UT-543

April 1989

# Character Formulas of an Extended Superconformal Algebra Relevant to String Compactification

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**Abstract**

We present the character formulas of the $c = 9$ algebra, which is an extended algebra obtained by adding the spectral flow generators to the $N = 2$ superconformal algebra.

We also discribe their modular transformation properties.

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# 1 Introduction

In recent years connections between the space-time symmmetry and underlying world-sheet symmetry have been receiving much attention. Especially relations between the space-time supersymmetry and world-sheet superconformal symmetry were studied in connection with string compactification [1,2,3,4,5]. It was shown that world-sheet $N = 2$ superconformal symmetry and $U(1)$ charge quantization are necessary and sufficient in order to realize the $N = 1$ space-time supersymmetry. Since $N = 2$ superconformal algebra has an automorphism (spectral flow) [6], Neveu-Schwarz(NS) and Ramond(R) sectors are mapped onto each other by the spectral flow. This spectral flow is considered as the space-time supersymmetry. $U(1)$ charge quantization condition is the generalization of the GSO projection. If compactified space can be considered as a manifold, $N = 1$ space-time supersymmetry implies that the compactified space is a $\tilde{n}$-dimensional complex manifold with $SU(\tilde{n})$ holonomy, in other words, Ricci-flat Kähler manifold, namely Calabi-Yau manifold[7,8]. This manifold possesses a unique covariantly constant spinor, which generates $N = 1$ space-time supersymmetry transformation, and an (anti-)holomorphic $\tilde{n}$-form.

The $N = 2$ superconformal symmetry plus the $U(1)$ charge quantization condition suggest that the $N = 2$ superconformal symmetry becomes enhanced. We have studied the extension of the $N = 2$ superconformal algebra by adding the spectral flow generators which correspond to the (anti-)holomorphic $\tilde{n}$-form[9,10]. For $\tilde{n} = 2$, the extended algebra becomes the $N = 4$ superconformal algebra[9]. In the case of $\tilde{n} = 3$, we have determined the structure of the extended algebra[10]. We call it the $c = 9$ algebra or $\tilde{n} = 3$ algebra. Since the $\tilde{n} = 3$ algebra is regarded as the generalization of the $\tilde{n} = 2$ algebra (i.e. $N = 4$ superconformal algebra), properties of the $c = 9$ algebra resemble those of the $N = 4$ superconformal algebra. Using this fact we have conjectured character formulas[11].

In this paper we present character formulas of the $c = 9$ algebra and prove them. We also discuss the modular transformation properties of characters. In section 2 we review the $c = 9$ algebra. In section 3 we present character formulas and prove them in the appendix. In section 4 we describe the modular transformation formulas and in section 5 we discuss the Gepner's models and curious modular properties of the characters.

## 2 Review of the $c = 9$ algebra

In this section we review the $c = 9$ algebra[10]. We follow the notation of [10].

We start with two properties of $N = 2$ superconformal algebra(SCA). One property is the decoupling of the $U(1)$ Kac-Moody current $I(z)$[3], namely, when $I(z)$ is bosonized

$$
\begin{aligned}
I(z) &= \sqrt{\frac{c}{3}} i\partial\phi(z) && (2.1)\\
\phi(z) &= q - ip\log z + i\sum_{n\in\mathbf{Z},n\neq 0}\frac{\alpha_n}{n}z^{-n}, && (2.2)
\end{aligned}
$$

other fields are decomposed into $\phi$-dependent part and $\phi$-independent part (we denote them by $\tilde{}$),

$$
\begin{aligned}
T(z) &= \frac{1}{2} : (i\partial\phi(z))^2 : + \tilde{T}(z)\\
G(z) &= : e^{i\sqrt{\frac{3}{c}}\phi(z)} : \tilde{G}(z)\\
\bar{G}(z) &= : e^{-i\sqrt{\frac{3}{c}}\phi(z)} : \tilde{\bar{G}}(z). && (2.3)
\end{aligned}
$$

Here $\tilde{G}(z)$ is the so-called parafermion field, a primary field with $\tilde{h}=\frac{3}{2}(1-\frac{1}{c})$ w.r.t. $\tilde{T}(z)$ with $\tilde{c} = c-1$. In general, $A(z)$ with $U(1)$ charge $Q$ is expressed as $A(z) =: e^{iQ\sqrt{\frac{3}{c}}\phi(z)}P_A(z) : \tilde{A}(z)$, where $P_A(z)$ is some differential polynomial of

$I(z)$. The other property is the spectral flow [6], i.e. under the transformation

$$\left\{ \begin{array}{lcl} L'_n &=& L_n + \eta I_n + \frac{c}{6}\eta^2 \delta_{n0} \\ I'_n &=& I_n + \frac{c}{3}\eta \delta_{n0} \\ G'_n &=& G_{n+\eta} \\ \bar{G}'_n &=& \bar{G}_{n-\eta}, \end{array} \right. \tag{2.4}$$

the algebra is invariant for an arbitrary real parameter $\eta$. This transformation is induced by the unitary transformation (momentum shift)

$$\begin{aligned} A'(z) &= U_\eta^\dagger A(z) U_\eta \qquad A = T, I, G, \bar{G} \\ U_\eta &= e^{i\eta\sqrt{\frac{c}{3}}q}. \end{aligned} \tag{2.5}$$

In general, $A(z)$=: $e^{iQ\sqrt{\frac{3}{c}}\phi(z)}$ : $\tilde{A}(z)$ is transformed to $A'(z)$=$U_\eta^\dagger A(z) U_\eta$=$z^{Q\eta}A(z)$, i.e. $A'_n$=$A_{n+Q\eta}$. For the half shift spectral flow ($\eta \in \mathbf{Z}+\frac{1}{2}$), NS sector is mapped to R sector and R to NS. For the integral shift spectral flow ($\eta \in \mathbf{Z}$), NS is mapped to NS and R to R, but the representation is changed. For example, in NS sector,

$$(h, Q)^{N=2} \xrightarrow{\eta=\pm 1} \left\{ \begin{array}{ll} (h \pm Q + \frac{1}{2}(\frac{c}{3} - 1), Q \pm (\frac{c}{3} - 1))^{N=2} & 2h \pm Q \neq 0 \\ (h \pm Q + \frac{c}{6}, Q \pm \frac{c}{3})^{N=2} & 2h \pm Q = 0. \end{array} \right. \tag{2.6}$$

In the case of $c = 3\tilde{n}$, vacuum state is mapped onto

$$(h = 0, Q = 0)^{N=2} = (0, 0)^{N=2}_{NS} \xrightarrow{\eta=\pm\frac{1}{2}} (\frac{\tilde{n}}{8}, \pm\frac{\tilde{n}}{2})^{N=2}_R \xrightarrow{\eta=\pm\frac{1}{2}} (\frac{\tilde{n}}{2}, \pm\tilde{n})^{N=2}_{NS}, \tag{2.7}$$

by the spectral flow with $\eta = \pm\frac{1}{2}, \pm 1$. So in [10] we considered the extension of the $N = 2$ SCA by the addition of the operators $X(z)$, $\bar{X}(z)$ with $(h, Q)^{N=2}$ =$(\frac{\tilde{n}}{2}, \pm\tilde{n})$, which we call the spectral flow generators, and their superpartners $Y, \bar{Y}$ with $(\frac{\tilde{n}+1}{2}, \pm(\tilde{n} - 1))$. For $\tilde{n} = 3$, associativity and closure determine the extended algebra and center $c$ uniquely:

