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Virasoro-type Symmetries in Solvable Models†

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Abstract

Virasoro-type symmetries and their roles in solvable models are reviewed. These symmetries are described by the two-parameter Virasoro-type algebra $\mathcal{V}ir_{p,q}$ by choosing the parameters p and q suitably.

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1 Introduction

Integrable models may have very huge symmetries, that help us to study various behaviors of the systems. For some well investigated models, we are able to calculate correlation functions of observables by using representation theories of the symmetries, that have been fruitfully studied within the language of infinite dimensional Lie algebras and their suitable deformation theories. We now have a lot of examples of these symmetry algebras and their applications to many problems. What is the natural candidate for that symmetry which is present in majority of the integrable models? In other words, what is the "universal" symmetry among all these?

The $1+1$-dimensional conformal field theories (CFT's) [1] describe the universality classes of massless field theories in $1+1$-dimensions. It is the guideline given in the celebrated paper [1] that all the CFT's must be regarded as representations of the "Virasoro algebra" regardless of the details of the models. Namely, the Virasoro algebra is the universal one in CFT. Indeed, through the Sugawara construction [2, 3], the Virasoro algebra exists in any Kac-Moody algebra. These models are successfully applied to the critical phenomena of two-dimensional classical statistical models, the low temperature behavior of one-dimensional electron systems and so on. In any sense, CFT is definitely quite well understood among other interacting field theories, because we have the infinite dimensional conformal symmetry.

We are gradually understanding the universal symmetry arising in off-critical models: massive integrable field theories in 1+1 dimension (sine-Gordon model etc.) [4], one-dimensional quantum spin chain systems (XYZ model etc.) [4], two-dimensional solvable lattice models (ABF models etc.) [5, 6, 7], deformations of the KdV hierarchy and discretized soliton equations [8, 9], Calogero-Sutherland(CS)-type quantum mechanical models [10, 11] and so on. It had been recognized that nontrivial Virasoro-type symmetries exist in these off-critical theories; in [5], the existence of two-parameter Virasoro-type symmetry was conjectured, and a one-parameter Virasoro-type Poisson structure was found in [8]. It was in the CS-type quantum mechanical model, that we finally obtained the definition and an explicit construction of the two-parameter Virasoro-type symmetry which we call $\mathcal{V}ir_{p,q}$ [12]. This is already extended to the case of $\mathcal{W}$-algebra [13, 14]. It is really astonishing that all the Virasoro-type algebras related with the off-critical integrable models are obtained by taking suitable limits from $\mathcal{V}ir_{p,q}$. However, we have not fully understand the meaning of "universality" played by this new Virasoro-type symmetry $\mathcal{V}ir_{p,q}$, so far. The algebra $\mathcal{V}ir_{p,q}$ is one of the simplest example of elliptic algebras [15, 13], since we have elliptic theta-functions in the operator product expansion (OPE) formulas. We strongly hope that this Virasoro-type algebra will be constructed in a canonical way from the elliptic algebra $\mathcal{A}_{q,p}$ [15][5] through a Sugawara-type construction and that will give us a clue for the total understanding of $\mathcal{V}ir_{p,q}$.

In this review, we will explain how this two parameter Virasoro-type algebra $\mathcal{V}ir_{p,q}$ arose in the CS-type model, and another aim is to accumulate as many problems and applications of $\mathcal{V}ir_{p,q}$ as possible. This paper is organized as follows. In Section 2, we study basic

[5] The relations between the parameters in $\mathcal{A}_{q,p}$ and $\mathcal{V}ir_{p,q}$ are $q_{\mathcal{V}ir} = p_{\mathcal{A}}$ and $p_{\mathcal{V}ir} = q_{\mathcal{A}}{}^2$.

results of the Virasoro-type algebras starting from the definition of the two-parameter Virasoro-type algebra $\mathcal{V}ir_{p,q}$. In Section 3, a Heisenberg realization of the Virasoro-type algebra and its applications to the CS-type models are presented. Based on this realization, we discuss representation theories, futher applications and relations with other several models, in Section 4. Section 5 is devoted to summary and comments.

