpha_{-n_1}\cdots\alpha_{-n_m}|\alpha_{l,k}\rangle_B, \tag{5.19}$$

where $|\alpha\rangle_B$ is given by (2.51) with replacing a_n by h_n,

$$a_0'|\alpha_{l,k}\rangle_B = (\alpha_{l,k}-\alpha_0)|\alpha_{l,k}\rangle_B, \quad \alpha_{l,k}-\alpha_0 = -\sqrt{\frac{r}{r-1}}l + \sqrt{\frac{r-1}{r}}k. \tag{5.20}$$

We use also free boson oscillator α_n' $(n \in \mathbb{Z}_{\neq 0})$,

$$\alpha_n' = (-1)^n\frac{[r^*n]_x}{[rn]_x}\alpha_n, \quad [\alpha_n', \alpha_m'] = \frac{[n]_x[2n]_x}{n}\frac{[r^*n]_x}{[rn]_x}\delta_{n+m,0}. \tag{5.21}$$

Operators $\hat{l}, \hat{k} : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k}$ are defined by

$$\hat{l}\,|_{\mathcal{F}_{l,k}} = l \times \mathrm{id}_{\mathcal{F}_{l,k}}, \quad \hat{k}\,|_{\mathcal{F}_{l,k}} = k \times \mathrm{id}_{\mathcal{F}_{l,k}}. \tag{5.22}$$

If $\hat{l}$ and $\hat{k}$ appear in arguments of $[u]^*$ and $[u]$ respectively, they can be realized by a_0',

$$[u+\hat{l}]^* = (-1)^{l-k}\Big[u - \sqrt{rr^*}a_0'\Big]^* \quad \text{on } \mathcal{F}_{l,k}, \tag{5.23}$$

$$[u+\hat{k}] = (-1)^{l-k}\Big[u - \sqrt{rr^*}a_0'\Big] \quad \text{on } \mathcal{F}_{l,k}. \tag{5.24}$$

Elliptic currents $x_\pm(z)$ for $U_x(\widehat{\mathfrak{sl}}_2)$ (or $\mathcal{B}_{x,\lambda}(\widehat{\mathfrak{sl}}_2)$) of level 1 ($c=1$) are obtained by a 'dressing' procedure described in 4.4 **5**

$$\begin{aligned} x_+(z) &: \mathcal{F}_{l,k} \to \mathcal{F}_{l-2,k} \\ & x_+(z) = :\exp\Big(-\sum_{n\neq 0}\frac{\alpha_n}{[n]_x}z^{-n}\Big): \times e^{\sqrt{\frac{r}{r^*}}Q} z^{\sqrt{\frac{r}{r^*}}a_0'+\frac{r}{r^*}}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} x_-(z) &: \mathcal{F}_{l,k} \to \mathcal{F}_{l,k-2} \\ & x_-(z) = :\exp\Big(\sum_{n\neq 0}\frac{\alpha_n'}{[n]_x}z^{-n}\Big): \times e^{-\sqrt{\frac{r^*}{r}}Q} z^{-\sqrt{\frac{r^*}{r}}a_0'+\frac{r^*}{r}}, \end{aligned} \tag{5.26}$$

and they are interpreted as screening currents in subsection 3.4.

The elliptic version of VO's (of type I and type II) are defined in terms of their trigonometric ones and a 'twistor' given by an infinite product of the universal R matrix described in subsection 4.4 **4**. They satisfy the commutation relations of the type (5.33)-(5.35) below. As we do not know how to evaluate the twistor in the bosonic realization, we have solved the relations (5.33)-(5.35) directly for $\Phi_\varepsilon(z), \Psi^*_\varepsilon(z)$ ($\varepsilon=\pm1$). We write $\Phi_\pm(z)=\Phi_{\pm1}(z)$ and $\Psi^*_\pm(z)=\Psi^*_{\pm1}(z)$. We obtain the following : [21, 68]

$$\text{type I} \quad \Phi_\varepsilon(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l,k-\varepsilon}$$

$$\Phi_-(z) = \sqrt{g}\, :\exp\Big(-\sum_{n\neq 0}\frac{\alpha_n'}{[2n]_x}z^{-n}\Big): \times e^{\frac{1}{2}\sqrt{\frac{r^*}{r}}Q} z^{\frac{1}{2}\sqrt{\frac{r^*}{r}}a_0'+\frac{r^*}{4r}}, \tag{5.27}$$

$$\Phi_+(z) = \oint_{C_\Phi(z)} \underline{dz'}\, \Phi_-(z)x_-(z')\frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]}\frac{1}{\sqrt{[\hat{k}][\hat{k}-1]}}, \tag{5.28}$$

$$\text{type II} \quad \Psi^*_\varepsilon(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l-\varepsilon,k}$$

$$\Psi^*_-(z) = \frac{1}{\sqrt{g^*}}\, :\exp\Big(\sum_{n\neq 0}\frac{\alpha_n}{[2n]_x}z^{-n}\Big): \times e^{-\frac{1}{2}\sqrt{\frac{r}{r^*}}Q} z^{-\frac{1}{2}\sqrt{\frac{r}{r^*}}a_0'+\frac{r}{4r^*}}, \tag{5.29}$$

$$\Psi^*_+(z) = \oint_{C_{\Psi^*}(z)} \underline{dz'}\, \Psi^*_-(z)x_+(z')\frac{[u-u'+\frac{1}{2}-\hat{l}]^*}{[u-u'-\frac{1}{2}]^*}\frac{1}{\sqrt{[\hat{l}]^*[\hat{l}-1]^*}}. \tag{5.30}$$

Here $z=x^{2u}$, $z'=x^{2u'}$, $\underline{dz'}=\frac{dz'}{2\pi i z'}$, and normalization constants g and g^* will be given in (5.48) and (5.49). For $k=1$ ($l=1$), we adopt the following prescription; set $k=1+\epsilon$ ($l=1+\epsilon$) and take a limit $\epsilon\to 0$ after all the calculation. (Another method of getting rid of this factor is a gauge transformation, see subsection 5.4.) The poles of the integrand of (5.27)-(5.30) and the integration contours are listed in the following table. For example, $C_\Phi(z)$ is a simple closed contour that encircles $x^{1+2rn}z$ ($n\geq 0$) but not $x^{-1-2rn}z$ ($n\geq 0$).

	inside	outside
$C_\Phi(z)$	$z'=x^{1+2rn}z$	$z'=x^{-1-2rn}z$
$C_{\Psi^*}(z)$	$z'=x^{-1+2r^*n}z$	$z'=x^{1-2r^*n}z$

$(n=0,1,2,\cdots)$. (5.31)

OPE formulas are given in subsection 5.a. In the conformal limit (3.23), $g^{-\frac{1}{2}}\Phi_-(z)$ reduces to a primary field $V_{\alpha_{1,2}}(z)$ and $g^{*\frac{1}{2}}\Psi^*_-(z)$ reduces to $V_{\alpha_{2,1}}(-z)$ (up to phase factor) where $V_\alpha(z)$ is given in (2.61). $g^{-\frac{1}{2}}\Phi_+(z)$ and $g^{*\frac{1}{2}}\Psi^*_+(z)$ reduce to primary fields with screening current (screened vertex operator [34]). We can show that type I VO $\Phi_\varepsilon(z)$ commutes with BRST operator (3.80)

$$[X_j, \Phi_\varepsilon(z)] = 0. \tag{5.32}$$

Type II VO $\Psi^*_\varepsilon(z)$ commutes with another BRST operator obtained by using(3.84), see subsection 5.b.

The VO's given above satisfy the following commutation relations ($\varepsilon_1, \varepsilon_2 = \pm 1$),

$$\Phi_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1) = \sum_{\substack{\varepsilon'_1,\varepsilon'_2=\pm 1\\ \varepsilon'_1+\varepsilon'_2=\varepsilon_1+\varepsilon_2}} W\left(\begin{matrix}\hat{k} & \hat{k}+\varepsilon'_1\\ \hat{k}+\varepsilon_2 & \hat{k}+\varepsilon_1+\varepsilon_2\end{matrix}\middle| u_1-u_2\right)\Phi_{\varepsilon'_1}(z_1)\Phi_{\varepsilon'_2}(z_2), \tag{5.33}$$

$$\Psi^*_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \sum_{\substack{\varepsilon'_1,\varepsilon'_2=\pm 1\\ \varepsilon'_1+\varepsilon'_2=\varepsilon_1+\varepsilon_2}} W^*\left(\begin{matrix}\hat{l} & \hat{l}+\varepsilon_1\\ \hat{l}+\varepsilon'_2 & \hat{l}+\varepsilon_1+\varepsilon_2\end{matrix}\middle| u_1-u_2\right)\Psi^*_{\varepsilon'_2}(z_2)\Psi^*_{\varepsilon'_1}(z_1), \tag{5.34}$$

$$\Phi_{\varepsilon_2}(z_2)\Psi^*_{\varepsilon_1}(z_1) = \tau(u_1-u_2)\Psi^*_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2), \tag{5.35}$$

where $z_i = x^{2u_i}$ and $\tau(u)$ is given in (5.136).

Proof. First let us show (5.33). For $(\varepsilon_1, \varepsilon_2) = (-1, -1)$, (5.33) is

$$\Phi_-(z_2)\Phi_-(z_1) = \rho(u_1-u_2)\Phi_-(z_1)\Phi_-(z_2).$$

This is an OPE rule (5.131) itself. For $(\varepsilon_1, \varepsilon_2) = (-1, 1)$, (5.33) is

$$\begin{aligned}\Phi_+(z_2)\Phi_-(z_1) \;=\; & W\left(\begin{matrix}\hat{k} & \hat{k}+1\\ \hat{k}+1 & \hat{k}\end{matrix}\middle| u_1-u_2\right)\Phi_+(z_1)\Phi_-(z_2)\\ & +W\left(\begin{matrix}\hat{k} & \hat{k}-1\\ \hat{k}+1 & \hat{k}\end{matrix}\middle| u_1-u_2\right)\Phi_-(z_1)\Phi_+(z_2).\end{aligned}$$

We set (see subsection 5.a)

$$\Phi_+(z) = \oint \underline{dz'}\, \Phi_-(z)x_-(z')\varphi_+(u,u';\hat{k}),$$

$$\varphi_+(u,u';\hat{k}) = \frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]}\frac{1}{\sqrt{[\hat{k}][\hat{k}-1]}}, \tag{5.36}$$

$$x_-(z_1)\Phi_-(z_2) = f(u_1-u_2)\Phi_-(z_2)x_-(z_1), \quad f(u) = \frac{[u+\frac{1}{2}]}{[-u+\frac{1}{2}]}, \tag{5.37}$$

$$x_-(z_1)x_-(z_2) = h(u_1-u_2)x_-(z_2)x_-(z_1), \quad h(u) = \frac{[u-1]}{[u+1]}. \tag{5.38}$$

By using OPE rules in subsection 5.a, the above equation becomes

$$
\begin{aligned}
& \oint \underline{dz'}\, \Phi_-(z_2)\Phi_-(z_1)x_-(z')\varphi_+(u_2,u';\hat{k}+1)f(u'-u_1) \\
= & \oint \underline{dz'}\, \Phi_-(z_2)\Phi_-(z_1)x_-(z')\overline{W}\Big(\begin{array}{cc} \hat{k} & \hat{k}+1 \\ \hat{k}+1 & \hat{k} \end{array}\Big| u_1-u_2\Big)\varphi_+(u_1,u';\hat{k}+1)f(u'-u_2) \\
& + \oint \underline{dz'}\, \Phi_-(z_2)\Phi_-(z_1)x_-(z')\overline{W}\Big(\begin{array}{cc} \hat{k} & \hat{k}-1 \\ \hat{k}+1 & \hat{k} \end{array}\Big| u_1-u_2\Big)\varphi_+(u_2,u';\hat{k}).
\end{aligned}
$$

Careful analysis of the location of poles shows that we can take a common integration contour. Therefore it is sufficient to compare the integrands,

$$
\begin{aligned}
\varphi_+(u_2,u';\hat{k}+1)f(u'-u_1) = & \ \overline{W}\Big(\begin{array}{cc} \hat{k} & \hat{k}+1 \\ \hat{k}+1 & \hat{k} \end{array}\Big| u_1-u_2\Big)\varphi_+(u_1,u';\hat{k}+1)f(u'-u_2) \\
& + \overline{W}\Big(\begin{array}{cc} \hat{k} & \hat{k}-1 \\ \hat{k}+1 & \hat{k} \end{array}\Big| u_1-u_2\Big)\varphi_+(u_2,u';\hat{k}).
\end{aligned}
$$

This equation does hold by the Riemann identity (A.11). $(\varepsilon_1,\varepsilon_2)=(1,-1)$ case is similar. For $(\varepsilon_1,\varepsilon_2)=(1,1)$, (5.33) is

$$
\Phi_+(z_2)\Phi_+(z_1)=\rho(u_1-u_2)\Phi_+(z_1)\Phi_+(z_2).
$$

By using OPE rules, this becomes

$$
\begin{aligned}
& \oint \underline{dz'_2} \oint \underline{dz'_1}\, \Phi_-(z_2)\Phi_-(z_1)x_-(z'_2)x_-(z'_1)\varphi_+(u_2,u'_2;\hat{k}-1)\varphi_+(u_1,u'_1;\hat{k})f(u'_2-u_1) \\
= & \oint \underline{dz'_1} \oint \underline{dz'_2}\, \Phi_-(z_2)\Phi_-(z_1)x_-(z'_2)x_-(z'_1) \\
& \qquad \times\varphi_+(u_1,u'_1;\hat{k}-1)\varphi_+(u_2,u'_2;\hat{k})f(u'_1-u_2)h(u'_1-u'_2).
\end{aligned}
$$

Detailed analysis of the location of poles shows that we can take a common integration contour which is symmetric in z'_1 and z'_2. Consequently it is sufficient to compare the integrands after symmetrization in z'_1 and z'_2. Here symmetrization means

$$
\begin{aligned}
& \oint \underline{dz'_1} \oint \underline{dz'_2}\, x_-(z'_1)x_-(z'_2)F(u'_1,u'_2) \\
= & \oint \underline{dz'_1} \oint \underline{dz'_2}\, x_-(z'_1)x_-(z'_2)\tfrac{1}{2}\Big(F(u'_1,u'_2)+F(u'_2,u'_1)h(u'_1-u'_2)\Big).
\end{aligned}
\tag{5.39}
$$

We remark that $h(-u)=h(u)^{-1}$. Therefore we are enough to show

$$
\begin{aligned}
& \varphi_+(u_2,u'_2;\hat{k}-1)\varphi_+(u_1,u'_1;\hat{k})f(u'_2-u_1)+(u'_1\leftrightarrow u'_2)\times h(u'_1-u'_2) \\
= & \ \varphi_+(u_1,u'_1;\hat{k}-1)\varphi_+(u_2,u'_2;\hat{k})f(u'_1-u_2)h(u'_1-u'_2)+(u'_1\leftrightarrow u'_2)\times h(u'_1-u'_2),
\end{aligned}
$$

and this is correct due to the Riemann identity (A.11).

Next let us show (5.34). We set (see subsection 5.a)

$$\Psi^*_+(z) = \oint \underline{dz'}\, \Psi^*_-(z)x_+(z')\varphi^*_+(u,u';\hat{l}\,),$$

$$\varphi^*_+(u,u';\hat{l}\,) = \frac{[u-u'+\frac{1}{2}-\hat{l}\,]^*}{[u-u'-\frac{1}{2}]^*}\frac{1}{\sqrt{[\hat{l}]^*[\hat{l}-1]^*}}, \tag{5.40}$$

$$x_+(z_1)\Psi^*_-(z_2) = f^*(u_1-u_2)\Psi^*_-(z_2)x_+(z_1), \quad f^*(u) = \frac{[u-\frac{1}{2}]^*}{[-u-\frac{1}{2}]^*}, \tag{5.41}$$

$$x_+(z_1)x_+(z_2) = h^*(u_1-u_2)x_+(z_2)x_+(z_1), \quad h^*(u) = \frac{[u+1]^*}{[u-1]^*}. \tag{5.42}$$

We remark that

$$\varphi^*_+(u,u';a) = \varphi_+(-u,-u';a)\Big|_{r\to r^*}, \tag{5.43}$$

$$f^*(u) = f(-u)\Big|_{r\to r^*}, \tag{5.44}$$

$$h^*(u) = h(-u)\Big|_{r\to r^*}, \tag{5.45}$$

and (5.17) and (5.7). Integrands of (5.34) are obtained from those of (5.33) by replacement $r\to r^*$, $u\to -u$ and $\hat{k}\to\hat{l}$. Careful analysis of the location of poles shows that we can take a common (symmetric) integration contour for each case. Therefore (5.34) holds.

(5.35) is easily shown, although we have to take care of integration contours. □

The dual VO's are realized in the following way,

$$\Phi^*_\varepsilon(z) \;=\; \sqrt{[\hat{k}]}^{-1}\,\Phi_{-\varepsilon}(x^{-2}z)\sqrt{[\hat{k}]}, \tag{5.46}$$

$$\Psi_\varepsilon(z) \;=\; \sqrt{[\hat{l}]^*}\Psi^*_{-\varepsilon}(x^{-2}z)\sqrt{[\hat{l}]^*}^{-1}, \tag{5.47}$$

and normalization constants g and g^* in (5.27) and (5.29) are

$$g^{-1} \;=\; x^{\frac{r^*}{2r}}\frac{1}{(x^2,x^{2r},x^{2r};x^{2r})_\infty}\frac{(x^4,x^{2r};x^4,x^{2r})_\infty}{(x^2,x^{2r+2};x^4,x^{2r})_\infty}, \tag{5.48}$$

$$g^* \;=\; x^{-\frac{r}{2r^*}}\frac{1}{(x^{-2},x^{2r^*},x^{2r^*};x^{2r^*})_\infty}\frac{(x^2,x^{2r^*+2};x^4,x^{2r^*})_\infty}{(x^4,x^{2r^*};x^4,x^{2r^*})_\infty}. \tag{5.49}$$

Then we have

$$\sum_{\varepsilon=\pm1}\Phi^*_\varepsilon(z)\Phi_\varepsilon(z) = \mathrm{id}, \tag{5.50}$$

$$\Phi_{\varepsilon_2}(z)\Phi^*_{\varepsilon_1}(z) = \delta_{\varepsilon_1,\varepsilon_2}\times\mathrm{id}, \tag{5.51}$$

$$\sum_{\varepsilon=\pm1}\Psi^*_\varepsilon(z_2)\Psi_\varepsilon(z_1) = \frac{1}{1-\frac{z_1}{z_2}}\times\mathrm{id}+\cdots \quad (z_1\to z_2), \tag{5.52}$$

$$\Psi_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \frac{\delta_{\varepsilon_1,\varepsilon_2}}{1-\frac{z_1}{z_2}}\times\mathrm{id}+\cdots \quad (z_1\to z_2), \tag{5.53}$$

and for d in (3.43)

$$w^d\mathcal{O}(z)w^{-d}=\mathcal{O}(wz),\quad \text{for } \mathcal{O}=\Phi_\varepsilon,\Phi^*_\varepsilon,\Psi^*_\varepsilon,\Psi_\varepsilon,x_\pm. \tag{5.54}$$

We identify type I VO's in subsection 4.2 and those here in the following way:

$$\Phi^{(a-\varepsilon,a)}(z) \;=\; \Phi_\varepsilon(z)\Big|_{\mathcal{L}_{l,a}}, \tag{5.55}$$

$$\Phi^{*(a+\varepsilon,a)}(z) \;=\; \Phi^*_\varepsilon(z)\Big|_{\mathcal{L}_{l,a}}. \tag{5.56}$$

Then eqs. (4.106)-(4.110) correspond to (5.46),(5.33),(5.54),(5.50) and (5.51) respectively.