$$
\left\{
\begin{array}{rcl}
c=9 & & \\
[L_n, L_m] & = & (n-m)L_{n+m} + \frac{3}{4}n(n^2-1)\delta_{n+m,0} \\
\{G_n, \bar{G}_m\} & = & 3(n^2-\frac{1}{4})\delta_{n+m,0} + (n-m)I_{n+m} + 2L_{n+m} \\
[I_n, I_m] & = & 3n\delta_{n+m,0} \\
\{G_n, G_m\} & = & \{\bar{G}_n, \bar{G}_m\} = 0
\end{array}
\right.
$$
$$
\left\{
\begin{array}{rclcccccccccc}
 & & & \phi & = & I & G & \bar{G} & X & \bar{X} & Y & \bar{Y} \\
[L_n, \phi_m] & = & ((h-1)n-m)\phi_{n+m} & h & = & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 2 & 2 \\
[I_n, \phi_m] & = & Q\phi_{n+m} & Q & = & * & 1 & -1 & 3 & -3 & 2 & -2
\end{array}
\right.
$$
$$
\left\{
\begin{array}{rclrcl}
\{G_n, X_m\} & = & 0 & \{G_n, \bar{X}_m\} & = & 2\bar{Y}_{n+m} \\
\{\bar{G}_n, X_m\} & = & 2Y_{n+m} & \{\bar{G}_n, \bar{X}_m\} & = & 0 \\
[G_n, Y_m] & = & (2n-m)X_{n+m} & [G_n, \bar{Y}_m] & = & 0 \\
[\bar{G}_n, Y_m] & = & 0 & [\bar{G}_n, \bar{Y}_m] & = & (2n-m)\bar{X}_{n+m}
\end{array}
\right.
$$
$$
\left\{
\begin{array}{rcl}
\{X_n, \bar{X}_m\} & = & (n^2-\frac{1}{4})\delta_{n+m,0} + (n-m)I_{n+m} + (I^2)_{n+m} \\
[X_n, \bar{Y}_m] & = & (n+\frac{1}{2})G_{n+m} + (IG)_{n+m} \\
[\bar{X}_n, Y_m] & = & (n+\frac{1}{2})\bar{G}_{n+m} - (I\bar{G})_{n+m} \\
[Y_n, \bar{Y}_m] & = & \frac{1}{2}n(n^2-1)\delta_{n+m,0} + \frac{1}{2}(n(n+1)+m(m+1))I_{n+m} \\
 & & +\frac{1}{4}(n-m)(I^2)_{n+m} - (m+1)L_{n+m} + (IT)_{n+m} - \frac{1}{2}(G\bar{G})_{n+m} \\
\{X_n, X_m\} & = & \{\bar{X}_n, \bar{X}_m\} = [X_n, Y_m] = [\bar{X}_n, \bar{Y}_m] = [Y_n, Y_m] = [\bar{Y}_n, \bar{Y}_m] = 0.
\end{array}
\right.
\tag{2.8}
$$

Here $(AB)_n = \sum_{p\le -h_A} A_p B_{n-p} + (-1)^{AB} \sum_{p>-h_A} B_{n-p}A_p$ except for $(G\bar{G})_n$ in R sector (see [10]). We call this the $c=9$ algebra or $\tilde{n}=3$ algebra. The $c=9$ algebra is not a Lie algebra but an analogue of the $W$-algebra[12]. We note that the spectral flow generator $X(z)$ is "square" of the "true" spectral flow generator $\Sigma(z)$ of [3] because the decopling of $U(1)$ current implies $X(z) = \sqrt{2} : e^{i\sqrt{3}\phi(z)} :$ and $\Sigma(z) =: e^{i\frac{\sqrt{3}}{2}\phi(z)} :$. $\Sigma$ maps NS to R and R to NS, so $X$ maps

NS to NS and R to R.

Properties of the $c = 9$ algebra:

1. "degenerate" conditions.

   Jacobi identity requires some operator relations and among them

   $$(IX)(z) = \partial X(z), \tag{2.9}$$

   and its hermitian conjugate $(I\bar{X})(z) = -\partial\bar{X}(z)$ are the most basic ones. Here $(AB)(z) = \oint_z \frac{dx}{2\pi i}\frac{1}{x-z}A(x)B(z)$.

2. subalgebras.

   The $c = 9$ algebra containes two $N = 2$ SCA as subalgebras

   $$c = 9\ N = 2 \text{ SCA generated by} \quad T, I, G, \bar{G} \tag{2.10}$$

   $$c = 1\ N = 2 \text{ SCA generated by} \quad \frac{1}{6}(I^2), \frac{1}{3}I, \frac{1}{\sqrt{3}}X, \frac{1}{\sqrt{3}}\bar{X}. \tag{2.11}$$

   The second subalgebra corresponds to $SU(2)$ Kac-Moody algebra of $N = 4$ SCA which is considerd as "$\tilde{n} = 2$ algebra".

3. spectral flow.

   The $c = 9$ algebra is invariant under the transformation

   $$\left\{\begin{array}{lcl} L'_n &=& L_n + \eta I_n + \frac{3}{2}\eta^2\delta_{n0} \\ I'_n &=& I_n + 3\eta\delta_{n0} \\ \phi'_n &=& \phi_{n+Q\eta} \end{array}\right. \qquad \begin{array}{lcrrrrrr} \\ \phi &=& G & \bar{G} & X & \bar{X} & Y & \bar{Y} \\ Q &=& 1 & -1 & 3 & -3 & 2 & -2. \end{array} \tag{2.12}$$

4. unitary irreducible representation.

   The highest weight states of the $c = 9$ algebra are labeled by conformal weight $h$ and $U(1)$ charge $Q$. Unitary irreducible representation are as follows: (see [10] about other zero mode conditions)

$$\begin{array}{llllll}
\text{massless} & \text{representation} & & & & \\
\bullet\text{NS1} & h=0 \;\; Q=0 & \bullet\text{R1} & h=\frac{3}{8} & Q=\pm\frac{3}{2} & \\
\bullet\text{NS2} & h=\frac{1}{2} \;\; Q=1 & \bullet\text{R2} & h=\frac{3}{8} & Q=-\frac{1}{2} & \\
\bullet\text{NS3} & h=\frac{1}{2} \;\; Q=-1 & \bullet\text{R3} & h=\frac{3}{8} & Q=\frac{1}{2} & \\
\text{massive} & \text{representation} & & & & \\
\bullet\text{NS4} & h>0 \;\; Q=0 & \bullet\text{R4} & h>\frac{3}{8} & Q=\pm\frac{3}{2},\pm\frac{1}{2} & \\
\bullet\text{NS5} & h>\frac{1}{2} \;\; Q=\pm 1 & \bullet\text{R5} & h>\frac{3}{8} & Q=\pm\frac{1}{2}. &
\end{array} \tag{2.13}$$

$Q$ is determined by subalgebra eq. (2.11) just like as $SU(2)$ Kac-Moody algebra of the $N=4$ SCA determines the isospin. Each state of the representation in NS(R) sector has integer(half odd integer) $U(1)$ charge respectively. The half shift flow ($\eta \in \mathbf{Z}+\frac{1}{2}$, eq. (2.12)) connects NS and R sector:NSi $\leftrightarrow$ Ri (i=1,$\cdots$,5). By the integral shift flow ($\eta \in \mathbf{Z}$, eq. (2.12)), representations come back to the same representations. This is the desired property because we consider the spectral flow as the space-time SUSY, which , by twice operations, takes the representation to the initial one.

