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Extension of $N = 2$ Superconformal Algebra and Calabi-Yau Compactification

Satoru Odake

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

We study an extension of $N = 2$ superconformal algebra by the addition of the spectral flow generators. We present the extended algebra corresponding to complex 3 dimensional Calabi-Yau manifold and discuss its representation theory. We also discuss some symmetry properties of the non-linear σ model on the Ricci-flat Kähler manifold.

1 Introduction

In recent years extentions of (super)conformal algebra have been investigated [1, 2, 3, 4] by adding primary fields with integer or half odd integer conformal weights. Extended algebras are constructed by requiring the closure and associativity of the operator algebra.

Concerning string compactification, the relation between $N = 1$ spsce-time supersymmery and $N = 2$ superconformal algebra on the world sheet has been clarified in [5, 6, 7]. Since $N = 2$ superconformal algebra has an automorphism (*spectral flow*) [8], Neveu-Schwarz sector and Ramond sector are transformed each other by the spectral flow. This means that the space-time bosons and fermions are paired. Representations of $N = 2$ superconformal algebra are labeled by conformal weight h and $U(1)$ charge Q. In the case of $c = 3\tilde{n}$, where c is the central charge of the Virasoro algebra, vacuum state$(h = 0, Q = 0)$ in the NS sector is mapped onto the states$(h = \tilde{n}/2, Q = \pm\tilde{n})$ in the NS sector and the states$(h = \tilde{n}/8, Q = \pm\tilde{n}/2)$ in the R sector. On the other hand, study of the low energy supergravity theory showed that compactification space of the string theory must be a complex $\tilde{n}$ dimensional manifold with $SU(\tilde{n})$ holonomy [9, 10]. The states with $(h = \tilde{n}/8, Q = \pm\tilde{n}/2)$ and $(h = \tilde{n}/2, Q = \pm\tilde{n})$ correspond to the covariantly constant spinors and (anti-)holomorphic $\tilde{n}$-forms of the manifold with $SU(\tilde{n})$ holonomy respectively. The former are related to space-time supersymmetry and the latter to the flow generators. Extended algebra, which is obtained by addition of the flow generators to $N = 2$ superconformal algebra, controls string compactification on the manifold with $SU(\tilde{n})$ holonomy. These are the idea of refs.[11, 12] and their authors extensively studied string compactification on K_3 surface ($\tilde{n} = 2$) in particular.

In this paper we study the extended algebra obtained by adding the spectral flow generators to $N = 2$ superconformal algebra. In particular we will concentrate on $\tilde{n} = 3$ case, which corresponds to Calabi-Yau compactification, and present the extended algebra and discuss its representation theory. We will also discuss the symmetry property of non-linear σ model on the Ricci-flat Kähler manifold.

2 Extension of $N = 2$ superconformal algebra

Let us first fix our notation. A field $A(z)$ is expanded as $A(z) = \sum_n A_n z^{-n-h}$ (except $T(z) = \sum_n L_n z^{-n-2}$), where h is conformal weight and n runs $\mathbf{Z}+\frac{1}{2}$ for NS sector ($h \in \mathbf{Z}+\frac{1}{2}$), $\mathbf{Z}$ for NS sector($h \in \mathbf{Z}$) or R sector and $A_n^\dagger = \bar{A}_{-n}$. Normal ordering $(AB)(z)$ of the two fields $A(z)$,$B(z)$ is defined by [2]

$$(AB)(z) = \oint_z \frac{dx}{2\pi i} \frac{1}{x-z} A(x)B(z). \tag{2.1}$$

This definition is natural from the operator product expansion(OPE) point of view but does not always agree with the usual normal ordering : :. In the rest of this section we consider NS sector only. Mode expansion of $(AB)(z)$ is

$$(AB)_n = \sum_{p \leq -h_A} A_p B_{n-p} + (-1)^{AB} \sum_{p > -h_A} B_{n-p} A_p. \tag{2.2}$$

Virasoro algebra is generated by energy-momentum tensor $T(z)$ and its OPE is

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular term} . \tag{2.3}$$

A primary field w.r.t. $T(z)$ with conformal weigth h, $\phi(z)$, is defined by

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w} + \text{reg.} \ . \tag{2.4}$$