Next let us see how the DVA current is obtained from VO's. Let introduce free boson oscillator h_n $(n\in\mathbb{Z}_{\neq 0})$,

$$h_n=(x-x^{-1})(-1)^n\frac{[r^*n]_x}{[2n]_x}\alpha_n=(x-x^{-1})\frac{[rn]_x}{[2n]_x}\alpha'_n, \tag{5.57}$$

$$[h_n,h_m]=(x-x^{-1})^2\frac{1}{n}\frac{[n]_x}{[2n]_x}[rn]_x[r^*n]_x\delta_{n+m,0}. \tag{5.58}$$

As explained in subsection 3.4, the DVA current $T(z)$ is realized as

$$\begin{aligned}&T(z)=\Lambda_+(z)+\Lambda_-(z),\\ &\Lambda_\pm(z)=:\exp\Big(\pm\sum_{n\neq 0}h_n(x^{\pm 1}z)^{-n}\Big):\times x^{\pm\sqrt{rr^*}a'_0}.\end{aligned} \tag{5.59}$$

This $T(z)$ is obtained from type I VO's by fusing them [69],

$$\begin{aligned}&\Phi_{\varepsilon_2}(x^{1+r}z')\Phi^*_{\varepsilon_1}(x^{1-r}z)\\ &=\Big(1-\frac{z}{z'}\Big)\delta_{\varepsilon_1,\varepsilon_2}T(z)\cdot x^{-\frac{r^*}{2}}\frac{(x^4,x^{4-2r};x^4)_\infty}{(x^2,x^{2-2r};x^4)_\infty}+\cdots\quad(z'\to z),\end{aligned} \tag{5.60}$$

or from type II VO's

$$\begin{aligned}&\Psi_{\varepsilon_1}(x^{1+r^*}z')\Psi^*_{\varepsilon_2}(x^{1-r^*}z)\\ &=\frac{1}{1-\frac{z'}{z}}\delta_{\varepsilon_1,\varepsilon_2}\Big(-T(-z)\Big)\cdot(-x^{-\frac{r}{2}})\frac{(x^2,x^{2-2r^*};x^4)_\infty}{(x^4,x^{-2r^*};x^4)_\infty}+\cdots\quad(z'\to z).\end{aligned} \tag{5.61}$$

Higher DVA currents (3.132) in subsection 3.5 are also obtained by fusion,

$$\Phi_{\varepsilon_2}(x^{1+rj}z')\Phi^*_{\varepsilon_1}(x^{1-rj}z)=\delta_{\varepsilon_1,\varepsilon_2}\begin{cases}\mathrm{id} & (j=0)\\ (1-\frac{z}{z'})T_{(j)}(z)A_j & (j\geq 1)\end{cases}+\cdots\quad(z'\to z), \tag{5.62}$$

where A_j is

$$A_j=x^{-\frac{1}{2}r^*j}\frac{(x^4,x^{4-2rj};x^4)_\infty}{(x^2,x^{2-2rj};x^4)_\infty}\prod_{i=1}^{j-1}\frac{(x^{4-2ri},x^{-2ri};x^4)_\infty}{(x^{2-2ri},x^{2-2ri};x^4)_\infty}, \tag{5.63}$$

and we have assumed that r is generic.

5.3 Local height probability

Let us consider local height probabilities $P_{a_n,\cdots,a_0}(l)$. As remarked in subsection 5.1, we have

$$P_{a_n,\cdots,a_0}(l)=\sum_{m=0}^{1}\frac{Z_{l,m}}{Z_l}P_{a_n,\cdots,a_0}(l,m)=\frac{1}{2}P_{a_n,\cdots,a_0}(l,l-a_0). \tag{5.64}$$

One-point LHP is already obtained in (4.44),

$$P_k(l)=Z_l^{-1}\,[k]\,\chi_{l,k}(x^4). \tag{5.65}$$

Here the partition function Z_l (4.74) is

$$Z_l=\sum_{k=1}^{r-1}[k]\,\chi_{l,k}(x^4), \tag{5.66}$$

where $\chi_{l,k}(q)$ is given in (2.37). Two-point LHP's satisfy recursion relations (4.66)

$$\sum_a P_{a,b}(l)=P_b(l),\quad \sum_b P_{a,b}(l)=P_a(l). \tag{5.67}$$

Since $P_{a,b}(l)$ vanishes unless $|a-b|=1$ and $1\le a,b\le r-1$, two-point LHP can be determined uniquely by these recursion relations and expressed in terms of one-point LHP. Starting from $P_{0,1}(l)=P_{1,0}(l)=0$, we obtain

$$P_{k+1,k}(l)=P_{k,k+1}(l)=\sum_{a=1}^{k}(-1)^{k-a}P_a(l)=Z_l^{-1}\sum_{a=1}^{k}(-1)^{k-a}\,[a]\,\chi_{l,a}(x^4), \tag{5.68}$$

and this satisfies $P_{r,r-1}(l)=P_{r-1,r}(l)=0$ (which is equivalent to $Z_{l,0}=Z_{l,1}$). $P_{a,b}(l)=P_{b,a}(l)$ agrees with physical requirement (4.112). For higher-point LHP's, however, the recursion relations can not determine them. So we will use vertex operator approach.

As explained in subsection 4.2, local height probabilities can be expressed in terms of corner transfer matrices and type I VO's. In this case (4.73) with (4.111) becomes

$$\begin{aligned}&P_{a_n,\cdots,a_0}(l) \\ =\ & Z_l^{-1}\,[a_0]\,\mathrm{tr}_{\mathcal{H}_l^{(a_0)}}\Big(x^{4H_C^{(a_0)}}\Phi^{*(a_0,a_1)}(z)\cdots\Phi^{*(a_{n-1},a_n)}(z)\Phi^{(a_n,a_{n-1})}(z)\cdots\Phi^{(a_1,a_0)}(z)\Big).\end{aligned} \tag{5.69}$$

We evaluate this LHP using a free field realization of VO. Our identification of the space of state and operators are (5.15),(5.16),(5.55) and (5.56). Then we have

$$P_{a_n,\cdots,a_0}(l)=Z_l^{-1}\,[a_0]\,\mathrm{tr}_{\mathcal{L}_{l,a_0}}\Big(x^{4d}\Phi^*_{\varepsilon_1}(z)\cdots\Phi^*_{\varepsilon_n}(z)\Phi_{\varepsilon_n}(z)\cdots\Phi_{\varepsilon_1}(z)\Big), \tag{5.70}$$

where $\varepsilon_i=a_{i-1}-a_i$. Since the type I VO has the property (5.32) (see also subsection 5.b), we can apply the formula (3.83) to the trace in the above LHP.

We illustrate this free field calculation by taking two-point LHP as an example. Two-point LHP is ($\varepsilon=\pm 1$)

$$\begin{aligned} P_{k-\varepsilon,k}(l) &= Z_l^{-1}[k]\,\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{4d}\Phi^*_\varepsilon(z)\Phi_\varepsilon(z)\Big) \\ &= Z_l^{-1}\sqrt{[k][k-\varepsilon]}\,\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{4d}\Phi_{-\varepsilon}(x^{-2}z)\Phi_\varepsilon(z)\Big). \end{aligned} \tag{5.71}$$

From (5.71), (5.48) and OPE in subsection 5.a, we have

$$\begin{aligned} &P_{k-1,k}(l) \\ &= Z_l^{-1}(x^2,x^2,x^{2r};x^{2r})_\infty \oint_C \underline{dz'}\frac{(x^{2r-1}\zeta,x^{2r+1}\zeta;x^{2r})_\infty}{(x\zeta,x^3\zeta;x^{2r})_\infty} \\ &\times\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{4d}g^{-1}:\Phi_-(x^{-2}z)\Phi_-(z)x_-(z'):z^{-\frac{3}{2}\frac{r^*}{r}}x^{\frac{r^*}{2r}}\sqrt{\frac{[k][k-1]}{[\hat{k}][\hat{k}-1]}}\frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]}\Big), \end{aligned} \tag{5.72}$$

where $z=x^{2u}$, $z'=x^{2u'}$ and $\zeta=\frac{z'}{z}$. Calculation of trace is separated into two parts, oscillator part and zero mode part. The oscillator parts are common for all Fock spaces

$$\begin{aligned} &\mathrm{tr}^{\mathrm{osc}}_{\mathcal{F}_{l',k}}\Big(x^{4d^{\mathrm{osc}}}:\exp\Big(\sum_{n\neq 0}\frac{\alpha'_n}{[n]_x}(z'^{-n}-x^nz^{-n})\Big):\Big) \\ &= \frac{1}{(x^4;x^4)_\infty}\frac{(x^4,x^4,x^{2r+3}\zeta,x^{2r+1}\zeta^{-1};x^2,x^{2r})_\infty}{(x^{2r+2},x^{2r+2},x^5\zeta,x^3\zeta^{-1};x^2,x^{2r})_\infty}, \end{aligned} \tag{5.73}$$

where $\zeta=\frac{z'}{z}$ and we have used (A.53) (Or use the formula in subsection 5.a). On the other hand, the zero mode part,

$$\mathrm{tr}^{\mathrm{zero}}_{\mathcal{L}_{l,k}}\Big(x^{4d^{\mathrm{zero}}}(\tfrac{z'}{z})^{-\sqrt{\frac{r^*}{r}}a'_0+\frac{r^*}{r}}x^{-\sqrt{\frac{r^*}{r}}a'_0}\sqrt{\frac{[k][k-1]}{[\hat{k}][\hat{k}-1]}}\frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]}\Big), \tag{5.74}$$

can be calculated by (3.83) with the Felder complex (3.79)

$$C_{2j}=\mathcal{F}_{l-2r^*j,k},\quad C_{2j+1}=\mathcal{F}_{-l-2r^*j,k}. \tag{5.75}$$

(5.74) becomes

$$\frac{[u-u'-\frac{1}{2}+k]}{[u-u'+\frac{1}{2}]}\mathcal{O}_{l,k}(\tfrac{z'}{z}), \tag{5.76}$$

where $\mathcal{O}_{l,k}(\zeta)$ is (this definition is different from [21], $\mathcal{O}^{LP}_{l,k}(\zeta^{-2})=\frac{\zeta^{-l+\frac{r^*}{r}(k-1)}x^{\frac{r^*}{r}}}{(x^4;x^4)_\infty}\mathcal{O}_{l,k}(x^{-1}\zeta)$)

$$\begin{aligned} \mathcal{O}_{l,k}(\zeta) &= \sum_{j\in\mathbb{Z}}(-1)^j\mathrm{tr}^{\mathrm{zero}}_{C_j}\Big(x^{4d}\zeta^{-\sqrt{\frac{r^*}{r}}a'_0+\frac{r^*}{r}}x^{-\sqrt{\frac{r^*}{r}}a'_0}\Big) \\ &= \sum_{j\in\mathbb{Z}}\Big(x^{4(h_{l-2r^*j,k}-\frac{c}{24})}(x\zeta)^{l-2r^*j-\frac{r^*}{r}k}-x^{4(h_{-l-2r^*j,k}-\frac{c}{24})}(x\zeta)^{-l-2r^*j-\frac{r^*}{r}k}\Big)\zeta^{\frac{r^*}{r}} \\ &= x^{4(h_{l,k}-\frac{c}{24})}(x\zeta)^{l-\frac{r^*}{r}(k-1)}x^{-\frac{r^*}{r}}\times\Big(\Theta_{8rr^*}(-x^{4(r^*k-rl+rr^*)}(x\zeta)^{-2r^*}) \\ &\qquad -\Theta_{8rr^*}(-x^{4(r^*k+rl+rr^*)}(x\zeta)^{-2r^*})x^{4lk}(x\zeta)^{-2l}\Big). \end{aligned} \tag{5.77}$$

So we have obtained an integral representation of the two-point LHP,

$$P_{k-1,k}(l)=Z_l^{-1}\frac{(x^2;x^2)_\infty^3}{(x^4;x^4)_\infty}I_{l,k}, \tag{5.78}$$

where $I_{l,k}$ is

$$I_{l,k}=\oint_{|z'|=|z|}\underline{dz}'F_{l,k}(\zeta)=\oint_{|\zeta|=1}\underline{d\zeta}F_{l,k}(\zeta), \tag{5.79}$$

$$F_{l,k}(\zeta)=\frac{\Theta_{x^{2r}}(x^{-1+2k}\zeta^{-1})}{\Theta_{x^{2r}}(x\zeta)}\mathcal{O}_{l,k}(\zeta)\zeta^{-\frac{1}{r}(k-1)}x^{\frac{1}{r}k(k-1)+1-k}. \tag{5.80}$$

Next let us evaluate this integral. $F_{l,k}(\zeta)$ has simple poles at $\zeta=x^{1+2n}$ $(n\in\mathbb{Z})$. Since $\mathcal{O}_{l,k}(\zeta)$ satisfies

$$\mathcal{O}_{l,k}(x^{4rm}\zeta)=\mathcal{O}_{l,k}(\zeta)\zeta^{-2r^*m}x^{-4rr^*m^2+2r^*m}\quad(m\in\mathbb{Z}), \tag{5.81}$$

$F_{l,k}(\zeta)$ has the property

$$F_{l,k}(x^{4r}\zeta)=F_{l,k}(\zeta). \tag{5.82}$$

Consequently $I_{l,k}$ requires some regularization. We regularize $I_{l,k}$ in the following way,

$$I_{l,k}=\lim_{\epsilon\to0}I^\epsilon_{l,k},\quad I^\epsilon_{l,k}=\oint_{|\zeta|=1}\underline{d\zeta}\,\zeta^\epsilon F_{l,k}(\zeta). \tag{5.83}$$

Deforming the contour

$$I^\epsilon_{l,k}=\oint_{\zeta=x,x^3,\cdots,x^{4r-1}}\underline{d\zeta}\,\zeta^\epsilon F_{l,k}(\zeta)+\oint_{|\zeta|=x^{4r}}\underline{d\zeta}\,\zeta^\epsilon F_{l,k}(\zeta), \tag{5.84}$$

and using (5.82), we obtain

$$I^\epsilon_{l,k}=\frac{1}{1-x^{4r\epsilon}}\oint_{\zeta=x,x^3,\cdots,x^{4r-1}}\underline{d\zeta}\,\zeta^\epsilon F_{l,k}(\zeta). \tag{5.85}$$

We remark that

$$\oint_{\zeta=x,x^3,\cdots,x^{4r-1}}\underline{d\zeta}F_{l,k}(\zeta)=0, \tag{5.86}$$

namely

$$\sum_{a=0}^{2r-1}(-1)^a[k-1-a]x^{\frac{r^*}{r}a^2}\mathcal{O}_{l,k}(x^{1+2a})=0. \tag{5.87}$$

Using these and picking up residues at $\zeta=x,x^3,\cdots,x^{4r-1}$, we obtain

$$P_{k-1,k}(l)=\frac{1}{Z_l(x^4;x^4)_\infty}\sum_{a=0}^{2r-1}\frac{-a}{2r}(-1)^a[k-1-a]x^{\frac{r^*}{r}a^2}\mathcal{O}_{l,k}(x^{1+2a}). \tag{5.88}$$

We can check that this answer (and $P_{k,k-1}(l)=P_{k-1,k}(l)$) satisfies (5.67) and $P_l(0,1)=0$ (use (5.87)). Therefore this answer obtained by free field approach agrees with (5.68).

For multi-point LHP (5.70), we can write down its integral representation in a similar way. (Evaluation of the integral is another problem.) See [21].

5.4 Form factor

Let us recall the translation invariant vacuum state (4.90)

$$|\mathrm{vac}\rangle = \sqrt{[\hat{k}]}\, x^{2d}, \tag{5.89}$$

and the action of the column-to-column transfermatrix (4.113)

$$\Big(\mathcal{T}_{\mathrm{col}}(u)|f\rangle\Big)^{(k)} = \sum_{\varepsilon=\pm 1} \Phi_\varepsilon(z)\cdot f \cdot \Phi_{-\varepsilon}(z)\Big|_{\mathcal{L}_{l,k}}, \tag{5.90}$$

where f is a linear map $f : \mathcal{L}_{l,a} \to \mathcal{L}_{l,a}$.

For the time being, we consider operators on the Fock spaces $\mathcal{F}_{l,k}$ not on the cohomology $\mathcal{L}_{l,k}$. Action of $\mathcal{T}_{\mathrm{col}}$ is $\mathcal{T}_{\mathrm{col}}(u)|f\rangle = \sum_{\varepsilon=\pm1} \Phi_\varepsilon(z)\cdot f\cdot \Phi_{-\varepsilon}(z)$. Let us define an 'excited' state

$$|w_m,\cdots,w_1\rangle_{\varepsilon_m,\cdots,\varepsilon_1} = \Psi^*_{\varepsilon_m}(w_m)\cdots\Psi^*_{\varepsilon_1}(w_1)|\mathrm{vac}\rangle \,:\, \mathcal{F}_{l,k} \to \mathcal{F}_{l-\sum_i \varepsilon_i,k}. \tag{5.91}$$

(We remark that the original description of this Ψ^* is $\Psi^* \otimes 1$, see (4.87).) This state is an eigenstate of $\mathcal{T}_{\mathrm{col}}$

$$\begin{aligned}\mathcal{T}_{\mathrm{col}}(u)|w_m,\cdots,w_1\rangle_{\varepsilon_m,\cdots,\varepsilon_1} &= \sum_{\varepsilon=\pm1}\Phi_\varepsilon(z)\cdot\Psi^*_{\varepsilon_m}(w_m)\cdots\Psi^*_{\varepsilon_1}(w_1)\sqrt{[\hat{k}]}\,x^{2d}\cdot\Phi_{-\varepsilon}(z)\\ &= \prod_{j=1}^m \tau(v_j-u)\cdot|w_m,\cdots,w_1\rangle_{\varepsilon_m,\cdots,\varepsilon_1},\end{aligned} \tag{5.92}$$

where $z = x^{2u}$, $w_j = x^{2v_j}$ and we have used (5.35). Therefore type II VO creates (or annihilates) 'particles'. But we remark that this state is a 'true' eigenstate of $\mathcal{T}_{\mathrm{col}}$ because it is not a linear map on $\mathcal{L}_{l,k}$ in general. We will show when this state becomes a true eigenstate.

To avoid $\frac{1}{\sqrt{[\hat{k}][\hat{k}-1]}}$ and $\frac{1}{\sqrt{[\hat{l}]^*[\hat{l}-1]^*}}$ factors we perform a gauge transformation (see (4.4))

$$\widetilde{W}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big|u\Big) = \frac{F(a,b)F(b,d)}{F(a,c)F(c,d)} W\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big|u\Big), \quad F(a,b) = ([a][b])^{-\frac14}, \tag{5.93}$$

$$\widetilde{W}^*\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big|u\Big) = \frac{F^*(a,b)F^*(b,d)}{F^*(a,c)F^*(c,d)} W^*\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big|u\Big), \quad F^*(a,b) = ([a]^*[b]^*)^{\frac14}, \tag{5.94}$$

$$\widetilde{\Phi}_-(z) = \Phi_-(z), \quad \widetilde{\Phi}_+(z) = \Phi_+(z)\sqrt{[\hat{k}][\hat{k}-1]}, \tag{5.95}$$

$$\widetilde{\Psi}^*_-(z) = \Psi^*_-(z), \quad \widetilde{\Psi}^*_+(z) = \Psi^*_+(z)\sqrt{[\hat{l}]^*[\hat{l}-1]^*}. \tag{5.96}$$

$\widetilde{W}$ differs from W only in the following component

$$\widetilde{W}\begin{pmatrix} a & a\pm1 \\ a\mp1 & a \end{pmatrix}\Big|u\Big) = \frac{[a\mp1]}{[a]}\frac{[-u]}{[1+u]}\rho(u). \tag{5.97}$$

In this gauge the reflection symmetry (5.7) is lost. $\widetilde{W}^*$ is

$$\widetilde{W}^*\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \left(\rho(u)^{-1}\widetilde{W}\left(\begin{matrix} a & c \\ b & d \end{matrix}\middle| u\right)\right)\Bigg|_{r\to r^*} \times \rho^*(u). \tag{5.98}$$

$\widetilde{\Phi}_\varepsilon(z)$ and $\widetilde{\Psi}^*_\varepsilon(z)$ satisfy (5.33)-(5.35) with (Φ, Ψ^*, W, W^*) replaced by $(\widetilde{\Phi}, \widetilde{\Psi}^*, \widetilde{W}, \widetilde{W}^*)$ respectively. (5.35) is also hold by replacing Ψ^* with $\widetilde{\Psi}^*$ and Φ unchanged. Therefore the state

$$|w_m, \cdots, w_1\rangle^{\sim}_{\varepsilon_m,\cdots,\varepsilon_1} = \widetilde{\Psi}^*_{\varepsilon_m}(w_m)\cdots\widetilde{\Psi}^*_{\varepsilon_1}(w_1)|\mathrm{vac}\rangle \,:\, \mathcal{F}_{l,k} \to \mathcal{F}_{l-\sum_i \varepsilon_i,k}, \tag{5.99}$$

which differs from (5.91) by a multiplicative constant factor dependent on $\hat{l}$, is also an eigenstate of $\mathcal{T}_{\mathrm{col}}$.

