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c=3d Conformal Algebra with Extended Supersymmetry

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Abstract

We define a superconformal algebra with the central charge $c = 3d$, which is the symmetry of the non-linear σ model on a

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complex d dimensional Calabi-Yau manifold. The $c = 3d$ algebra is an extended superconformal algebra obtained by adding the spectral flow generators to the $N = 2$ superconformal algebra. We study the representation theory and show that its representations are invariant under the integer-shift spectral flow. We present the character formulas and their modular transformation properties. We also discuss the relation to the $N = 4$ superconformal algebra.

1 Introduction

In the search for four dimensional realistic string theories, space-time supersymmetry and world-sheet superconformal symmetry are the most important requirements and internal sector is restricted severely by them. If internal sector is considered as the non-linear σ model on certain manifold, from the study of the low energy supergravity, $N = 1$ space-time supersymmetry implies the manifold must be a Calabi-Yau manifold [1,2]. This manifold possesses a unique covariantly constant spinor and an (anti-)holomorphic d-form, where d is the complex dimension of the manifold and, in the case of four dimensional string theory, d is equal to 3. On the other hand, world-sheet $N = 2$ superconformal symmetry and $U(1)$ charge quantization are equivalent to space-time $N = 1$ supersymmetry[3,4,5,6,7,8]. The $N = 2$ superconformal algebra(SCA) has an automorphism (so-called spectral flow) due to the $U(1)$ Kac-Moody subalgebra[9], so that Neveu-Schwarz(NS) and Ramond(R) sectors are mapped onto each other by the spectral flow which are considered as space-time supersymmetry transformation. In previous papers[10,11], we studied the extension of the $N = 2$ SCA by adding the flow generators which generate the integer-shift spectral flow. Its representations are invariant under the integer-shift spectral flow because such flow corresponds to twice operation of space-time supersymmetry transformation. In this context the covariantly constant spinor corresponds to Ramond ground state and the (anti-)holomorphic d-forms correspond to the spectral flow generators.

In this paper we generalize the previous $d = 3$ result (the $c = 9$ algebra) to arbitrary d case (the $c = 3d$ algebra). Since space-time dimension of string theory is $10 - 2d$, d more than three case is not relevant to string compactification. However, study of $d > 3$ case is interesting because it is the symmetry of the non-linear σ model on a complex d dimensional Calabi-Yau manifold (i.e. manifold with $SU(d)$ holonomy, i.e. Ricci-flat Kähler manifold) and finding modular invariant partition functions will give one method to study the properties of Calabi-Yau manifold itself[6,8]. The $N = 2$ SCA, which is the symmetry of the non-linear σ model on a Kähler manifold, is invariant under the spectral flow but its representations are not so. We want to find the extended algebra which representations are invariant under the integer-shift spectral flow. For $c > 3$, its representation contains infinite many representations of the $N = 2$ SCA because representation of the $N = 2$ SCA never comes back to itself under the integer-shift spectral flow in contrast to the rational case $c < 3$. The $N = 4$ SCA, which is the symmetry of the non-linear σ model on a hyper-Kähler manifold, has this property but it is too large from this point of view and we want the smallest one, i.e. its representation contains only one highest weight state of the $N = 2$ SCA modulo the spectral flow of the $N = 2$ SCA. This is the $c = 3d$ algebra. The $c = 3d$ algebra is obtained from the $N = 2$ SCA by the addition of the spectra flow generators and by the requirement that the cental charge c is equal to $3d$.

This paper is organized as follows. We begin with the definition of the $c = 3d$ algebra and its properties (degeneracy conditions, subalgebras, realization and the spectral flow). Then, in section 3, we discuss the representation theory. We present the conditions for the irreducible unitary highest weight representations. These representations are invariant under the integer-shift spectral flow and the structures of their representation spaces are derived. Using this structure, in section 4, we present the character formulas and derive their modular transformation properties. In section 5 we discuss the decomposition of characters of one algebra into those of another algebra. Representations of the $N = 4$ SCA with $c > 6$ are infinitely reducible with respect to the $c = 3d$ algebra.

2 $c = 3d$ Algebra

In this section we present the $c = 3d$ algebra. We start from the notations. A field operator $A(z)$ with conformal weight h has a mode expansion: $A(z) = \sum_n A_n z^{-n-h}$ (except $T(z) = \sum_n L_n z^{-n-2}$), where $n \in \mathbf{Z} - h$ for NS sector and $n \in \mathbf{Z}$ for R sector and $A_n^\dagger = \bar{A}_{-n}$. Normal ordering $(AB)(z)$ of the mutually local two fields $A(z)$ and $B(z)$ is defined by

$$(AB)(z) = \oint_z \frac{dx}{2\pi i} \frac{1}{x-z} A(x)B(z). \tag{2.1}$$

In NS sector or in R sector with $h_A \in \mathbf{Z}$, mode expansion of $(AB)(z)$ is

$$(AB)_n = \sum_{p \leq -h_A} A_p B_{n-p} + (-1)^{AB} \sum_{p > -h_A} B_{n-p} A_p. \tag{2.2}$$

In the following we will consider NS sector only unless it is explicitly mentioned. The state $|A\rangle$ is given by $|A\rangle = \lim_{z\to 0} A(z)|\mathbf{0}\rangle = A_{-h}|\mathbf{0}\rangle$, where the vacuum $|\mathbf{0}\rangle$ is defined by $A_n|\mathbf{0}\rangle = 0$ for $n > -h$.

In [10], for each positive integer d which was denoted as $\tilde{n}$, we studied the extension of the $N = 2$ SCA by the addition of the spectral flow generators $X(z)$ and $\bar{X}(z)$, which has $h = \frac{d}{2}$ and $U(1)$ charge $Q = \pm d$, and their superpartners $Y(z)$ and $\bar{Y}(z)$. There we treated central charge c and dimension d as independent parameters. Assumptions of [10] are (i) $X(z)$ is a primary field with $(h, Q) = (\frac{d}{2}, d)$ with respect to Virasoro generator $T(z)$ and $U(1)$ Kac-Moody current $I(z)$, (ii) operator product expansion(OPE) of supercurrent G and X is regular and OPE of $\bar{G}$ and X has only simple pole term whose residue will be denoted as Y, (iii) the $N = 2$ SCA generators and $X, \bar{X}, Y$ and $\bar{Y}$ generate a closed associative algebra. OPE of X and $\bar{Y}$ or Y and $\bar{Y}$ is obtained from OPE of X and $\bar{X}$ by supertransformation. If OPE of X and $\bar{X}$ is expressed by the $N = 2$ SCA generators, closure of the extended algebra is ensured by this fact. Unknown OPE is OPE of X and $\bar{X}$ only.

In this paper we require center c is equal to $3d$. By bosonizing the $U(1)$ current $I(z)$ as $I(z) = \sqrt{d} i\partial\phi(z)$, the spectral flow generator $X(z)$ with $U(1)$ charge d is expressed as $: e^{i\sqrt{d}\phi(z)} : \tilde{X}(z)$. Since $X(z)$ has conformal weight $\frac{d}{2}$ and $: e^{i\sqrt{d}\phi(z)} :$ has already conformal weight $\frac{d}{2}$, $\tilde{X}(z)$ is a constant. We take the normalization $\tilde{X} = 1$ (this normalization is different from [10,11]). OPE of $G(z) =: e^{i\frac{1}{\sqrt{d}}\phi(z)} : \tilde{G}(z)$ and $X(z)$ are regular and OPE of $\bar{G}(z) =$