$N = 2$ superconformal algebra is generated by $T(z)$, supercurrents $G(z), \bar{G}(z)$ and $U(1)$ current $I(z)$, which are primary fields w.r.t. $T(z)$ with $h = \frac{3}{2}, \frac{3}{2}, 1$. Their OPEs are

$$\begin{aligned}
I(z)I(w) &= \frac{c}{3(z-w)^2} + \text{reg.} \\
I(z)G(w) &= \frac{1}{z-w}G(w) + \text{reg.}, \qquad I(z)\bar{G}(w) = \frac{-1}{z-w}\bar{G}(w) + \text{reg.} \\
G(z)\bar{G}(w) &= \frac{2c}{3(z-w)^3} + \frac{2I(w)}{(z-w)^2} + \frac{1}{z-w}(\partial I(w) + 2T(w)) + \text{reg.} \\
G(z)G(w) &= \text{reg.}, \qquad \bar{G}(z)\bar{G}(w) = \text{reg.} \ .
\end{aligned} \tag{2.5}$$

Now consider the field having $h = \tilde{n}/2$ and $U(1)$ charge $Q = \tilde{n}$ ($\tilde{n} = 1, 2, 3, \cdots$). Let us call this the flow generator $X(z)$, which is a primary field w.r.t. $T(z)$ with $h = \tilde{n}/2$ and

$$I(z)X(w) = \frac{\tilde{n}}{z-w}X(w) + \text{reg. } . \tag{2.6}$$

We define the supertransformation of $X(z)$ as follows:

$$G(z)X(w) = \text{reg.}, \qquad \bar{G}(z)X(w) = \frac{2Y(w)}{z-w} + \text{reg. } . \tag{2.7}$$

Then $Y(z)$ is a primary field w.r.t. $T(z)$ with $h = \frac{\tilde{n}+1}{2}$ and OPEs are

$$\begin{aligned} I(z)Y(w) &= \frac{\tilde{n}-1}{z-w}Y(w) + \text{reg.} \\ G(z)Y(w) &= \frac{\tilde{n}X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w} + \text{reg.}, \qquad \bar{G}(z)Y(w) = \text{reg. } . \end{aligned} \tag{2.8}$$

Our question is *whether the operator algebra generated by* $T(z), I(z), G(z), \bar{G}(z), X(z), \bar{X}(z), Y(z)$ *and* $\bar{Y}(z)$ *is closed and associative or not* (fig.1). This algebra contains as a subalgebra $N = 2$ superconformal algebra generated by $T(z), I(z), G(z)$ and $\bar{G}(z)$. Calculations are done by using the technique of [13, 1, 2]. Normalization of $X(z)$ is $X(z)\bar{X}(w) = \frac{2}{(z-w)^{\tilde{n}}} + \cdots$.

$\tilde{n} = 1$ case is almost trivial. Algebra becomes (one pair of free complex boson and fermion) $\oplus$ ($N = 2$ superconformal algebra with center= $c - 3$). In $\tilde{n} = 2$ case, algebra becomes the $N = 4$ superconformal algebra:

$$\begin{array}{llllllll} c = 6k & & & & & & & \\ J^+(z) &=& \sqrt{\frac{k}{2}}X(z), & G^{(1)}(z) &=& -\sqrt{2k}\bar{Y}(z), & G^{(2)}(z) &=& G(z) \\ J^-(z) &=& \sqrt{\frac{k}{2}}\bar{X}(z), & \bar{G}^{(1)}(z) &=& -\sqrt{2k}Y(z), & \bar{G}^{(2)}(z) &=& \bar{G}(z) \\ J^3(z) &=& \frac{1}{2}I(z), & & & & & \end{array} \tag{2.9}$$

where k is the level of $SU(2)$ Kac-Moody algebra ($k = 1, 2, 3, \cdots$), so c is multiple of 6. This case is extensively studied by [11][12] ($k = 1$).

$\tilde{n} = 3$ case is the main result of this paper. *There is a unique closed associative algebra* (not Lie algebra but W algebra like), *whose center c is* *equal to* 9 . Algebra is as follows (we present it in commutator form):

$$\begin{cases} \quad c = 9 & \\ [L_n, L_m] &= (n-m)L_{n+m} + \frac{3}{4}n(n^2-1)\delta_{n+m,0} \\ \{G_n, \bar{G}_m\} &= 3(n^2-\frac{1}{4})\delta_{n+m,0} + (n-m)I_{n+m} + 2L_{n+m} \\ [I_n, I_m] &= 3n\delta_{n+m,0} \\ \{G_n, G_m\} &= \{\bar{G}_n, \bar{G}_m\} = 0 \end{cases}$$

$$\begin{cases} & & & \phi = I \quad G \quad \bar{G} \quad X \quad \bar{X} \quad Y \quad \bar{Y} \\ [L_n, \phi_m] &= ((h-1)n - m)\phi_{n+m} & & h = 1 \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2} \quad \frac{3}{2} \quad 2 \quad 2 \\ [I_n, \phi_m] &= Q\phi_{n+m} & & Q = * \quad 1 \quad -1 \quad 3 \quad -3 \quad 2 \quad -2 \end{cases}$$