2 Quantum deformed Virasoro algebra $\mathcal{V}ir_{p,q}$

In this section we examine some of the fundamental properties of the Virasoro-type algebra $\mathcal{V}ir_{p,q}$ [12] which can be directly derived from the defining relation. A Heisenberg realization and its application to various problems are studied in the next sections.

2.1 definition of $\mathcal{V}ir_{p,q}$

Let p be a generic complex parameter with $|p| < 1$. Let us consider an associative algebra generated by $\{T_n | n \in \mathbf{Z}\}$ with the relation

$$f(w/z)T(z)T(w) - T(w)T(z)f(z/w) = \text{const.}\left[\delta\Big(\frac{pw}{z}\Big) - \delta\Big(\frac{p^{-1}w}{z}\Big)\right],$$

where $T(z) = \sum_{n\in\mathbf{Z}} T_n z^{-n}$, $\delta(x) = \sum_{n\in\mathbf{Z}} x^n$ and $f(z) = \sum_{l=0}^{\infty} f_l z^l$ is a structure function. We show that the commutativity of the diagram (Yang-Baxter equation for $T(z)$)

$$\begin{array}{ccc}
T(x)T(y)T(z) & \xrightarrow{f(z/y)\times} & \begin{array}{c} T(x)T(z)T(y)f(y/z) \\ +T(x)\ \delta\text{−function} \end{array} \\
\Big\downarrow{\scriptstyle f(y/x)\times} & & \Big\downarrow{\scriptstyle f(z/x)\times} \\
\begin{array}{c} T(y)T(x)T(z)f(x/y) \\ +T(z)\ \delta\text{−function} \end{array} & & \begin{array}{c} T(z)T(x)T(y)f(y/z)f(x/z) \\ +T(x)\ \delta\text{−function} \\ +T(y)\ \delta\text{−function} \end{array} \\
\Big\downarrow{\scriptstyle f(z/x)\times} & & \Big\downarrow{\scriptstyle f(y/x)\times} \\
\begin{array}{c} T(y)T(z)T(x)f(x/y)f(x/z) \\ +T(z)\ \delta\text{−function} \\ +T(y)\ \delta\text{−function} \end{array} & \xrightarrow{f(z/y)\times} & \begin{array}{c} T(z)T(y)T(x)f(x/y)f(x/z)f(y/z) \\ +T(z)\ \delta\text{−function} \\ +T(y)\ \delta\text{−function} \\ +T(x)\ \delta\text{−function} \end{array}
\end{array} \tag{1}$$

determines this structure function $f(z)$ completely. Here the terms denoted by "δ-function" mean some combinations of the δ-functions and the structure functions. The

commutativity of this diagram means

$$
\begin{aligned}
0 &= T(x)\Big(\delta(pz/y)g(x/z) - \delta(py/z)g(x/y)\Big) \\
&+ T(y)\Big(\delta(px/z)g(y/x) - \delta(pz/x)g(y/z)\Big) \\
&+ T(z)\Big(\delta(py/x)g(z/y) - \delta(px/y)g(z/x)\Big),
\end{aligned}
\tag{2}
$$

where $g(x) \equiv f(x)f(x/p) - f(1/x)f(p/x)$. Note that $g(x) = -g(p/x)$. It is an interesting exercise to see that the general solution to the eq. (2) is $g(x) = c_1\Big(\delta(x/p) - \delta(x)\Big)$, where c_1 is a constant. Hereafter, we just set $c_1 = 1$, since there is no loss of generality. Note that the Yang-Baxter equation for $T(z)$ is "not" trivially satisfied even if the current $T(z)$ has no spin degrees of freedom, because we have the δ-function term in the relation (4). From this we have

$$
f(x)f(xp) = \alpha + \sum_{n=1}^{\infty}(1-p^n)x^n,
$$

where α is a constant. It should be noted that this is the place where one more parameter comes in. If we parameterize α by introducing another parameter q as

$$
\alpha = \frac{1-p}{(1-q)(1-t^{-1})}, \qquad t = qp^{-1},
$$

we have the difference equation

$$
f(x)f(xp) = \alpha\frac{(1-qx)(1-t^{-1}x)}{(1-x)(1-px)}.
$$

This can be solved as

$$
f(x) = \exp\left\{\sum_{n=1}^{\infty}\frac{(1-q^n)(1-t^{-n})}{1+p^n}\frac{x^n}{n}\right\}. \tag{3}
$$