How to realize the type II VO on the cohomology is discussed in subsection 5.b. In order to interpret the state (5.99) as a linear map on $\mathcal{L}_{l,k}$, the first requirement is $\sum_{i=1}^{m} \varepsilon_i = 0$. Hence m is an even integer, $m = 2n$. Then $\prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{\varepsilon_i}(w_i)$ and the BRST charge satisfy the intertwining property

$$Q_m \cdot \prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{\varepsilon_i}(w_i) = (-1)^n \prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{-\varepsilon_i}(w_i) \cdot Q_m \quad (m = l, r^* - l). \tag{5.100}$$

Therefore this product of type II VO's is a well-defined operator on the cohomology $\mathcal{L}_{l,k}$. Explicitly it is

$$\prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{\varepsilon_i}(w_i) \ \text{ on } C_{2j}, \quad (-1)^n \prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{-\varepsilon_i}(w_i) \ \text{ on } C_{2j+1}. \tag{5.101}$$

Consequently the state $|w_{2n}, \cdots, w_1\rangle^{\sim}_{\varepsilon_{2n},\cdots,\varepsilon_1}$ with $\sum_{i=1}^{2n} \varepsilon_i = 0$ is a true eigenstate of $\mathcal{T}_{\mathrm{col}}$. (If m is an odd integer or $\sum_i \varepsilon_i$ does not vanish, then $|w_m, \cdots, w_1\rangle_{\varepsilon_m,\cdots,\varepsilon_1}$ is a map from $\mathcal{F}_{l,k}$ to $\mathcal{F}_{l',k}$ ($l' = l - \sum_i \varepsilon_i \neq l$), namely in the original description it corresponds to a state whose half is in the ground state l and the other half in l'. As remarked in subsection 4.2, to obtain a complete set of excited states, we need not only the translation invariant vacuum state (5.89) but also the translation non-invariant vacuum states.) The form factors of local operator $\widetilde{\mathcal{O}}$ are defined by

$$\langle \mathrm{vac}|\widetilde{\mathcal{O}}|w_m, \cdots, w_1\rangle_{\varepsilon_m,\cdots,\varepsilon_1}. \tag{5.102}$$

Next let us study (4.75) with the following $\mathcal{O} = \mathcal{O}_{\Psi^*}$

$$\mathcal{O}_{\Psi^*} = \prod_{i=1}^{\overset{\frown}{2n}} \widetilde{\Psi}^*_{\varepsilon_i}(w_i) \qquad (w_j = x^{2v_j}, \ \sum_{i=1}^{2n} \varepsilon_i = 0). \tag{5.103}$$

In this model we have

$$Q_{a_n,\cdots,a_0}(l\,;z|\mathcal{O}_{\Psi^*})=\sum_{m=0}^{1}\frac{Z_{l,m}}{Z_l}Q_{a_n,\cdots,a_0}(l,m\,;z|\mathcal{O}_{\Psi^*})=\frac{1}{2}Q_{a_n,\cdots,a_0}(l,l-a_0\,;z|\mathcal{O}_{\Psi^*}). \quad (5.104)$$

This is the form factor of $\Phi^*_{a_0-a_1}(z)\cdots\Phi^*_{a_{n-1}-a_n}(z)\Phi_{a_{n-1}-a_n}(z)\cdots\Phi_{a_0-a_1}(z)$. (4.76) and (4.77) with $n=1$ become

$$\sum_{\varepsilon=\pm1}Q_{a+\varepsilon,a}(l\,;z|\mathcal{O}_{\Psi^*}) \;=\; Q_a(l|\mathcal{O}_{\Psi^*}), \quad (5.105)$$

$$\sum_{\varepsilon=\pm1}Q_{a,a+\varepsilon}(l\,;z|\mathcal{O}_{\Psi^*}) \;=\; G(z)Q_a(l|\mathcal{O}_{\Psi^*}), \quad (5.106)$$

where $Q_a(l|\mathcal{O}_{\Psi^*})=Q_a(l\,;z|\mathcal{O}_{\Psi^*})$ and $G(z)$ is

$$G(z)=\prod_{j=1}^{2n}\tau(w_j-u). \quad (5.107)$$

This $Q_{a,b}(l\,;z|\mathcal{O}_{\Psi^*})$ is uniquely determined by these recursion relations and it is expressed in terms of $Q_a(l|\mathcal{O}_{\Psi^*})$. Starting from $Q_{0,1}(l\,;z|\mathcal{O}_{\Psi^*})=Q_{1,0}(l\,;z|\mathcal{O}_{\Psi^*})=0$, we obtain

$$Q_{a+1,a}(l\,;z|\mathcal{O}_{\Psi^*}) \;=\; \sum_{\substack{b=1\\ b\equiv a\,(\mathrm{mod}\,2)}}^{a}Q_b(l|\mathcal{O}_{\Psi^*})-G(z)\sum_{\substack{b=1\\ b\not\equiv a\,(\mathrm{mod}\,2)}}^{a}Q_b(l|\mathcal{O}_{\Psi^*}), \quad (5.108)$$

$$Q_{a,a+1}(l\,;z|\mathcal{O}_{\Psi^*}) \;=\; -\sum_{\substack{b=1\\ b\not\equiv a\,(\mathrm{mod}\,2)}}^{a}Q_b(l|\mathcal{O}_{\Psi^*})+G(z)\sum_{\substack{b=1\\ b\equiv a\,(\mathrm{mod}\,2)}}^{a}Q_b(l|\mathcal{O}_{\Psi^*}). \quad (5.109)$$

This answer should satisfy $Q_{r,r-1}(l\,;z|\mathcal{O}_{\Psi^*})=Q_{r-1,r}(l\,;z|\mathcal{O}_{\Psi^*})=0$.

As an example let us calculate $Q_k(l|\widetilde{\Psi}^*_+(w_2)\widetilde{\Psi}^*_-(w_1))$ by free field realization. By using (3.83) it becomes

$$\begin{aligned}&Q_k(l|\widetilde{\Psi}^*_+(w_2)\widetilde{\Psi}^*_-(w_1))=Z_l^{-1}[k]\,\mathrm{tr}_{\mathcal{L}_{l,k}}\left(x^{4d}\widetilde{\Psi}^*_+(w_2)\widetilde{\Psi}^*_-(w_1)\right)\\ =&\; Z_l^{-1}[k]\sum_{j\in\mathbb{Z}}\left(\mathrm{tr}_{C_{2j}}\left(x^{4d}\widetilde{\Psi}^*_+(w_2)\widetilde{\Psi}^*_-(w_1)\right)+\mathrm{tr}_{C_{2j+1}}\left(x^{4d}\widetilde{\Psi}^*_-(w_2)\widetilde{\Psi}^*_+(w_1)\right)\right). \quad (5.110)\end{aligned}$$

By using OPE and trace rule listed in subsection 5.a, we obtain

$$\begin{aligned}&Q_k(l|\widetilde{\Psi}^*_+(w_2)\widetilde{\Psi}^*_-(w_1))\\ =&\; Z_l^{-1}[k]g^{*-1}\Bigg(\oint_{C_{\Psi^*}(w_2)}\underline{dw'}(w_1w_2)^{\frac{r}{2r^*}l}w'^{-\frac{r}{r^*}l}\langle\!\langle x_+(w')\Psi^*_-(w_1)\rangle\!\rangle\frac{[v_2-v'-\frac{1}{2}-l]^*}{[v_2-v'-\frac{1}{2}]^*}\\ &\qquad\times\Theta_{x^{8rr^*}}(-(\tfrac{w'^2}{w_1w_2})^r x^{4(-rl+r^*k+rr^*)})\\ &\qquad+\oint_{C_{\Psi^*}(w_1)}\underline{dw'}(w_1w_2)^{-\frac{r}{2r^*}l}w'^{\frac{r}{r^*}l}\langle\!\langle\Psi^*_-(w_1)x_+(w')\rangle\!\rangle\frac{[v_1-v'+\frac{1}{2}+l]^*}{[v_1-v'-\frac{1}{2}]^*}\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times\Theta_{x^{8rr^*}}\big(-(\tfrac{w'^2}{w_1w_2})^r x^{4(rl+r^*k+rr^*)}\big)x^{4lk}\Big) \\
&\times(w_1w_2)^{-\frac{1}{2}k+\frac{r}{4r^*}}w'^{k+\frac{r}{r^*}}x^{4h_{l,k}-\frac{c}{24}}\langle\!\langle\Psi_-^*(w_2)\Psi_-^*(w_1)\rangle\!\rangle\langle\!\langle\Psi_-^*(w_2)x_+(w')\rangle\!\rangle \\
&\times(x^4;x^4)_\infty^{-1}\prod_{i,j=1}^{2}F_{\Psi_-^*,\Psi_-^*}(\tfrac{w_i}{w_j})\cdot\prod_{i=1}^{2}F_{\Psi_-^*,x_+}(\tfrac{w_i}{w'})F_{x_+,\Psi_-^*}(\tfrac{w'}{w_i}),
\end{aligned}
\tag{5.111}
$$

where $F_{A,B}(z)$ is given in subsection 5.a. $Q_{k-1,k}(l\,;z|\widetilde{\Psi}_+^*(w_2)\widetilde{\Psi}_-^*(w_1))$ can be obtained by using this result and (5.109). If you want to calculate it directly, you will follow

$$
\begin{aligned}
&Q_{k-1,k}(l\,;z|\widetilde{\Psi}_+^*(w_2)\widetilde{\Psi}_-^*(w_1)) = Z_l^{-1}[k]\,\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{4d}\Phi_+^*(z)\Phi_+(z)\widetilde{\Psi}_+^*(w_2)\widetilde{\Psi}_-^*(w_1)\Big) \\
&= Z_l^{-1}[k]\sum_{j\in\mathbb{Z}}\Big(\mathrm{tr}_{C_{2j}}\Big(x^{4d}\Phi_+^*(z)\Phi_+(z)\widetilde{\Psi}_+^*(w_2)\widetilde{\Psi}_-^*(w_1)\Big) \\
&\qquad\qquad +\mathrm{tr}_{C_{2j+1}}\Big(x^{4d}\Phi_+^*(z)\Phi_+(z)\widetilde{\Psi}_-^*(w_2)\widetilde{\Psi}_+^*(w_1)\Big)\Big) \\
&= \cdots
\end{aligned}
\tag{5.112}
$$

5.a OPE and trace

OPE

We list the normal ordering relations used in section 5. Recall

$$r^* = r-1.$$

For operators $A(z), B(w)$ that have the form : exp(linear in boson) :, we use the notation

$$A(z)B(w) = \langle\!\langle A(z)B(w)\rangle\!\rangle : A(z)B(w) :, \tag{5.113}$$

and write down only the part $\langle\!\langle A(z)B(w)\rangle\!\rangle$:

$$\langle\!\langle x_+(z_1)x_+(z_2)\rangle\!\rangle = z_1^{\frac{2r}{r^*}}(1-\zeta)\frac{(x^{-2}\zeta;x^{2r^*})_\infty}{(x^{2r^*+2}\zeta;x^{2r^*})_\infty}, \tag{5.114}$$

$$\langle\!\langle x_-(z_1)x_-(z_2)\rangle\!\rangle = z_1^{\frac{2r^*}{r}}(1-\zeta)\frac{(x^{2}\zeta;x^{2r})_\infty}{(x^{2r-2}\zeta;x^{2r})_\infty}, \tag{5.115}$$

$$\langle\!\langle x_\pm(z_1)x_\mp(z_2)\rangle\!\rangle = z_1^{-2}\frac{1}{(1+x\zeta)(1+x^{-1}\zeta)}, \tag{5.116}$$

$$\langle\!\langle \Phi_-(z_1)x_+(z_2)\rangle\!\rangle = \langle\!\langle x_+(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1+z_2, \tag{5.117}$$

$$\langle\!\langle \Phi_-(z_1)x_-(z_2)\rangle\!\rangle = \langle\!\langle x_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{-\frac{r^*}{r}}\frac{(x^{2r-1}\zeta;x^{2r})_\infty}{(x\zeta;x^{2r})_\infty}, \tag{5.118}$$

$$\langle\!\langle \Psi_-^*(z_1)x_+(z_2)\rangle\!\rangle = \langle\!\langle x_+(z_1)\Psi_-^*(z_2)\rangle\!\rangle = z_1^{-\frac{r}{r^*}}\frac{(x^{2r^*+1}\zeta;x^{2r^*})_\infty}{(x^{-1}\zeta;x^{2r^*})_\infty}, \tag{5.119}$$

$$\langle\!\langle \Psi_-^*(z_1)x_-(z_2)\rangle\!\rangle = \langle\!\langle x_-(z_1)\Psi_-^*(z_2)\rangle\!\rangle = z_1+z_2, \tag{5.120}$$

$$\langle\!\langle \Phi_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{\frac{r^*}{2r}}\frac{(x^2\zeta, x^{2r+2}\zeta; x^4, x^{2r})_\infty}{(x^4\zeta, x^{2r}\zeta; x^4, x^{2r})_\infty}, \tag{5.121}$$

$$\langle\!\langle \Psi^*_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1^{\frac{r}{2r^*}}\frac{(\zeta, x^{2r^*+4}\zeta; x^4, x^{2r^*})_\infty}{(x^2\zeta, x^{2r^*+2}\zeta; x^4, x^{2r^*})_\infty}, \tag{5.122}$$

$$\langle\!\langle \Phi_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = \langle\!\langle \Psi^*_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{-\frac{1}{2}}\frac{(-x^3\zeta; x^4)_\infty}{(-x\zeta; x^4)_\infty}, \tag{5.123}$$

where $\zeta = \frac{z_2}{z_1}$ and we have used (A.50).

As meromorphic functions we have ($z_i = x^{2u_i}$)

$$x_+(z_1)x_+(z_2) = x_+(z_2)x_+(z_1)\frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}, \tag{5.124}$$

$$x_-(z_1)x_-(z_2) = x_-(z_2)x_-(z_1)\frac{[u_1-u_2-1]}{[u_1-u_2+1]}, \tag{5.125}$$

$$x_\pm(z_1)x_\mp(z_2) = x_\mp(z_2)x_\pm(z_1), \tag{5.126}$$

$$\Phi_-(z_1)x_+(z_2) = x_+(z_2)\Phi_-(z_1), \tag{5.127}$$

$$\Phi_-(z_1)x_-(z_2) = x_-(z_2)\Phi_-(z_1)\frac{[u_1-u_2+\frac{1}{2}]}{[-u_1+u_2+\frac{1}{2}]}, \tag{5.128}$$

$$\Psi^*_-(z_1)x_+(z_2) = x_+(z_2)\Psi^*_-(z_1)\frac{[u_1-u_2-\frac{1}{2}]^*}{[-u_1+u_2-\frac{1}{2}]^*}, \tag{5.129}$$

$$\Psi^*_-(z_1)x_-(z_2) = x_-(z_2)\Psi^*_-(z_1), \tag{5.130}$$

$$\Phi_-(z_1)\Phi_-(z_2) = \Phi_-(z_2)\Phi_-(z_1)\rho(u_2-u_1), \tag{5.131}$$

$$\Psi^*_-(z_1)\Psi^*_-(z_2) = \Psi^*_-(z_2)\Psi^*_-(z_1)\rho^*(u_1-u_2), \tag{5.132}$$

$$\Phi_-(z_1)\Psi^*_-(z_2) = \Psi^*_-(z_2)\Phi_-(z_1)\tau(u_2-u_1). \tag{5.133}$$

Here $\rho(u)$, $\rho^*(u)$ and $\tau(u)$ are given by

$$z^{\frac{r^*}{2r}}\rho(u) = \frac{\rho_+(u)}{\rho_+(-u)}, \quad \rho_+(u) = \frac{(x^2z, x^{2r+2}z; x^4, x^{2r})_\infty}{(x^4z, x^{2r}z; x^4, x^{2r})_\infty}, \tag{5.134}$$

$$z^{-\frac{r}{2r^*}}\rho^*(u) = \frac{\rho^*_+(u)}{\rho^*_+(-u)}, \quad \rho^*_+(u) = \frac{(x^2z, x^{2r^*+2}z; x^4, x^{2r^*})_\infty}{(z, x^{2r^*+4}z; x^4, x^{2r^*})_\infty}, \tag{5.135}$$

$$\tau(u) = z^{\frac{1}{2}}\frac{\Theta_{x^4}(-xz^{-1})}{\Theta_{x^4}(-xz)}. \tag{5.136}$$

Note that

$$\rho^*(u) = -\rho(u)\Big|_{r\to r^*}. \tag{5.137}$$

Trace

For an operator $A(z)$ that have the form : exp(linear in boson) :, we write

$$A(z) = A^{\mathrm{osc}}(z)A^{\mathrm{zero}}(z), \quad A^{\mathrm{osc}}(z) = \exp\Big(\sum_{n>0} f^A_{-n}\alpha_{-n}z^n\Big)\exp\Big(\sum_{n>0} f^A_n\alpha_n z^{-n}\Big), \tag{5.138}$$

where f_n^A is a coefficient. In this model A is $x_\pm, \Phi_-, \Psi_-^*$. For example,

$$\Phi_-^{\rm osc}(z) = \exp\Big(\sum_{n>0} \frac{\alpha_n'}{[2n]_x} z^n\Big) \exp\Big(-\sum_{n>0} \frac{\alpha_n'}{[2n]_x} z^{-n}\Big), \quad \Phi_-^{\rm zero}(z) = \sqrt{g} e^{\frac{1}{2}\sqrt{\frac{r^*}{r}}Q} z^{\frac{1}{2}\sqrt{\frac{r^*}{r}}a_0' + \frac{r^*}{4r}}.$$

For such operators Wick's theorem tells us

$$A_1(z_1)\cdots A_n(z_n) = \prod_{i<j} \langle\!\langle A_i(z_i)A_j(z_j)\rangle\!\rangle \times :\prod_i A_i(z_i): . \tag{5.139}$$

The trace of oscillator parts over the Fock space $\mathcal{F} = \mathcal{F}_{l,k}$ can be calculated by using the trace technique (A.53). We have

$$\mathrm{tr}_{\mathcal{F}}\Big(x^{4d^{\rm osc}} :\prod_i A_i^{\rm osc}(z_i):\Big) = \frac{1}{(x^4;x^4)_\infty} \prod_{i,j} F_{A_i,A_j}(\tfrac{z_i}{z_j}), \tag{5.140}$$

where $F_{A,B}(z)$ is given by

$$F_{A,B}(z) = \exp\Big(\sum_{n>0} \frac{1}{n}[n]_x[2n]_x \frac{[rn]_x}{[r^*n]_x} \frac{x^{4n}}{1-x^{4n}} f_{-n}^A f_n^B z^n\Big). \tag{5.141}$$

We write down $F_{A,B}(z)$ (Remark $F_{A,B}(z) = F_{B,A}(z)$) :

$$\begin{aligned}
F_{x_+,x_+}(z) &= \frac{(x^2z;x^2,x^{2r^*})_\infty}{(x^{2r^*+4}z;x^2,x^{2r^*})_\infty}, && (5.142)\\
F_{x_-,x_-}(z) &= \frac{(x^4z;x^2,x^{2r})_\infty}{(x^{2r+2}z;x^2,x^{2r})_\infty}, && (5.143)\\
F_{x_+,x_-}(z) &= \frac{1}{(-x^3z,-x^5z;x^4)_\infty}, && (5.144)\\
F_{\Phi_-,x_+}(z) &= (-x^4z;x^4)_\infty, && (5.145)\\
F_{\Phi_-,x_-}(z) &= \frac{(x^{2r+3}z;x^4,x^{2r})_\infty}{(x^5z;x^4,x^{2r})_\infty}, && (5.146)\\
F_{\Psi_-^*,x_+}(z) &= \frac{(x^{2r^*+5}z;x^4,x^{2r^*})_\infty}{(x^3z;x^4,x^{2r^*})_\infty}, && (5.147)\\
F_{\Psi_-^*,x_-}(z) &= (-x^4z;x^4)_\infty, && (5.148)\\
F_{\Phi_-,\Phi_-}(z) &= \frac{(x^6z,x^{2r+6}z;x^4,x^4,x^{2r})_\infty}{(x^8z,x^{2r+4}z;x^4,x^4,x^{2r})_\infty}, && (5.149)\\
F_{\Psi_-^*,\Psi_-^*}(z) &= \frac{(x^4z,x^{2r^*+8}z;x^4,x^4,x^{2r^*})_\infty}{(x^6z,x^{2r^*+6}z;x^4,x^4,x^{2r^*})_\infty}, && (5.150)\\
F_{\Phi_-,\Psi_-^*}(z) &= \frac{(-x^7z;x^4,x^4)_\infty}{(-x^5z;x^4,x^4)_\infty}. && (5.151)
\end{aligned}$$

5.b Screening operators and vertex operators

All the formulas in this subsection are understood as meromorphic functions.

Let us recall the screening operators (3.72) and (3.84)

$$X(z) = \oint_{C_X(z)} \underline{dz'}\, x_+(z') \frac{[u-u'+\frac{1}{2}-\hat{l}\,]^*}{[u-u'-\frac{1}{2}]^*} \,:\, \mathcal{F}_{l,k} \to \mathcal{F}_{l-2,k}, \tag{5.152}$$

$$X'(z) = \oint_{C_{X'}(z)} \underline{dz'}\, x_-(z') \frac{[u-u'-\frac{1}{2}+\hat{k}]}{[u-u'+\frac{1}{2}]} \,:\, \mathcal{F}_{l,k} \to \mathcal{F}_{l,k-2}, \tag{5.153}$$

where integration contours are

	inside	outside
$C_X(z)$	$z' = x^{-1+2r^*n}z$	$z' = x^{-1-2r^*(n+1)}z$
$C_{X'}(z)$	$z' = x^{1+2rn}z$	$z' = x^{1-2r(n+1)}z$

$(n = 0, 1, 2, \cdots)$. (5.154)

These screening operators satisfy the following exchange relations [63]

$$X(z_1)X(z_2)\frac{[u_1-u_2-1]^*}{[u_1-u_2]^*} - X(z_2)^2\frac{[u_1-u_2-\hat{l}+2]^*}{[u_1-u_2]^*}\frac{[1]^*}{[\hat{l}-2]^*} = (u_1 \leftrightarrow u_2), \tag{5.155}$$

$$X'(z_1)X'(z_2)\frac{[u_1-u_2+1]}{[u_1-u_2]} - X'(z_2)^2\frac{[u_1-u_2+\hat{k}-2]}{[u_1-u_2]}\frac{[1]}{[\hat{k}-2]} = (u_1 \leftrightarrow u_2), \tag{5.156}$$

$$X(z_1)X'(z_2) = X'(z_2)X(z_1). \tag{5.157}$$

$\widetilde{\Phi}_+$ and $\widetilde{\Psi}^*_+$ can be expressed as (see (5.28),(5.30),(5.95),(5.96))

$$\widetilde{\Phi}_+(z) = \widetilde{\Phi}_-(z)X'(z), \quad \widetilde{\Psi}^*_+(z) = \widetilde{\Psi}^*_-(z)X(z). \tag{5.158}$$

Here RHS's are understood as the analytic continuation of $A(z)B(z')$ from the region $|z| \gg |z'|$ and hence integration contours become (5.31).