$$\begin{cases} \{G_n, X_m\} = 0 & \{G_n, \bar{X}_m\} = 2\bar{Y}_{n+m} \\ \{\bar{G}_n, X_m\} = 2Y_{n+m} & \{\bar{G}_n, \bar{X}_m\} = 0 \\ [G_n, Y_m] = (2n-m)X_{n+m} & [G_n, \bar{Y}_m] = 0 \\ [\bar{G}_n, Y_m] = 0 & [\bar{G}_n, \bar{Y}_m] = (2n-m)\bar{X}_{n+m} \end{cases}$$

$$\begin{cases} \{X_n, \bar{X}_m\} &= (n^2-\frac{1}{4})\delta_{n+m,0} + (n-m)I_{n+m} + (I^2)_{n+m} \\ [X_n, \bar{Y}_m] &= (n+\frac{1}{2})G_{n+m} + (IG)_{n+m} \\ [\bar{X}_n, Y_m] &= (n+\frac{1}{2})\bar{G}_{n+m} - (I\bar{G})_{n+m} \\ [Y_n, \bar{Y}_m] &= \frac{1}{2}n(n^2-1)\delta_{n+m,0} + \frac{1}{2}(n(n+1)+m(m+1))I_{n+m} \\ & \quad +\frac{1}{4}(n-m)(I^2)_{n+m} - (m+1)L_{n+m} + (IT)_{n+m} - \frac{1}{2}(G\bar{G})_{n+m} \\ \{X_n, X_m\} &= \{\bar{X}_n, \bar{X}_m\} = [X_n, Y_m] = [\bar{X}_n, \bar{Y}_m] = [Y_n, Y_m] = [\bar{Y}_n, \bar{Y}_m] = 0. \end{cases} \tag{2.10}$$

Associativity of algebra eq. (2.10), i.e. Jacobi identity, requires some operator relations

$$\begin{aligned} (IX)(z) &= \partial X(z), \quad (IY)(z) = \partial Y(z) - \frac{1}{2}(\bar{G}X)(z) \\ (Y^2)(z) &= 0, \quad \text{etc. } . \end{aligned} \tag{2.11}$$

In these relations the first one is the most basic and the others are derived from it. Using $(IX)(z) = \partial X(z)$, we can show

$$[(I^2)_n, X_m] = 6(\frac{n}{2} - m)X_{n+m}. \tag{2.12}$$

Therefore the algebra eq. (2.10) contains $c = 1$ $N = 2$ superconformal algebra generated by

$$\begin{aligned} \tilde{T}(z) &= \tfrac{1}{6}(I^2)(z), & \tilde{I}(z) &= \tfrac{1}{3}I(z) \\ \tilde{G}(z) &= \tfrac{1}{\sqrt{3}}X(z), & \tilde{\bar{G}}(z) &= \tfrac{1}{\sqrt{3}}\bar{X}(z) \end{aligned} \tag{2.13}$$

as a subalgebra.

Check of the associativity is more easily done by the free field realization. If the free field realization exist, associativity is automatically satisfied and eq. (2.11) is also checked easily. One method is given by taking three copies of $c=3$ theory $(c=3)^3$

$$\begin{aligned} G(z) &= \sqrt{2}\sum_{j=1}^{3}\psi^j(z)i\partial\varphi^j(z) \\ X(z) &= \sqrt{2}\frac{1}{3!}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\psi^i(z)\psi^j(z)\psi^k(z)=\sqrt{2}\psi^1\psi^2\psi^3, \end{aligned} \tag{2.14}$$

where $\varphi^i(z)$ are complex free bosons ($\langle\varphi^i(z)\varphi^j(w)\rangle=-\delta^{ij}\log(z-w)$), $\psi^i(z)$ are complex free fermions ($\langle\psi^i(z)\psi^j(w)\rangle=\delta^{ij}\frac{1}{z-w}$). Another method is given by $(c=1)^9$

$$G(z)=\sqrt{\frac{2}{3}}\sum_{j=1}^{9}\gamma^j:e^{i\sqrt{3}\phi^j(z)}:,\qquad X(z)=\sqrt{2}:e^{\frac{i}{\sqrt{3}}\sum_{j=1}^{9}\phi^j(z)}:, \tag{2.15}$$

where $\phi^i(z)$ are real free bosons ($\langle\phi^i(z)\phi^j(w)\rangle=-\delta^{ij}\log(z-w)$) and γ^i are gamma matrices $\{\gamma^i,\gamma^j\}=2\delta^{ij}$. $(c=\frac{3}{2})^6$ is also constructed similarly and all of the Gepner's tensor product constructions [14] must fit this algebra eq. (2.10).