$: e^{-i\frac{1}{\sqrt{d}}\phi(z)} : \tilde{\bar{G}}(z)$ and $X(z)$ has only simple pole term, so that assumptions (i) and (ii) are satisfied. From this bosonized form, OPE of X and $\bar{X}$ is uniquely expressed by the differential polynomial of $I(z)$, so that closure of the algebra is satisfied. Therefore the extended algebra is uniquely determined. We call this extended algebra as the $c = 3d$ algebra. OPEs of generators of the $c = 3d$ algebra are as follows:

$$
\begin{cases}
c = 3d \\
T(z)T(w) = \dfrac{c}{2(z-w)^4} + \dfrac{2T(w)}{(z-w)^2} + \dfrac{\partial T(w)}{z-w} + \text{regular term} \\
I(z)I(w) = \dfrac{c}{3(z-w)^2} + \text{reg.} \\
G(z)\bar{G}(w) = \dfrac{2c}{3(z-w)^3} + \dfrac{2I(w)}{(z-w)^2} + \dfrac{\partial I(w) + 2T(w)}{z-w} + \text{reg.}
\end{cases}
$$

$$
\begin{cases}
 & A = I \quad \begin{matrix} G & X & Y \\ \bar{G} & \bar{X} & \bar{Y} \end{matrix} \\
T(z)A(w) = \dfrac{hA(w)}{(z-w)^2} + \dfrac{\partial A(w)}{z-w} + \text{reg.} & h = 1 \quad \tfrac{3}{2} \quad \tfrac{d}{2} \quad \tfrac{d+1}{2} \\
I(z)A(w) = \dfrac{QA(w)}{z-w} + \text{reg.} & Q = * \quad \pm 1 \quad \pm d \quad \pm(d-1)
\end{cases}
$$

$$
\begin{cases}
\bar{G}(z)X(w) = \dfrac{2Y(w)}{z-w} + \text{reg.} \\
G(z)Y(w) = \dfrac{dX(w)}{(z-w)^2} + \dfrac{\partial X(w)}{z-w} + \text{reg.}
\end{cases}
$$

$$
\begin{cases}
X(z)|\bar{X}\rangle = \displaystyle\sum_{p=1}^{d} z^{-p} \sum_{\substack{\{i_k \geq 0\}_{k\geq 1} \\ \sum_{k\geq 1} k i_k = d-p}} \prod_{k\geq 1} \frac{1}{i_k! k^{i_k}} I_{-k}^{i_k} |\mathbf{0}\rangle + \text{reg.} \\
X(z)|\bar{Y}\rangle = -\tfrac{1}{2} G_{-\frac{1}{2}} X(z)|\bar{X}\rangle \\
Y(z)|\bar{Y}\rangle = \tfrac{1}{2} X(z)|\partial\bar{X}\rangle + \tfrac{1}{2}\bar{G}_{-\frac{1}{2}} X(z)|\bar{Y}\rangle
\end{cases}
\tag{2.3}
$$

and their hermitian conjugates. Other OPE is regular. Last two equations mean that OPE of X and $\bar{Y}$ or Y and $\bar{Y}$ is obtained by moving $G_{-\frac{1}{2}}$ or $\bar{G}_{-\frac{1}{2}}$ right until they annihilate the vacuum $|\mathbf{0}\rangle$, for example, in the case of $d=3$,

$$\begin{aligned} X(z)|\bar{Y}\rangle &= -\tfrac{1}{2}G_{-\frac{1}{2}}(z^{-3}+z^{-2}I_{-1}+z^{-1}\tfrac{1}{2}(I_{-1}^2+I_{-2}))|\mathbf{0}\rangle + \text{reg.} \\ &= \tfrac{1}{2}(z^{-2}G_{-\frac{3}{2}}+z^{-1}I_{-1}G_{-\frac{3}{2}})|\mathbf{0}\rangle + \text{reg.} \end{aligned}$$

i.e. $X(z)\bar{Y}(w) = \frac{1}{2}(\frac{G(w)}{(z-w)^2}+\frac{(IG)(w)}{z-w}) + \text{reg.}$ The $c=3$ algebra is an algebra generated by one pair of complex free boson and fermion and the $c=6$ algebra is the $N=4$ SCA with $c=6$[12,8,10]. For $d\geq 3$, the $c=3d$ algebra is not a Lie algebra but an analogue of the W algebra[13].

Associativity requires some operator relations (degeneracy conditions) and among them

$$(IX)(z) = \partial X(z) \tag{2.4}$$

and its hermitian conjugate are the most important. Other relations, for example,

$$\begin{aligned} &(IY)(z) - \partial Y(z) + \tfrac{1}{2}(\bar{G}X)(z) = 0 && (2.5) \\ &(GY)(z) + \tfrac{1}{2}(\partial IX)(z) - (TX)(z) = 0 \\ &(\partial^j XX)(z) = 0 \quad j=0,1,\cdots,d-1 \\ &(\partial^j XY)(z) = (\partial^j YY)(z) = 0 \quad j=0,1,\cdots,d-2 && (2.6) \end{aligned}$$

are derived from eq. (2.4).

The $c=3d$ algebra has free field realizations. One of them is given by d pairs of complex free bosons and fermions:

$$G(z) = \sqrt{2}\sum_{j=1}^{d}\psi^j(z)i\partial\varphi^j(z)$$

$$X(z) = \frac{1}{d!}\sum_{i_1,\cdots,i_d=1}^{d} \varepsilon_{i_1\cdots i_d}\psi^{i_1}(z)\cdots\psi^{i_d}(z) = \psi^1\psi^2\cdots\psi^d(z). \quad (2.7)$$

Existence of a free field realization ensures associativity of the $c = 3d$ algebra and degeneracy conditions are also checked. The non-linear σ model on a complex d dimensional Calabi-Yau manifold has a symmetry characterized by the presence of an (anti-)holomorphic d-form[10]. Spectral flow generator X corresponds to a holomorphic d-form and the $c = 3d$ algebra is just this symmetry.

The $c = 3d$ algebra contains two important subalgebras. One is the $N = 2$ SCA with $c = 3d$ generated by T, I, G and $\bar{G}$. The other is generated by $\frac{1}{2d}(I^2), I, X$ and $\bar{X}$. Using eq. (2.4), we can show that X is a primary field with conformal weight $\frac{d}{2}$ with respect to $\frac{1}{2d}(I^2)$, which is the Sugawara type Virasoro generator with $c = 1$. We will call this algebra as the $c = 1$ subalgebra. From the OPE of X and $\bar{X}$ or Y and $\bar{Y}$, (anti-)commutator of X and $\bar{X}$ contains I only and (anti-)commutator of Y and $\bar{Y}$ contains $N = 2$ generators only. Using this (anti-)commutators and combinatorial identities

$$\sum_{\substack{\{i_k\geq 0\}_{k\geq 1} \\ \sum_{k\geq 1} k i_k = n}} \prod_{k\geq 1} \frac{x^{i_k}}{i_k! k^{i_k}} = \binom{x+n-1}{n}$$

$$\sum_{p=0}^{n} \binom{x+p}{p}\binom{y+n-p}{n-p} = \binom{x+y+n+1}{n},$$

we can show that

$$\langle Q|[X_n, \bar{X}_{-n}\}|Q\rangle = \binom{Q+n+\frac{d}{2}-1}{d-1} \quad (2.8)$$

$$ {}^{N=2}\langle h,Q|[Y_n,\bar{Y}_{-n}\}|h,Q\rangle^{N=2} = \begin{cases} \frac{1}{2}(h+\frac{n(n+Q)}{d-1}-\frac{d-1}{4})\binom{Q+n+\frac{d-3}{2}}{d-2} & d\geq 2 \\ \frac{1}{2}n & d=1, \end{cases} \tag{2.9} $$

where $|Q\rangle$ is the highest weight state of $I(z)$ and $|h,Q\rangle^{N=2}$ is the highest weight state of the $N=2$ SCA and $\binom{x}{r}=\frac{x(x-1)\cdots(x-r+1)}{r!}$.