We arrive at the definition of the quantum deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ [12].

Definition 1. *Let p and q be complex parameters with the conditions $|p| < 1$ and $|q| < 1$, and set $t = qp^{-1}$. The associative algebra $\mathcal{V}ir_{p,q}$ is generated by the current $T(z) = \sum_{n\in\mathbf{Z}} T_n z^{-n}$ satisfying the relation*

$$
f(w/z)T(z)T(w) - T(w)T(z)f(z/w) = -\frac{(1-q)(1-t^{-1})}{1-p}\left[\delta\Big(\frac{pw}{z}\Big) - \delta\Big(\frac{w}{pz}\Big)\right], \tag{4}
$$

with the structure function (3).

Note that the constant factor in the R.H.S. is so chosen that our Heisenberg realization of this algebra becomes simple.

We now have the associative algebra $\mathcal{V}ir_{p,q}$ which exists in a nontrivial way, since we have so chosen the structure function $f(z)$ that the Yang-Baxter equation for the $\mathcal{V}ir_{p,q}$ current (1) will not give us any more relations than the quadratic relation (4) for $T(z)$.

The defining relation (4) can be written in terms of T_n as

$$[T_n\,,\,T_m] = -\sum_{l=1}^{\infty} f_l\,(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p}(p^n - p^{-n})\delta_{m+n,0}, \tag{5}$$

where $[A,B] = AB - BA$.

The relation (4) is invariant under the transformations

$$\text{(I)} \qquad T_n \to -T_n, \tag{6}$$

$$\text{(II)} \qquad (q,t) \to (q^{-1},t^{-1}), \tag{7}$$

$$\text{(III)} \qquad q \leftrightarrow t. \tag{8}$$

In what follows, we will frequently use the notation

$$t = qp^{-1} = q^{\beta}. \tag{9}$$

This parameter β plays the role of the coupling constant of the Calogero-Sutherland model. See Section 3.3.

2.2 special limits of $\mathcal{V}ir_{p,q}$

Here we study some special limits of $\mathcal{V}ir_{p,q}$, which explains the connections among known examples of the Virasoro-type algebras.

2.2.1 limit of $q \to 1$: ordinary Virasoro algebra

Let us study the limit $q \to 1$ (β: fixed) by parameterizing $q = e^h$. Suppose that $T(z)$ has the following expansion in h

$$T(z) = 2 + \beta\left(z^2 L(z) + \frac{(1-\beta)^2}{4\beta}\right)h^2 + T^{(2)}(z)h^4 + \cdots. \tag{10}$$

This expansion is consistent with the invariance under transformation (7). The defining relation (4) gives us the well known relations for the ordinary Virasoro current $L(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}$, namely

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{11}$$

where the central charge c is

$$c = 1 - \frac{6(1-\beta)^2}{\beta}. \tag{12}$$

This relation between the central charge of the Virasoro algebra and the coupling constant of CS model is discussed in Section 3.3.

2.2.2 Frenkel-Reshetikhin's q-Virasoro algebra ($\beta \to 0$)

Let us consider the limit $\beta \to 0$ (q: fixed). In this limit, we obtain the classical q-Virasoro algebra found by Frenkel and Reshetikhin [8]. Their algebra is the first one among many q-deformed Virasoro algebras, which was obtained through the study of the quantum affine algebra $U_q(\widehat{sl}_2)$ [16]. Other nice features of their algebra are that it is no more a Lie algebra but a quadratic algebra which resembles the relation $\{L(u), L(v)\} = [r(u-v), L(u)L(v)]$ in the quantum inverse scattering method, and there exists a mysterious resemblance between their bosonic realization and the Baxter's dressed vacuum form in the Bethe ansatz method. See Section 4.1.