# 3 Character Formulas of the $c=9$ Algebra

Characters of the $c=9$ algebra are defined by

$$ch_*(\theta,\tau) = \mathrm{tr}_* q^{L_0-\frac{c}{24}} z^{I_0} \;, \;\; (*=\text{NSi,Ri}) \;, \quad q=e^{2\pi i\tau} \;, \;\; z=e^{i\theta}, \tag{3.1}$$

where $\tau$ is a modular parameter of torus ($\mathrm{Im}\tau > 0$) and $\theta$ is a complex parameter. Due to the spectral flow, characters have quasi-periodicity in $\theta$. The spectral flow with $\eta=\frac{1}{2}$ implies

$$ch_{\mathrm{NSi}}(\theta+\pi\tau,\tau) = q^{-\frac{3}{8}} z^{-\frac{3}{2}} ch_{\mathrm{Ri}}(\theta,\tau) \quad (\mathrm{i}=1,\cdots,5), \tag{3.2}$$

so in the following we will consider NS sector only. Considering the sign of the $U(1)$ charge it is clear that

$$\begin{aligned} ch_{\mathrm{NSi}}(-\theta,\tau) &= ch_{\mathrm{NSi}}(\theta,\tau) \quad (\mathrm{i}=1,4,5) \\ ch_{\mathrm{NS2}}(-\theta,\tau) &= ch_{\mathrm{NS3}}(\theta,\tau). \end{aligned} \tag{3.3}$$

Since bosonic(fermionic) generators have even(odd) $U(1)$ charge, i.e. $(-1)^F = (-1)^{I_0}$, characters of $\widetilde{\mathrm{NS}}$ and $\widetilde{\mathrm{R}}$ sector are

$$\begin{aligned} ch_{\widetilde{\mathrm{NS}}}(\theta,\tau) &= \mathrm{tr}_{\mathrm{NS}}(-1)^F q^{L_0-\frac{c}{24}} z^{I_0} = ch_{\mathrm{NS}}(\theta+\pi,\tau) \\ ch_{\widetilde{\mathrm{R}}}(\theta,\tau) &= ch_{\mathrm{R}}(\theta+\pi,\tau). \end{aligned} \tag{3.4}$$

The spectral flow with $\eta = 1$ implies

$$ch_{\mathrm{NSi}}(\theta+2\pi\tau,\tau) = q^{-\frac{3}{2}} z^{-3} ch_{\mathrm{NSi}}(\theta,\tau) \quad (\mathrm{i}=1,\cdots,5), \tag{3.5}$$

because of invariance of representations under the integral shift spectral flow. Characters in NS sector are Laurant serieses in $z$ variable since each state has integral $U(1)$ charge. Applying these two properties to Hermite's lemma[1] $ch_{\mathrm{NSi}}(\theta,\tau)$ are expanded into three basis functions $f_Q(\theta,\tau)$:

$$f_Q(\theta,\tau) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbf{Z}} q^{\frac{3}{2}(n+\frac{Q}{3})^2} z^{3(n+\frac{Q}{3})} \quad (Q = 0, \pm 1) \tag{3.6}$$

as function in $z$. Here $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n)$. In other words, this expansion is nothing but the decomposition of $ch_{\mathrm{NSi}}$ into characters of subalgebra eq. (2.11) because $f_0(\theta,\tau) = ch^{N=2,c=1}_{h=0,Q=0}(3\theta,\tau)$ and $f_{\pm 1}(\theta,\tau) = ch^{N=2,c=1}_{h=\frac{1}{6},Q=\pm\frac{1}{3}}(3\theta,\tau)$. Using this fact, properties of the modular invariant partition functions have been studied [9]. Authors of [9] showed that cosmological constant vanishes

[1]Fix $a = 1,2,3,\cdots$ and $0 \leq b \leq a$ and $\delta = \pm 1$. If $\Phi(z) = \Phi(z,q)$ is a Laurant series in $z$ and $\Phi$ satisfies $\Phi(zq,q) = \frac{\delta}{z^a q^{\frac{b}{2}}}\Phi(z,q)$, then $\{\Phi(z)\}$ is a $a$-dimensional vector space and we can take basis, e.g. $z^\rho \sum_{n\in\mathbf{Z}} \delta^n z^{an} q^{\frac{a}{2}n^2+(\rho+\frac{b-a}{2})n} \quad (\rho = 0,1,2,\cdots,a-1)$.

This is easily proved by expressing $\Phi$ as $\Phi(z) = \sum a_n z^n$.

in heterotic string compactification due to supersymmetry in the right moving sector. They also showed in SST II (non-linear $\sigma$ model) one can vary Euler numbers by 4. Since there is no geometrical meaning for such a procedure, so moduli of the string compactification with $N=1$ space-time supersymmetry (i.e. modular invariant partition function of the $c=9$ algebra) seems to be larger than that of Calabi-Yau manifold.

Next we will consider the decomposition of $ch_{\rm NSi}$ into characters of the another subalgebra eq. (2.10). In the following $N=2$ means $N=2$ SCA with $c=9$. We take the representation NS4 for illustration. First we note that the highest weight state of the $c=9$ algebra $|h,0\rangle$ is also the highest weight state of the $N=2$ SCA $|h,0\rangle^{N=2}$, i.e. $|h,0\rangle^{N=2}$=$|h,0\rangle$, and this state $|h,0\rangle^{N=2}$ is mapped to

$$|h,0\rangle^{N=2} \xrightarrow{\eta_{N=2}=m} |h+m^2,2m\rangle^{N=2}, \tag{3.7}$$

by the spectral flow of $N=2$ SCA with $\eta_{N=2}=m(\in \mathbf{Z})$ (see eq. (2.6)). In the $N=2$ SCA these states $|h+m^2,2m\rangle^{N=2}$ are not connected by any generators. In the $c=9$ algebra, however, these are connected by $X,\bar{X}$ $(Y,\bar{Y})$. In fact these state with unit norm are given by

$$|h+m^2,2m\rangle^{N=2} = \begin{cases} \prod_{j=1}^{m} \frac{Y_{-(2j-1)}}{\sqrt{h+j(j-1)}}|h,0\rangle & m\geq 1 \\ |h,0\rangle & m=0 \\ \prod_{j=1}^{-m} \frac{\bar{Y}_{-(2j-1)}}{\sqrt{h+j(j-1)}}|h,0\rangle & m\leq -1. \end{cases} \tag{3.8}$$

From this property we have called $X,\bar{X}$ $(Y,\bar{Y})$ as the spectral flow generators. eq. (3.8) means representation space of the $c=9$ algebra, $V_{\rm NS4}$, contains representation space of the $N=2$ SCA $V^{N=2}_{h+m^2,2m}$. Moreover we can prove the following result:

**Theorem** $$V_{\rm NS4} = \bigoplus_{m\in\mathbf{Z}} V^{N=2}_{h+m^2,2m}. \tag{3.9}$$

This has been conjectured [13,11] because for $\tilde{n}=2$ algebra, i.e. $N=4$ SCA with $c=6$, $N=4$ character with $(h,I)$ is sum of $N=2$ characters with $(h', Q')$, where $(h', Q')$ is mapped from $(h, Q = 2I)$ by the integral shift $N = 2$ spectral flow[14]. To prove eq. (3.9), it is sufficient to show that any state of $V_{\mathrm{NS4}}$, $(N=2)X\cdots XY\cdots Y\bar{X}\cdots\bar{X}\bar{Y}\cdots\bar{Y}|h,0\rangle$, is a descendant of $|h+m^2, 2m\rangle^{N=2}$, i.e. $(N=2)|h+m^2, 2m\rangle^{N=2}$ or has zero norm. Here $(N=2)$ denotes creation operators composed of $T, I, G, \bar{G}$. The proof is given in the appendix.