The $c=3d$ algebra is invariant under the spectral flow, i.e. (anti-) commutators obtained from eq. (2.3) are invariant under the transformation:

$$ \begin{cases} L'_n = L_n+\eta I_n+\frac{d}{2}\eta^2\delta_{n0} & \\ I'_n = I_n+d\eta\delta_{n0} & A = G \quad \bar{G} \quad X \quad \bar{X} \quad Y \quad \bar{Y} \\ A'_n = A_{n+Q\eta} & Q = 1 \quad -1 \quad d \quad -d \quad d-1 \quad -(d-1), \end{cases} \tag{2.10} $$

where η is an arbitrary real parameter and this transformation is achieved by the momentum shift of bosonized $U(1)$ current. Normal ordering $(\partial^i G'\partial^j\bar{G}')(z)$ is defined by OPE of $\partial^i(z^\eta G(z))$ and $\partial^j(w^{-\eta}\bar{G}(w))$ and eq. (2.1).

3 Representation Theory

The highest weight state of the $c=3d$ algebra $|h,Q\rangle$ is an eigenstate of L_0 and I_0 and annihilated by generators of positive mode:

$$ \begin{aligned} L_0|h,Q\rangle &= h|h,Q\rangle, \qquad I_0|h,Q\rangle = Q|h,Q\rangle \\ A_n|h,Q\rangle &= 0 \quad (n>0) \quad A=L,I,G,\bar{G},X,\bar{X},Y,\bar{Y} \end{aligned} \tag{3.1} $$

and conditions on the other zero mode are determined by consistency and irreducibility. $|h,Q\rangle$ is also the highest weight state of the two subalgebras, the $N=2$ SCA $|h,Q\rangle^{N=2}$ and the $c=1$ subalgebra $|Q\rangle^{c=1}$ ($h^{c=1}=\frac{Q^2}{2d}$).

We start from the representation theory of the $c=1$ subalgebra. For even d, operating the zero mode of eq. (2.4) to $|Q\rangle^{c=1}$, we obtain

$$\begin{aligned} 0 &= ((IX)-\partial X)_0|Q\rangle^{c=1} \\ &= (\sum_{p\leq -1} I_p X_{-p} + \sum_{p>-1} X_{-p} I_p + \frac{d}{2} X_0)|Q\rangle^{c=1} \\ &= (Q+\frac{d}{2})X_0|Q\rangle^{c=1}, \end{aligned}$$

i.e. $X_0|Q\rangle^{c=1}=0$ for $Q\neq -\frac{d}{2}$. Similarly, hermitian conjugate of eq. (2.4) implies $\bar{X}_0|Q\rangle^{c=1}=0$ for $Q\neq \frac{d}{2}$. Eq. (2.8) says

$$^{c=1}\langle Q|[X_0,\bar{X}_0]|Q\rangle^{c=1} \;=\; \frac{1}{(d-1)!}\prod_{r=-(\frac{d}{2}-1)}^{r=\frac{d}{2}-1}(Q-r).$$

If $Q\neq \pm\frac{d}{2}$, these relations require that $U(1)$ charge Q must be quantized to $0,\pm1,\pm2,\cdots,\ \pm(\frac{d}{2}-1)$. If $Q=\pm\frac{d}{2}$, these relations mean that norms of the states $\bar{X}_0|\frac{d}{2}\rangle^{c=1}$ and $X_0|-\frac{d}{2}\rangle^{c=1}$ are equal to 1 and these states are mapped each other $|-\frac{d}{2}\rangle^{c=1}=\bar{X}_0|\frac{d}{2}\rangle^{c=1}$. Consequently irreducible unitary highest weight representations of the $c=1$ subalgebra for even d exist if and only if $Q=0,\pm1,\cdots,\pm(\frac{d}{2}-1),\frac{d}{2}$. Sufficiency is easily checked by a free field realization. For odd d, by operating the $-\frac{1}{2}$ mode of eq. (2.4) to $|Q\rangle^{c=1}$, similar argument show that necessary and sufficient condition is $Q=0,\pm1,\cdots,\pm\frac{d-1}{2}$. We remark that Q takes integer value and number of the representations is d.

Now we come back to the $c=3d$ algebra and we first consider $d\geq 2$ case and denote $k=d-1$. By the $c=1$ subalgebra, allowed $U(1)$ charges have been determined. Norms of states $G_{-\frac{1}{2}}|h,Q\rangle$ and $\bar{G}_{-\frac{1}{2}}|h,Q\rangle$, which must be non-negative, require $h \geq \frac{|Q|}{2}$. For even d and $Q = \frac{d}{2}$, operating $-\frac{1}{2}$ mode of eq. (2.5) to $|h,\frac{d}{2}\rangle$, we obtain $Y_{-\frac{1}{2}}|h,\frac{d}{2}\rangle = 0$. On the other hand, from eq. (2.9), square norm of $Y_{-\frac{1}{2}}|h,\frac{d}{2}\rangle$ is equal to $\frac{1}{2}(h-\frac{d}{4})$. Therefore h must be $\frac{d}{4}$ for $Q = \frac{d}{2}$. For odd d, operating the zero mode of eq. (2.5) and its hermitian conjugate to $|h,Q\rangle$, we obtain $Y_0|h,Q\rangle = 0$ for $Q \neq -\frac{k}{2}$ and $\bar{Y}_0|h,Q\rangle = 0$ for $Q \neq \frac{k}{2}$. $|h,\frac{k}{2}\rangle$ and $|h,-\frac{k}{2}\rangle$ are mapped each other by Y_0 and $\bar{Y}_0$. Consequently, irreducible unitary highest weight representations of the $c = 3d$ algebra exist if and only if in the following cases:

- massive representations $h > \frac{|Q|}{2}$

$$\begin{aligned} Q &= 0, \pm 1, \cdots, \pm([\tfrac{d}{2}]-1) \qquad A_0|h,Q\rangle = 0 \\ \text{and} \qquad & \tfrac{k}{2} \text{ (for odd } d) \quad |h,-\tfrac{k}{2}\rangle \underset{\frac{1}{a}\bar{Y}_0}{\overset{\frac{1}{a}Y_0}{\rightleftarrows}} |h,\tfrac{k}{2}\rangle \end{aligned} \tag{3.2}$$

- massless representations $h = \frac{|Q|}{2}$

$$\begin{aligned} Q &= 0, \pm 1, \cdots, \pm[\tfrac{k}{2}] \qquad A_0|\tfrac{|Q|}{2},Q\rangle = 0 \\ \text{and} \qquad & \tfrac{d}{2} \text{ (for even } d) \quad |\tfrac{d}{4},-\tfrac{d}{2}\rangle \underset{\bar{X}_0}{\overset{X_0}{\rightleftarrows}} |\tfrac{d}{4},\tfrac{d}{2}\rangle, \end{aligned} \tag{3.3}$$

where A_0 stands for X_0 and $\bar{X}_0$ for even d, Y_0 and $\bar{Y}_0$ for odd d and $[x]$ is the greatest integer not exceeding x and $a = \sqrt{\frac{1}{2}(h-\frac{Q}{2})}$. Sufficiency is checked by the free field realization eq. (2.7). Since the Fock space of eq. (2.7) is very large, there exist infinite many highest weight states of the $c = 3d$ algebra, for example, $: \psi^1 \cdots \psi^j e^{i\alpha(\varphi^d + \bar{\varphi}^d)} : (z)$ creates the state with $(h, Q) = (\alpha^2 + \frac{j}{2}, j)$ for $0 \le j \le [\frac{k}{2}]$ and $: \psi^1 \cdots \psi^{\frac{d}{2}} : (z)$ creates the state with $(h, Q) = (\frac{d}{4}, \frac{d}{2})$.