Since screening operators and VO's satisfy

$$X(z)\widetilde{\Phi}_\varepsilon(w) = \widetilde{\Phi}_\varepsilon(w)X(z), \tag{5.159}$$

$$X'(z)\widetilde{\Psi}^*_\varepsilon(w) = \widetilde{\Psi}^*_\varepsilon(w)X'(z), \tag{5.160}$$

$\prod_i \widetilde{\Phi}_{\varepsilon_i}(w_i)$ and $\prod_i \widetilde{\Psi}^*_{\varepsilon_i}(w_i)$ ($\sum_i \varepsilon_i = 0$) are well-defined operators on the cohomology $H^0(C_{l,k})$ and $H^0(C'_{l,k})$ respectively where the BRST charges are $Q_m = X(1)^m$ and $Q'_m = X'(1)^m$ (see subsection 3.4.3). On the other hand $\widetilde{\Phi}_\varepsilon(w)$ and $\widetilde{\Psi}^*_\varepsilon(w)$ do not commute with $X'(z)$ and $X(z)$ respectively, so they are not well-defined operators on the cohomology $H^0(C'_{l,k})$ and $H^0(C_{l,k})$ in general. However we can find the following intertwining properties.

We can show that

$$X'(z)\widetilde{\Phi}_-(w) = -\widetilde{\Phi}_-(w)X'(z)\frac{[u-v+1]}{[u-v]}\frac{[\hat{k}]}{[\hat{k}-1]} + \widetilde{\Phi}_+(w)\frac{[u-v+\hat{k}]}{[u-v]}\frac{[1]}{[\hat{k}-1]}, \quad (5.161)$$

$$X(z)\widetilde{\Psi}^*_-(w) = -\widetilde{\Psi}^*_-(w)X(z)\frac{[u-v-1]^*}{[u-v]^*}\frac{[\hat{l}]^*}{[\hat{l}-1]^*} + \widetilde{\Psi}^*_+(w)\frac{[u-v-\hat{l}]^*}{[u-v]^*}\frac{[1]^*}{[\hat{l}-1]^*}, \quad (5.162)$$

where $z = x^{2u}$ and $w = x^{2v}$. By induction we obtain

$$X'(z)^n\widetilde{\Phi}_-(w) = \widetilde{\Phi}_-(w)X'(z)^n A_n(u-v,\hat{k}) + \widetilde{\Phi}_+(w)X'(z)^{n-1}B_n(u-v,\hat{k}), \quad (5.163)$$

$$X'(z)^n\widetilde{\Phi}_+(w) = \widetilde{\Phi}_-(w)X'(z)^{n+1}B_n(v-u,\hat{k}-2) + \widetilde{\Phi}_+(w)X'(z)^n A_n(v-u,\hat{k}-2), \quad (5.164)$$

$$X(z)^n\widetilde{\Psi}^*_-(w) = \widetilde{\Psi}^*_-(w)X(z)^n A^*_n(u-v,\hat{l}) + \widetilde{\Psi}^*_+(w)X(z)^{n-1}B^*_n(u-v,\hat{l}), \quad (5.165)$$

$$X(z)^n\widetilde{\Psi}^*_+(w) = \widetilde{\Psi}^*_-(w)X(z)^{n+1}B^*_n(v-u,\hat{l}-2) + \widetilde{\Psi}^*_+(w)X(z)^n A^*_n(v-u,\hat{l}-2), \quad (5.166)$$

where coefficients are

$$A_n(u,k) = (-1)^n\frac{[u+n]}{[u]}\frac{[k-n+1]}{[k-2n+1]}, \quad (5.167)$$

$$B_n(u,k) = (-1)^{n-1}\frac{[u+k-n+1]}{[u]}\frac{[n]}{[k-2n+1]}, \quad (5.168)$$

$$A^*_n(u,l) = (-1)^n\frac{[u-n]^*}{[u]^*}\frac{[l-n+1]^*}{[l-2n+1]^*}, \quad (5.169)$$

$$B^*_n(u,l) = (-1)^{n-1}\frac{[u-k+n-1]^*}{[u]^*}\frac{[n]^*}{[l-2n+1]^*}. \quad (5.170)$$

Noticing the following properties of these coefficients

$$A_{k+1}(u,k') = 0, \quad B_{k+1}(u,k') = (-1)^{k+1} \quad \text{for } k' \equiv k \pmod r, \quad (5.171)$$

$$A^*_{l+1}(u,l') = 0, \quad B^*_{l+1}(u,l') = (-1)^{l+1} \quad \text{for } l' \equiv l \pmod{r^*}, \quad (5.172)$$

we obtain the intertwining properties ($1 \le l \le r^*-1$, $1 \le k \le r-1$)

$$X'(z)^{k-\varepsilon}\widetilde{\Phi}_\varepsilon(w) = (-1)^{k-\varepsilon}\widetilde{\Phi}_{-\varepsilon}(w)X'(z)^k \quad \text{on } \mathcal{F}_{l',k'}\ k' \equiv k \pmod r, \quad (5.173)$$

$$X'(z)^{r-k-\varepsilon}\widetilde{\Phi}_\varepsilon(w) = (-1)^{r-k-\varepsilon}\widetilde{\Phi}_{-\varepsilon}(w)X'(z)^{r-k} \quad \text{on } \mathcal{F}_{l',k'}\ k' \equiv -k \pmod r, \quad (5.174)$$

$$X(z)^{l-\varepsilon}\widetilde{\Psi}^*_\varepsilon(w) = (-1)^{l-\varepsilon}\widetilde{\Psi}^*_{-\varepsilon}(w)X(z)^l \quad \text{on } \mathcal{F}_{l',k'}\ l' \equiv l \pmod{r^*}, \quad (5.175)$$

$$X(z)^{r^*-l-\varepsilon}\widetilde{\Psi}^*_\varepsilon(w) = (-1)^{r^*-l-\varepsilon}\widetilde{\Psi}^*_{-\varepsilon}(w)X(z)^l \quad \text{on } \mathcal{F}_{l',k'}\ l' \equiv -l \pmod{r^*}. \quad (5.176)$$

Therefore $\prod_i \widetilde{\Phi}_{\varepsilon_i}(w_i)$ and $\prod_i \widetilde{\Psi}^*_{\varepsilon_i}(w_i)$ ($\sum_i \varepsilon_i = 0$) are well-defined operators on the cohomology $H^0(C'_{l,k})$ and $H^0(C_{l,k})$ respectively. However we remark that on C'_{2j+1} and C_{2j+1} those operators should be replaced by $\pm\prod_i \widetilde{\Phi}_{-\varepsilon_i}(w_i)$ and $\pm\prod_i \widetilde{\Psi}^*_{-\varepsilon_i}(w_i)$ respectively.

6 DVA ($A_2^{(2)}$ type) and Dilute A_L Models

In this section we introduce another deformed Virasoro algebra and study dilute A_L models by free field approach [48]. Both of them are associated to $A_2^{(2)}$ algebra.

r, β, $x_\pm(z)$, $T(z)$, α_n, α'_n, h_n, $C_{l,k}$, etc. in this section are different form those in section 5.

6.1 DVA ($A_2^{(2)}$)

Brazhnikov and Lukyanov [22] pointed out that one can associate to the algebra $A_2^{(2)}$ a deformed Virasoro algebra which is different from the one discussed in section 3. We denote it DVA($A_2^{(2)}$).

DVA($A_2^{(2)}$) is an associative algebra over $\mathbb{C}$ generated by T_n ($n \in \mathbb{Z}$) with two parameters x and r, and their relation is [22]

$$\begin{aligned}[T_n, T_m] \;=\; & -\sum_{\ell=1}^{\infty} f_\ell (T_{n-\ell}T_{m+\ell} - T_{m-\ell}T_{n+\ell}) \\ & +(x-x^{-1})^2 \frac{[r+\frac{1}{2}]_x [r]_x [r-1]_x [r-\frac{3}{2}]_x}{[\frac{1}{2}]_x [\frac{3}{2}]_x} [3n]_x \delta_{n+m,0} \\ & +(x-x^{-1})^2 \frac{[r]_x [r-\frac{1}{2}]_x [r-1]_x}{[\frac{1}{2}]_x} [n-m]_x T_{n+m},\end{aligned} \tag{6.1}$$

where the structure constants f_ℓ is given by

$$\begin{aligned}f(z) \;=\; & \sum_{\ell=0}^{\infty} f_\ell z^\ell = \exp\Bigl(-\sum_{n>0} (x-x^{-1})^2 \frac{1}{n} \frac{[n]_x [rn]_x [(r-1)n]_x}{[2n]_x - [n]_x} z^n \Bigr) \\ = \; & \frac{1}{1-z} \frac{(x^{2-2r}z, x^{3-2r}z, x^4 z, x^5 z, x^{2r}z, x^{2r+1}z; x^6)_\infty}{(x^{5-2r}z, x^{6-2r}z, xz, x^2 z, x^{2r+3}z, x^{2r+4}z; x^6)_\infty}.\end{aligned} \tag{6.2}$$

By introducing DVA current $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$, the above relation can be written as a formal power series,

$$\begin{aligned}& f(\tfrac{w}{z})T(z)T(w) - T(w)T(z)f(\tfrac{z}{w}) \\ =\; & (x-x^{-1}) \frac{[r+\frac{1}{2}]_x [r]_x [r-1]_x [r-\frac{3}{2}]_x}{[\frac{1}{2}]_x [\frac{3}{2}]_x} \Bigl(\delta(x^3 \tfrac{w}{z}) - \delta(x^{-3} \tfrac{w}{z}) \Bigr) \\ & +(x-x^{-1}) \frac{[r]_x [r-\frac{1}{2}]_x [r-1]_x}{[\frac{1}{2}]_x} \Bigl(\delta(x^2 \tfrac{w}{z}) T(xw) - \delta(x^{-2} \tfrac{w}{z}) T(x^{-1}w) \Bigr).\end{aligned} \tag{6.3}$$

The notation of [22] is related to ours by $x_{BL} = x^{\frac{3}{2}}$, $\frac{b}{Q} = r$, $\frac{1}{Qb} = 1-r$, $g(z) = f(z)$, $\mathbf{V}(z) = T(z)$. For later use we add a grading operator d,

$$[d, T_n] = -nT_n. \tag{6.4}$$

The above relation (6.1) is invariant under

$$\text{(i)} \qquad x \mapsto x^{-1}, \quad r \mapsto r, \quad T(z) \mapsto T(z), \tag{6.5}$$

$$\text{(ii)} \qquad x \mapsto x, \quad r \mapsto 1-r, \quad T(z) \mapsto -T(z). \tag{6.6}$$

In the case of (i) $f(z)$ is understood as the first line of (6.2). Let us introduce β as

$$\beta = \frac{r}{2(r-1)}, \tag{6.7}$$

then (2.19) becomes

$$\alpha_0 = \frac{2-r}{\sqrt{2r(r-1)}}. \tag{6.8}$$

In the conformal limit ($x = e^{\hbar} \to 1$, r : fixed), (6.1) admits two limits [22] related by (6.6),

$$T(z) = 3 - 2r + 8r(r-1)\hbar^2\Big(z^2L(z) + \tfrac{1}{48}(1-2r) + \tfrac{(2-r)^2}{8r(r-1)}\Big) + O(\hbar^4), \tag{6.9}$$

$$T(z) = -1 - 2r - 8r(r-1)\hbar^2\Big(z^2\widetilde{L}(z) - \tfrac{1}{48}(1-2r) + \tfrac{(1+r)^2}{8r(r-1)}\Big) + O(\hbar^4), \tag{6.10}$$

where $L(z), \widetilde{L}(z)$ are the Virasoro currents with the central charges $c, \widetilde{c}$ respectively,

$$c = 1 - \frac{3(2-r)^2}{r(r-1)} = 1 - 6\alpha_0^2, \quad \widetilde{c} = 1 - \frac{3(1+r)^2}{r(r-1)}. \tag{6.11}$$

The highest weight representation is defined by the same manner presented in subsection 3.3. The Kac determinant at level N is [48]

$$\begin{aligned} &\det\Big(\langle\lambda; N, i|\lambda; N, j\rangle\Big)_{1\le i,j\le p(N)} \\ &= \prod_{\substack{l,k\ge1 \\ lk\le N}} \left(\frac{(x^{rl} - x^{-rl})(x^{(r-1)l} - x^{-(r-1)l})}{x^l - 1 + x^{-l}}(\lambda - \lambda_{l,k})(\lambda - \widetilde{\lambda}_{l,k})\right)^{p(N-lk)}, \end{aligned} \tag{6.12}$$

where $\lambda_{l,k}$ and $\widetilde{\lambda}_{l,k}$ are given by

$$\lambda_{l,k} = x^{-lr+2k(r-1)} + x^{lr-2k(r-1)} - \frac{[r-\frac{1}{2}]_x}{[\frac{1}{2}]_x}, \tag{6.13}$$

$$\widetilde{\lambda}_{l,k} = -x^{l(r-1)-2kr} - x^{-l(r-1)+2kr} - \frac{[r-\frac{1}{2}]_x}{[\frac{1}{2}]_x}, \tag{6.14}$$

and they are related by (6.6). In the conformal limit this Kac determinant reduces to the Virasoro one with the proportional constant expected from (6.9) or (6.10).

6.2 Free field realization

6.2.1 free field realization

Let us introduce free boson oscillator h_n $(n \in \mathbb{Z}_{\neq 0})$,

$$[h_n, h_m] = (x - x^{-1})^2 \frac{1}{n} \frac{[n]_x [rn]_x [(r-1)n]_x}{[2n]_x - [n]_x} \delta_{n+m,0}, \tag{6.15}$$

and use zero mode a_0 and Q defined in (2.49) (or a'_0 in (2.72)). Notice that $[2n]_x - [n]_x = [3n]_x [\frac{n}{2}]_x / [\frac{3n}{2}]_x$. The Fock space $\mathcal{F}_\alpha$ is defined as before.

The DVA($A_2^{(2)}$) current $T(z)$ is realized as follows:

$$\begin{aligned}
&T(z) = \Lambda_+(z) + \Lambda_0(z) + \Lambda_-(z), \\
&\Lambda_\pm(z) = : \exp\Big(\pm \sum_{n \neq 0} h_n (x^{\pm \frac{3}{2}} z)^{-n} \Big) : \times x^{\pm \sqrt{2r(r-1)} a'_0}, \\
&\Lambda_0(z) = - \frac{[r - \frac{1}{2}]_x}{[\frac{1}{2}]_x} : \exp\Big(\sum_{n \neq 0} h_n (x^{-n/2} - x^{n/2}) z^{-n} \Big) : .
\end{aligned} \tag{6.16}$$

To prove this we need (A.23) and the OPE formula,

$$\begin{aligned}
f(\tfrac{w}{z}) \Lambda_\pm(z) \Lambda_\pm(w) &= : \Lambda_\pm(z) \Lambda_\mp(w) :, \\
f(\tfrac{w}{z}) \Lambda_\pm(z) \Lambda_\mp(w) &= : \Lambda_\pm(z) \Lambda_\mp(w) : \gamma(x^{\mp 1} \tfrac{w}{z}) \gamma(x^{\mp 2} \tfrac{w}{z}), \\
f(\tfrac{w}{z}) \Lambda_0(z) \Lambda_0(w) &= : \Lambda_0(z) \Lambda_0(w) : \gamma(\tfrac{w}{z}), \\
f(\tfrac{w}{z}) \Lambda_\pm(z) \Lambda_0(w) &= : \Lambda_\pm(z) \Lambda_0(w) : \gamma(x^{\mp 1} \tfrac{w}{z}), \\
f(\tfrac{w}{z}) \Lambda_0(z) \Lambda_\pm(w) &= : \Lambda_0(z) \Lambda_\pm(w) : \gamma(x^{\pm 1} \tfrac{w}{z}),
\end{aligned} \tag{6.17}$$

where $\gamma(z)$ is given in (3.41). The grading operator d is realized by

$$\begin{aligned}
&d = d^{\mathrm{osc}} + d^{\mathrm{zero}}, \\
&d^{\mathrm{osc}} = \sum_{n>0} \frac{n^2 ([2n]_x - [n]_x)}{(x - x^{-1})^2 [n]_x [rn]_x [(r-1)n]_x} h_{-n} h_n, \quad d^{\mathrm{zero}} = \frac{1}{4} {a'_0}^2 - \frac{1}{24},
\end{aligned} \tag{6.18}$$

which satisfies

$$[d, h_n] = -n h_n, \quad [d, Q] = a'_0, \quad d |\alpha_{l,k}\rangle_B = (h_{l,k} - \frac{c}{24}) |\alpha_{l,k}\rangle_B, \tag{6.19}$$

where c and $h_{l,k}$ are given by (2.17) and (2.18) respectively. We remark that $\widetilde{T}(z) = -\Lambda_+(z) + \Lambda_0(z) - \Lambda_-(z)$ also satisfies (6.1).

$|\alpha\rangle_B$ is the highest weight state of DVA with $\lambda = \lambda(\alpha)$,

$$|\alpha\rangle_B = |\lambda(\alpha)\rangle, \tag{6.20}$$

$$\lambda(\alpha) = x^{\sqrt{2r(r-1)}(\alpha - \alpha_0)} + x^{-\sqrt{2r(r-1)}(\alpha - \alpha_0)} - \frac{[r - \frac{1}{2}]_x}{[\frac{1}{2}]_x}. \tag{6.21}$$

The dual space $\mathcal{F}_\alpha^*$ becomes a DVA module by (2.63) with

$$
\begin{aligned}
{}^tT_n &= T_{-n}, && (6.22)\\
{}^th_n &= -h_{-n}, \quad \text{eq.(2.66)}, && (6.23)
\end{aligned}
$$

and (2.67). By (6.23), $\mathcal{F}_\alpha^*$ is isomorphic to $\mathcal{F}_{2\alpha_0-\alpha}$ as DVA module,

$$
\begin{aligned}
\mathcal{F}_\alpha^* &\cong \mathcal{F}_{2\alpha_0-\alpha} \quad \text{(DVA module)}. && (6.24)\\
|\alpha\rangle_B^* &\leftrightarrow |2\alpha_0-\alpha\rangle_B. && (6.25)
\end{aligned}
$$

Note that $\lambda(\alpha)=\lambda(2\alpha_0-\alpha)$ by (6.21).

In the conformal limit ($x=e^\hbar \to 1$, r : fixed), oscillator h_n is expressed by a_n in (2.49) as follows:

$$h_n = \hbar\sqrt{2r(r-1)}\sqrt{\frac{1}{x^n-1+x^{-n}}\frac{x^{rn}-x^{-rn}}{2rn\hbar}\frac{x^{(r-1)n}-x^{-(r-1)n}}{2(r-1)n\hbar}}a_n. \tag{6.26}$$

Substituting this expression into (6.16) and expanding in $\hbar$, we get (6.9) with $L(z)$ in (2.55) (or (2.73)).

The representation with $\lambda=\lambda_{l,k}$ is realized on $\mathcal{F}_{l,k}=\mathcal{F}_{\alpha_{l,k}}$,

$$\lambda_{l,k}=\lambda(\alpha_{l,k}), \tag{6.27}$$

where $\alpha_{l,k}$ is given in (2.79).

6.2.2 Kac determinant

In the free boson realization the singular vectors can be expressed by screening currents. A screening current $x_+(z)$ is defined by

$$x_+(z) =: \exp\Big(-\sum_{n\neq 0}\frac{\alpha_n}{[n]_x}z^{-n}\Big) : \times e^{\sqrt{\frac{r}{2(r-1)}}Q} z^{\sqrt{\frac{r}{2(r-1)}}a_0'+\frac{r}{2(r-1)}}, \tag{6.28}$$

where oscillators α_n ($n\in\mathbb{Z}_{\neq 0}$) are related to h_n as

$$h_n = (x-x^{-1})(-1)^n\frac{[(r-1)n]_x}{[2n]_x-[n]_x}\alpha_n. \tag{6.29}$$

Like as (3.61) singular vector is obtained by the BRST operator. Screening charges and BRST charges will be discussed in the next subsection.