Characters of the $N = 2$ SCA with $c > 3$ are classified to three types: massive and two massless[15,16,14]. $ch^{N=2}_{h+m^2,2m}$ is a massive character. Now the character of representation NS4 is easily derived from eq. (3.9)

$$
\begin{aligned}
& ch_{\mathrm{NS4}}(\theta,\tau) \\
= & \sum_{m\in\mathbf{Z}} q^{h+m^2-\frac{9}{24}} z^{2m} \prod_{n\geq 1} \frac{(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2} \\
= & \frac{q^{h-\frac{1}{4}}}{\eta(\tau)} \frac{\vartheta_3(2\theta,2\tau)}{\eta(\tau)} \frac{\vartheta_3(\theta,\tau)}{\eta(\tau)} \\
= & \frac{q^{h-\frac{1}{4}}}{\eta(\tau)} \left( \frac{\sum_{n\in\mathbf{Z}} q^{3n^2}}{\eta(\tau)} f_0(\theta,\tau) + \frac{\sum_{n\in\mathbf{Z}} q^{3(n-\frac{1}{3})^2}}{\eta(\tau)} f_+(\theta,\tau) \right),
\end{aligned}
\tag{3.10}
$$

where $f_\pm(\theta,\tau) = (f_1 \pm f_{-1})(\theta,\tau)$. Last equation is checked by Schröter's lemma:

$$
\begin{aligned}
& \sum_{m\in\mathbf{Z}} z_1^m q^{\frac{a}{2}m^2} \times \sum_{n\in\mathbf{Z}} z_2^n q^{\frac{b}{2}n^2} \qquad\qquad (a,b = 1,2,3,\cdots) \\
= & \sum_{\rho=0}^{a+b-1} z_2^\rho q^{\frac{b}{2}\rho^2} \sum_{m\in\mathbf{Z}} z_1^{-bm} z_2^{am} q^{\frac{1}{2}ab(a+b)m^2+ab\rho m} \times \sum_{n\in\mathbf{Z}} z_1^n z_2^n q^{\frac{a+b}{2}n^2+b\rho n}.
\end{aligned}
\tag{3.11}
$$

For all other representations, we can prove similar results like as eq. (3.9). For NS5, the highest state $|h,1\rangle = |h,1\rangle^{N=2}$ is mapped to $|h+m^2+m, 2m+1\rangle^{N=2}$ by the spectral flow of $N = 2$ SCA with $\eta_{N=2} = m (\in \mathbf{Z})$. In the representation space of the $c = 9$ algebra, these states are expressed as

$$
|h+m^2+m, 2m+1\rangle^{N=2} = \begin{cases} \prod_{j=1}^{m} \frac{Y_{-2j}}{\sqrt{h+j^2-\frac{1}{2}}} |h,1\rangle & m \geq 0 \\ \prod_{j=1}^{-m-1} \frac{\bar{Y}_{-2j}}{\sqrt{h+j^2-\frac{1}{2}}} \times \frac{\bar{Y}_0}{\sqrt{h-\frac{1}{2}}} |h,1\rangle & m \leq -1, \end{cases}
\tag{3.12}
$$

where $\prod_{j=n}^{n-1} * = 1$. These are massive characters of $N=2$ SCA. $ch_{\mathrm{NS5}}$ is

$$
\begin{aligned}
& ch_{\mathrm{NS5}}(\theta,\tau) \\
= & \sum_{m\in\mathbf{Z}} q^{h+m^2+m-\frac{9}{24}} z^{2m+1} \prod_{n\geq 1} \frac{(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2} \\
= & \frac{q^{h-\frac{1}{2}}}{\eta(\tau)} \frac{\vartheta_2(2\theta,2\tau)}{\eta(\tau)} \frac{\vartheta_3(\theta,\tau)}{\eta(\tau)} \\
= & \frac{q^{h-\frac{1}{2}}}{\eta(\tau)} \left( \frac{\sum_{n\in\mathbf{Z}+\frac{1}{2}} q^{3n^2}}{\eta(\tau)} f_0(\theta,\tau) + \frac{\sum_{n\in\mathbf{Z}+\frac{1}{2}} q^{3(n-\frac{1}{3})^2}}{\eta(\tau)} f_+(\theta,\tau) \right).
\end{aligned}
\tag{3.13}
$$

For NS1, the highest weight state $|0,0\rangle = |0,0\rangle^{N=2}$ is mapped to

$$
\begin{cases}
|m^2+m-\frac{1}{2}, 2m+1\rangle^{N=2} = \prod_{j=2}^{m} \frac{Y_{-2j}}{\sqrt{j^2-1}} \times \frac{X_{-3/2}}{\sqrt{2}}|0,0\rangle & m\geq 1 \\
|0,0\rangle^{N=2} = |0,0\rangle & m=0 \\
|m^2-m-\frac{1}{2}, 2m-1\rangle^{N=2} = \prod_{j=2}^{-m} \frac{\bar{Y}_{-2j}}{\sqrt{j^2-1}} \times \frac{\bar{X}_{-3/2}}{\sqrt{2}}|0,0\rangle & m\leq -1
\end{cases}
\tag{3.14}
$$

and these are massless characters of $N=2$ SCA connected to $(0,0)^{N=2}$. $ch_{\mathrm{NS1}}$ is

$$
\begin{aligned}
ch_{\mathrm{NS1}}(\theta,\tau) &= \sum_{m\in\mathbf{Z}} \frac{(1-q)q^{m^2+m-\frac{1}{2}-\frac{9}{24}} z^{2m+1}}{(1+zq^{m-\frac{1}{2}})(1+zq^{m+\frac{1}{2}})} \prod_{n\geq 1} \frac{(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2} \\
&= ch_{\mathrm{NS4}}^{h=0}(\theta,\tau) - ch_{\mathrm{NS5}}^{h=\frac{1}{2}}(\theta,\tau).
\end{aligned}
\tag{3.15}
$$

For NS2, $|\frac{1}{2},1\rangle = |\frac{1}{2},1\rangle^{N=2}$ is mapped to

$$
\begin{cases}
|m^2+m+\frac{1}{2}, 2m+1\rangle^{N=2} = \prod_{j=1}^{m} \frac{Y_{-2j}}{j}|\frac{1}{2},1\rangle & m\geq 0 \\
|m^2, 2m\rangle^{N=2} = \prod_{j=2}^{-m} \frac{\bar{Y}_{-(2j-1)}}{\sqrt{j(j-1)}} \times \frac{\bar{X}_{-1/2}}{\sqrt{2}}|\frac{1}{2},1\rangle & m\leq -1
\end{cases}
\tag{3.16}
$$

and these are massless characters of $N=2$ SCA disconnected to $(0,0)^{N=2}$. $ch_{\mathrm{NS2}}$ is

$$
\begin{aligned}
ch_{\mathrm{NS2}}(\theta,\tau) &= \sum_{m\in\mathbf{Z}} \frac{q^{m^2+m+\frac{1}{2}-\frac{9}{24}} z^{2m+1}}{1+zq^{m+\frac{1}{2}}} \prod_{n\geq 1} \frac{(1+zq^{n-\frac{1}{2}})(1+z^{-1}q^{n-\frac{1}{2}})}{(1-q^n)^2} \\
&= \frac{1}{2} f_-(\theta,\tau) + \frac{1}{2} ch_{\mathrm{NS5}}^{h=\frac{1}{2}}(\theta,\tau).
\end{aligned}
\tag{3.17}
$$

Last equation is proved by comparing the modular property[17].