Since the $c = 3d$ algebra is invariant under the spectral flow, NS and R sector are mapped each other by the spectral flow with $\eta \in \mathbf{Z} + \frac{1}{2}$. The highest weight state of R sector are obtained from those of NS sector by $\eta = \frac{1}{2}$ spectral flow:

$$\begin{aligned} \mathrm{R} \xleftarrow{\eta=\frac{1}{2}} \mathrm{NS} \qquad & Q \le 0 \quad |h^R = h + \tfrac{Q}{2} + \tfrac{d}{8}, Q^R = Q + \tfrac{d}{2}\rangle^R = |h, Q\rangle^{NS} \\ & Q > 0 \quad |h^R = h - \tfrac{Q}{2} + \tfrac{d}{8}, Q^R = Q - \tfrac{d}{2}\rangle^R = \bar{X}_{-(\frac{d}{2} - Q)} |h, Q\rangle^{NS}. \end{aligned} \tag{3.4}$$

Similarly NS sector is mapped from R sector by $\eta = \frac{1}{2}$ spectral flow as follows:

$$\begin{aligned} \mathrm{NS} \xleftarrow{\eta=\frac{1}{2}} \mathrm{R} \qquad & Q \le 0 \quad |h, Q\rangle^{NS} = \bar{X}^R_Q |h^R = h + \tfrac{Q}{2} + \tfrac{d}{8}, Q^R = Q + \tfrac{d}{2}\rangle^R \\ & Q > 0 \quad |h, Q\rangle^{NS} = |h^R = h - \tfrac{Q}{2} + \tfrac{d}{8}, Q^R = Q - \tfrac{d}{2}\rangle^R. \end{aligned} \tag{3.5}$$

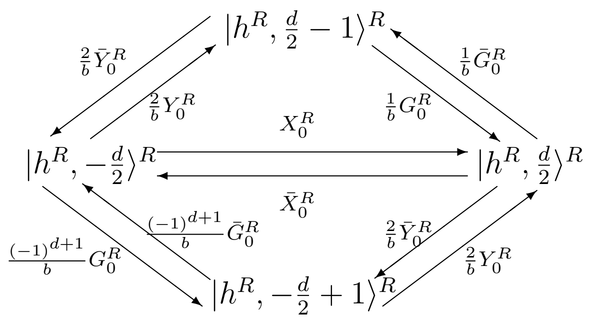
Thus, representations come back to the same representations under the $\eta = 1$ spectral flow. The highest weight states in R sector, which are defined by eq. (3.4), have following zero mode conditions:

- massive representations

$$Q < 0 \qquad |h^R, Q^R - 1\rangle^R \underset{\frac{1}{b}\bar{G}^R_0}{\overset{\frac{1}{b}G^R_0}{\rightleftarrows}} |h^R, Q^R\rangle^R \tag{3.6}$$

$$Q > 0 \qquad |h^R, Q^R + 1\rangle^R \;\underset{\frac{1}{b}G_0^R}{\overset{\frac{1}{b}\bar{G}_0^R}{\rightleftarrows}}\; |h^R, Q^R\rangle^R \tag{3.7}$$

$Q = 0$



(3.8)

- massless representations

$$Q \neq 0 \qquad |\tfrac{d}{8}, Q^R\rangle^R \tag{3.9}$$

$$Q = 0 \qquad |\tfrac{d}{8}, -\tfrac{d}{2}\rangle^R \;\underset{\bar{X}_0^R}{\overset{X_0^R}{\rightleftarrows}}\; |\tfrac{d}{8}, \tfrac{d}{2}\rangle^R, \tag{3.10}$$

where $b = \sqrt{2(h^R - \frac{d}{8})}$. From these, only massless representations with $Q \neq 0$ and massless representation with $Q = 0$ for even d possess non-vanishing Witten indices and their absolute values are 1 and 2 respectively.

Next we consider the structure of the representation spaces. We take the massive representations for illustration. The highest weight state of the $N = 2$ SCA, $|h, Q\rangle = |h, Q\rangle^{N=2}$, is mapped to the $N = 2$ highest weight state $|h_m, Q_m\rangle^{N=2}$ by the spectral flow of the $N = 2$ SCA with $\eta_{N=2} = m \in \mathbf{Z}$:

$$|h, Q\rangle^{N=2} \xrightarrow{\eta_{N=2}=m} |h_m = h - \tfrac{Q^2}{2k} + \tfrac{k}{2}(m + \tfrac{Q}{k})^2, Q_m = k(m + \tfrac{Q}{k})\rangle^{N=2} \tag{3.11}$$

and these states are not connected by the $N=2$ SCA generators. In the representation space of the $c=3d$ algebra, however, these states are connected by $X, \bar{X}(Y, \bar{Y})$, which is the reason why we call X and $\bar{X}$ as the spectral flow generators. In fact these states with unit norm are given by

$$|m\rangle = |h_m, Q_m\rangle^{N=2} = \begin{cases} \prod_{j=1}^{m} \frac{Y_{-k(j-\frac{1}{2})-Q}}{\sqrt{\frac{1}{2}(h - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{Q}{k})^2)}} |h, Q\rangle & m \geq 0 \\ \prod_{j=1}^{-m} \frac{\bar{Y}_{-k(j-\frac{1}{2})+Q}}{\sqrt{\frac{1}{2}(h - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} - \frac{Q}{k})^2)}} |h, Q\rangle & m < 0, \end{cases} \tag{3.12}$$

where $\prod_{j=n}^{n-1} * = 1$. Just like as the $c=9$ algebra, we can prove that representation space of the $c=3d$ algebra is a direct sum of representation spaces of the $N=2$ SCA, which are mapped from $|h, Q\rangle = |h, Q\rangle^{N=2}$ by the spectral flow of the $N=2$ SCA with $\eta_{N=2} \in \mathbf{Z}$. The proof is an easy generalization of [11] and key steps are (i) proposition 1 of [11], (ii) $X_{-n}, Y_{-n}|m\rangle = 0$ $(m \geq 0, n < k(m+\frac{1}{2})+Q)$, (iii) $|m\rangle$ is the highest weight state of the $N=2$ SCA, (iv) $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle$ $(m \geq 0, n \geq k(m+\frac{1}{2})+Q)$ (v) proposition 3 of [11]. For massless representations, similar results hold.

The highest weight state of massless representation with $Q > 0$, $|\frac{Q}{2}, Q\rangle = |\frac{Q}{2}, Q\rangle^{N=2}$, is mapped to

$$|h_m, Q_m\rangle^{N=2} = \begin{cases} \prod_{j=1}^{m} \frac{Y_{-k(j-\frac{1}{2})-Q}}{\sqrt{\frac{1}{2}(\frac{Q}{2} - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{Q}{k})^2)}} |\frac{Q}{2}, Q\rangle & m \geq 0 \\ \prod_{j=2}^{-m} \frac{Y_{-k(j-\frac{1}{2})+Q-1}}{\sqrt{\frac{1}{2}(\frac{Q-1}{2} - \frac{k}{8} - \frac{(Q-1)^2}{2k} + \frac{k}{2}(j - \frac{1}{2} - \frac{Q-1}{k})^2)}} \bar{X}_{-\frac{d}{2}+Q} |\frac{Q}{2}, Q\rangle & m < 0 \end{cases}$$

$$(h_m, Q_m) = \begin{cases} (\frac{Q}{2} - \frac{Q^2}{2k} + \frac{k}{2}(m + \frac{Q}{k})^2, k(m + \frac{Q}{k})) & m \geq 0 \\ (\frac{Q-1}{2} - \frac{(Q-1)^2}{2k} + \frac{k}{2}(m + \frac{Q-1}{k})^2, k(m + \frac{Q-1}{k})) & m < 0 \end{cases} \tag{3.13}$$

by the spectral flow of the $N = 2$ SCA with $\eta_{N=2} = m \in \mathbf{Z}$ and massless representation with $Q < 0$ is mapped to

$$\begin{aligned} &|h_m, Q_m\rangle^{N=2} \\ &= \begin{cases} \prod_{j=2}^{m} \frac{Y_{-k(j-\frac{1}{2})-Q-1}}{\sqrt{\frac{1}{2}(-\frac{Q+1}{2} - \frac{k}{8} - \frac{(Q+1)^2}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{Q+1}{k})^2)}} X_{-\frac{d}{2}-Q}|-\frac{Q}{2}, Q\rangle & m > 0 \\ \prod_{j=1}^{-m} \frac{\bar{Y}_{-k(j-\frac{1}{2})+Q}}{\sqrt{\frac{1}{2}(-\frac{Q}{2} - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} - \frac{Q}{k})^2)}} |-\frac{Q}{2}, Q\rangle & m \leq 0 \end{cases} \end{aligned}$$

$$(h_m, Q_m) = \begin{cases} (-\frac{Q+1}{2} - \frac{(Q+1)^2}{2k} + \frac{k}{2}(m + \frac{Q+1}{k})^2, k(m + \frac{Q+1}{k})) & m > 0 \\ (-\frac{Q}{2} - \frac{Q^2}{2k} + \frac{k}{2}(m + \frac{Q}{k})^2, k(m + \frac{Q}{k})) & m \leq 0. \end{cases} \tag{3.14}$$