Like as in subsection 3.4.2 let us introduce matrices $C(N,\alpha)$ and $C'(N,\alpha)$ by (3.63) and (3.64). Then their determinants are given by

$$\det C(N,\alpha)_{I,J} = \prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(\Big(z^{\frac{1}{2}}x^{\frac{1}{2}lr-k(r-1)} - z^{-\frac{1}{2}}x^{-\frac{1}{2}lr+k(r-1)}\Big)$$

$$\times\Big(z^{\frac{1}{2}}x^{-\frac{1}{2}l(r-1)+kr}+z^{-\frac{1}{2}}x^{\frac{1}{2}l(r-1)-kr}\Big)\Big)^{p(N-lk)}, \qquad (6.30)$$

$$\begin{aligned}\det C'(N,\alpha)_{I,J} &= \det C(N,2\alpha_0-\alpha)\cdot\det D\\ &= \prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(\Big(z^{\frac{1}{2}}x^{-\frac{1}{2}lr+k(r-1)}-z^{-\frac{1}{2}}x^{\frac{1}{2}lr-k(r-1)}\Big)\\ &\qquad\times\Big(z^{\frac{1}{2}}x^{\frac{1}{2}l(r-1)-kr}+z^{-\frac{1}{2}}x^{-\frac{1}{2}l(r-1)+kr}\Big)\Big)^{p(N-lk)}, \end{aligned}\qquad (6.31)$$

where $z=x^{\sqrt{2r(r-1)}(\alpha-\alpha_0)}$. The inner product of two states in the Verma module is given by (3.67) where $G_{K,L}$ is

$$\begin{aligned}G_{K,L} &= <({}^th)_K|\alpha\rangle_B^*, h_{-L}|\alpha\rangle_B>\\ &= \delta_{K,L}\prod_i\Big(\frac{1}{i}\frac{(x^{ri}-x^{-ri})(x^{(r-1)i}-x^{-(r-1)i})}{x^i-1+x^{-i}}\Big)^{k_i}k_i!,\end{aligned}\qquad (6.32)$$

and its determinant is (use (A.28) and (A.30))

$$\begin{aligned}\det G_{K,L} &= \prod_{\substack{\{k_i\}\\ \sum_i ik_i=N}}\prod_i\Big(\frac{1}{i}\frac{(x^{ri}-x^{-ri})(x^{(r-1)i}-x^{-(r-1)i})}{x^i-1+x^{-i}}\Big)^{k_i}k_i!\\ &= \prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(\frac{(x^{rl}-x^{-rl})(x^{(r-1)l}-x^{-(r-1)l})}{x^l-1+x^{-l}}\Big)^{p(N-lk)}.\end{aligned}\qquad (6.33)$$

Therefore we obtain the Kac determinant (6.12),

$$\begin{aligned}&\det\langle h|T_IT_J|h\rangle=\det C'(N,\alpha)\cdot\det G\cdot\det{}^tC(N,\alpha)\\ &=\prod_{\substack{l,k\geq 1\\ lk\leq N}}\Big(\frac{(x^{rl}-x^{-rl})(x^{(r-1)l}-x^{-(r-1)l})}{x^l-1+x^{-l}}(\lambda-\lambda_{l,k})(\lambda-\widetilde{\lambda}_{l,k})\Big)^{p(N-lk)}.\end{aligned}\qquad (6.34)$$

6.2.3 Felder complex

We consider the representation of $\lambda=\lambda_{l,k}$ in (6.27) with (2.20) and (2.23), i.e.

$$r=\frac{2p''}{2p''-p'}. \qquad (6.35)$$

Let us consider the Felder complex $C_{l,k}$,

$$\cdots\xrightarrow{X_{-3}}C_{-2}\xrightarrow{X_{-2}}C_{-1}\xrightarrow{X_{-1}}C_0\xrightarrow{X_0}C_1\xrightarrow{X_1}C_2\xrightarrow{X_2}\cdots, \qquad (6.36)$$

where C_j and $X_j\ :\ C_j\to C_{j+1}$ $(j\in\mathbb{Z})$ are

$$C_{2j}=\mathcal{F}_{l-2p'j,k},\quad C_{2j+1}=\mathcal{F}_{-l-2p'j,k}, \qquad (6.37)$$

$$X_{2j}=Q_l,\quad X_{2j+1}=Q_{p'-l}. \qquad (6.38)$$

We assume that p' is odd (we have not obtained the result for general even p'). BRST charge Q_m $(1\leq m\leq p'-1)$ is defined by

$$Q_{2m+1} = Q_1 Q_2^{(1)} Q_2^{(2)} \cdots Q_2^{(m)}, \tag{6.39}$$
$$Q_{2m} = Q_2^{(\frac{p'+1}{2}-m)} \cdots Q_2^{(\frac{p'-3}{2})} Q_2^{(\frac{p'-1}{2})}. \tag{6.40}$$

Here Q_1 and $Q_2^{(a)}$ are

$$Q_1 = \oint_{|z|=1} \underline{dz}\, x_+(z) \frac{[u+\frac{1}{2}\hat{l}]^*}{[u+\frac{1}{2}]^*}, \tag{6.41}$$
$$\begin{aligned} Q_2^{(a)} &= \oint\oint_{|z_1|=|z_2|=1} \underline{dz}_1 \underline{dz}_2\, x_+(z_1)x_+(z_2) \frac{1}{[u_1+\frac{1}{2}]^*[u_2+\frac{1}{2}]^*} \\ &\times \frac{[u_1-u_2]^*}{[u_1-u_2+1]^*[u_1-u_2-\frac{1}{2}]^*} f_2^{(a)}(u_1+\tfrac{1}{2}\hat{l}, u_2+\tfrac{1}{2}\hat{l}), \end{aligned} \tag{6.42}$$

where $[u]^*$ is given in (A.6) with $r^*=r-1$ and $z=x^{2u}$, $z_i=x^{2u_i}$ and

$$\begin{aligned} f_2^{(a)}(u_1,u_2) &= [2a+1]^*[a-\tfrac{1}{2}]^*[u_1-a]^*[u_2+a-1]^*[u_1-u_2+a-\tfrac{1}{2}]^* \\ &-[2a-1]^*[a+\tfrac{1}{2}]^*[u_1+a]^*[u_2-a-1]^*[u_1-u_2-a-\tfrac{1}{2}]^*. \end{aligned} \tag{6.43}$$

X satisfies the BRST property,

$$X_j X_{j-1} = 0. \tag{6.44}$$

We assume that this Felder complex has the same structure as the Virasoro case because it formally tends to Virasoro one in the conformal limit ($x\to 1$ and r and $z=x^{2u}$ fixed kept). Then the cohomology groups of the complex $C_{l,k}$ are

$$H^j(C_{l,k}) = \text{Ker } X_j/\text{Im } X_{j-1} = \begin{cases} 0 & j\neq 0, \\ \mathcal{L}_{l,k} & j=0, \end{cases} \tag{6.45}$$

where $\mathcal{L}_{l,k}$ is the irreducible DVA module of $\lambda=\lambda_{l,k}$. The trace of operator $\mathcal{O}$ over $\mathcal{L}_{l,k}$ can be written as (3.83).

The BRST charge commutes with DVA [48]

$$[T_n, Q_l] = 0 \quad \text{on } \mathcal{F}_{l',k} \quad l'\equiv l \ (\text{mod } p'). \tag{6.46}$$

Singular vector can be obtained by Q_l like as (3.61).

6.3 Dilute A_L models

In the following we fix a positive integer $L\geq 3$. The dilute A_L model [70, 71] is an integrable RSOS model obtained by restricting the face model of type $A_2^{(2)}$ [72]. In the dilute A_L model, the local fluctuation variables $a, b, \cdots$ take one of the L states $1, 2, \cdots, L$, and those on neighboring lattice sites are subject to the condition $a - b = 0, \pm 1$. The Boltzmann weights can be found in [71], eq.(3.1). For our purpose it is convenient to use the parametrization given in Appendix A of [71], which is suitable in the 'low-temperature' regime. With some change of notation we recall the formula below [48].

Let $x = e^{-2\pi\lambda/\varepsilon}$, $r = \pi/(2\lambda)$ and $u = -u_{orig}/(2\lambda)$, where λ, ε are the variables used in [71] and u_{orig} stands for 'u' there. We shall restrict ourselves to the 'regime 2^+' defined by

$$0 < x < 1, \quad r = 2\,\frac{L+1}{L+2}, \quad -\frac{3}{2} < u < 0. \tag{6.47}$$

We have taken $p' = L$, $p'' = L + 1$ in (6.35), which corresponds to minimal unitary series. (6.7) and (6.8) become

$$\beta = \frac{L+1}{L}, \quad \alpha_0 = \frac{1}{\sqrt{L(L+1)}}, \quad p' = L, \quad p'' = L+1, \tag{6.48}$$

and we set

$$r^* = r - 1. \tag{6.49}$$

On the critical point ($x \to 1$), this model is described by CFT, i.e. the minimal unitary series with $\beta = \frac{L+1}{L}$. On the off-critical point ($x < 1$), which corresponds to the $(1, 2)$-perturbation of the minimal unitary CFT, the Virasoro symmetry is lost but the DVA($A_2^{(2)}$) symmetry remains. Changing an overall scalar factor we put the Boltzmann weights in the form

$$W\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big) = \rho(u)\overline{W}\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big), \tag{6.50}$$

where $\rho(u)$ is given in (6.126) and chosen so that the partition function per site of the model equals to 1. $\overline{W}$ is

$$\begin{aligned}
&\overline{W}\Big(\begin{matrix} a\pm 1 & a \\ a & a\mp 1 \end{matrix}\Big| u\Big) = 1,\\
&\overline{W}\Big(\begin{matrix} a & a\pm 1 \\ a & a\pm 1 \end{matrix}\Big| u\Big) = \overline{W}\Big(\begin{matrix} a\pm 1 & a\pm 1 \\ a & a \end{matrix}\Big| u\Big) = -\left(\frac{[\pm a+\frac{3}{2}]_+[\pm a-\frac{1}{2}]_+}{[\pm a+\frac{1}{2}]_+^2}\right)^{\frac{1}{2}}\frac{[u]}{[1+u]},\\
&\overline{W}\Big(\begin{matrix} a\pm 1 & a \\ a & a \end{matrix}\Big| u\Big) = \overline{W}\Big(\begin{matrix} a & a \\ a & a\pm 1 \end{matrix}\Big| u\Big) = \frac{[\pm a+\frac{1}{2}+u]_+}{[\pm a+\frac{1}{2}]_+}\frac{[1]}{[1+u]},\\
&\overline{W}\Big(\begin{matrix} a & a\mp 1 \\ a\pm 1 & a \end{matrix}\Big| u\Big) = (G_a^+G_a^-)^{\frac{1}{2}}\frac{[\frac{1}{2}+u]}{[\frac{3}{2}+u]}\frac{[u]}{[1+u]},\\
&\overline{W}\Big(\begin{matrix} a & a \\ a\pm 1 & a \end{matrix}\Big| u\Big) = \overline{W}\Big(\begin{matrix} a & a\pm 1 \\ a & a \end{matrix}\Big| u\Big) = -(G_a^\pm)^{\frac{1}{2}}\frac{[\pm a-1-u]_+}{[\pm a+\frac{1}{2}]_+}\frac{[1]}{[1+u]}\frac{[u]}{[\frac{3}{2}+u]},\\
&\overline{W}\Big(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\Big| u\Big) = \frac{[\pm 2a+1-u]}{[\pm 2a+1]}\frac{[1]}{[1+u]} - G_a^\pm\frac{[\pm 2a-\frac{1}{2}-u]}{[\pm 2a+1]}\frac{[u]}{[\frac{3}{2}+u]}\frac{[1]}{[1+u]},
\end{aligned} \tag{6.51}$$

$$\overline{W}\Big(\begin{matrix} a & a \\ a & a \end{matrix}\Big| u\Big) = \frac{[3+u]}{[3]}\frac{[1]}{[1+u]}\frac{[\frac{3}{2}-u]}{[\frac{3}{2}+u]} + H_a\frac{[1]}{[3]}\frac{[u]}{[1+u]}.$$

Here

$$G_a^{\pm} = \frac{S(a\pm 1)}{S(a)}, \quad S(a) = (-1)^a\frac{[2a]}{[a]_+}, \quad H_a = G_a^{+}\frac{[a-\frac{5}{2}]_+}{[a+\frac{1}{2}]_+} + G_a^{-}\frac{[a+\frac{5}{2}]_+}{[a-\frac{1}{2}]_+}, \tag{6.52}$$

where $[u]$ and $[u]_+$ are given in (A.5) and (A.7).

This Boltzmann weight W enjoys YBE (4.3), initial condition (4.38), unitarity (4.39) and crossing symmetry (4.99) ($\lambda = -\frac{3}{2}$, $G_a = S(a)$),

$$W\Big(\begin{matrix} b & d \\ a & c \end{matrix}\Big| -\frac{3}{2}-u\Big) = \sqrt{\frac{S(a)S(d)}{S(b)S(c)}}W\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big). \tag{6.53}$$

In this gauge W enjoys also a reflection symmetry

$$W\Big(\begin{matrix} a & c \\ b & d \end{matrix}\Big| u\Big) = W\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big). \tag{6.54}$$

Along with the additive variable u, we often use the multiplicative variable $z = x^{2u}$. For later use we define W^*,

$$W^*\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big) = \overline{W}\Big(\begin{matrix} a & b \\ c & d \end{matrix}\Big| u\Big)\Big|_{r\to r^*} \times \rho^*(u), \tag{6.55}$$

where $\rho^*(u)$ is given in (6.127).

Hereafter we assume that L is odd. The model has ground states labeled by odd integers $l = 1, 3, \cdots, L-2$ [71]. They are characterized as configurations in which all heights take the same value b. If $L = 4n\pm 1$, then the possible values are $b = l$ ($1 \le l \le 2n-1$, $l:$ odd) or $b = l+1$ ($2n+1 \le l \le L-2$, $l:$ odd). Therefore in the notation of subsection 4.2 we have $(i_1) = (b)$, $m = 0 \in \mathbb{Z}/\mathbb{Z}$ and $\mathcal{H}_l^{(k)} = \mathcal{H}_{l,0}^{(k)}$. In the thermodynamic limit CTM's (4.100) become

$$A^{(k)}(u) = C^{(k)}(u) = x^{-2uH_C^{(k)}}, \quad B^{(k)}(u) = D^{(k)}(u) = \sqrt{S(k)}x^{2(u+\frac{3}{2})H_C^{(k)}}. \tag{6.56}$$

Careful study of the corner Hamiltonian shows that the character (4.43) coincides with the Virasoro minimal unitary character [71],

$$\chi_{l,0,k}(q) = \chi_{l,k}^{\mathrm{Vir}}(q), \tag{6.57}$$

where $\chi_{l,k}^{\mathrm{Vir}}(q)$ is given in (2.37). Comparing the free filed realization given in subsection 6.2, we make an identification

$$\mathcal{H}_l^{(k)} = \mathcal{L}_{l,k}\,, \tag{6.58}$$

$$H_C^{(k)} = d\,\Big|_{\mathcal{L}_{l,k}}, \tag{6.59}$$

where $\mathcal{L}_{l,k}$ is given in (6.45) as a cohomology of the Felder complex and d is realized in (6.18).

6.4 Free field approach

Although bosons are already introduced in subsection 6.2, we present their definitions again. Recall that

$$r=2\,\frac{L+1}{L+2},\quad r^*=r-1.$$

Let us introduce free boson oscillator α_n $(n\in\mathbb{Z}_{\neq 0})$

$$[\alpha_n,\alpha_m]=\frac{[n]_x([2n]_x-[n]_x)}{n}\frac{[rn]_x}{[r^*n]_x}\delta_{n+m,0}, \tag{6.60}$$

and use zero mode a_0 and Q defined in (2.49) (or a'_0 in (2.72)). The Fock space $\mathcal{F}_{l,k}$ is defined by

$$\mathcal{F}_{l,k}=\bigoplus_{m\geq 0}\bigoplus_{n_1\geq\cdots\geq n_m>0}\mathbb{C}\alpha_{-n_1}\cdots\alpha_{-n_m}|\alpha_{l,k}\rangle_B, \tag{6.61}$$

where $|\alpha\rangle_B$ is given by (2.51) with replacing a_n by h_n,

$$a'_0|\alpha_{l,k}\rangle_B=(\alpha_{l,k}-\alpha_0)|\alpha_{l,k}\rangle_B,\quad \alpha_{l,k}-\alpha_0=-\sqrt{\frac{L+1}{L}}l+\sqrt{\frac{L}{L+1}}k. \tag{6.62}$$

We use also free boson oscillator α'_n $(n\in\mathbb{Z}_{\neq 0})$

$$\alpha'_n=(-1)^n\frac{[r^*n]_x}{[rn]_x}\alpha_n,\quad [\alpha'_n,\alpha'_m]=\frac{[n]_x([2n]_x-[n]_x)}{n}\frac{[r^*n]_x}{[rn]_x}\delta_{n+m,0}. \tag{6.63}$$

Operators $\hat{l},\hat{k}:\mathcal{F}_{l,k}\to\mathcal{F}_{l,k}$ are defined by (5.22).

Elliptic currents $x_\pm(z)$ for $U_x(A_2^{(2)})$ (or $\mathcal{B}_{x,\lambda}(A_2^{(2)})$) of level 1 ($c=1$) are obtained by a 'dressing' procedure described in subsection 4.4 **5**

$$\begin{aligned}
x_+(z)\ &:\ \mathcal{F}_{l,k}\to\mathcal{F}_{l-2,k}\\
&x_+(z)=:\exp\Big(-\sum_{n\neq 0}\frac{\alpha_n}{[n]_x}z^{-n}\Big):\times e^{\sqrt{\frac{r}{2r^*}}Q}z^{\sqrt{\frac{r}{2r^*}}a'_0+\frac{r}{2r^*}},
\end{aligned}\tag{6.64}$$

$$\begin{aligned}
x_-(z)\ &:\ \mathcal{F}_{l,k}\to\mathcal{F}_{l,k-1}\\
&x_-(z)=:\exp\Big(\sum_{n\neq 0}\frac{\alpha'_n}{[n]_x}z^{-n}\Big):\times e^{-\sqrt{\frac{r^*}{2r}}Q}z^{-\sqrt{\frac{r^*}{2r}}a'_0+\frac{r^*}{2r}}.
\end{aligned}\tag{6.65}$$

$x_+(z)$ is interpreted as a screening current in subsection 6.2. Note that $x_-(z)$ is not a screening current.