In this limit, T_n's become commutative. However, we can define the Poisson bracket structure defined by $\{\ ,\ \}_{\text{P.B.}} = -\lim_{\beta\to 0}[\ ,\ ]/(\beta \ln q)$. Thus we have

$$\{T_n, T_m\}_{\text{P.B.}} = \sum_{l\in\mathbf{Z}} \frac{1-q^l}{1+q^l} T_{n-l}T_{m+l} + (q^n - q^{-n})\delta_{n+m,0}, \tag{13}$$

which is the relations for the classical q-Virasoro algebra [8].

In the paper [9], deformations of the KdV hierarchy is studied. The N-th Korteweg-de Vries (KdV) hierarchy is a bihamiltonian integrable system with the N-th order differential operators. They are deformed to the q-shift operators

$$D^N - t_1(z)D^{N-1} + \cdots + (-1)^{N-1}t_{N-2}(z)D + (-1)^N t_{N-1}(z),$$

where $D \cdot f(z) = f(qz)$. It was shown there that the deformed bihamiltonian structure is given by the Poisson bracket for the q-$\mathcal{W}$ algebra of Frenkel and Reshetikhin [8].

Proposition 1. *[8] In the case of $N = 2$, the bihamiltonian structure is given by*

$$\{t_i(z), t_j(w)\}_1 = \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right), \tag{14}$$

$$\{t_i(z), t_j(w)\}_2 = \sum_{m\in\mathbf{Z}} \left(\frac{w}{z}\right)^m \frac{1-q^m}{1+q^m} t(z)t(w) + \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right). \tag{15}$$

These Poisson brackets coincide with that of the q-Virasoro algebra of Frenkel and Reshetikhin (13) with two different choices of the structure function $f(z)$: *i.e.*, $f(z) = 1$ and $f(z)$ given by (3) with $\beta \to 0$.

2.2.3 Zamolodchikov-Faddeev algebra of sine-Gordon and XYZ models

The q-Virasoro algebra can be interpreted as the Zamolodchikov-Faddeev (ZF) algebra satisfied by a particle excitation operator. The structure function $f(x)$ determined by the associativity relates with the S matrix characterized by the factorization property.

First, we can rewrite the defining relation of the q-Virasoro algebra (4) as follows;

$$T(z_1)T(z_2) = S\left(\frac{z_1}{z_2}\right) T(z_2)T(z_1) + C\left(\delta\left(\frac{z_1}{pz_2}\right) + \delta\left(\frac{pz_1}{z_2}\right)\right), \tag{16}$$

with

$$S(z) = \frac{f(z)}{f(z^{-1})} = \frac{\vartheta_1(zt^{-1};p)\,\vartheta_0(zt;p)}{\vartheta_1(zt;p)\,\vartheta_0(zt^{-1};p)}, \tag{17}$$

$$C = \frac{(1-q)(1-t^{-1})}{(1-p)f(p)} = \frac{(q;p^2)_\infty(t^{-1};p^2)_\infty}{(pq;p^2)_\infty(pt^{-1};p^2)_\infty}, \tag{18}$$

for $|p|<1$. Here, $\vartheta_1(z;p) = -ip^{\frac{1}{4}}z^{\frac{1}{2}}(p^2z;p^2)_\infty(z^{-1};p^2)_\infty(p^2;p^2)_\infty$ and $\vartheta_0(z;p) = (pz;p^2)_\infty\times$ $\times(pz^{-1};p^2)_\infty(p^2;p^2)_\infty$ with $(z;q)_\infty \equiv \prod_{n\geq 0}(1-zq^n)$.