The highest weight state of massless representation with $Q = 0$, $|0,0\rangle = |0,0\rangle^{N=2}$, is mapped to

$$|h_m, Q_m\rangle^{N=2} = \begin{cases} \prod_{j=2}^{m} \frac{Y_{-k(j-\frac{1}{2})-1}}{\sqrt{\frac{1}{2}(-\frac{1}{2} - \frac{k}{8} - \frac{1}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{1}{k})^2)}} X_{-\frac{d}{2}}|0,0\rangle & m > 0 \\ |0,0\rangle & m = 0 \\ \prod_{j=2}^{-m} \frac{\bar{Y}_{-k(j-\frac{1}{2})-1}}{\sqrt{\frac{1}{2}(-\frac{1}{2} - \frac{k}{8} - \frac{1}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{1}{k})^2)}} \bar{X}_{-\frac{d}{2}}|0,0\rangle & m < 0 \end{cases}$$

$$(h_m, Q_m) = \begin{cases} (-\frac{1}{2} - \frac{1}{2k} + \frac{k}{2}(m + \frac{1}{k})^2, k(m + \frac{1}{k})) & m > 0 \\ (0,0) & m = 0 \\ (-\frac{1}{2} - \frac{1}{2k} + \frac{k}{2}(m - \frac{1}{k})^2, k(m - \frac{1}{k})) & m < 0. \end{cases} \tag{3.15}$$

In the case of $d = 1$, the $c = 3$ algebra is one pair of complex free boson and fermion ($G = \sqrt{2}\psi i\partial\varphi$, $X = \psi$, $Y = \frac{1}{\sqrt{2}} i\partial\bar{\varphi}$). Although the highest weight state is the simultaneous eigenstate of L_0, I_0 and momentum of boson α_0 and $\bar{\alpha}_0$, we abbreviate this to $|h,Q\rangle$. Irreducible unitary highest weight representation exist if and only if $Q=0$ and $h(=p\bar{p}=|p|^2)\geq 0$ and these are invariant under the spectral flow with $\eta\in\mathbf{Z}$. Moreover massive representation $h>0$ is invariant under the spectral flow of the $N=2$ SCA with $\eta_{N=2}\in\mathbf{Z}$. The highest weight state of massless representation, $|0,0\rangle=|0,0\rangle^{N=2}$, is mapped to

$$
\begin{aligned}
|h_m,Q_m\rangle^{N=2} &= \begin{cases} \prod_{j=2}^{m}\frac{Y_{-1}}{\sqrt{\frac{1}{2}(j-1)}}X_{-\frac{1}{2}}|0,0\rangle & m>0 \\ |0,0\rangle & m=0 \\ \prod_{j=2}^{-m}\frac{\bar{Y}_{-1}}{\sqrt{\frac{1}{2}(j-1)}}\bar{X}_{-\frac{1}{2}}|0,0\rangle & m<0 \end{cases} \\
(h_m,Q_m) &= \begin{cases} (m-\frac{1}{2},1) & m>0 \\ (0,0) & m=0 \\ (-m-\frac{1}{2},-1) & m<0, \end{cases}
\end{aligned}
\tag{3.16}
$$

by the spectral flow of the $N=2$ SCA with $\eta_{N=2}=m\in\mathbf{Z}$.

Representation space of the $c=3d$ algebra contains only one highest weight state of the $N=2$ SCA modulo the spectral flow of the $N=2$ SCA. In this sense the $c=3d$ algebra is the smallest one among the algebras which contain $N=2$ SCA as subalgebra and have integer-shift spectral flow invariant representations.

4 Character Formulas

First we consider $d \geq 2$ case. Characters of the $c = 3d$ algebra in the various sectors are defined by

$$\begin{aligned} ch_*^{\mathrm{Sec}}(\theta,\tau) &= \mathrm{tr}_* q^{L_0-\frac{c}{24}} z^{I_0} \qquad\qquad \mathrm{Sec} = \mathrm{NS,R} \\ ch_*^{\widetilde{\mathrm{Sec}}}(\theta,\tau) &= \mathrm{tr}_*(-1)^F q^{L_0-\frac{c}{24}} z^{I_0} \quad * = \text{representation}, \end{aligned} \tag{4.1}$$

where $q = e^{2\pi i\tau}$ ($\mathrm{Im}\tau > 0$) and $z = e^{i\theta}$ and F is a fermion number operator. Due to the spectral flow and $(-1)^F = (-1)^{I_0}$ for generators, characters have quasi-periodicity in θ and their relations are

$$\begin{aligned} ch_*^{\widetilde{\mathrm{NS}}}(\theta,\tau) &= ch_*^{\mathrm{NS}}(\theta+\pi,\tau) \\ ch_{*'}^{\mathrm{R}}(\theta,\tau) &= q^{\frac{d}{8}} z^{\frac{d}{2}} ch_*^{\mathrm{NS}}(\theta+\pi\tau,\tau) \\ ch_{*'}^{\widetilde{\mathrm{R}}}(\theta,\tau) &= e^{-i\pi\frac{d}{2}} ch_{*'}^{\mathrm{R}}(\theta+\pi,\tau), \end{aligned} \tag{4.2}$$

where $*'$ is the representation corresponding to $*$ and we take the convention of fermion number such that

$$\begin{aligned} (-1)^F|h,Q\rangle^{NS} &= (-1)^Q|h,Q\rangle^{NS} \\ (-1)^F|h^R,Q^R\rangle^R &= \begin{cases} (-1)^Q|h^R,Q^R\rangle^R & Q \leq 0 \\ (-1)^{d-Q}|h^R,Q^R\rangle^R & Q > 0 \end{cases} \end{aligned} \tag{4.3}$$

(in [11] we omitted the factor $e^{-i\pi\frac{d}{2}}$). In the following we denote ch^{NS} as ch.

For $k = d-1 = 1,2,3,\cdots$ and $Q \in \mathbf{Z}$, we define the following functions:

$$f_{k,Q}(\theta,\tau) = \frac{1}{\eta(\tau)} \sum_{m\in\mathbf{Z}} q^{\frac{k}{2}(m+\frac{Q}{k})^2} z^{k(m+\frac{Q}{k})} \tag{4.4}$$

$$f'_{k,Q}(\theta,\tau) = \frac{1}{\eta(\tau)} \sum_{m\in\mathbf{Z}} \frac{(zq^{m+\frac{1}{2}})^{Q-\frac{k}{2}}}{1+zq^{m+\frac{1}{2}}} q^{\frac{k}{2}(m+\frac{1}{2})^2} z^{k(m+\frac{1}{2})}, \tag{4.5}$$

where $\eta(\tau)=q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)$. $f_{h,Q}$ and $f'_{h,Q}$ have properties

$$\begin{aligned}
f_{k,Q}(\theta,\tau) &= f_{k,Q+k\mathbf{Z}}(\theta,\tau) \\
f_{k,Q}(-\theta,\tau) &= f_{k,-Q}(\theta,\tau) \qquad (4.6) \\
f_{k,Q}(\theta+2\pi\tau,\tau) &= q^{-\frac{k}{2}}z^{-k}f_{k,Q}(\theta,\tau) \\
f'_{k,Q}(\theta+2\pi\tau,\tau) &= q^{-\frac{k}{2}}z^{-k}f'_{k,Q}(\theta,\tau) \qquad (4.7)
\end{aligned}$$

and we abbreviate $f_1=f_{1,0}$. $f_{k,Q}$ and $f_1 f'_{k,Q}$ are Laurant series of z.