VO's are obtained by solving the relations (6.74)-(6.76) below directly for $\Phi_\varepsilon(z),\Psi^*_\varepsilon(z)$ $(\varepsilon=0,\pm1)$. We write $\Phi_\pm(z)=\Phi_{\pm1}(z)$ and $\Psi^*_\pm(z)=\Psi^*_{\pm1}(z)$. Results are

type I $\quad\Phi_\varepsilon(z):\mathcal{F}_{l,k}\to\mathcal{F}_{l,k-\varepsilon}$

$$\Phi_-(z)\ =\ \sqrt{g}:\exp\Big(-\sum_{n\neq 0}\frac{\alpha'_n}{[2n]_x-[n]_x}z^{-n}\Big):\times e^{\sqrt{\frac{r^*}{2r}}Q}z^{\sqrt{\frac{r^*}{2r}}a'_0+\frac{r^*}{2r}}, \tag{6.66}$$

$$\Phi_0(z) = x^{-\frac{r^*}{2r}} \oint_{C_0(z)} \underline{dz}_1 \Phi_-(z) x_-(z_1) \frac{1}{\sqrt{[\hat{k}+\frac{1}{2}]_+[\hat{k}-\frac{1}{2}]_+}} \frac{[u-u_1+\hat{k}]_+}{[u-u_1+\frac{1}{2}]}, \tag{6.67}$$

$$\begin{aligned}\Phi_+(z) &= x^{-\frac{r^*}{r}} \oint\oint_{C_+(z)} \underline{dz}_1 \underline{dz}_2 \Phi_-(z) x_-(z_1) x_-(z_2) \\ &\times \sqrt{\frac{S(\hat{k}-1)}{S(\hat{k})}} \frac{1}{[\hat{k}-\frac{1}{2}]_+[2\hat{k}-2]} \frac{[u-u_1+2\hat{k}-\frac{3}{2}]}{[u-u_1+\frac{1}{2}]} \frac{[u_1-u_2+\hat{k}]_+}{[u_1-u_2+\frac{1}{2}]},\end{aligned} \tag{6.68}$$

type II $\quad \Psi^*_\varepsilon(z) : \mathcal{F}_{l,k} \to \mathcal{F}_{l-2\varepsilon,k}$

$$\Psi^*_-(z) = \frac{1}{\sqrt{g^*}} : \exp\Big(\sum_{n\neq 0} \frac{\alpha_n}{[2n]_x-[n]_x} z^{-n}\Big) : \times e^{-\sqrt{\frac{r}{2r^*}}Q} z^{-\sqrt{\frac{r}{2r^*}}a'_0 + \frac{r}{2r^*}}, \tag{6.69}$$

$$\Psi^*_0(z) = i x^{\frac{r}{2r^*}} \oint_{C^*_0(z)} \underline{dz}_1 \Psi^*_-(z) x_+(z_1) \frac{1}{\sqrt{[\frac{1}{2}(\hat{l}+1)]^*_+[\frac{1}{2}(\hat{l}-1)]^*_+}} \frac{[u-u_1-\frac{1}{2}\hat{l}]^*_+}{[u-u_1-\frac{1}{2}]^*}, \tag{6.70}$$

$$\begin{aligned}\Psi^*_+(z) &= x^{\frac{r}{r^*}} \oint\oint_{C^*_+(z)} \underline{dz}_1 \underline{dz}_2 \Psi^*_-(z) x_+(z_1) x_+(z_2) \\ &\times \sqrt{\frac{S^*(\frac{1}{2}\hat{l}-1)}{S^*(\frac{1}{2}\hat{l})}} \frac{1}{[\frac{1}{2}(\hat{l}-1)]^*_+[\hat{l}-2]^*} \frac{[u-u_1-\hat{l}+\frac{3}{2}]^*}{[u-u_1-\frac{1}{2}]^*} \frac{[u_1-u_2-\frac{1}{2}\hat{l}]^*_+}{[u_1-u_2-\frac{1}{2}]^*}.\end{aligned} \tag{6.71}$$

Here $z = x^{2u}$, $z_j = x^{2u_j}$, $\underline{dz}_j = \frac{dz_j}{2\pi i z_j}$, and normalization constants g and g^* will be given in (6.79) and (6.80). The poles of the integrand of (6.67)-(6.71) and the integration contours are listed in the following table. For example, $C_0(z)$ is a simple closed contour that encircles $x^{1+2rn}z$ $(n \geq 0)$ but not $x^{-1-2rn}z$ $(n \geq 0)$.

	inside	outside
$C_0(z)$	$z_1 = x^{1+2rn}z$	$z_1 = x^{-1-2rn}z$
$C_+(z)$	$z_1 = x^{1+2rn}z$ $z_2 = x^{1+2rn}z_1$	$z_1 = x^{-1-2rn}z$ $z_2 = x^{-1-2rn}z, x^{-1-2rn}z_1, x^{2-2r(n+1)}z_1$
$C^*_0(z)$	$z_1 = x^{-1+2r^*n}z$	$z_1 = x^{1-2r^*n}z$
$C^*_+(z)$	$z_1 = x^{-1+2r^*n}z$ $z_2 = x^{-1+2r^*n}z_1$	$z_1 = x^{1-2r^*n}z$ $z_2 = x^{1-2r^*n}z, x^{1-2r^*n}z_1, x^{-2-2r^*(n+1)}z_1$

$(n = 0, 1, 2, \cdots)$. (6.72)

OPE formulas are given in subsection 6.a. In the conformal limit ($x = e^\hbar \to 1$, r : fixed), $x_+(z) \to zV_{\alpha_{-1,1}}(-z)$ (up to phase), $x_-(z) \to V_{\alpha_{1,0}}(z)$ (up to power of z), $g^{-\frac{1}{2}}\Phi_-(z) \to V_{\alpha_{1,2}}(z)$ (up to power of z), and $g^{*\frac{1}{2}}\Psi^*_-(z) \to V_{\alpha_{3,1}}(-z)$ (up to phase and power of z), where $V_\alpha(z)$ is given in (2.61). Type I VO $\Phi_\varepsilon(z)$ commutes with BRST operator (6.38)

$$[X_j, \Phi_\varepsilon(z)] = 0. \tag{6.73}$$

The VO's given above satisfy the following commutation relations ($\varepsilon_1, \varepsilon_2 = 0, \pm 1$),

$$\Phi_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1) = \sum_{\substack{\varepsilon_1', \varepsilon_2' = 0, \pm 1 \\ \varepsilon_1' + \varepsilon_2' = \varepsilon_1 + \varepsilon_2}} W\left(\begin{matrix} \hat{k} & \hat{k} + \varepsilon_1' \\ \hat{k} + \varepsilon_2 & \hat{k} + \varepsilon_1 + \varepsilon_2 \end{matrix} \Big| u_1 - u_2 \right) \Phi_{\varepsilon_1'}(z_1)\Phi_{\varepsilon_2'}(z_2), \tag{6.74}$$

$$\Psi^*_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \sum_{\substack{\varepsilon_1', \varepsilon_2' = 0, \pm 1 \\ \varepsilon_1' + \varepsilon_2' = \varepsilon_1 + \varepsilon_2}} W^*\left(\begin{matrix} \frac{1}{2}\hat{l} & \frac{1}{2}\hat{l} + \varepsilon_1 \\ \frac{1}{2}\hat{l} + \varepsilon_2' & \frac{1}{2}\hat{l} + \varepsilon_1 + \varepsilon_2 \end{matrix} \Big| u_1 - u_2 \right) \Psi^*_{\varepsilon_2'}(z_2)\Psi^*_{\varepsilon_1'}(z_1), \tag{6.75}$$

$$\Phi_{\varepsilon_2}(z_2)\Psi^*_{\varepsilon_1}(z_1) = \tau(u_1 - u_2)\Psi^*_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2), \tag{6.76}$$

where $z_i = x^{2u_i}$ and $\tau(u)$ is given in (6.128). We do not present the tedious but straightforward verification of (6.74)-(6.76).

For the description of correlation functions we need also the 'dual' VO's. Define

$$\Phi^*_\varepsilon(z) = \sqrt{S(\hat{k})}^{-1} \Phi_{-\varepsilon}(x^{-3}z)\sqrt{S(\hat{k})}, \tag{6.77}$$

$$\Psi_\varepsilon(z) = \sqrt{S^*(\tfrac{1}{2}\hat{l})}\Psi^*_{-\varepsilon}(x^{-3}z)\sqrt{S^*(\tfrac{1}{2}\hat{l})}^{-1}, \tag{6.78}$$

and normalization constants g and g^* in (6.66) and (6.69) are

$$g^{-1} = \frac{(x;x^{2r})_\infty}{(x^2;x^{2r})^2_\infty (x^{2r-1};x^{2r})_\infty (x^{2r};x^{2r})^4_\infty} \frac{(x^5, x^6, x^{2r}, x^{2r+1}; x^6, x^{2r})_\infty}{(x^2, x^3, x^{2r+3}, x^{2r+4}; x^6, x^{2r})_\infty}, \tag{6.79}$$

$$g^* = \frac{(x^{-1};x^{2r^*})_\infty}{(x^{-2};x^{2r^*})^2_\infty (x^{2r^*+1};x^{2r^*})_\infty (x^{2r^*};x^{2r^*})^5_\infty} \frac{(x^3, x^4, x^{2r^*+2}, x^{2r^*+3}; x^6, x^{2r^*})_\infty}{(x, x^6, x^{2r^*+5}, x^{2r^*+6}; x^6, x^{2r^*})_\infty}. \tag{6.80}$$

Then we have

$$\sum_{\varepsilon=0,\pm1} \Phi^*_\varepsilon(z)\Phi_\varepsilon(z) = \mathrm{id}, \tag{6.81}$$

$$\Phi_{\varepsilon_2}(z)\Phi^*_{\varepsilon_1}(z) = \delta_{\varepsilon_1,\varepsilon_2} \times \mathrm{id}, \tag{6.82}$$

$$\sum_{\varepsilon=0,\pm1} \Psi^*_\varepsilon(z_2)\Psi_\varepsilon(z_1) = \frac{1}{1-\frac{z_1}{z_2}} + \cdots \quad (z_1 \to z_2), \tag{6.83}$$

$$\Psi_{\varepsilon_1}(z_1)\Psi^*_{\varepsilon_2}(z_2) = \frac{\delta_{\varepsilon_1,\varepsilon_2}}{1-\frac{z_1}{z_2}} + \cdots \quad (z_1 \to z_2), \tag{6.84}$$

and for d in (6.18)

$$w^d \mathcal{O}(z) w^{-d} = \mathcal{O}(wz), \quad \text{for } \mathcal{O} = \Phi_\varepsilon, \Phi^*_\varepsilon, \Psi^*_\varepsilon, \Psi_\varepsilon, x_\pm. \tag{6.85}$$

We identify type I VO's in subsection 4.2 and those here in the following way:

$$\Phi^{(a-\varepsilon,a)}(z) = \Phi_\varepsilon(z)\Big|_{\mathcal{L}_{l,a}}, \tag{6.86}$$

$$\Phi^{*(a+\varepsilon,a)}(z) = \Phi^*_\varepsilon(z)\Big|_{\mathcal{L}_{l,a}}. \tag{6.87}$$

Then eqs. (4.106)-(4.110) correspond to (6.77),(6.74),(6.85),(6.81) and (6.82) respectively.

Next let us see how the DVA($A_2^{(2)}$) current is obtained from VO's. Let introduce free boson oscillator h_n ($n \in \mathbb{Z}_{\neq 0}$),

$$\lambda_n = (-1)^n (x - x^{-1}) \frac{[r^* n]_x}{[2n]_x - [n]_x} \alpha_n = (x - x^{-1}) \frac{[rn]_x}{[2n]_x - [n]_x} \alpha'_n, \tag{6.88}$$

$$[\lambda_n, \lambda_m] = (x - x^{-1})^2 \frac{1}{n} \frac{[n]_x [rn]_x [r^* n]_x}{[2n]_x - [n]_x} \delta_{m+n,0}. \tag{6.89}$$

As explained in subsection 6.2, the DVA($A_2^{(2)}$) current $T(z)$ is realized as

$$\begin{aligned} T(z) &= \Lambda_+(z) + \Lambda_0(z) + \Lambda_-(z), \\ \Lambda_\pm(z) &= : \exp\Big(\pm \sum_{n \neq 0} h_n (x^{\pm\frac{3}{2}} z)^{-n} \Big) : \times x^{\pm\sqrt{2rr^*} a'_0}, \\ \Lambda_0(z) &= -\frac{[r - \frac{1}{2}]_x}{[\frac{1}{2}]_x} : \exp\Big(\sum_{n \neq 0} h_n (x^{-n/2} - x^{n/2}) z^{-n} \Big) : . \end{aligned} \tag{6.90}$$

This $T(z)$ is obtained from type I VO's by fusing them,

$$\begin{aligned} & \Phi_{\varepsilon_2}(x^{\frac{3}{2}+r} z') \Phi^*_{\varepsilon_1}(x^{\frac{3}{2}-r} z) \\ = \ & \Big(1 - \frac{z}{z'}\Big) (-1)^{\varepsilon_1 + 1} \delta_{\varepsilon_1, \varepsilon_2} T(z) \cdot x^{1-r} \frac{(x, x^6, x^{5-2r}, x^{6-2r}; x^6)_\infty}{(x^3, x^4, x^{2-2r}, x^{3-2r}; x^6)_\infty} + \cdots \quad (z' \to z). \end{aligned} \tag{6.91}$$

$\widetilde{T}(z) = -\Lambda_+(z) + \Lambda_0(z) - \Lambda_-(z)$ also satisfies (6.1), and it is obtained from type II VO's,

$$\begin{aligned} & \Psi_{\varepsilon_1}(x^{\frac{3}{2}+r^*} z') \Psi^*_{\varepsilon_2}(x^{\frac{3}{2}-r^*} z) \\ = \ & \frac{1}{1 - \frac{z'}{z}} (-1)^{\varepsilon_1 + 1} \delta_{\varepsilon_1, \varepsilon_2} \widetilde{T}(-z) \cdot (-x^{-r}) \frac{(x^2, x^3, x^{3-2r^*}, x^{4-2r^*}; x^6)_\infty}{(x^5, x^6, x^{-2r^*}, x^{1-2r^*}; x^6)_\infty} + \cdots \quad (z' \to z). \end{aligned} \tag{6.92}$$

Using these bosonized VO's let us calculate LHP (4.73) with (4.111),

$$\begin{aligned} & P_{a_n, \cdots, a_0}(l) \\ = \ & Z_l^{-1} S(a_0) \, \mathrm{tr}_{\mathcal{H}_l^{(a_0)}} \Big(x^{6H_C^{(a_0)}} \Phi^{*(a_0, a_1)}(z) \cdots \Phi^{*(a_{n-1}, a_n)}(z) \Phi^{(a_n, a_{n-1})}(z) \cdots \Phi^{(a_1, a_0)}(z) \Big). \\ = \ & Z_l^{-1} S(a_0) \, \mathrm{tr}_{\mathcal{L}_{l, a_0}} \Big(x^{6d} \Phi^*_{\varepsilon_1}(z) \cdots \Phi^*_{\varepsilon_n}(z) \Phi_{\varepsilon_n}(z) \cdots \Phi_{\varepsilon_1}(z) \Big), \end{aligned} \tag{6.93}$$

where $\varepsilon_i = a_{i-1} - a_i$. Here we have identified the space of state and operators as (6.58),(6.59),(6.86) and (6.87).

One-point LHP is already obtained in (4.44),

$$P_k(l) = Z_l^{-1} S(k) \, \chi_{l,k}(x^6), \tag{6.94}$$

where $\chi_{l,k}(q)$ is given in (2.37) and Z_l (4.74) is

$$Z_l = \sum_{k=1}^{L} S(k) \, \chi_{l,k}(x^6), \tag{6.95}$$

which can be expressed in product of theta functions with conjugate modulus [71]. Two-point LHP satisfies (4.112) and (4.66)

$$P_{a,b}(l) = P_{b,a}(l), \quad \sum_b P_{a,b}(l) = P_a(l). \tag{6.96}$$

In contrast to subsection 5.3, however, this recursion relation does not determine $P_{a,b}(l)$ uniquely. So we will use free field realization of vertex operator approach. Two-point LHP $P_{k-\varepsilon,k}(l)$ is ($\varepsilon = 0, \pm 1$)

$$\begin{aligned} P_{k-\varepsilon,k}(l) &= Z_l^{-1}\, S(k)\, \mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{6d}\Phi^*_\varepsilon(z)\Phi_\varepsilon(z)\Big) \\ &= Z_l^{-1}\sqrt{S(k)S(k-\varepsilon)}\, \mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{6d}\Phi_{-\varepsilon}(x^{-3}z)\Phi_\varepsilon(z)\Big). \end{aligned} \tag{6.97}$$

Since this is independent on z, we take $z = 1$. The evaluation of the trace yields the following expressions:

$$\begin{aligned} P_{k-1,k}(l) &= -\frac{g}{Z_l}\frac{S(k-1)\, x^{-\frac{r^*}{r}}}{[k-\frac{1}{2}]_+[2k-2]}\oint\oint_{C_+(1)} \underline{dw}_1\, \underline{dw}_2\, \mathcal{I}(w_1,w_2) \\ &\qquad \times\frac{[v_1-2k+\frac{3}{2}][v_1-v_2+k]_+}{[v_1+\frac{1}{2}][v_1-v_2+\frac{1}{2}]}, \end{aligned} \tag{6.98}$$

$$\begin{aligned} P_{k,k}(l) &= \frac{g}{Z_l}\frac{S(k)\, x^{-\frac{r^*}{r}}}{[k+\frac{1}{2}]_+[k-\frac{1}{2}]_+}\oint_{C_0(x^{-3})} \underline{dw}_1 \oint_{C_0(1)} \underline{dw}_2\, \mathcal{I}(w_1,w_2) \\ &\qquad \times\frac{[v_1-k+\frac{3}{2}]_+[v_2-k]_+}{[v_1+1][v_2-\frac{1}{2}]}, \end{aligned} \tag{6.99}$$

$$\begin{aligned} P_{k+1,k}(l) &= -\frac{g}{Z_l}\frac{S(k)\, x^{\frac{-r^*}{r}}}{[k+\frac{1}{2}]_+[2k]}\oint\oint_{C_+(x^{-3})} \underline{dw}_1\, \underline{dw}_2\, \mathcal{I}(w_1,w_2) \\ &\qquad \times\frac{[v_1-2k+1][v_1-v_2+k+1]_+[v_2+\frac{1}{2}]}{[v_1+1][v_1-v_2+\frac{1}{2}][v_2-\frac{1}{2}]}, \end{aligned} \tag{6.100}$$

where $w_i = x^{2v_i}$ $(i=1,2)$ and $\mathcal{I}(w_1,w_2)$ is

$$\begin{aligned} \mathcal{I}(w_1,w_2) &= g^{-1}\mathrm{tr}_{\mathcal{L}_{l,k}}\Big(x^{6d}\Phi_-(x^{-3})x_-(w_1)\Phi_-(1)x_-(w_2)\Big) \\ &= \mathcal{I}^{\mathrm{OPE}}(w_1,w_2)\mathcal{I}^{\mathrm{osc}}(w_1,w_2)\mathcal{I}^{\mathrm{zero}}(w_1,w_2). \end{aligned} \tag{6.101}$$

Here $\mathcal{I}^{\mathrm{OPE}}(w_1,w_2)$ is the OPE contribution

$$\begin{aligned} \mathcal{I}^{\mathrm{OPE}}(w_1,w_2) &= \langle\!\langle\Phi_-(x^{-3})x_-(w_1)\rangle\!\rangle\langle\!\langle\Phi_-(x^{-3})\Phi_-(1)\rangle\!\rangle\langle\!\langle\Phi_-(x^{-3})x_-(w_2)\rangle\!\rangle \\ &\quad\times\langle\!\langle x_-(w_1)\Phi_-(1)\rangle\!\rangle\langle\!\langle x_-(w_1)x_-(w_2)\rangle\!\rangle\langle\!\langle\Phi_-(1)x_-(w_2)\rangle\!\rangle, \end{aligned} \tag{6.102}$$

and $\mathcal{I}^{\mathrm{osc}}(w_1,w_2)$ is the oscillator contribution

$$\mathcal{I}^{\mathrm{osc}}(w_1,w_2) = \mathrm{tr}_{\mathcal{F}}\Big(x^{6d^{\mathrm{osc}}} :\Phi^{\mathrm{osc}}_-(x^{-3})x^{\mathrm{osc}}_-(w_1)\Phi^{\mathrm{osc}}_-(1)x^{\mathrm{osc}}_-(w_2):\Big)$$

$$
\begin{aligned}
&= \frac{1}{(x^6;x^6)_\infty} F_{\Phi_-,\Phi_-}(1)^2 F_{\Phi_-,\Phi_-}(x^3) F_{\Phi_-,\Phi_-}(x^{-3}) F_{x_-,x_-}(1)^2 \\
&\quad \times \prod_{i=1,2} F_{\Phi_-,x_-}(w_i) F_{\Phi_-,x_-}(w_i^{-1}) F_{\Phi_-,x_-}(x^3 w_i) F_{\Phi_-,x_-}(x^{-3} w_i^{-1}) \\
&\quad \times F_{x_-,x_-}(\tfrac{w_2}{w_1}) F_{x_-,x_-}(\tfrac{w_1}{w_2}),
\end{aligned} \tag{6.103}
$$

and $\mathcal{I}^{\text{zero}}(w_1,w_2)$ is the zero mode contribution

$$
\begin{aligned}
\mathcal{I}^{\text{zero}}(w_1,w_2) &= g^{-1}\mathrm{tr}^{\text{zero}}_{\mathcal{L}_{l,k}}\Big(x^{6d^{\text{zero}}} :\Phi_-^{\text{zero}}(x^{-3}) x_-^{\text{zero}}(w_1) \Phi_-^{\text{zero}}(1) x_-^{\text{zero}}(w_2):\Big) \\
&= \mathrm{tr}^{\text{zero}}_{\mathcal{L}_{l,k}}\Big(x^{6d^{\text{zero}}} (x^3 w_1 w_2)^{-\sqrt{\frac{r^*}{2r}}a_0'} (x^{-3} w_1 w_2)^{\frac{r^*}{2r}}\Big) \\
&= \sum_{j\in\mathbb{Z}} \Big(x^{6(h_{l-2Lj,k}-\frac{c}{24})} (x^3 w_1 w_2)^{\frac{1}{2}(l-2Lj)-\frac{r^*}{r}k} \\
&\qquad\quad - x^{6(h_{-l-2Lj,k}-\frac{c}{24})} (x^3 w_1 w_2)^{\frac{1}{2}(-l-2Lj)-\frac{r^*}{r}k}\Big) (x^{-3} w_1 w_2)^{\frac{r^*}{2r}} \\
&= x^{6(h_{l,k}-\frac{c}{24})} x^{-3\frac{r^*}{r}} (x^3 w_1 w_2)^{\frac{1}{2}l-\frac{r^*}{r}k+\frac{r^*}{2r}} \\
&\quad \times \Big(\Theta_{x^{12L(L+1)}}(-x^{6(-(L+1)l+Lk+L(L+1))}(x^3 w_1 w_2)^{-L}) \\
&\qquad - \Theta_{x^{12L(L+1)}}(-x^{6((L+1)l+Lk+L(L+1))}(x^3 w_1 w_2)^{-L}) x^{6lk} (x^3 w_1 w_2)^{-l}\Big).
\end{aligned} \tag{6.104}
$$

Using formulas in subsection 6.a we have

$$
\begin{aligned}
&\mathcal{I}^{\text{OPE}}(w_1,w_2)\mathcal{I}^{\text{osc}}(w_1,w_2) \\
&= x^{3\frac{r^*}{r}} \frac{(x^5,x^5,x^6,x^6,x^8,x^8,x^{2r+5},x^{2r+5};x^6,x^{2r})_\infty}{(x^7,x^7,x^{2r+3},x^{2r+3},x^{2r+4},x^{2r+4},x^{2r+4},x^{2r+4};x^6,x^{2r})_\infty} \\
&\quad \times \frac{(x^{2r-1}w_1^{-1},x^{2r+2}w_1;x^3,x^{2r})_\infty}{(xw_1^{-1},x^4 w_1;x^3,x^{2r})_\infty} \frac{(x^{2r-1}w_2,x^{2r+2}w_2^{-1};x^3,x^{2r})_\infty}{(xw_2,x^4 w_2^{-1};x^3,x^{2r})_\infty} \\
&\quad \times \Theta_{x^6}(z) \frac{(x^2 z, x^8 z^{-1}, x^{2r-1} z, x^{2r+5} z^{-1}; x^6, x^{2r})_\infty}{(xz, x^7 z^{-1}, x^{2r-2} z, x^{2r+4} z^{-1}; x^6, x^{2r})_\infty}\Bigg|_{z=\frac{w_2}{w_1}}.
\end{aligned} \tag{6.105}
$$

The contours $C_+(1)$, $C_0(x^{-3})\cup C_0(1)$, $C_+(x^{-3})$ are chosen as follows $(n,m\geq 0)$; For all the contours, the poles $w_1 = x^{4+3m+2rn}$, $w_2 = x^{4+3m+2rn}$, $x^{4+6m+2r(n+1)}w_1$, $x^{1+6(m+1)+2rn}w_1$ are inside and the poles $w_1 = x^{-4-3m-2rn}$, $w_2 = x^{-1-3m-2rn}$, $x^{2-6m-2r(n+1)}w_1$, $x^{-1-6m-2rn}w_1$ are outside. In addition,

	inside	outside
$C_+(1)$	$w_1 = x^{-1+2r(n+1)}$ $w_2 = x^{1+2rn}w_1$	$w_1 = x^{-1-2rn}$ $w_2 = x^{-1-2rn}, x^{-1-2rn}w_1, x^{2-2r(n+1)}w_1$
$C_0(x^{-3})\cup C_0(1)$	$w_1 = x^{-2+2rn}$ $w_2 = x^{1+2rn}$	$w_1 = x^{-4-2rn}$ $w_2 = x^{-1-2rn}$
$C_+(x^{-3})$	$w_1 = x^{-2+2rn}$ $w_2 = x^{1+2rn}, x^{1+2rn}w_1$	$w_1 = x^{-4-2rn}$ $w_2 = x^{1-2r(n+1)}, x^{-1-2rn}w_1, x^{2-2r(n+1)}w_1$

.