3 Representation theory of algebra eq.(2.10)

In the previous section we have considered the algebra eq. (2.10) in the NS sector. In this section we will consider it in the R sector and discuss its representation theory.

Algebra eq. (2.10) has an automorphism (spectral flow) just like as $N=2$ superconformal algebra[8]. Algebra eq. (2.10) is invariant under the transformation

$$\left\{\begin{array}{lcl} I'_n &=& I_n+3\eta\delta_{n0} \\ L'_n &=& L_n+\eta I_n+\frac{3}{2}\eta^2\delta_{n0} \\ \phi'_n &=& \phi_{n+Q\eta} \end{array}\right. \quad \begin{array}{cccccccc} \phi &=& G & \bar{G} & X & \bar{X} & Y & \bar{Y} \\ Q &=& 1 & -1 & 3 & -3 & 2 & -2, \end{array} \tag{3.1}$$

where $\eta \in \mathbf{R}$, so eq. (2.10) is also valid in the R sector. But normal ordering $(AB)_n$ is not always given by eq. (2.2). In eq. (2.10), only $(G\bar{G})_n$ is changed to

$$\begin{aligned}(G\bar{G})_n &= \sum_{\substack{p\leq -3/2+\varepsilon \\ p\in\mathbf{Z}+1/2+\varepsilon}} G_p\bar{G}_{n-p} - \sum_{\substack{p> -3/2+\varepsilon \\ p\in\mathbf{Z}+1/2+\varepsilon}} \bar{G}_{n-p}G_p \\ &\quad -2\varepsilon L_n + \varepsilon(n+2-\varepsilon)I_n - \varepsilon(1-\varepsilon)(2-\varepsilon)\delta_{n0}, \qquad (3.2)\end{aligned}$$

for G_m ($m \in \mathbf{Z} + \frac{1}{2} + \varepsilon, -\frac{1}{2} < \varepsilon \leq \frac{1}{2}$). This is derived from OPE of $z^{\eta}G(z)$ and $w^{-\eta}\bar{G}(w)$ and eq. (2.1).

Now consider irreducible unitary highest weight representations of algebra eq. (2.10). A highest weight state $\mid h\ Q\rangle$ is an eigenstate of L_0 and I_0 and annihilated by generators of positive mode:

$$\begin{aligned} L_0 \mid h\ Q\rangle &= h \mid h\ Q\rangle, \qquad I_0 \mid h\ Q\rangle = Q \mid h\ Q\rangle \\ A_n \mid h\ Q\rangle &= 0 \quad (n>0) \quad A = L, I, G, \bar{G}, X, \bar{X}, Y, \bar{Y} \qquad (3.3)\end{aligned}$$

and conditions on the other zero mode are determined later by consistency and irreducibility. Irreducible unitary highest weight representations of algebra eq. (2.10) exist if and only if in the following cases:

NS sector:

$$\begin{array}{llllll}
(\mathrm{NS1}) & h = 0 & Q = 0 & Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{NS2}) & h = \frac{1}{2} & Q = 1 & Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{NS3}) & h = \frac{1}{2} & Q = -1 & Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{NS4}) & h > 0 & Q = 0 & Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{NS5}) & h > \frac{1}{2} & Q = 1 & Y_0 \mid h\ Q\rangle = 0, \quad \bar{Y}_0 \mid h\ Q\rangle \neq 0 \\
 & & Q = -1 & \bar{Y}_0 \mid h\ Q\rangle = 0, \quad Y_0 \mid h\ Q\rangle \neq 0
\end{array} \qquad (3.4)$$

R sector:

$$
\begin{array}{llllllll}
(\mathrm{R1}) & h & = & \frac{3}{8} & Q & = & \frac{3}{2} & G_0, \bar{G}_0, X_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad \bar{X}_0 \mid h\ Q\rangle \neq 0 \\
 & & & & Q & = & -\frac{3}{2} & G_0, \bar{G}_0, \bar{X}_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad X_0 \mid h\ Q\rangle \neq 0 \\
(\mathrm{R2}) & h & = & \frac{3}{8} & Q & = & -\frac{1}{2} & G_0, \bar{G}_0, X_0, \bar{X}_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{R3}) & h & = & \frac{3}{8} & Q & = & \frac{1}{2} & G_0, \bar{G}_0, X_0, \bar{X}_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0 \\
(\mathrm{R4}) & h & > & \frac{3}{8} & Q & = & \frac{3}{2} & G_0, X_0, Y_0 \mid h\ Q\rangle = 0, \quad \bar{G}_0, \bar{X}_0, \bar{Y}_0 \mid h\ Q\rangle \neq 0 \\
 & & & & Q & = & -\frac{3}{2} & \bar{G}_0, \bar{X}_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad G_0, X_0, Y_0 \mid h\ Q\rangle \neq 0 \\
 & & & & Q & = & -\frac{1}{2} & G_0, X_0, \bar{X}_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad \bar{G}_0, Y_0 \mid h\ Q\rangle \neq 0 \\
 & & & & Q & = & \frac{1}{2} & \bar{G}_0, X_0, \bar{X}_0, Y_0 \mid h\ Q\rangle = 0, \quad G_0, \bar{Y}_0 \mid h\ Q\rangle \neq 0 \\
(\mathrm{R5}) & h & > & \frac{3}{8} & Q & = & -\frac{1}{2} & \bar{G}_0, X_0, \bar{X}_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad G_0 \mid h\ Q\rangle \neq 0 \\
 & & & & Q & = & \frac{1}{2} & G_0, X_0, \bar{X}_0, Y_0, \bar{Y}_0 \mid h\ Q\rangle = 0, \quad \bar{G}_0 \mid h\ Q\rangle \neq 0.
\end{array}
\tag{3.5}
$$

(NS1-3) and (R1-3) are called massless representation and (NS4,5) and (R4,5) are called massive representation. We give the sketch of proof in NS sector. Since algebra eq. (2.10) contains $c = 1$ $N = 2$ superconformal algebra eq. (2.13) as a subalgebra, allowed $U(1)$ charges are $\tilde{Q} = 0, \pm\frac{1}{3}$, i.e. $Q = 0, \pm 1$ [5]. Norms of states $G_{-1/2} \mid h\ Q\rangle$ and $\bar{G}_{-1/2} \mid h\ Q\rangle$, which must be non-negative, require $h \geq \mid Q \mid /2$. Operator relation eq. (2.11) $(IY)(z) = \partial Y(z) - \frac{1}{2}(\bar{G}X)(z)$ and its hermitian conjugate determine $Y_0, \bar{Y}_0$ conditions. So (NS1-5) are necessary conditions. They are also sufficient conditions because in the free field realization eq. (2.14),

$$
\begin{array}{l}
V_{\vec{\alpha}}(z) =: e^{i\sum_{j=1}^{3}\alpha_j(\varphi^j(z)+\bar{\varphi}^j(z))} :, \qquad (\vec{\alpha}\cdot\vec{\beta} = 0, \quad \vec{\beta} \neq 0) \\
\sum_{i=1}^{3}\beta_i\psi^i(z)V_{\vec{\alpha}}(z), \qquad \sum_{i=1}^{3}\beta_i\bar{\psi}^i(z)V_{\vec{\alpha}}(z)
\end{array}
\tag{3.6}
$$

create highest weight states with $(h, Q) = (\vec{\alpha}^2, 0), (\vec{\alpha}^2 + \frac{1}{2}, 1), (\vec{\alpha}^2 + \frac{1}{2}, -1)$. In (NS5), we can show that $\bar{Y}_0^2 \mid h,\ 1\rangle = Y_0^2 \mid h, -1\rangle = 0$. So $Q = \pm 1$ sectors are intertwined (fig.2). In R sector, similar arguments hold (fig.3). (R2) and (R3) have non-zero Witten index.

Spectral flow eq. (3.1) connects NS sector and R sector : (NSi)$\leftrightarrow$(Ri) (i$= 1, \cdots, 5$). Explicitly the correspondence is given as follows:

$$\begin{array}{llcll}
\mathrm{NS}\to\mathrm{R} & & & \eta=\frac{1}{2} & \eta=-\frac{1}{2}\\
(1) & \mid\frac{3}{8},\ \frac{3}{2}\rangle^{R} & = & \mid 0,\ 0\rangle^{NS} & ,\ \frac{1}{\sqrt{2}}X_{-3/2}\mid 0,\ 0\rangle^{NS}\\
(2) & \mid\frac{3}{8},\ -\frac{1}{2}\rangle^{R} & = & \frac{1}{\sqrt{2}}\bar{X}_{-1/2}\mid\frac{1}{2},\ 1\rangle^{NS} & ,\ \mid\frac{1}{2},\ 1\rangle^{NS}\\
(3) & \mid\frac{3}{8},\ \frac{1}{2}\rangle^{R} & = & \mid\frac{1}{2},\ -1\rangle^{NS} & ,\ \frac{1}{\sqrt{2}}X_{-1/2}\mid\frac{1}{2},\ -1\rangle^{NS}\\
(4) & \mid h+\frac{3}{8},\ \frac{3}{2}\rangle^{R} & = & \mid h>0,\ 0\rangle^{NS} & ,\ \frac{1}{\sqrt{2}}X_{-3/2}\mid h>0,\ 0\rangle^{NS}\\
(5) & \mid h-\frac{1}{8},\ -\frac{1}{2}\rangle^{R} & = & \frac{1}{\sqrt{2}}\bar{X}_{-1/2}\mid h>\frac{1}{2},\ 1\rangle^{NS} & ,\ \mid h>\frac{1}{2},\ 1\rangle^{NS}\\
\mathrm{R}\to\mathrm{NS} & & & \eta=\frac{1}{2} & \eta=-\frac{1}{2}\\
(1) & \mid 0,\ 0\rangle^{NS} & = & \frac{1}{\sqrt{2}}\bar{X}_{0}\mid\frac{3}{8},\ \frac{3}{2}\rangle^{R} & ,\ \mid\frac{3}{8},\ \frac{3}{2}\rangle^{R}\\
(2) & \mid\frac{1}{2},\ 1\rangle^{NS} & = & \mid\frac{3}{8},\ -\frac{1}{2}\rangle^{R} & ,\ \frac{1}{\sqrt{2}}X_{-1}\mid\frac{3}{8},\ -\frac{1}{2}\rangle^{R}\\
(3) & \mid\frac{1}{2},\ -1\rangle^{NS} & = & \frac{1}{\sqrt{2}}\bar{X}_{-1}\mid\frac{3}{8},\ \frac{1}{2}\rangle^{R} & ,\ \mid\frac{3}{8},\ \frac{1}{2}\rangle^{R}\\
(4) & \mid h-\frac{3}{8},\ 0\rangle^{NS} & = & \frac{1}{\sqrt{2}}\bar{X}_{0}\mid h>\frac{3}{8},\ \frac{3}{2}\rangle^{R} & ,\ \mid h>\frac{3}{8},\ \frac{3}{2}\rangle^{R}\\
(5) & \mid h+\frac{1}{8},\ 1\rangle^{NS} & = & \mid h>\frac{3}{8},\ -\frac{1}{2}\rangle^{R} & ,\ \frac{1}{\sqrt{2}}X_{-1}\mid h>\frac{3}{8},\ -\frac{1}{2}\rangle^{R}.
\end{array}\tag{3.7}$$

From this, X_n and $\bar{X}_n$ are called as the flow generators. Representation is unchanged under a shift $\eta \in \mathbf{Z}$.

4 Non-linear σ model on Ricci-flat Kähler manifold

In the introduction we mentioned that the flow generators correspond to (anti-)holomorphic $\tilde{n}$-form of the manifold with $SU(\tilde{n})$ holonomy. In this section we will make this relation clear by considering the non-linear σ model on the manifold with $SU(\tilde{n})$ holonomy, i.e. Ricci-flat Kähler manifold, i.e. Calabi-Yau manifold. Although we don't know whether the non-linear σ model on the Ricci-flat Kähler manifold has vanishing β function or not [15, 16, 17], we put this matter aside for the time being.

Non-linear σ model on the complex $\tilde{n}$ dimensional Kähler manifold has $(2,2)$ supersymmetry [18]. Its action $S = \int dx^0 dx^1 \mathcal{L}$ is

$$\begin{aligned}
\mathcal{L} \;=\;& 2g_{\alpha\bar{\beta}}(\partial_+\phi^\alpha\partial_-\phi^{\bar{\beta}} + \partial_-\phi^\alpha\partial_+\phi^{\bar{\beta}})\\
&+2g_{\alpha\bar{\beta}}(i\psi_+^\alpha(\partial_-\psi_+^{\bar{\beta}} + \Gamma^{\bar{\beta}}_{\bar{\gamma}\bar{\delta}}\partial_-\phi^{\bar{\gamma}}\psi_+^{\bar{\delta}}) + i\psi_+^{\bar{\beta}}(\partial_-\psi_+^\alpha + \Gamma^\alpha_{\gamma\delta}\partial_-\phi^\gamma\psi_+^\delta)\\
&\quad+i\psi_-^\alpha(\partial_+\psi_-^{\bar{\beta}} + \Gamma^{\bar{\beta}}_{\bar{\gamma}\bar{\delta}}\partial_+\phi^{\bar{\gamma}}\psi_-^{\bar{\delta}}) + i\psi_-^{\bar{\beta}}(\partial_+\psi_-^\alpha + \Gamma^\alpha_{\gamma\delta}\partial_+\phi^\gamma\psi_-^\delta))
\end{aligned}$$