By the structure of representation space discussed in section 3 eq. (3.12), character of massive representation with (h,Q), which we denote as $ch_{h,Q}(\theta,\tau)$, is a sum of the $N=2$ SCA characters with (h_m,Q_m), which are the $N=2$ SCA massive characters[14,12],

$$\begin{aligned}
ch_{h,Q}(\theta,\tau) &= \sum_m ch^{N=2}_{h_m,Q_m}(\theta,\tau) \\
&= \sum_m q^{h_m-\frac{k}{8}}z^{Q_m}\frac{f_1(\theta,\tau)}{\eta(\tau)^2} \\
&= \frac{q^{h-\frac{k}{8}-\frac{Q^2}{2k}}}{\eta(\tau)}f_1(\theta,\tau)f_{k,Q}(\theta,\tau). \qquad (4.8)
\end{aligned}$$

For massless representation with $Q>0$, we denote the character as $ch_Q(\theta,\tau)$ and it is a sum of the $N=2$ SCA characters with (h_m,Q_m) eq. (3.13), which are the $N=2$ SCA massless characters not connected to $(h=0,Q=0)$,

$$\begin{aligned}
ch_Q(\theta,\tau) &= \sum_m ch^{N=2}_{h_m,Q_m}(\theta,\tau) \\
&= \sum_m \frac{q^{h_m-\frac{k}{8}}z^{Q_m}}{1+zq^{m+\frac{1}{2}}}\frac{f_1(\theta,\tau)}{\eta(\tau)^2}
\end{aligned}$$

$$= \frac{1}{\eta(\tau)} f_1(\theta,\tau) f'_{k,Q}(\theta,\tau). \qquad (4.9)$$

For massless representation with $Q<0$, its character ch_Q is, by eq. (3.14),

$$ch_Q(\theta,\tau) = \sum_m ch^{N=2}_{h_m,Q_m}(\theta,\tau) = ch_{-Q}(-\theta,\tau). \qquad (4.10)$$

For massless representation with $Q=0$, we denote the character as $ch_{vac}(\theta,\tau)$ and it is a sum of the $N=2$ SCA characters with (h_m,Q_m) eq. (3.15), which are the $N=2$ SCA massless characters connected to $(h=0,Q=0)$,

$$\begin{aligned} ch_{vac}(\theta,\tau) &= \sum_m ch^{N=2}_{h_m,Q_m}(\theta,\tau) \\ &= \sum_m \frac{(1-q)q^{h_m-\frac{k}{8}} z^{Q_m}}{(1+zq^{m-\frac{1}{2}})(1+zq^{m+\frac{1}{2}})} \frac{f_1(\theta,\tau)}{\eta(\tau)^2} \\ &= (ch_{0,0} - ch_1 - ch_{-1})(\theta,\tau). \end{aligned} \qquad (4.11)$$

When h of massive representation reaches unitarity bound $\frac{|Q|}{2}$, massive representation splits into massless representations, which are expected from the zero mode conditions of R sector,

$$\begin{aligned} ch_{\frac{Q}{2},Q} &= ch_Q + ch_{Q+1} && Q>0 \\ ch_{0,0} &= ch_{vac} + ch_1 + ch_{-1} && Q=0. \end{aligned} \qquad (4.12)$$

Moreover we can show that $ch_{h,Q} = q^{h-\frac{Q}{2}}(ch_Q + ch_{Q+1})$ for $Q>0$ and $ch_{h,0} = q^h(ch_{vac} + ch_1 + ch_{-1})$. $Q<0$ cases are obtained from above results and $ch_{h,Q}(-\theta,\tau) = ch_{h,-Q}(\theta,\tau)$ and $ch_Q(-\theta,\tau) = ch_{-Q}(\theta,\tau)$.

Witten index is defined by

$$index_* = ch^{\widetilde{\mathrm{R}}}_{*'}(0,\tau) = q^{\frac{d}{8}} ch_*(\pi+\pi\tau,\tau) \qquad (4.13)$$

and their values are

$$index = \begin{cases} 0 & \text{massive} \\ (-1)^{d-Q} & \text{massless} \quad Q>0 \\ (-1)^{Q} & \text{massless} \quad Q<0 \\ 1+(-1)^{d} & \text{massless} \quad Q=0. \end{cases} \tag{4.14}$$

Next we will consider the modular transformation properties of characters. Under the S and T transformations, $f_{k,Q}$ and $f'_{k,Q}$ transform as[15]

$$\begin{aligned} f_{k,Q}(\theta,\tau)|_S &= e^{\frac{ik\theta^2}{4\pi\tau}} \sum_{Q'=0}^{k-1} \frac{1}{\sqrt{k}} e^{-2\pi i \frac{QQ'}{k}} f_{k,Q'}(\theta,\tau) \\ f_{k,Q}(\theta,\tau)|_T &= e^{2\pi i(\frac{Q^2}{2k}-\frac{Q}{2}-\frac{1}{24})} f_{k,Q}(\theta+\pi,\tau) \end{aligned} \tag{4.15}$$

$$\begin{aligned} f'_{k,Q}(\theta,\tau)|_S &= \sqrt{-i\tau}e^{\frac{ik\theta^2}{4\pi\tau}}\theta(k:\text{even})(-1)^Q\tfrac{i}{2}(2f'_{k,\frac{k}{2}}-f_{k,\frac{k}{2}})(\theta,\tau) \\ &\quad +\sqrt{-i\tau}e^{\frac{ik\theta^2}{4\pi\tau}}\theta(k:\text{odd})(-1)^Q f'_{k,\frac{k+1}{2}}(\theta,\tau) \\ &\quad +\sqrt{-i\tau}e^{\frac{ik\theta^2}{4\pi\tau}}\sum_{Q'=0}^{k-1} e^{-2\pi i\frac{QQ'}{k}} f_{k,Q'}(\theta,\tau) \\ &\qquad \times \int_{-\infty}^{\infty} dt \frac{1}{2}\Big(\frac{e^{2\pi(Q-\frac{k}{2})t}}{1+e^{-2\pi i\frac{Q'}{k}}e^{2\pi t}} + \frac{e^{-2\pi(Q-\frac{k}{2})t}}{1+e^{-2\pi i\frac{Q'}{k}}e^{-2\pi t}}\Big) q^{\frac{k}{2}t^2} \\ f'_{k,Q}(\theta,\tau)|_T &= e^{2\pi i(-\frac{k}{8}-\frac{1}{24})} f'_{k,Q}(\theta+\pi,\tau), \end{aligned} \tag{4.16}$$

where $f(\theta,\tau)|_S = f(\frac{\theta}{\tau},\frac{-1}{\tau})$ and $f(\theta,\tau)|_T = f(\theta,\tau+1)$ and $\theta(P)$ is a step function, $\theta(P)=1$ if the proposition P is true and $\theta(P)=0$ if P is false.