For integral representations of general LHP, see [48].

Excited states are obtained by using type II VO and traces of type I and type II VO's are calculated similarly like as subsection 5.4.

6.a OPE and trace

OPE

We list the normal ordering relations used in section 6. r^* is

$$r^* = r - 1.$$

Notation $\langle\!\langle A(z)B(w)\rangle\!\rangle$ is given in (5.113) :

$$\langle\!\langle x_+(z_1)x_+(z_2)\rangle\!\rangle = z_1^{\frac{r}{r^*}}(1-\zeta)\frac{(x^{-2}\zeta, x^{2r^*+1}\zeta; x^{2r^*})_\infty}{(x^{-1}\zeta, x^{2r^*+2}\zeta; x^{2r^*})_\infty}, \tag{6.106}$$

$$\langle\!\langle x_-(z_1)x_-(z_2)\rangle\!\rangle = z_1^{\frac{r^*}{r}}(1-\zeta)\frac{(x^{2}\zeta, x^{2r-1}\zeta; x^{2r})_\infty}{(x\zeta, x^{2r-2}\zeta; x^{2r})_\infty}, \tag{6.107}$$

$$\langle\!\langle x_\pm(z_1)x_\mp(z_2)\rangle\!\rangle = z_1^{-1}\frac{1+\zeta}{(1+x\zeta)(1+x^{-1}\zeta)}, \tag{6.108}$$

$$\langle\!\langle \Phi_-(z_1)x_+(z_2)\rangle\!\rangle = \langle\!\langle x_+(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1 + z_2, \tag{6.109}$$

$$\langle\!\langle \Phi_-(z_1)x_-(z_2)\rangle\!\rangle = \langle\!\langle x_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{-\frac{r^*}{r}}\frac{(x^{2r-1}\zeta; x^{2r})_\infty}{(x\zeta; x^{2r})_\infty}, \tag{6.110}$$

$$\langle\!\langle \Psi^*_-(z_1)x_+(z_2)\rangle\!\rangle = \langle\!\langle x_+(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1^{-\frac{r}{r^*}}\frac{(x^{2r^*+1}\zeta; x^{2r^*})_\infty}{(x^{-1}\zeta; x^{2r^*})_\infty}, \tag{6.111}$$

$$\langle\!\langle \Psi^*_-(z_1)x_-(z_2)\rangle\!\rangle = \langle\!\langle x_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1 + z_2, \tag{6.112}$$

$$\langle\!\langle \Phi_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{\frac{r^*}{r}}\frac{(x^2\zeta, x^3\zeta, x^{2r+3}\zeta, x^{2r+4}\zeta; x^6, x^{2r})_\infty}{(x^5\zeta, x^6\zeta, x^{2r}\zeta, x^{2r+1}\zeta; x^6, x^{2r})_\infty}, \tag{6.113}$$

$$\langle\!\langle \Psi^*_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = z_1^{\frac{r}{r^*}}\frac{(\zeta, x\zeta, x^{2r^*+5}\zeta, x^{2r^*+6}\zeta; x^6, x^{2r^*})_\infty}{(x^3\zeta, x^4\zeta, x^{2r^*+2}\zeta, x^{2r^*+3}\zeta; x^6, x^{2r^*})_\infty}, \tag{6.114}$$

$$\langle\!\langle \Phi_-(z_1)\Psi^*_-(z_2)\rangle\!\rangle = \langle\!\langle \Psi^*_-(z_1)\Phi_-(z_2)\rangle\!\rangle = z_1^{-1}\frac{(-x^4\zeta, -x^5\zeta; x^6)_\infty}{(-x\zeta, -x^2\zeta; x^6)_\infty}, \tag{6.115}$$

where $\zeta = \frac{z_2}{z_1}$ and we have used (A.50).

As meromorphic functions we have ($z_i = x^{2u_i}$)

$$x_+(z_1)x_+(z_2) = x_+(z_2)x_+(z_1)\frac{[u_1-u_2+1]^*}{[u_1-u_2-1]^*}\frac{[u_1-u_2-\frac{1}{2}]^*}{[-u_1+u_2-\frac{1}{2}]^*}, \tag{6.116}$$

$$x_-(z_1)x_-(z_2) = x_-(z_2)x_-(z_1)\frac{[u_1-u_2-1]}{[u_1-u_2+1]}\frac{[u_1-u_2+\frac{1}{2}]}{[-u_1+u_2+\frac{1}{2}]}, \tag{6.117}$$

$$x_\pm(z_1)x_\mp(z_2) = x_\mp(z_2)x_\pm(z_1), \tag{6.118}$$

$$\Phi_-(z_1)x_+(z_2) = x_+(z_2)\Phi_-(z_1), \tag{6.119}$$

$$\Phi_-(z_1)x_-(z_2) = x_-(z_2)\Phi_-(z_1)\frac{[u_1-u_2+\frac{1}{2}]}{[-u_1+u_2+\frac{1}{2}]}, \tag{6.120}$$

$$\Psi^*_-(z_1)x_+(z_2) = x_+(z_2)\Psi^*_-(z_1)\frac{[u_1-u_2-\frac{1}{2}]^*}{[-u_1+u_2-\frac{1}{2}]^*}, \tag{6.121}$$

$$\Psi^*_-(z_1)x_-(z_2) = x_-(z_2)\Psi^*_-(z_1), \tag{6.122}$$

$$\Phi_-(z_1)\Phi_-(z_2) = \Phi_-(z_2)\Phi_-(z_1)\rho(u_2-u_1), \tag{6.123}$$

$$\Psi^*_-(z_1)\Psi^*_-(z_2) = \Psi^*_-(z_2)\Psi^*_-(z_1)\rho^*(u_1-u_2), \tag{6.124}$$

$$\Phi_-(z_1)\Psi^*_-(z_2) = \Psi^*_-(z_2)\Phi_-(z_1)\tau(u_2-u_1). \tag{6.125}$$

Here $\rho(u)$, $\rho^*(u)$ and $\tau(u)$ are given by

$$z^{\frac{r^*}{r}}\rho(u) = \frac{\rho_+(u)}{\rho_+(-u)}, \quad \rho_+(u) = \frac{(x^2z, x^3z, x^{2r+3}z, x^{2r+4}z; x^6, x^{2r})_\infty}{(x^5z, x^6z, x^{2r}z, x^{2r+1}z; x^6, x^{2r})_\infty}, \tag{6.126}$$

$$z^{-\frac{r}{r^*}}\rho^*(u) = \frac{\rho^*_+(u)}{\rho^*_+(-u)}, \quad \rho^*_+(u) = \frac{(x^3z, x^4z, x^{2r^*+2}z, x^{2r^*+3}z; x^6, x^{2r^*})_\infty}{(z, xz, x^{2r^*+5}z, x^{2r^*+6}z; x^6, x^{2r^*})_\infty}, \tag{6.127}$$

$$\tau(u) = z\frac{\Theta_{x^6}(-xz^{-1})\Theta_{x^6}(-x^2z^{-1})}{\Theta_{x^6}(-xz)\Theta_{x^6}(-x^2z)}. \tag{6.128}$$

Note that

$$\rho^*(u) = -\rho(u)\Big|_{r\to r^*} \times z\frac{\Theta_{x^6}(xz^{-1})\Theta_{x^6}(x^2z^{-1})}{\Theta_{x^6}(xz)\Theta_{x^6}(x^2z)}. \tag{6.129}$$

Trace

We use the same notation as the second part of subsection 5.a. The trace of oscillator parts over the Fock space $\mathcal{F} = \mathcal{F}_{l,k}$ is

$$\mathrm{tr}_{\mathcal{F}}\Big(x^{6d^{\mathrm{osc}}} : \prod_i A_i^{\mathrm{osc}}(z_i) :\Big) = \frac{1}{(x^6;x^6)_\infty}\prod_{i,j} F_{A_i,A_j}(\tfrac{z_i}{z_j}), \tag{6.130}$$

where $F_{A,B}(z)$ is given by

$$F_{A,B}(z) = \exp\Big(\sum_{n>0}\frac{1}{n}[n]_x([2n]_x-[n]_x)\frac{[rn]_x}{[r^*n]_x}\frac{x^{6n}}{1-x^{6n}}f^A_{-n}f^B_n z^n\Big). \tag{6.131}$$

We write down $F_{A,B}(z)$ (Remark $F_{A,B}(z) = F_{B,A}(z)$) :

$$F_{x_+,x_+}(z) = (x^6z;x^6)_\infty\frac{(x^4z, x^{2r^*+7}z; x^6, x^{2r^*})_\infty}{(x^5z, , x^{2r^*+8}; x^6, x^{2r^*})_\infty}, \tag{6.132}$$

$$F_{x_-,x_-}(z) = (x^6z;x^6)_\infty\frac{(x^8z, x^{2r+5}z; x^6, x^{2r})_\infty}{(x^7z, x^{2r+4}z; x^6, x^{2r})_\infty}, \tag{6.133}$$

$$F_{x_+,x_-}(z) = \frac{(-x^6z;x^6)_\infty}{(-x^5z, -x^7z; x^6)_\infty}, \tag{6.134}$$

$$F_{\Phi_-,x_+}(z) = (-x^6z;x^6)_\infty, \tag{6.135}$$

$$F_{\Phi_-,x_-}(z) = \frac{(x^{2r+5}z;x^6,x^{2r})_\infty}{(x^7z;x^6,x^{2r})_\infty}, \tag{6.136}$$

$$F_{\Psi^*_-,x_+}(z) = \frac{(x^{2r^*+7}z;x^6,x^{2r^*})_\infty}{(x^5z;x^6,x^{2r^*})_\infty}, \tag{6.137}$$

$$F_{\Psi^*_-,x_-}(z) = (-x^6z;x^6)_\infty, \tag{6.138}$$

$$F_{\Phi_-,\Phi_-}(z) = \frac{(x^8z,x^9z,x^{2r+9}z,x^{2r+10}z;x^6,x^6,x^{2r})_\infty}{(x^{11}z,x^{12}z,x^{2r+6}z,x^{2r+7}z;x^6,x^6,x^{2r})_\infty}, \tag{6.139}$$

$$F_{\Psi^*_-,\Psi^*_-}(z) = \frac{(x^6z,x^7z,x^{2r^*+11}z,x^{2r^*+12}z;x^6,x^6,x^{2r^*})_\infty}{(x^9z,x^{10}z,x^{2r^*+8}z,x^{2r^*+9}z;x^6,x^6,x^{2r^*})_\infty}, \tag{6.140}$$

$$F_{\Phi_-,\Psi^*_-}(z) = \frac{(-x^{10}z,-x^{11}z;x^6,x^6)_\infty}{(-x^7z,-x^8z;x^6,x^6)_\infty}. \tag{6.141}$$

7 Conclusion

In this lecture we have explained deformed Virasoro algebras ($A_1^{(1)}$ type and $A_2^{(2)}$ type) and elliptic quantum groups (face type algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and vertex type algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$) and studied solvable lattice models (ABF model in regime III and dilute A_L model in regime 2^+) by using vertex operators and free field realizations.

We close this lecture by mentioning some related topics.

deformed W algebras (DWA's): In CFT there are several extensions of the Virasoro algebra, e.g. superconformal algebras, current algebras (affine Lie algebras), W algebras, parafermions, which contain the Virasoro algebra as a subalgebra. For W algebras, see review [73]. W_N algebra is a W algebra associated to A_{N-1} algebra. Deformation of W_N algebra, which we denote DWA(A_{N-1}), was obtained in [18](see also [74]) and [20] by using correspondence between singular vectors and Macdonald symmetric polynomials or quantization of the deformed W_N Poisson algebra respectively. The deformed W_n Poisson algebra was obtained from the Wakimoto realization of $U_q(\widehat{\mathfrak{sl}}_N)$ at the critical level by E. Frenkel and Reshetikhin [19] and they pointed out that deformed W currents in a free field realization have the same forms of transfer matrices in analytic Bethe ansatz, dressed vacuum form. (Bethe ansatz is also a powerful method to study solvable models [6, 75]) Based on this observation, DWA's for arbitrary simple Lie algebras were constructed [76]. See [77] for screening currents, [78] for relation to $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$, [79] for connection to q-difference version of the Drinfeld-Sokorov reduction, and see [80, 81, 48] for higher currents. DWA(A_{N-1}) appears in the $A_{N-1}^{(1)}$ face model [82, 77] and also in the ABF model in regime II [83]. Since $A_1^{(1)}$ and $A_2^{(2)}$ are the only affine Lie algebras of rank 1, those DWA closes for one current $T(z)$, i.e. DVA. In CFT, W_N algebra does not contain W_n ($n \leq N$) algebra as a subalgebra explicitly except for $n=2$ case which corresponds to the Virasoro algebra. In deformed case, DWA(A_{N-1}) does not contain even DVA($A_1^{(1)}$) explicitly. As shortly explained in subsection 2.2.3, singular vectors of Virasoro and W algebras imply that the correlation functions containing corresponding primary fields satisfy the differential equations. For singular vectors of DVA and DWA, do the correlation functions satisfy some difference equations?

KZ and q-KZ equations : In CFT Wess-Zumino-Novikov-Witten (WZNW) model has gauge symmetries, i.e. affine Lie algebra symmetries [2]. Virasoro current is realized as a quadratic form of affine Lie algebra currents (Sugawara construction). Consequently conformal Ward identity has rich structure, which is known as the Knizhnik-Zamolodchikov (KZ) equation [84, 85]. Since the Virasoro algebra is a Lie algebra, we know a rule for its tensor product representation. Moreover we obtain new realizations and character formulas of the Virasoro algebra by coset construction [86]. On the other hand DVA is not a Lie algebra and its tensor product representation is unknown. (We remark that tensor product representations of W_N algebras in CFT are also unknown.) If there exists some deformation of the Sugawara construction, it could give tensor product representations of DVA but we do not know it at present. Rather we derive DVA or DWA from (elliptic) quantum groups by fusion of VO's [69]. q-deformation of KZ equation were presented by I. Frenkel and Reshetikhin [61]. q-KZ equations are holonomic q-difference equations for the matrix coefficients of the products of intertwining operators for representations of quantum affine algebra. Connection matrix of their solutions gives a face type elliptic solution of YBE. See [61, 87, 88, 16, 62].

massive integrable models : Integrable perturbations of CFT were studied in [12, 13, 14, 15] and $(1,3)$-perturbed CFT is described by the sine-Gordon model. Sine-Gordon model is a typical massive integrable model. S-matrix was obtained by Zamlodochikov's bootstrap approach [4]. For form factors see Smirnov's bootstrap approach [89] (see also [90]). Lukyanov pointed out that DVA($A_1^{(1)}$) current $T(z)$ in certain scaling limit gives the Zamolodchikov-Faddeev (ZF) algebra of sine-Gordon model (before taking a scaling limit, $T(z)$ is interpreted as the ZF algebra for basic scalar excitation of XYZ spin chain) [91]. In section 3, z of $T(z)$ is introduced as a formal parameter, but here z of $T(z)$ is related to the spectral parameter of the particle, like as lattice models in section 4, 5 ($z = x^{2u}$). This is contrasted with the CFT case, where z of $L(z)$ is interpreted as a complex coordinate of the Riemann surface. We can obtain integrable massive field theory models from solvable lattice models by taking appropriate scaling limit. Particles in a field theory are created by type II VO's. For XXZ model, sine-Gordon model, Bullough-Dodd model and affine Toda model see [92, 68, 22, 93, 94]. Field theory analog of transfer matrix and Baxter's Q-operator is studied in [95].

eight vertex model : ABF model was studied by Lukyanov and Pugai by bosonization of type I VO's. An algebraic approach to the fusion ABF models was presented in [57, 63] on the basis of the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Bosonization of VO's for the $A^{(1)}_{N-1}$ face model was given in [82]. Another interesting direction is to study Baxter's eight vertex model and Belavin's generalization. Lashkevich and Pugai proposed a remarkable bosonization formula of the type I VO for the eight vertex model [96]. They succeeded in reducing the problem to the already known bosonization for the ABF model through the use of intertwining vectors and Lukyanov's screening operators. To understand their bosonization scheme, it seems necessary to clarify the relationship between the intertwining vectors and the two twistors $F(\lambda)$ and $E(r)$, which define $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ respectively. It is also interesting to seek a more direct bosonization, which is intrinsically connected with the quasi-Hopf structure of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ and does not rely on the bosonization of the ABF model.

supersymmetry : In string theory supersymmetries are essential to cancelation of divergence and consistency of theory, and $N = 2$ superconformal algebra is related to many interesting topics, e.g. chiral ring, mirror symmetry, topological field theory. Are there 'good' deformations of superconformal algebras? Super version ($\mathbb{Z}_2$ graded algebra) of elliptic quantum group was formulated in [97] along the line of [26] (See also [98]). Can we obtain deformed superconformal currents by fusion of VO's of this elliptic superalgebra or higher level VO's of $U_q(\widehat{\mathfrak{sl}}_2)$ [99] ? See also [100].

As explained in the introduction our motivation is to find the symmetry of massive integrable models, but present status is far from satisfactory. We hope that this lecture can help the study in this (and also other) field.

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A Some Formulas

In this appendix we give a summary of notations and formulas used throughout this lecture.