$$+4R_{\alpha\bar{\beta}\gamma\bar{\delta}}\psi_+^\alpha\psi_+^{\bar{\beta}}\psi_-^\gamma\psi_-^{\bar{\delta}} \tag{4.1}$$

and S is invariant under the supertransformations

$$\left\{\begin{array}{rcl} \delta_{G_\pm}\phi^{\bar{\alpha}} & = & -i\sqrt{2}\varepsilon\psi_\pm^{\bar{\alpha}} \\ \delta_{G_\pm}\psi_\pm^\alpha & = & \sqrt{2}\varepsilon\partial_\pm\phi^\alpha \\ \delta_{G_\pm}\psi_\mp^{\bar{\alpha}} & = & -i\sqrt{2}\varepsilon\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}\psi_\mp^{\bar{\beta}}\psi_\pm^{\bar{\gamma}} \\ \delta_{G_\pm}\text{others} & = & 0 \end{array}\right. \quad \left\{\begin{array}{rcl} \delta_{\bar{G}_\pm}\phi^{\alpha} & = & -i\sqrt{2}\varepsilon\psi_\pm^{\alpha} \\ \delta_{\bar{G}_\pm}\psi_\pm^{\bar{\alpha}} & = & \sqrt{2}\varepsilon\partial_\pm\phi^{\bar{\alpha}} \\ \delta_{\bar{G}_\pm}\psi_\mp^{\alpha} & = & -i\sqrt{2}\varepsilon\Gamma^{\alpha}_{\beta\gamma}\psi_\mp^{\beta}\psi_\pm^{\gamma} \\ \delta_{\bar{G}_\pm}\text{others} & = & 0. \end{array}\right. \tag{4.2}$$

When the manifold is Ricci-flat Kähler, S has an additional symmetry. For a complex $\tilde{n}$ dimensional Ricci-flat Kähler manifold, there is a holomorphic (antiholomorphic) and covariantly constant $(\tilde{n},0)$-form ($(0,\tilde{n})$-form) : $E_{\alpha_1\alpha_2\cdots\alpha_{\tilde{n}}}$ ($E_{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{\tilde{n}}}$) [10]. S is invariant under the following variations

$$\begin{array}{l} \left\{\begin{array}{rcl} \delta_{X_\pm}\psi_\pm^\alpha & = & \sqrt{2}\varepsilon\frac{(-1)^{\tilde{n}-1}}{(\tilde{n}-1)!}g^{\alpha\bar{\alpha}_1}E_{\bar{\alpha}_1\cdots\bar{\alpha}_{\tilde{n}}}\psi_\pm^{\bar{\alpha}_2}\cdots\psi_\pm^{\bar{\alpha}_{\tilde{n}}} \\ \delta_{X_\pm}\text{others} & = & 0 \end{array}\right. \\ \left\{\begin{array}{rcl} \delta_{\bar{X}_\pm}\psi_\pm^{\bar{\alpha}} & = & \sqrt{2}\varepsilon\frac{(-1)^{\tilde{n}-1}}{(\tilde{n}-1)!}g^{\alpha_1\bar{\alpha}}E_{\alpha_1\cdots\alpha_{\tilde{n}}}\psi_\pm^{\alpha_2}\cdots\psi_\pm^{\alpha_{\tilde{n}}} \\ \delta_{\bar{X}_\pm}\text{others} & = & 0. \end{array}\right. \end{array} \tag{4.3}$$

For example, δ_{X_+} is generated by $X_+ = \sqrt{2}\frac{1}{\tilde{n}!}E_{\bar{\alpha}_1\cdots\bar{\alpha}_{\tilde{n}}}\psi_+^{\bar{\alpha}_1}\cdots\psi_+^{\bar{\alpha}_{\tilde{n}}}$ and using $\partial_\beta E_{\bar{\alpha}_1\cdots\bar{\alpha}_{\tilde{n}}} = 0$ we can show that

$$\begin{array}{rcl} \delta_{X_+}\mathcal{L}_{\text{fermion term}} & = & -i4\sqrt{2}\varepsilon\dfrac{1}{\tilde{n}!}\partial_-\phi^{\bar{\beta}}\nabla_{\bar{\beta}}E_{\bar{\alpha}_1\cdots\bar{\alpha}_{\tilde{n}}}\psi_+^{\bar{\alpha}_1}\cdots\psi_+^{\bar{\alpha}_{\tilde{n}}} \\ & & +\text{total derivative term} \\ \delta_{X_+}\mathcal{L}_{\text{curvature term}} & = & 4\varepsilon(-1)^{\tilde{n}-1}R_{\alpha\bar{\beta}}\psi_-^\alpha\psi_-^{\bar{\beta}}X_+ \quad . \end{array} \tag{4.4}$$