Using these and transformation formula for Dedekind eta function and Gaussian integral, massive characters transform as

$$ch_{\frac{\alpha^2}{2}+\frac{k}{8}+\frac{Q^2}{2k},Q}(\theta,\tau)|_S = e^{\frac{id\theta^2}{4\pi\tau}} \sum_{1-[\frac{d}{2}]\le Q'\le[\frac{k}{2}]} \frac{1}{\sqrt{k}} e^{-2\pi i\frac{QQ'}{k}}$$

$$
\begin{aligned}
& \times \int_{-\infty}^{\infty} d\beta \cos(2\pi\alpha\beta) ch_{\frac{\beta^2}{2}+\frac{k}{8}+\frac{Q'^2}{2k},Q'}(\theta,\tau) \\
ch_{-\frac{a^2}{2}+\frac{k}{8}+\frac{Q^2}{2k},Q}(\theta,\tau)|_S \;=\; & e^{\frac{id\theta^2}{4\pi\tau}} \sum_{1-[\frac{d}{2}]\le Q'\le[\frac{k}{2}]} \frac{1}{\sqrt{k}} e^{-2\pi i\frac{QQ'}{k}} \\
& \times \int_{-\infty}^{\infty} d\beta \cosh(2\pi a\beta) ch_{\frac{\beta^2}{2}+\frac{k}{8}+\frac{Q'^2}{2k},Q'}(\theta,\tau) \\
ch_{h,Q}(\theta,\tau)|_T \;=\; & e^{2\pi i(h-\frac{Q}{2}-\frac{d}{8})} ch^{\widetilde{\mathrm{NS}}}_{h,Q}(\theta,\tau), \qquad (4.17)
\end{aligned}
$$

where α and a are real and $\frac{a^2}{2} < \frac{k}{8} + \frac{Q^2}{2k} - \frac{|Q|}{2}$. Massless characters with $Q > 0$ transform as

$$
\begin{aligned}
ch_Q(\theta,\tau)|_S \;=\; & e^{\frac{id\theta^2}{4\pi\tau}} \theta(d:\mathrm{odd})(-1)^Q \tfrac{i}{2}(ch_{\frac{k}{2}} - ch_{-\frac{k}{2}})(\theta,\tau) \\
& + e^{\frac{id\theta^2}{4\pi\tau}} \theta(d:\mathrm{even})(-1)^Q ch_{\frac{d}{2}}(\theta,\tau) \\
& + e^{\frac{id\theta^2}{4\pi\tau}} \sum_{1-[\frac{d}{2}]\le Q'\le[\frac{k}{2}]} \frac{1}{\sqrt{k}} e^{-2\pi i\frac{QQ'}{k}} \int_{-\infty}^{\infty} d\alpha\; ch_{\frac{\alpha^2}{2}+\frac{k}{8}+\frac{Q'^2}{2k},Q'}(\theta,\tau) \\
& \times \frac{1}{2}\Big(\frac{e^{2\pi(Q-\frac{k}{2})\frac{\alpha}{\sqrt{k}}}}{1+e^{-2\pi i\frac{Q'}{k}} e^{2\pi\frac{\alpha}{\sqrt{k}}}} + \frac{e^{-2\pi(Q-\frac{k}{2})\frac{\alpha}{\sqrt{k}}}}{1+e^{-2\pi i\frac{Q'}{k}} e^{-2\pi\frac{\alpha}{\sqrt{k}}}} \Big) \\
ch_Q(\theta,\tau)|_T \;=\; & e^{2\pi i(-\frac{d}{8})} ch^{\widetilde{\mathrm{NS}}}_Q(\theta,\tau). \qquad (4.18)
\end{aligned}
$$

Transformation formulas for massless characters with $Q \le 0$ are obtained from above formulas and massive-massless relations eq. (4.12). Massive characters are divided into two classes: $h \ge \frac{k}{8} + \frac{Q^2}{2k}$ and $h < \frac{k}{8} + \frac{Q^2}{2k}$. Massive characters with $h < \frac{k}{8} + \frac{Q^2}{2k}$ do not appear in the right hand sides of above formulas and this is the common feature of the irrational theory[11].

In the case of $d = 1$, massive character is the $N = 2$ SCA character itself $ch_{h,0}(\theta,\tau) = q^h \frac{f_1(\theta,\tau)}{\eta(\tau)^2}$. Massless character is a sum of the $N = 2$ SCA massless characters with (h_m, Q_m) eq. (3.16),

$$
\begin{aligned}
ch_{vac}(\theta,\tau) &= \sum_m ch^{N=2}_{h_m,Q_m}(\theta,\tau) \\
&= \sum_m \frac{(1-q)q^{h_m}z^{Q_m}}{(1+zq^{m-\frac{1}{2}})(1+zq^{m+\frac{1}{2}})}\frac{f_1(\theta,\tau)}{\eta(\tau)^2} \\
&= \frac{f_1(\theta,\tau)}{\eta(\tau)^2}\lim_{M\to\infty}\sum_{m=-M}^{M}\left(\frac{zq^{m-\frac{1}{2}}}{1+zq^{m-\frac{1}{2}}}-\frac{zq^{m+\frac{1}{2}}}{1+zq^{m+\frac{1}{2}}}\right) \\
&= \frac{f_1(\theta,\tau)}{\eta(\tau)^2}
\end{aligned}
\tag{4.19}
$$

In this case, massive-massless relation is $ch_{0,0} = ch_{vac}$. These characters are easily calculated by a free field realization and there is no distinction between massive and massless representations.

5 Discussion

We have obtained the characters of the $c = 3d$ algebra as a sum of the $N = 2$ SCA characters. Next we will consider the decomposition of characters into characters of another subalgebra, the $c = 1$ subalgebra. Since representations of the $c = 3d$ algebra are invariant under the spectral flow with $\eta \in \mathbf{Z}$, $\eta = 1$ flow means that characters have the following property

$$
ch_*(\theta + 2\pi\tau, \tau) = q^{-\frac{d}{2}}z^{-d}ch_*(\theta,\tau). \tag{5.1}
$$

This property is checked by eq. (4.7). Hermite's lemma [11] says that, as a function of θ, Laurent series in z variable with property eq. (5.1) form a d-dimensional vector space and one can take $\{f_{d,Q}(\theta,\tau)\}$ as basis. Therefore characters are expanded into $f_{d,Q}$ as a function of z. Moreover their expansion coefficients are q-series with non-negative integer coefficients because $f_{d,Q}$ are characters of the $c=1$ subalgebra. In fact characters of the $c=1$ subalgebra are easily calculated by the free boson realization ($I(z)=\sqrt{d}i\partial\phi(z), X(z)= :e^{i\sqrt{d}\phi(z)}:$) and agree with $f_{d,Q}$:

$$ch_Q^{c=1}(\theta,\tau)=\mathrm{tr}q^{\frac{1}{2d}(I^2)_0-\frac{1}{24}}z^{I_0}=f_{d,Q}(\theta,\tau). \tag{5.2}$$

Due to the massive-massless relation eq. (4.12), we have only to know decomposition of $d-1$ massive characters $ch_{h,Q}$ and one massless character $ch_{[\frac{d}{2}]}$. Using the formula

$$f_{k_1,Q_1}(\theta,\tau)f_{k_2,Q_2}(\theta,\tau)=\sum_{j=0}^{k_1+k_2-1}f_{k_1k_2(k_1+k_2),k_1k_2j+k_1Q_2-k_2Q_1}(0,\tau)f_{k_1+k_2,k_2j+Q_1+Q_2}(\theta,\tau), \tag{5.3}$$

$ch_{h,Q}$ are expanded into $f_{d,Q}$ as

$$ch_{h,Q}(\theta,\tau)=\frac{q^{h-\frac{k}{8}-\frac{Q^2}{2k}}}{\eta(\tau)}\sum_{j=0}^{k}f_{kd,kj+Q}(0,\tau)f_{d,kj+Q}(\theta,\tau). \tag{5.4}$$