Relations between massive and massless characters are

$$\begin{aligned} ch^{h}_{\mathrm{NS4}}(\theta,\tau) &= q^{h}(ch_{\mathrm{NS1}}+ch_{\mathrm{NS2}}+ch_{\mathrm{NS3}})(\theta,\tau) \\ ch^{h}_{\mathrm{NS5}}(\theta,\tau) &= q^{h-\frac{1}{2}}(ch_{\mathrm{NS2}}+ch_{\mathrm{NS3}})(\theta,\tau). \end{aligned} \tag{3.18}$$

In the case of $h \to 0(\frac{1}{2})$ for NS4(NS5), these have been expected from figure 3 in [10].

# 4 Modular Transformation Formulas

We will discusss the modular properties of characters of the $c = 9$ algebra. Using Gaussian integral and the modular transformation formulas of theta functions and Dedekind's eta function, we can show that under the $S$ transformation ($\tau \mapsto \frac{-1}{\tau}, \theta \mapsto \frac{\theta}{\tau}$) massive characters transform as

$$ch^{h=\frac{1}{4}+\frac{\alpha^2}{2}}_{\mathrm{NS4}}(\theta,\tau)|_S = e^{\frac{3i\theta^2}{4\pi\tau}}\int_{-\infty}^{\infty} d\beta \cos(2\pi\alpha\beta)\frac{1}{\sqrt{2}}(ch^{h=\frac{1}{4}+\frac{\beta^2}{2}}_{\mathrm{NS4}} + ch^{h=\frac{1}{2}+\frac{\beta^2}{2}}_{\mathrm{NS5}})(\theta,\tau) \tag{4.1}$$

$$ch^{h=\frac{1}{4}-\frac{a^2}{2}}_{\mathrm{NS4}}(\theta,\tau)|_S = e^{\frac{3i\theta^2}{4\pi\tau}}\int_{-\infty}^{\infty} d\beta \cosh(2\pi a\beta)\frac{1}{\sqrt{2}}(ch^{h=\frac{1}{4}+\frac{\beta^2}{2}}_{\mathrm{NS4}} + ch^{h=\frac{1}{2}+\frac{\beta^2}{2}}_{\mathrm{NS5}})(\theta,\tau) \tag{4.2}$$

$$ch^{h=\frac{1}{2}+\frac{\alpha^2}{2}}_{\mathrm{NS5}}(\theta,\tau)|_S = e^{\frac{3i\theta^2}{4\pi\tau}}\int_{-\infty}^{\infty} d\beta \cos(2\pi\alpha\beta)\frac{1}{\sqrt{2}}(ch^{h=\frac{1}{4}+\frac{\beta^2}{2}}_{\mathrm{NS4}} - ch^{h=\frac{1}{2}+\frac{\beta^2}{2}}_{\mathrm{NS5}})(\theta,\tau), \tag{4.3}$$

where $f(\theta,\tau)|_S = f(\frac{\theta}{\tau},\frac{-1}{\tau})$ and $\alpha$ and $a$ are real. NS4 characters are divided into two classes:$h \geq \frac{1}{4}$ and $h < \frac{1}{4}$. NS4 characters with $h < \frac{1}{4}$ don't appear in the right hand sides of above formulas (this fact will be discussed in section 5). Using the transformation formulas of the massive characters and $f_{-}(\theta,\tau)$, massless characters transform as

$$ch_{\mathrm{NS1}}(\theta,\tau)|_S = (ch^{h=0}_{\mathrm{NS4}} - ch^{h=\frac{1}{2}}_{\mathrm{NS5}})(\theta,\tau)|_S \tag{4.4}$$

$$ch_{\mathrm{NS2}}(\theta,\tau)|_S = -ie^{\frac{3i\theta^2}{4\pi\tau}}\frac{1}{2}(ch_{\mathrm{NS2}} - ch_{\mathrm{NS3}})(\theta,\tau) + \frac{1}{2}ch^{h=\frac{1}{2}}_{\mathrm{NS5}}(\theta,\tau)|_S. \tag{4.5}$$

Under the $T$ transformation $(\tau \mapsto \tau+1, \theta \mapsto \theta)$, characters transform to $\widetilde{\mathrm{NS}}$ sector as

$$\begin{aligned} ch_{\mathrm{NSi}}(\theta,\tau)|_T &= e^{2\pi i(h-\delta-\frac{3}{8})} ch_{\widetilde{\mathrm{NSi}}}(\theta,\tau), \\ & \delta = \begin{cases} 0 & \mathrm{i}=1,4 \\ \frac{1}{2} & \mathrm{i}=2,3,5 \end{cases} \end{aligned} \tag{4.6}$$

where $f(\theta,\tau)|_T = f(\theta,\tau+1)$.

Above formulas realize the representation of the modular group. In fact we can check that

$$\begin{aligned} ch_{\mathrm{NSi}}(\theta,\tau)|_{S^2} &= ch_{\mathrm{NSi}}(-\theta,\tau) \\ ch_{\mathrm{NSi}}(\theta,\tau)|_{(ST)^3} &= ch_{\mathrm{NSi}}(-\theta,\tau) \quad (\mathrm{i}=1,\cdots,5). \end{aligned} \tag{4.7}$$

For NS4 characters with $h < \frac{1}{4}$, we must take care of the order of integration, see discussion.

Modular properties of R sector characters is derived from NS formulas and eq. (3.2) easily.

# 5 Discussion

We have obtained the character formulars of the $c=9$ algebra and their modular properties. Now we are ready to challenge the construction of the modular invariant partition functions. Unfortunately this is very difficult because the $c=9$ algebra has infinite many primary fields, i.e. *irrational* theory. This irrational theory, however, can be converted to *rational* one by combining infinite many characters to some orbits. Gepner's model[4] is one of this method. By construction Gepner's model is rational but $U(1)$ charge quantization is imposed on by hand. On the other hand the $c=9$ algebra is irrational but $U(1)$ charge quantization is automatically satisfied. Each orbit of Gepner's model is expanded into the characters of the $c=9$ algebra with positive integer coefficients.

We give an example taking the model $(k=1)^9$. The subtheory $k=1$ has three representations $(h,Q)^{N=2}=(0,0),(\frac{1}{6},\pm\frac{1}{3})$ in NS sector. We denote their characters as $A,B$ and $C$ respectively following the notation of [9]. For example $A^9+B^9+C^9$ (graviton orbit, which contains massless representation NS1), $A^6B^3+B^6C^3+C^6A^3$ (massless matter orbit, which contains massless representation NS2) and $A^5B^2C^2+B^5C^2A^2+C^5A^2B^2$ (massive orbit, which contains massive representations only) are expanded into