Therefore S is invariant for a Ricci-flat Kähler manifold because $\nabla_{\bar{\beta}}E_{\bar{\alpha}_1\cdots\bar{\alpha}_{\tilde{n}}}$ and $R_{\alpha\bar{\beta}}$ vanish. It is also checked that δ_{X_+} and δ_{G_+} are commute and δ_{X_+} and $\delta_{\bar{G}_+}$ are not commute. This corresponds to eq. (2.7). Free field realizations eq. (2.14), eq. (5.3) are flat background case.

5 Discussion

We have given the algebra eq. (2.10) and a part of its representation theory. In order to study Calabi-Yau compactification and clarify the Gepner's models, we need character formulas. Character formula and Gepner's models are under investigation. In [12], properties of $c=9$ modular invariant partition functions have been studied.

For $\tilde{n}=4$ case, associativity requires that c is either 12 or $-15\pm\frac{3}{7}\sqrt{889}$ (<0). Since we are interested in positive c , c is equal to 12 and we can show that the algebra is unique. For general $\tilde{n}(\geq 1)$, when we restrict to $c=3\tilde{n}$, there is at least one algebra. OPE of X and $\bar{X}$ is

$$X(z)\mid\bar{X}\rangle = 2\sum_{m=1}^{\tilde{n}} z^{-m} \sum_{\substack{\{i_k\geq 0\}_{k\geq 1}\\ \sum_{k\geq 1} k i_k=\tilde{n}-m}} \prod_{k\geq 1}\frac{1}{i_k! k^{i_k}} I_{-k}^{i_k}\mid 0\rangle + \text{reg.} \tag{5.1}$$

and other OPEs are derived from this easily:

$$\begin{aligned} X(z)\mid\bar{Y}\rangle &= -\frac{1}{2}G_{-1/2}X(z)\mid\bar{X}\rangle \\ Y(z)\mid\bar{Y}\rangle &= \frac{1}{2}X(z)\mid\partial\bar{X}\rangle + \frac{1}{2}\bar{G}_{-1/2}X(z)\mid\bar{Y}\rangle. \end{aligned} \tag{5.2}$$

I, X and $\bar{X}$ generate a subalgebra and operator relation $(IX)(z)=\partial X(z)$ holds. Like as eq. (2.14) this algebra eq. (5.1) is realized by $\tilde{n}$ pairs of complex free bosons and fermions:

$$\begin{aligned} G(z) &= \sqrt{2}\sum_{j=1}^{\tilde{n}}\psi^j(z) i\partial\varphi^j(z) \\ X(z) &= \sqrt{2}\frac{1}{\tilde{n}!}\sum_{i_1,\cdots,i_{\tilde{n}}=1}^{\tilde{n}}\varepsilon_{i_1\cdots i_{\tilde{n}}}\psi^{i_1}(z)\cdots\psi^{i_{\tilde{n}}}(z) = \sqrt{2}\psi^1\psi^2\cdots\psi^{\tilde{n}}. \end{aligned} \tag{5.3}$$

We have not checked the other possibilities yet.

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errata

page4 eq.(2.7): $Y(z) \rightarrow Y(w)$,

page4 line 17: $X(z)\bar{X}(z) \rightarrow X(z)\bar{X}(w)$.

note added

We can show that, for $c = 9$ compactification, $N = 1$ space-time supersymmetry implies algebra eq.(2.10). Authors of ref.[7] have shown that internal spin operator $\Sigma(z)$ is expressed as $: e^{i\frac{\sqrt{3}}{2}H(z)} :$ and OPE of two $\Sigma(z)$'s yields an $h = \frac{3}{2}$ operator. Our $X(z)$ is this $h = \frac{3}{2}$ operator $\sqrt{2} : e^{i\sqrt{3}H(z)} :$ and OPE of X and $\bar{X}$ is eq.(5.1). Since $G(z)$ is expressed as $: e^{\frac{i}{\sqrt{3}}H(z)} : \Psi(z)$, where Ψ is some $h = \frac{4}{3}$ operator, we can show that supertransformation of X is given by eq.(2.7). By eq.(5.2) whole algebra closes and it is identical to eq.(2.10). We are indebted to Professor H. Kawai, Dr. Y. Kitazawa, Mr. N. Ishibashi and Mr. A. Kato for discussions on this point.