A.1 Some functions

Let us fix x, r, r^*. Following functions are used in this lecture:

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}, \tag{A.1}$$

$$(z; p_1, \cdots, p_k)_\infty = \prod_{n_1, \cdots, n_k = 0}^{\infty} (1 - p_1^{n_1} \cdots p_k^{n_k} z), \tag{A.2}$$

$$(z_1, \cdots, z_n; p_1, \cdots, p_k)_\infty = \prod_{j=1}^{n} (z_j; p_1, \cdots, p_k)_\infty, \tag{A.3}$$

$$\Theta_p(z) = (p, z, pz^{-1}; p)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n z^n p^{\frac{1}{2}n(n-1)}, \tag{A.4}$$

$$[u] = x^{\frac{u^2}{r} - u} \Theta_{x^{2r}}(x^{2u}), \tag{A.5}$$

$$[u]^* = x^{\frac{u^2}{r^*} - u} \Theta_{x^{2r^*}}(x^{2u}). \tag{A.6}$$

$$[u]_+ = x^{\frac{u^2}{r} - u} \Theta_{x^{2r}}(-x^{2u}), \tag{A.7}$$

$$[u]_+^* = x^{\frac{u^2}{r^*} - u} \Theta_{x^{2r^*}}(-x^{2u}). \tag{A.8}$$

$[u]$ and $[u]^*$, $[u]_+$ and $[u]_+^*$, are related by

$$[u]^* = [u]\Big|_{r \to r^*}, \quad [u]_+^* = [u]_+\Big|_{r \to r^*}. \tag{A.9}$$

$[u]$ satisfies

$$[-u] = -[u], \quad [u + r] = -[u], \quad [u + \tau] = -[u] e^{\frac{2\pi i}{r}(u + \frac{\tau}{2})} \quad (x = e^{\frac{\pi i}{\tau}}), \tag{A.10}$$

and the Riemann identity

$$\begin{aligned} & [2u_1][2u_2][2u_3][2u_4] \qquad \text{(A.11)} \\ = \ & [u_1 + u_2 + u_3 + u_4][u_1 - u_2 - u_3 + u_4][u_1 + u_2 - u_3 - u_4][u_1 - u_2 + u_3 - u_4] \\ & + [-u_1 + u_2 + u_3 + u_4][u_1 - u_2 + u_3 + u_4][u_1 + u_2 - u_3 + u_4][u_1 + u_2 + u_3 - u_4]. \end{aligned}$$

Lemma 4 in [43] is

$$\begin{aligned} & \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} [v_{\sigma(i)} - 2i + 2] \cdot \prod_{\substack{1 \le i < j \le m \\ \sigma(i) > \sigma(j)}} \frac{[v_{\sigma(i)} - v_{\sigma(j)} - 1]}{[v_{\sigma(i)} - v_{\sigma(j)} + 1]} \\ = \ & \frac{1}{m!} \prod_{i=1}^{m} \frac{[i]}{[1]} \cdot \prod_{1 \le i < j \le m} \frac{[v_i - v_j]}{[v_i - v_j - 1]} \cdot \prod_{i=1}^{m} [v_i - m + 1]. \end{aligned} \tag{A.12}$$

Along with the additive variable u, we often use the multiplicative variable

$$z = x^{2u}, \quad z_j = x^{2u_j}, \tag{A.13}$$

and the following abbreviation for an integration measure

$$\underline{dz} = \frac{dz}{2\pi i z}, \quad \underline{dz}_j = \frac{dz_j}{2\pi i z_j}. \tag{A.14}$$

$(z;p)_\infty$, $\Theta_p(z)$ and $[u]$ have no poles and have simple zeros,

$$\begin{aligned} (z;p)_\infty \quad &: \quad z = p^{-n} \quad (n \in \mathbb{Z}_{\geq 0}), && \text{(A.15)} \\ \Theta_p(z) \quad &: \quad z = p^m \quad (m \in \mathbb{Z}), && \text{(A.16)} \\ [u] \quad &: \quad u = rm \quad (m \in \mathbb{Z}) \quad (\text{i.e. } z = x^{2rm}). && \text{(A.17)} \end{aligned}$$

For $m \in \mathbb{Z}$ we have

$$\oint_{u=rm+a} \underline{dz} \frac{1}{[u-a]} f(u) = \frac{(-1)^{m-1}}{(x^{2r};x^{2r})_\infty^3} f(rm+a), \tag{A.18}$$

where $f(u)$ is regular at $u = rm + a$, and $\frac{1}{[u-a]} f(u)$ does not contain a fractional power of z.

A.2 Delta function

Delta function $\delta(z)$ is a formal power series

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n, \tag{A.19}$$

and has the property

$$\oint_0 \frac{dz}{2\pi i z} f(z) \delta(\tfrac{z}{a}) = f(a). \tag{A.20}$$

Let $F(z;q)$ be the following Taylor series in z,

$$F(z;q) = \prod_{i=1}^{n} \frac{1 - q^{\beta_i} z}{1 - q^{\alpha_i} z}. \tag{A.21}$$

Here $\frac{1}{1-z}$ means $\sum_{m \geq 0} z^m$ and parameters α_i, β_i satisfy

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i, \quad \alpha_i \text{ are all distinct.} \tag{A.22}$$

Then we have a formula,

$$F(z;q) - F(z^{-1};q^{-1}) = \sum_{i=1}^{n} a_i \delta(q^{\alpha_i} z), \qquad a_i = \frac{\displaystyle\prod_{j=1}^{n} (1 - q^{\beta_j - \alpha_i})}{\displaystyle\prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - q^{\alpha_j - \alpha_i})}. \tag{A.23}$$

Example: For $a \neq a'$, we have

$$\begin{aligned}
&\frac{(1-q^{2b}z)(1-q^{2(a+a'-b)}z)}{(1-q^{2a}z)(1-q^{2a'}z)} - \frac{(1-q^{-2b}z^{-1})(1-q^{-2(a+a'-b)}z^{-1})}{(1-q^{-2a}z^{-1})(1-q^{-2a'}z^{-1})} \\
= \; & -(q-q^{-1})\frac{[b-a]_q[b-a']_q}{[a-a']_q}\Big(\delta(q^{2a}z) - \delta(q^{2a'}z)\Big). \qquad \text{(A.24)}
\end{aligned}$$

By taking a limit $a' \to a$, we get

$$\begin{aligned}
&\frac{(1-q^{2b}z)(1-q^{2(2a-b)}z)}{(1-q^{2a}z)^2} - \frac{(1-q^{-2b}z^{-1})(1-q^{-2(2a-b)}z^{-1})}{(1-q^{-2a}z^{-1})^2} \\
= \; & -(q-q^{-1})^2[b-a]_q^2\, q^{2a}z\delta'(q^{2a}z). \qquad \text{(A.25)}
\end{aligned}$$

A.3 Some summations

As it is well known in statistical mechanics, in order to calculate the following summation

$$\sum_{\substack{\{k_i \geq 0\} \\ \sum_{i=1}^{N} ik_i = N}} f(k_1, k_2, \cdots), \qquad \text{(A.26)}$$

it is convenient to introduce its generating function

$$\sum_{N=0}^{\infty} \sum_{\substack{\{k_i\} \\ \sum_i ik_i = N}} f(k_1, k_2, \cdots) y^N = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \cdots f(k_1, k_2, \cdots) y^{k_1+2k_2+\cdots}. \qquad \text{(A.27)}$$

We have two formulas:

$$\begin{aligned}
\prod_{\substack{\{k_i\} \\ \sum_i ik_i = N}} \prod_i k_i! \;&=\; \prod_{\substack{\{k_i\} \\ \sum_i ik_i = N}} \prod_i i^{k_i}, \qquad \text{(A.28)} \\
\sum_{\substack{\{k_i\} \\ \sum_i ik_i = N}} k_l \;&=\; \sum_{\substack{k \geq 1 \\ lk \leq N}} p(N-lk) \qquad (1 \leq l \leq N), \qquad \text{(A.29)}
\end{aligned}$$

where $p(N)$ is the number of partition (see (2.13)). Generating function of log(A.28) is $\prod_{n>0}\frac{1}{1-y^n} \times \sum_{i>0}\frac{y^i}{1-y^i}\log i$, and that of (A.29) is $\prod_{n>0}\frac{1}{1-y^n} \times \frac{y^l}{1-y^l}$. From (A.29) we obtain

$$\prod_{\substack{\{k_i\} \\ \sum_i ik_i = N}} \prod_i F(i)^{k_i} = \prod_{\substack{l,k \geq 1 \\ lk \leq N}} F(l)^{p(N-lk)}, \qquad \text{(A.30)}$$

where $F(z)$ is any function.

A.4 Some integrals

We summarize the relations among the following integral with various contours [31, 34, 101, 30],

$$I = \int dz_1 \cdots dz_m F, \tag{A.31}$$

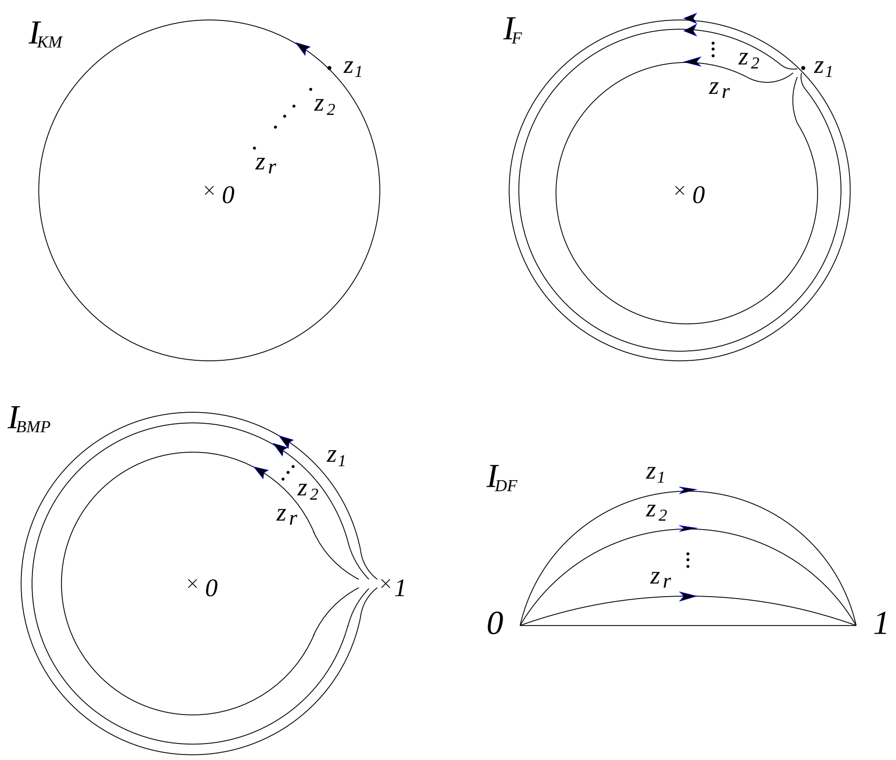
$$F = \prod_{1\le i<j\le m} (z_i - z_j)^{2\alpha} \cdot \prod_{i=1}^{m} z_i^{\alpha'} \cdot f(z_1, \cdots, z_m).$$

Here $f(z_1, \cdots, z_m)$ is a symmetric function and has no pole at $z_i = z_j$. Parameters α and α' are assumed to satisfy the condition (i) (and (ii) for (A.47)),

$$\text{(i)} \quad m(m-1)\alpha + m\alpha' \in \mathbb{Z}, \tag{A.32}$$

$$\text{(ii)} \quad m(m-1)\alpha + m\alpha' \in m\mathbb{Z}. \tag{A.33}$$

Let us consider the following contours:



$$I_{KM} = \oint dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \cdots \int_0^{z_{m-1}} dz_m F, \tag{A.34}$$

$$I_F = \oint dz_1 \int dz_2 \cdots dz_m F, \tag{A.35}$$

$$I_{BMP} = \int dz_1 \cdots dz_m F, \tag{A.36}$$

$$I'_{BMP} = \int dz_1 \cdots dz_m F, \quad (z_j = r_j e^{i\theta_j}, \quad 0 < \theta_1 < \theta_2 < \cdots < \theta_m < 2\pi), \tag{A.37}$$

$$I_{DF} = \int dz_1 \cdots dz_m F, \tag{A.38}$$

$$I'_{DF} = \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \cdots \int_0^{z_{m-1}} dz_m F. \tag{A.39}$$

For I_{KM} and I_F the contour for z_1 closes.

By analytic continuation these integrals are related as follows:

$$\begin{aligned} I_F &= (-1)^{m-1} \prod_{j=0}^{m-2} (1 - a'a^j) \cdot \prod_{j=1}^{m-1} (1 + a + a^2 + \cdots + a^{j-1}) \cdot I_{KM}, \\ &= \prod_{j=0}^{m-2} (a'a^j - 1) \cdot \prod_{j=1}^{m-1} \frac{a^j - 1}{a - 1} \cdot I_{KM}, \end{aligned} \tag{A.40}$$

$$I_{BMP} = \prod_{j=1}^{m} (1 + a + a^2 + \cdots + a^{j-1}) \cdot I'_{BMP} = \prod_{j=1}^{m} \frac{a^j - 1}{a - 1} \cdot I'_{BMP}, \tag{A.41}$$

$$I_{BMP} = (-1)^m \prod_{j=0}^{m-1} (1 - a'a^j) \cdot I_{DF} = \prod_{j=0}^{m-1} (a'a^j - 1) \cdot I_{DF}, \tag{A.42}$$

$$I_{DF} = \prod_{j=1}^{m} (1 + a + a^2 + \cdots + a^{j-1}) \cdot I'_{DF} = \prod_{j=1}^{m} \frac{a^j - 1}{a - 1} \cdot I'_{DF}, \tag{A.43}$$

$$\begin{aligned} I_F &= \prod_{j=1}^{m-1} (1 + a + a^2 + \cdots + a^{j-1}) \\ &\quad \times \left(1 + a'a^{m-1} + (a'a^{m-1})^2 + \cdots + (a'a^{m-1})^{m-1}\right) \cdot I'_{BMP}, \end{aligned} \tag{A.44}$$

where a and a' are

$$a = e^{2\pi i \alpha}, \quad a' = e^{2\pi i \alpha'}, \tag{A.45}$$

and we remark that

$$a'a^{m-1} = 1 \text{ for (ii)}, \quad a'a^{m-1} \neq 1 \text{ for (i) but not (ii)}. \tag{A.46}$$

Under the condition (ii), from (A.41) and (A.44), we have

$$I_{BMP} = \frac{1}{m} \frac{a^m - 1}{a - 1} I_F. \tag{A.47}$$

We give an example of this kind of integral:

$$f(z_1, \cdots, z_m) = \prod_{i=1}^{m} (1 - z_i)^{\alpha''},$$

$$I'_{DF} = \prod_{j=1}^{m} \frac{\Gamma(j\alpha)}{\Gamma(\alpha)} \frac{\Gamma((j-1)\alpha+\alpha'+1)\Gamma((j-1)\alpha+\alpha''+1)}{\Gamma((m-2+j)\alpha+\alpha'+\alpha''+2)}. \tag{A.48}$$

A.5 Hausdorff formula

For two operators A and B, we have a formula

$$e^{A}Be^{-A} = e^{\mathrm{ad}A}B, \quad (\mathrm{ad}A)B = [A,B], \tag{A.49}$$

because two functions of t, $e^{tA}Be^{-tA}$ and $e^{t\mathrm{ad}A}B$, satisfy the same differential equation $f'(t) = Af(t) - f(t)A$ and the same initial condition $f(0) = B$. When A and $[A,B]$ commute each other, we have $e^{A}Be^{-A} = B + [A,B]$ and so $e^{A}e^{B}e^{-A} = e^{B+[A,B]}$. Therefore we have a formula

$$[A,[A,B]] = [B,[A,B]] = 0 \implies e^{A}e^{B} = e^{[A,B]}e^{B}e^{A}, \tag{A.50}$$

which is a special case of the Campbell-Baker-Hausdorff formula $e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]+\cdots}$. A harmonic oscillator $[a,a^{\dagger}] = 1$ and this formula are basic tools of free field realization.

A.6 trace technique

Let us consider one free boson oscillator, $[a,a^{\dagger}] = 1$. The Fock space $\mathcal{F}$ is generated by $|0\rangle$ ($a|0\rangle = 0$), and its orthonormal basis is $|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle$ ($n \geq 0$). A trace over the Fock space ($\mathrm{tr}_{\mathcal{F}}\mathcal{O} = \sum_{n\geq 0}\langle n|\mathcal{O}|n\rangle$) can be expressed as a vacuum to vacuum amplitude. Let us introduce another oscillator b, $[b,b^{\dagger}] = 1$, which commute with a and satisfy $b|0\rangle = 0$. Then we have the following Clavelli-Shapiro's trace formula [102],

$$\mathrm{tr}_{\mathcal{F}}\Big(y^{a^{\dagger}a}\mathcal{O}(a,a^{\dagger})\Big) = \frac{1}{1-y}\langle 0|\mathcal{O}(d,\bar{d})|0\rangle. \tag{A.51}$$

Here $d, \bar{d}$ are

$$d = \frac{1}{1-y}a + b^{\dagger}, \quad \bar{d} = a^{\dagger} + \frac{y}{1-y}b, \quad [d,\bar{d}] = 1. \tag{A.52}$$

Especially we have

$$\mathrm{tr}_{\mathcal{F}}\Big(y^{a^{\dagger}a}e^{Aa^{\dagger}}e^{Ba}\Big) = \frac{1}{1-y}e^{AB\frac{y}{1-y}}, \tag{A.53}$$

where A, B are constants.

For reader's convenience, we give three direct proofs of (A.53).
(i) The first method uses a cyclic property of trace. We have

$$\begin{aligned} f(A,B) &= \mathrm{tr}_{\mathcal{F}}\Big(y^{a^{\dagger}a}e^{Aa^{\dagger}}e^{Ba}\Big) \quad (|y|<1) \\ &= \mathrm{tr}_{\mathcal{F}}\Big(e^{Aya^{\dagger}}y^{a^{\dagger}a}e^{Ba}\Big) = \mathrm{tr}_{\mathcal{F}}\Big(y^{a^{\dagger}a}e^{Ba}e^{Aya^{\dagger}}\Big) \\ &= e^{ABy}\,\mathrm{tr}_{\mathcal{F}}\Big(y^{a^{\dagger}a}e^{Aya^{\dagger}}e^{Ba}\Big) = e^{ABy}f(Ay,B), \end{aligned} \tag{A.54}$$

and

$$f(A,B) \;=\; e^{-AB}\,\mathrm{tr}_{\mathcal{F}}\Big(y^{a^\dagger a}e^{Ba}e^{Aa^\dagger}\Big) = e^{-AB}\,\mathrm{tr}_{\mathcal{F}}\Big(e^{By^{-1}a}y^{a^\dagger a}e^{Aa^\dagger}\Big)$$
$$\;=\; e^{-AB}\,\mathrm{tr}_{\mathcal{F}}\Big(y^{a^\dagger a}e^{Aa^\dagger}e^{By^{-1}a}\Big) = e^{-AB}f(A,By^{-1}). \qquad \text{(A.55)}$$

(A.54) implies

$$f(A,B) = e^{AB(y+y^2+\cdots)}f(Ay^\infty,B) = e^{AB\frac{y}{1-y}}f(0,B),$$

and (A.55) implies

$$f(0,B) = f(0,By) = f(0,By^\infty) = f(0,0).$$

Since $f(0,0)=\frac{1}{1-y}$, we obtain (A.53).
(ii) The second method uses a coherent state $|\alpha) = e^{\alpha a^\dagger}|0\rangle$ ($\alpha\in\mathbb{C}$), which satisfies $a|\alpha) = \alpha|\alpha)$, $(\alpha|\alpha') = e^{\bar{\alpha}\alpha'}$ and the completeness condition $1=\sum_{n\geq 0}|n\rangle\langle n| = \int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}|\alpha)(\alpha|$, where $d^2\alpha = d\alpha_1 d\alpha_2$ with $\alpha=\alpha_1+i\alpha_2$. The trace becomes

$$\mathrm{tr}_{\mathcal{F}}\Big(y^{a^\dagger a}e^{Aa^\dagger}e^{Ba}\Big) = \int\frac{d^2\alpha}{\pi}e^{-|\alpha|^2}(\alpha|y^{a^\dagger a}e^{Aa^\dagger}e^{Ba}|\alpha).$$

Since $(\alpha|y^{a^\dagger a}e^{Aa^\dagger}e^{Ba}|\alpha) = e^{B\alpha}(\alpha|y^{a^\dagger a}e^{Aa^\dagger}|\alpha) = e^{B\alpha}(\alpha|y^{a^\dagger a}|\alpha+A) = e^{B\alpha}(\alpha|y(\alpha+A)) = e^{B\alpha}e^{\bar{\alpha}y(\alpha+A)}$, by completing the square and performing the Gauss integral, we obtain the result.
(iii) The third method is a direct calculation.

$$\mathrm{tr}_{\mathcal{F}}\Big(y^{a^\dagger a}e^{Aa^\dagger}e^{Ba}\Big) = \sum_{n=0}^\infty\frac{1}{n!}\langle 0|a^n y^{a^\dagger a}e^{Aa^\dagger}e^{Ba}a^{\dagger n}|0\rangle$$
$$= \sum_{n=0}^\infty\frac{1}{n!}y^n\langle 0|a^n e^{Aa^\dagger}(a^\dagger+B)^n|0\rangle = \sum_{n=0}^\infty\frac{1}{n!}y^n\langle 0|a^n\sum_{l=0}^n\frac{A^{n-l}}{(n-l)!}\binom{n}{l}B^{n-l}a^{\dagger n}|0\rangle$$
$$= \sum_{n=0}^\infty y^n\sum_{l=0}^n\frac{1}{(n-l)!}\binom{n}{l}(AB)^{n-l}.$$

By interchanging the order of summations $\sum_{n=0}^\infty\sum_{l=0}^n = \sum_{l=0}^\infty\sum_{n=l}^\infty$ and shifting $n=m+l$, it becomes

$$\sum_{m=0}^\infty\frac{1}{m!}(ABy)^m\sum_{l=0}^\infty\binom{m+l}{l}y^l.$$

Since $\sum_{l=0}^\infty\binom{m+l}{l}y^l$ equals to $\frac{1}{m!}(y\partial_y+m)\cdots(y\partial_y+1)\sum_{l=0}^\infty y^l = \frac{1}{(1-y)^{m+1}}$, we obtain the result.

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List of Revised Points (misprints etc.)

hep-th/9910226

- page 11 : $\cup \Rightarrow +$ (5 places)
- below eq.(2.153) : add references
- $d \Rightarrow d^{\rm osc} + d^{\rm zero}$
  eqs.(3.43),(5.73),(5.74),(5.140),(6.18),(6.103),(6.104),(6.130).

These points are revised in hep-th/9910226v2.

hep-th/9910226v2

I will update this list (misprints etc.). See the following page:

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