$$\begin{aligned}
& (A^9+B^9+C^9)(\theta,\tau) \\
= \; & ch_{\mathrm{NS1}}(\theta,\tau)+\sum_{n\geq 1} a_n ch_{\mathrm{NS4}}^{h=n}(\theta,\tau)+\sum_{n\geq 2} b_n ch_{\mathrm{NS5}}^{h=n+\frac{1}{2}}(\theta,\tau) \\
& (A^6B^3+B^6C^3+C^6A^3)(\theta,\tau) \\
= \; & ch_{\mathrm{NS2}}(\theta,\tau)+\sum_{n\geq 1} a'_n ch_{\mathrm{NS4}}^{h=n}(\theta,\tau)+\sum_{n\geq 1} b'_n ch_{\mathrm{NS5}}^{h=n+\frac{1}{2}}(\theta,\tau) \\
& (A^5B^2C^2+B^5C^2A^2+C^5A^2B^2)(\theta,\tau) \\
= \; & \sum_{n\geq 0} a''_n ch_{\mathrm{NS4}}^{h=n+\frac{2}{3}}(\theta,\tau)+\sum_{n\geq 0} b''_n ch_{\mathrm{NS5}}^{h=n+\frac{1}{2}+\frac{2}{3}}(\theta,\tau).
\end{aligned}$$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | | 8 | 27 | 224 | 1071 | 4320 | 15596 | 50600 | $\cdots$ |
| $b_n$ | | | 56 | 334 | 1512 | 6064 | 21096 | 66960 | $\cdots$ |
| $a'_n$ | | 2 | 33 | 216 | 1062 | 4344 | 15579 | 50570 | $\cdots$ |
| $b'_n$ | | 6 | 54 | 331 | 1530 | 6048 | 21078 | 67020 | $\cdots$ |
| $a''_n$ | 1 | 16 | 120 | 640 | 2762 | 10304 | 34485 | 106000 | $\cdots$ |
| $b''_n$ | 2 | 28 | 186 | 940 | 3880 | 14072 | 45980 | 138800 | $\cdots$ |

Other orbits and other models are also expanded with positive integer coefficients.

We turn to the next topic: modular transformation property of $ch_{\mathrm{NS4}}$ with

$h < \frac{1}{4}$. Under $S$ transformation, $ch_{\mathrm{NS4}}$ with $h < \frac{1}{4}$ seems to disappear. This is not a special property of the $c = 9$ algebra but a common feature of the irratioinal theory: massive representaion with $h < \frac{1}{8}$ of $c = 6$ $N = 4$ SCA[14], representation with $h < \frac{c-1}{24}$ of $c > 1$ Virasoro algebra. We take $c > 1$ Virasoro algebra as illustration. $c > 1$ Virasoro characters with $h = \frac{c-1}{24} + \frac{\alpha^2}{2}$, $h = \frac{c-1}{24} - \frac{a^2}{2} > 0$ and $h = 0$ are $\chi^+_\alpha(\tau) = \frac{q^{\frac{\alpha^2}{2}}}{\eta(\tau)}$, $\chi^-_a(\tau) = \frac{q^{-\frac{a^2}{2}}}{\eta(\tau)}$ and $\chi^0(\tau) = (1-q)\frac{q^{-\frac{c-1}{24}}}{\eta(\tau)}$ respectively. We note that $\chi^0 = \chi^-_{\sqrt{\frac{c-1}{12}}} - \chi^+_{\sqrt{\frac{25-c}{12}}}$ for $c < 25$ and $\chi^0 = \chi^-_{\sqrt{\frac{c-1}{12}}} - \chi^-_{\sqrt{\frac{c-25}{12}}}$ for $c \geq 25$ and $c = 1, 25$ are special values in the Kac determinant. Due to Gaussian integral and $\eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\eta(\tau)$, characters transform as

$$
\begin{aligned}
\chi^+_\alpha(\tau)|_S &= \int_{-\infty}^{\infty} d\beta e^{2\pi i\alpha\beta}\chi^+_\beta(\tau) = \int_{-\infty}^{\infty} d\beta \cos(2\pi\alpha\beta)\chi^+_\beta(\tau) \\
\chi^-_a(\tau)|_S &= \int_{-\infty}^{\infty} d\beta e^{2\pi a\beta}\chi^+_\beta(\tau) = \int_{-\infty}^{\infty} d\beta \cosh(2\pi a\beta)\chi^+_\beta(\tau) \\
\chi^0(\tau)|_S &= \int_{-\infty}^{\infty} d\beta \rho_c(\beta)\chi^+_\beta(\tau) \\
&\rho_c(\beta) = \cosh(2\pi\sqrt{\frac{c-1}{12}}\beta) - \begin{cases} \cos(2\pi\sqrt{\frac{25-c}{12}}\beta) & c < 25 \\ \cosh(2\pi\sqrt{\frac{c-25}{12}}\beta) & c \geq 25. \end{cases}
\end{aligned}
\tag{5.1}
$$

$\chi^-_a$ and $\chi^0$ don't appear in the right hand side. In some sense these are degenerate. If one generalize finite dimensional formula $\chi_i(\tau)|_S = \sum_j S_i^{\ j}\chi_j(\tau)$ to infinite dimensional case $\chi_A(\tau)|_S = \int dB S_A^{\ B}\chi_B(\tau)$, $S_A^{\ B}$ is formaly ($A$ stand for $(+,\alpha)$ , $(-,a)$ and 0)

$$
S_A^{\ B} = \begin{pmatrix} \cos(2\pi\alpha\beta) & 0 & 0 \\ \cosh(2\pi a\beta) & 0 & 0 \\ \rho_c(\beta) & 0 & 0 \end{pmatrix} . \tag{5.2}
$$

Formal calculation of $S^2$ results in a failure:

$$
S^2{}_A^{\ B} = \int dC S_A^{\ C} S_C^{\ B} = \begin{pmatrix} \mathbf{1}(\alpha,\beta) & 0 & 0 \\ \text{divergent} & 0 & 0 \\ \text{divergent} & 0 & 0 \end{pmatrix} , \tag{5.3}
$$

where $\mathbf{1}(\alpha,\beta)=\frac{1}{2}(\delta(\alpha+\beta)+\delta(\alpha-\beta))$. Although $++$ component is desired result, $-$ and $0$ sector are wrong results or meaningless. $S^2$, however, acts characters as $\mathbf{1}$ if we keep characters: for example

$$\begin{aligned}
\chi_a^-(\tau)|_{S^2} &= \int_{-\infty}^{\infty} d\beta e^{2\pi a\beta}(\int_{-\infty}^{\infty} d\gamma e^{2\pi i\beta\gamma}\chi_\gamma^+(\tau)) \\
&= \int_{-\infty}^{\infty} d\beta e^{2\pi a\beta} e^{2\pi i\frac{-1}{\tau}\frac{\beta^2}{2}} \frac{1}{\eta(\tau)}\frac{1}{\sqrt{-i\tau}} \\
&= \chi_a^-(\tau).
\end{aligned} \tag{5.4}$$

The distinction of $h>\frac{c-1}{24}$ and $h<\frac{c-1}{24}$ appears also in the Feigin-Fuchs construction. Manifestly unitary Feigin-Fuchs construction with $c>1$ is given by

$$\begin{aligned}
&T(z)=\frac{1}{2}:(i\partial\tilde{\phi}(z))^2:-\lambda\partial^2\tilde{\phi}(z), \qquad \lambda\in\mathbf{R} \\
&L_n^\dagger=L_{-n}\ , \quad c=1+12\lambda^2,
\end{aligned} \tag{5.5}$$

where $\tilde{\phi}(z)=\phi(z)+\lambda\log z$ and $\phi$ is given by eq. (2.2). Momentum eigenstate $|\alpha\rangle$ ( $\alpha\in\mathbf{R}$ becuase of hermiticity of $p$) is the highest weight state w.r.t $T(z)$ with $h=\frac{\alpha^2}{2}+\frac{\lambda^2}{2}\geq\frac{\lambda^2}{2}=\frac{c-1}{24}$. So in this realization, only the highest weight states with $h\geq\frac{c-1}{24}$ exist.