Coefficients of $f_{d,Q'}(\theta,\tau)$ in above expansion are q-series with non-negative integer coefficients. Since the expansion is ensured by Hermite's lemma, decomposition of massless character is obtained by comparing coefficients of z^Q. For even d, $ch_{[\frac{d}{2}]}$ is expanded as,

$$\begin{aligned} ch_{\frac{d}{2}}(\theta,\tau) &= A_0^{(k)}(\tau)f_{d,0}(\theta,\tau)+\sum_{Q=1}^{\frac{d}{2}-1}A_Q^{(k)}(\tau)(f_{d,Q}+f_{d,-Q})(\theta,\tau)+A_{\frac{d}{2}}^{(k)}(\tau)f_{d,\frac{d}{2}}(\theta,\tau) \\ A_Q^{(k)}(\tau) &= \frac{1}{\eta(\tau)^2}\sum_{\substack{m\in\mathbf{Z}+\frac{1}{2}-\frac{Q}{d}\\ n\in\mathbf{Z}+\frac{1}{2}\\ (m+\frac{Q}{d})(n-dm)>0}} sgn(n)(-1)^{n-dm-\frac{1}{2}}q^{\frac{1}{2}n^2-\frac{d}{2}m^2} \end{aligned} \tag{5.5}$$

and for odd d

$$
\begin{aligned}
ch_{\frac{k}{2}}(\theta,\tau) &= \frac{1}{2}ch_{\frac{k}{4},\frac{k}{2}}(\theta,\tau)+\frac{1}{2}\sum_{Q=1}^{\frac{k}{2}}A_Q^{(k)}(\tau)(f_{d,Q}-f_{d,-Q})(\theta,\tau) \\
A_Q^{(k)}(\tau) &= \frac{1}{\eta(\tau)^2}\sum_{\substack{m\in\mathbf{Z}+\frac{1}{2}-\frac{Q}{d}\\ n\in\mathbf{Z}+\frac{1}{2}\\ (m+\frac{Q}{d})(n-dm+\frac{1}{2})>0}} sgn(n)(-1)^{n-dm}q^{\frac{1}{2}n^2-\frac{d}{2}m^2} \\
&\quad -(Q\to -Q).
\end{aligned}
\tag{5.6}
$$

$A_Q^{(k)}$ resembles the Hecke's indefinite quadratic form[16]. Although $A_Q^{(k)}(\tau)$ seems to have negative coefficients, it has non-negative coefficients if one expands $A_Q^{(k)}(\tau)$ into q-series. (Note that substituting $\theta=\pi-\pi\tau$ shows $A_1^{(2)}(\tau)$ is equal to 1.) We remark that, instead of $f_{d,Q}$, we can take $d-1$ massive characters $ch_{h,Q}$ and one massless character $ch_{[\frac{d}{2}]}$ as basis functions, so that characters of the certain algebra, which have integer-shift spectral flow invariant representations (i.e. property eq. (5.1)), can be expanded into the $c=3d$ characters. Futhermore, if it contains the $N=2$ SCA as subalgebra, its expansion coefficients are non-negative integers because representation of the $c=3d$ algebra contains only one highest weight state of the $N=2$ SCA modulo the spectral flow of the $N=2$ SCA.

Next we will consider the decomposition of characters of the $N=4$ SCA with $c=6\cdot\frac{d}{2}$ (d:even) into the $c=3d$ characters. The $N=4$ SCA contains the $N=2$ SCA as subalgebra and its representations are invariant under the integer-shift spectral flow. By the remark in the last paragraph, the $N=4$ SCA characters are expanded into the $c=3d$ characters with non-negative integer coefficients. This is interpreted as follows: the $N=4$

SCA and the $c = 3d$ algebra are symmetries of the non-linear σ model on hyper-Kähler and Ricci-flat Kähler manifold respectively and hyper-Kähler manifolds are Ricci-flat Kähler manifolds, so that the $N = 4$ SCA contains the $c = 3d$ algebra as subalgebra implicitly. The $c = 6 \cdot \frac{d}{2}$ $N = 4$ SCA characters [12] are defined by $ch_*^{N=4}(\theta, \tau) = \mathrm{tr}_* q^{L_0 - \frac{c}{24}} z^{2J_0^3}$ and we have only to consider $\frac{d}{2}$ massive characters $ch_{h,l}^{N=4}$ (l stands for isospin) and one massless character $ch_l^{N=4}$ with $l = \frac{d}{4}$ because of massive-massless relation of the $N = 4$ SCA characters. These functions span the $\frac{d}{2} + 1$ dimensional subspace which consists of even function of θ.(For $d = 2$, $\frac{d}{2} + 1$ is equal to d. In fact the $N = 4$ SCA with $c = 6$ is isomorphic to the $c = 6$ algebra.) For example, in the case of $d = 4$, the $N = 4$ SCA massive characters are expanded as

$$\begin{aligned}
ch_{h,0}^{N=4}(\theta, \tau) &= \sum_{n\geq 0} a_n ch_{h+n,0}(\theta, \tau) + \sum_{n \geq 0} b_n (ch_{h+n+\frac{1}{2},1} + ch_{h+n+\frac{1}{2},-1})(\theta, \tau) \\
ch_{h,\frac{1}{2}}^{N=4}(\theta, \tau) &= \sum_{n\geq 0} a'_n ch_{h+n+\frac{1}{2},0}(\theta, \tau) + \sum_{n \geq 0} b'_n (ch_{h+n,1} + ch_{h+n,-1})(\theta, \tau),
\end{aligned} \tag{5.7}$$

where positive integers a_n, b_n, a'_n and b'_n are given by

$$\begin{aligned}
\sum_{n\geq 0} a_n q^n &= \frac{\sum_m q^{3m^2}}{\prod_{n\geq 1}(1-q^n)}, \qquad \sum_{n\geq 0} b_n q^n = \frac{\sum_m q^{3m^2+2m}}{\prod_{n\geq 1}(1-q^n)} \\
\sum_{n\geq 0} a'_n q^n &= \frac{\sum_m q^{3m^2+3m}}{\prod_{n\geq 1}(1-q^n)}, \qquad \sum_{n\geq 0} b'_n q^n = \frac{\sum_m q^{3m^2+m}}{\prod_{n\geq 1}(1-q^n)}.
\end{aligned}$$

Massless character is expanded as

$$ch_1^{N=4}(\theta, \tau) = ch_2(\theta, \tau) + \sum_{n\geq 1} a''_n ch_{n,0}(\theta, \tau) + \sum_{n\geq 2} b''_n (ch_{n+\frac{1}{2},1} + ch_{n+\frac{1}{2},-1})(\theta, \tau)$$

$$\sum_{n\geq 1} a''_n q^n = \frac{q^{\frac{3}{8}}}{\eta(\tau)} \sum_{\delta=0,1} \sum_{\substack{m\in \mathbf{Z}+\frac{1}{3} \\ n\in\mathbf{Z}+\frac{3}{2}m \\ m(n-\frac{3}{2}m-\delta+\frac{1}{2})>0}} sgn(m) q^{3n^2-\frac{3}{4}m^2}$$

$$\sum_{n\geq 2} b''_n q^n = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \sum_{\delta=1,2} \sum_{\substack{m\in \mathbf{Z}+\frac{1}{3} \\ n\in\mathbf{Z}+\frac{3}{2}m+\frac{\delta}{3} \\ m(n-\frac{3}{2}m-\frac{\delta}{3}+\frac{1}{2})>0}} sgn(m) q^{3n^2-\frac{3}{4}m^2}, \tag{5.8}$$

where a''_n and b''_n are positive integers. The $N=4$ SCA characters are infinite sums of the $c=3d$ characters, namely, representations of the $N=4$ SCA with $c\geq 12$ are infinitely reducible with respect to the $c=3d$ algebra.

In order to discuss the properties of Calabi-Yau manifold, we must find the modular invariant partition functions. Unfortunately this is very hard because the $c=3d$ algebra has infinite many primary fields and modular transformation formulas contain integrals. This irrational theory, however, can be converted to rational one by combining infinite many characters to some orbits[6,17,8]. These issues and fusion rules are now under investigation.

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