Next, let $p = e^{\tau\pi i}$ and perform a modular transformation $\tau \to -1/\tau$ for theta functions and take a limit $-1/\tau \to i\infty$, *i.e.* $p \to 1$. Changing the parameterization as $z = p^{i\frac{\theta}{\pi}}$ and $t = p^{\xi}$, we have

Proposition 2. *[4][6] In the limit of $p \to 1$, the q-Virasoro generator satisfies the following Zamolodchikov-Faddeev algebra[7] of the sine-Gordon model,*

$$\mathcal{T}(\theta_1)\mathcal{T}(\theta_2) = S\left(\theta_1-\theta_2\right)\mathcal{T}(\theta_2)\mathcal{T}(\theta_1), \qquad S(\theta) = \frac{\sinh\theta + i\sin\pi\xi}{\sinh\theta - i\sin\pi\xi}, \tag{19}$$

where $\mathcal{T}(\theta) = \lim_{-1/\tau\to i\infty} T(z)$.

Namely, the $p=1$ q-Virasoro generator $\mathcal{T}(z)$ creates the basic particle (the first breather) with a rapidity θ of the sine-Gordon model, defined by the following Lagrangian density

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu\phi)^2 + \left(\frac{m_0}{b}\right)^2\cos\left(b\phi\right), \qquad b^2 = 8\pi\beta = 8\pi\frac{\xi}{1+\xi}, \tag{20}$$

at the attractive range $0<\xi<1$. This identification is based on the coincidence of the S-matrix for the basic scalar particles in sine-Gordon model with $f(z)/f(1/z)$ in the limit of $p\to 1$, and the existence of simple poles at the points $\theta_1 = \theta_2 \pm i\pi$ in the delta function terms. For the axiom of the ZF operator, see [18]. Lukyanov also proposed in [4] that, for generic p, eq. (16) can be interpreted as the ZF relation of the XYZ model, *i.e.*, the q-Virasoro generator is the basic scalar creation operator of this model.

2.2.4 limit of $q \to 0$

Next, study the $q\to 0$ limit (t: fixed). This limit is interesting from the point of view of the representation theory of the Hall-Littlewood polynomials [19] in terms of the Heisenberg algebra. It is known by the work of Jing [20] that the Hall-Littlewood polynomials are realized by a multiple integral formula. The integration kernel is simply given by multiplying vertex operators on a vacuum state. As for the number of the integration variables and the vertex operators, it is related with the shape of the Young diagram of each Hall-Littlewood polynomial. His realization can be regarded as a deformation of the determinant representation of the Schur polynomials. Recently, similar integral

[6] The notations in [4] are $x = p^{\frac{1}{2}}$, $\xi = \beta/(1-\beta)$ and $\epsilon = i\tau$.

[7] This ZF-equation should be understood in the sense of analytic continuation.

representations are studied for the Jack polynomials and the Macdonald polynomials [21, 19, 23, 10, 11]. However, the number of the integrals there is much greater in general than the case of the Schur or Hall-Littlewood polynomials. We will discuss this in Section 3.6 by using the Heisenberg realization of $\mathcal{V}ir_{p,q}$ in the limit $q \to 0$.

So as to obtain well behaving generators at $q \to 0$, let us scale T_n as

$$\tilde{T}_n = T_n p^{\frac{|n|}{2}}. \tag{21}$$

Using this notation and taking the limit ($q \to 0$) of the relation (4), we have the commutation relation for the deformed Virasoro algebra in this limit.