Curiousity of massive representation with $h<\frac{1}{8}$ of $c=6$ $N=4$ SCA has the same origin as $c>1$ Virasoro algebra. $N=4$ SCA with $c=6k$ is realized manifestly unitary by the level $k-1$ $SU(2)$ Kac-Moody algebra, two complex free fermions and one Feigin-Fuchs boson[17,18]. In this realizaton unitary reprezentations have conformal weight $h\geq\frac{k^2}{4(k+1)}$ because the boson has energy momentum tensor eq. (5.5) with $\lambda^2=\frac{k^2}{2(k+1)}$. For this range of conformal weight, fusion rules of the $N=4$ SCA are fusion rules of $SU(2)$ Kac-Moody algebra and momentum conservation of the Feigin-Fuchs boson[13].

## Acknowledgments

I would like to thank Professor T. Eguchi for useful suggestions and discussions and careful reading of this manuscript. I also thank Dr. K. Miki for useful discussions. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan No.01790191.

# Appendix

We present a proof of eq. (3.9) which tells us the structure of the representation space. We prove following lemmas, propositions and theorems by induction and using the operator relations derived from eq. (2.9): (see [11] about useful calculation methods)

$$
\begin{aligned}
&(IY)(z) + \frac{1}{2}(\bar{G}X)(z) - \partial Y(z) = 0 \\
&(GY)(z) + \frac{1}{2}(\partial IX)(z) - (TX)(z) = 0 \\
&(X^2)(z) = (Y^2)(z) = (XY)(z) = (X\partial Y)(z) = 0 \ \text{ etc.}
\end{aligned}
\tag{5.1}
$$

In the following $N = 2$ means $c = 9$ $N = 2$ SCA and $(N = 2)$ stand for creation operators of the $c = 9$ $N = 2$ SCA.

First we show the following lemma for the representation NSi (i=1, $\cdots$, 5).

**Lemma 1** $A_{-r}\bar{B}_{-s}|h\ Q\rangle = (N = 2)|h\ Q\rangle \qquad A, B = X, Y.$

**(proof)** Using eq. (2.9), we can show the following operator relations:

$$(A\partial^l \bar{B})(z) = (N = 2 \text{ generators})(z) \quad (l \geq 0). \tag{5.2}$$

For the case $A = B = X$, this is easily checked because from eq. (2.1) $X(z)$ is $\sqrt{2} : e^{i\sqrt{3}\phi(z)}$ :, so $(X\partial^l \bar{X})(z)$ is expressed by $I(z)$. eq. (5.2) says

$$(A\partial^l \bar{B})_{-r-s}|h\ Q\rangle = (N = 2)|h\ Q\rangle \quad (l \geq 0). \tag{5.3}$$

Combining these relations with various $l$ proves this lemma.□

From this, the next proposition holds.

**Proposition 1**

$$\overbrace{X \cdots X Y \cdots Y}^{\frac{n+m}{2}} \overbrace{\bar{X} \cdots \bar{X} \bar{Y} \cdots \bar{Y}}^{\frac{n-m}{2}} |h\ Q\rangle$$

$$
= \begin{cases} \sum(N=2)\overbrace{X\cdots XY\cdots Y}^{m}|h\ Q\rangle & m\geq 0 \\ \sum(N=2)\underbrace{\bar{X}\cdots\bar{X}\bar{Y}\cdots\bar{Y}}_{-m}|h\ Q\rangle & m<0. \end{cases}
$$

Here we abbreviate $X_{-n_1}\cdots X_{-n_k}$ to $X\cdots X$.

**1. representation NS4.**

We define the states $|m\rangle = Y_{-(2m-1)}\cdots Y_{-3}Y_{-1}|h,0\rangle$ $(m\geq 0)$.

**Lemma 2** $X_{-n}, Y_{-n}|m\rangle = 0 \quad n < 2m+1 \quad (m\geq 0)$.

**(proof)** For example we will show $Y_{-2m}|m\rangle = Y_{-(2m-1)}|m\rangle = 0$.

(0) $m=0$ OK.

(i) $m=1$

Operating $(Y^2)_{-3}$ and $(Y^2)_{-2}$ to $|h,0\rangle$, we obtain $Y_{-1}Y_{-2}|h,0\rangle = Y_{-1}^2|h,0\rangle$ $=0$ because $(Y^2)(z)=0$. (Or direct calculations show that $Y_{-1}Y_{-2}|h,0\rangle$ and $Y_{-1}^2|h,0\rangle$ have zero norm.) So $m=1$ case is OK.

(ii) Assume that $m$ case is OK.

Using $(Y^2)(z)=0$, $Y_{-(2m+2)}|m+1\rangle$ and $Y_{-(2m+1)}|m+1\rangle$ are

$$
\underbrace{Y_{-(2m+2)}Y_{-(2m+1)}}_{\substack{\| \\ -\sum_{p=1}^{2m} Y_{-4m-3+p}Y_{-p}}}|m\rangle,\quad \underbrace{Y_{-(2m+1)}Y_{-(2m+1)}}_{\substack{\| \\ -2\sum_{p=1}^{2m} Y_{-4m-2+p}Y_{-p}}}|m\rangle. \tag{5.4}
$$

By assumption, $m+1$ case is also OK.

From (i)(ii), the claim hold by induction. Similarly we can show that $X_{-(2m\pm\frac{1}{2})}|m\rangle = 0$ using $(XY)(z)=0$. $\square$

**Proposition 2** $|m\rangle$ *are the highest weight states of the* $N=2$ *SCA with norm* $=\prod_{j=1}^{m}(h+j(j-1))>0 \quad (m\geq 0)$.

**(proof)** (i) $m = 0, 1$ OK.

(ii) Assume that $m$ case is OK.

Then,

$$\begin{aligned} & A_n|m+1\rangle \quad A = T, I, G, \bar{G} \\ = \; & [A_n, Y_{-(2m+1)}]|m\rangle \quad (n > 0), \end{aligned} \tag{5.5}$$

so we will consider only $n < 2m+1$. $A = \bar{G}$ case is trivial. $A_n|m+1\rangle$ is proportional to $Y_{n-(2m+1)}|m\rangle$ for $A = T, I$ and $X_{n-(2m+1)}|m\rangle$ for $G$. These states are null states because of Lemma2. So $m+1$ case is also OK. $\square$

**Lemma 3** $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \qquad n \geq 2m+1 \quad (m \geq 0)$.

**(proof)** Operating $-(n+\frac{1}{2})$ mode of $(GY)(z) + \frac{1}{2}(\partial IX)(z) - (TX)(z) = 0$ and $-(n+1)$ mode of $(IY)(z) + \frac{1}{2}(\bar{G}X)(z) - \partial Y(z) = 0$ to $|m\rangle$, we obtain

$$\begin{aligned} (h + m^2 + m)X_{-n-\frac{1}{2}}|m\rangle &= \Big( \sum_{p \leq -\frac{1}{2}} G_p Y_{-n-\frac{1}{2}-p} - \sum_{p \leq -1} L_p X_{-n-\frac{1}{2}-p} \\ &\quad -\frac{1}{2} \sum_{p \leq -2} (p+1) I_p X_{-n-\frac{1}{2}-p} \Big)|m\rangle \\ (n - 2m)Y_{-n-1}|m\rangle &= \Big( \sum_{p \leq -1} I_p Y_{-n-1-p} + \frac{1}{2} \sum_{p \leq -\frac{1}{2}} \bar{G}_p X_{-n-1-p} \Big)|m\rangle. \end{aligned} \tag{5.6}$$

(i) $m = 0$

From eq. (5.6), $X_{-\frac{3}{2}}|0\rangle$, $Y_{-2}|0\rangle$, $X_{-\frac{5}{2}}|0\rangle$, $Y_{-3}|0\rangle$, $\cdots$ are expressed as $(N=2)|0\rangle$ in turn.