Proposition 3. *The commutation relations for the deformed Virasoro generators $\tilde{T}_n$ are*

$$\begin{aligned}
\left[\tilde{T}_n, \tilde{T}_m\right] &= -(1-t^{-1}) \sum_{\ell=1}^{n-m} \tilde{T}_{n-\ell}\tilde{T}_{m+\ell} \quad \textit{for} \quad n > m > 0 \quad \textit{or} \quad 0 > n > m, \\
\left[\tilde{T}_n, \tilde{T}_0\right] &= -(1-t^{-1}) \sum_{\ell=1}^{n} \tilde{T}_{n-\ell}\tilde{T}_{\ell} - (t-t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell}\tilde{T}_{-\ell}\tilde{T}_{n+\ell} \quad \textit{for} \quad n > 0, \\
\left[\tilde{T}_0, \tilde{T}_m\right] &= -(1-t^{-1}) \sum_{\ell=1}^{-m} \tilde{T}_{-\ell}\tilde{T}_{m+\ell} - (t-t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell}\tilde{T}_{m-\ell}\tilde{T}_{\ell} \quad \textit{for} \quad 0 > m, \\
\left[\tilde{T}_n, \tilde{T}_m\right] &= -(1-t^{-1})\tilde{T}_m\tilde{T}_n - (t-t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell}\tilde{T}_{m-\ell}\tilde{T}_{n+\ell} \\
&\quad +(1-t^{-1})\mathrm{sign}(n)\delta_{n+m,0} \quad \textit{for} \quad n > 0 > m,
\end{aligned} \tag{22}$$

where the function $\mathrm{sign}(n)$ *is* 1, 0 *and* -1 *for* $n > 0$, $n = 0$ *and* $n < 0$, *respectively.*

2.3 highest weight modules of $\mathcal{V}ir_{p,q}$

Let us define the Verma module of $\mathcal{V}ir_{p,q}$. Let $|\lambda\rangle$ be the highest weight vector such that $T_0|\lambda\rangle = \lambda|\lambda\rangle$, $\lambda \in \mathbf{C}$ and $T_n|\lambda\rangle = 0$ for $n > 0$. The Verma module $M(\lambda)$ is defined by $M(\lambda) = \mathcal{V}ir_{p,q}\,|\lambda\rangle$. The irreducible highest module $V(\lambda)$ is obtained from $M(\lambda)$ by removing all singular vectors and their descendants. Right modules are defined in a similar way from the lowest weight vector $\langle\lambda|$ s.t. $\langle\lambda|T_0 = \lambda\langle\lambda|$ and $\langle\lambda|T_n = 0$ for $n < 0$. A unique invariant paring is defined by setting $\langle\lambda|\lambda\rangle = 1$. The Verma module $M(\lambda)$ may have singular vectors same as that of the ordinary Virasoro algebra. Let us introduce the (outer) grading operator d which satisfies $[d, T_n] = nT_n$ and set $d|\lambda\rangle = 0$. We call a vector $|v\rangle \in M(\lambda)$ of level n if $d|v\rangle = -n|v\rangle$.

Whether there exist the singular vectors or not is checked by calculating the Kac determinant. Here, we give some explicit forms of f_n which we will use for the calculations

$$\begin{aligned}
f_1 &= \frac{(1-q)(1-t^{-1})}{1+p}, \\
f_2 &= \frac{(1-q^2)(1-t^{-2})}{2(1+p^2)} + \frac{(1-q)^2(1-t^{-1})^2}{2(1+p)^2}.
\end{aligned}$$

At level 1, the Kac determinant is the 1×1 matrix as follows

$$\langle\lambda|T_1T_{-1}|\lambda\rangle = \frac{(1-q)(1-t)}{q+t}(\lambda^2 - (p^{1/2}+p^{-1/2})^2). \tag{23}$$

Therefore, there exist a singular vector at level 1 iff $\lambda = \pm\left(p^{1/2}+p^{-1/2}\right)$, since q and t are generic. The signs $\pm$ in the RHS are due to the symmetry (6).