(ii) Assume that $m$ case is OK.

From eq. (5.6), $X_{-2m-\frac{3}{2}}|m\rangle$, $Y_{-2m-2}|m\rangle$, $X_{-2m-\frac{5}{2}}|m\rangle$, $Y_{-2m-3}|m\rangle$, $\cdots$ are expressed as $(N=2)|m+1\rangle$ in turn. So $m+1$ case is also OK. $\square$

From these the next proposition holds.

**Proposition 3** $\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m\geq 0)$.

We can show the similar results for $|m\rangle = \bar{Y}_{2m+1}\cdots\bar{Y}_{-3}\bar{Y}_{-1}|h\ 0\rangle$ $(m<0)$. Therefore we obtain the theorem:

**Theorem 1** $V_{\mathrm{NS4}} = \bigoplus_{m\in\mathbf{Z}} V^{N=2}_{h+m^2,2m}$.

Here $V$ denotes the representation space.

For other representations, we only mention the statements.

### 2. representation NS5.

We define the states $|m\rangle = Y_{-2m}\cdots Y_{-4}Y_{-2}|h,1\rangle$ $(m\geq 0)$.

**Lemma 4** $X_{-n}, Y_{-n}|m\rangle = 0 \qquad n<2m+2 \quad (m\geq 0)$.

**Proposition 4** *$|m\rangle$ are the highest weight states of the $N=2$ SCA with norm* $=\prod_{j=1}^{m}(h+j^2-\frac{1}{2})>0 \qquad (m\geq 0)$.

**Lemma 5** $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \qquad n\geq 2m+2 \quad (m\geq 0)$.

**Proposition 5** $\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m\geq 0)$.

**Theorem 2** $V_{\mathrm{NS5}} = \bigoplus_{m\in\mathbf{Z}} V^{N=2}_{h+m^2+m,2m+1}$.

### 3. representation NS1.

We define the states $|m\rangle = Y_{-2m}\cdots Y_{-6}Y_{-4}X_{-\frac{3}{2}}|0,0\rangle$ $(m>0)$ and $|0\rangle = |0,0\rangle$.

**Lemma 6** $X_{-n}, Y_{-n}|m\rangle = 0 \qquad n<2m+2 \quad (m>0) \ \ ; \quad n<\frac{3}{2} \quad (m=0)$.

**Proposition 6** *$|m\rangle$ are the highest weight states of the $N=2$ SCA with norm $= 2\prod_{j=2}^{m}(j^2-1) > 0 \;\; (m>0), \quad 1 \;\; (m=0)$.*

**Lemma 7** *$X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \quad n \geq 2m+2 \quad (m>0) \;\; ; \;\; n \geq \frac{3}{2} \quad (m=0)$.*

**Proposition 7** *$\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m \geq 0)$.*

**Theorem 3** *$V_{\mathrm{NS1}} = \bigoplus_{m>0} V^{N=2}_{m^2+m-\frac{1}{2},2m+1} \oplus V^{N=2}_{0,0} \bigoplus_{m<0} V^{N=2}_{m^2-m-\frac{1}{2},2m-1}$.*

**4. representation NS2.**

We define the states $|m\rangle = Y_{-2m}\cdots Y_{-4}Y_{-2}|\frac{1}{2},1\rangle \; (m \geq 0)$.

**Lemma 8** *$X_{-n}, Y_{-n}|m\rangle = 0 \quad n < 2m+2 \quad (m \geq 0)$.*

**Proposition 8** *$|m\rangle$ are the highest weight states of the $N=2$ SCA with norm $= \prod_{j=1}^{m} j^2 > 0 \quad (m \geq 0)$.*

**Lemma 9** *$X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \quad n \geq 2m+2 \quad (m \geq 0)$.*

**Proposition 9** *$\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m \geq 0)$.*

We define the states $|m\rangle = \bar{Y}_{2m+1}\cdots \bar{Y}_{-5}\bar{Y}_{-3}\bar{X}_{-\frac{1}{2}}|\frac{1}{2},1\rangle \; (m<0)$.

**Lemma 10** *$\bar{X}_{-n}, \bar{Y}_{-n}|m\rangle = 0 \quad n < -2m+1 \quad (m<0) \;\; ; \;\; n < \frac{1}{2} \quad (m=0)$.*

**Proposition 10** *$|m\rangle$ are the highest weight states of the $N=2$ SCA with norm $= 2\prod_{j=2}^{-m} j(j-1) > 0 \quad (m<0)$.*

**Lemma 11** *$\bar{X}_{-n}, \bar{Y}_{-n}|m\rangle = (N=2)|m-1\rangle \quad n \geq -2m+1 \quad (m<0) \;\; ; \;\; n \geq \frac{1}{2} \quad (m=0)$.*

**Proposition 11** *$\underbrace{\bar{X}\cdots \bar{X}\bar{Y}\cdots \bar{Y}}_{-m}|0\rangle = (N=2)|m\rangle \qquad (m<0)$.*

**Theorem 4** *$V_{\mathrm{NS2}} = \bigoplus_{m\geq 0} V^{N=2}_{m^2+m+\frac{1}{2},2m+1} \bigoplus_{m<0} V^{N=2}_{m^2,2m}$.*

## References


[1] W.Boucher, D.Friedan and A.Kent, Phys.Lett. **172B** (1986)316.

[2] A.Sen, Nucl.Phys. **B278** (1986)289; **B284** (1987)423.

[3] T.Banks, L.Dixon, D.Friedan and E.Martinec, Nucl.Phys. **B299** (1988) 613.

[4] D.Gepner, Nucl.Phys. **B296** (1988)757; Phys.Lett. **199B** (1987)380.

[5] N.Seiberg, Nucl.Phys. **B303** (1988)286.

[6] A.Schwimmer and N.Seiberg, Phys.Lett. **184B** (1987)191.

[7] P.Candelas, G.Horowitz, A.Strominger and E.Witten, Nucl.Phys. **B258** (1985)46.

[8] M.Green, J.Schwarz and E.Witten, *Superstring Theory 2*, Cambridge University Press(1987).

[9] T.Eguchi, H.Ooguri, A.Taormina and S.K.Yang, Nucl.Phys. **B315** (1989) 193.

[10] S.Odake, Mod.Phys.Lett. **4** (1989)557.

[11] S.Odake, PhD Thesis, Tokyo University, Dec.1988; Soryushiron Kenkyu (Kyoto) **78** (1989)201. (in Japanese)

[12] A.B.Zamolodchikov, Theor.Math.Phys. **65** (1986)1205.

[13] T.Eguchi, *private communication.*

[14] T.Eguchi and A.Taormina, Phys.Lett. **210B** (1988)125.

[15] V.K.Dobrev, Phys.Lett. **B186** (1987) 43.

[16] E.B.Kiritsis, Int.J.Mod.Phys. **A3** (1988) 1871.

[17] K.Miki, *private communication*; PhD Thesis *"$N = 3$ superconformal algbra"*, Tokyo University, Dec.1988.

[18] P.Mathieu, *"Representation of the $SO(N)$ and $U(N)$ Superconformal Algebra via Miura Transformations"*, Laval Univ. preprint (1988).