At level 2, the Kac determinant is

$$\begin{vmatrix} \langle\lambda|T_1T_1T_{-1}T_{-1}|\lambda\rangle & \langle\lambda|T_1T_1T_{-2}|\lambda\rangle \\ \langle\lambda|T_2T_{-1}T_{-1}|\lambda\rangle & \langle\lambda|T_2T_{-2}|\lambda\rangle \end{vmatrix} = \frac{(1-q^2)(1-q)^2q^{-4}(1-t^2)(1-t)^2t^{-4}}{(q+t)^2(q^2+t^2)}$$
$$\times(\lambda^2qt-(q+t)^2)(\lambda^2q^2t-(q^2+t)^2)(\lambda^2qt^2-(q+t^2)^2). \tag{24}$$

The vanishing conditions of the Kac determinant are

$$\text{(i)} \quad \lambda = \pm\left(p^{1/2}+p^{-1/2}\right), \tag{25}$$

$$\text{(ii)} \quad \lambda = \pm\left(p^{1/2}q^{1/2}+p^{-1/2}q^{-1/2}\right), \tag{26}$$

$$\text{(iii)} \quad \lambda = \pm\left(p^{1/2}t^{-1/2}+p^{-1/2}t^{1/2}\right). \tag{27}$$

In the case (i), there is a singular vector at level 1. In the cases (ii) and (iii), we have a singular vector at level 2. The singular vector for the case (ii) is

$$\frac{qt^{-1/2}(q+t)}{(1-q)^2(1+q)}T_{-1}T_{-1}|\lambda\rangle \mp T_{-2}|\lambda\rangle, \tag{28}$$

and for (iii) is

$$\frac{q^{-1/2}t(q+t)}{(1-t)^2(1+t)}T_{-1}T_{-1}|\lambda\rangle \mp T_{-2}|\lambda\rangle. \tag{29}$$

To calculate the Kac determinant becomes difficult task when N increases. We have calculated up to level 4, and write down the conjectural form at level N.

Conjecture 1. *The Kac determinant at level-N is written as*

$$\det{}_N = \det\Big(\langle i|j\rangle\Big)_{1\le i,j\le p(N)} = \prod_{\substack{r,s\ge 1\\ rs\le N}} \left(\lambda^2-\lambda_{r,s}^2\right)^{p(N-rs)} \left(\frac{(1-q^r)(1-t^r)}{q^r+t^r}\right)^{p(N-rs)}, \tag{30}$$

where $\lambda_{r,s} = t^{r/2}q^{-s/2}+t^{-r/2}q^{+s/2}$ and the basis at level N is defined $|1\rangle = T_{-N}|\lambda\rangle$, $|2\rangle = T_{-N+1}T_{-1}|\lambda\rangle, \cdots, |p(N)\rangle = T_{-1}^N|\lambda\rangle$, and $p(N)$ is the number of the partition of N.

We remark that the λ dependence has essentially the same structure as the case of the usual Virasoro algebra. Therefore, if q and t are generic, the character of the quantum Virasoro algebra $\mathcal{V}ir_{p,q}$, which counts the degeneracy at each level, exactly coincides with that of the usual Virasoro algebra. The λ-independent factor in the RHS will play an important role when we study the case that q is a root of unity.

2.4 problem of obtaining a geometric interpretation of $\mathcal{V}ir_{p,q}$

It is remarkable that $\mathcal{V}ir_{p,q}$ arises in a variety of off-critical models in a universal way. As for the geometric interpretation, however, we have not have a satisfactory answer yet. For the ordinary Virasoro algebra with $c = 0$, we have the differential operator realization, $L_n = -z^{n+1}\partial_z$, which explains that the Virasoro algebra describes the Lie algebra structure of the tangent space of the conformal group. As a natural deformation of this differential operator realization, is it possible to have a difference operator representation of the deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ for some parameters q and p? It may be possible to study a connection between $\mathcal{V}ir_{p,q}$ and the analysis over the local fields in the limit of $q \to 0$ with fixed t. See Section 3.6.

3 Free boson realization of $\mathcal{V}ir_{p,q}$

In this section, we present the Heisenberg realization of $\mathcal{V}ir_{p,q}$ and its applications to Calogero-Sutherland-type models.

3.1 conformal field theory

One of the simplest example of the conformal field theory is the massless Klein-Gordon field in $1+1$ dimensions. We briefly review how we can treat the Virasoro current in terms of the Klein-Gordon field at the conformal point to prepare basic ideas and notations for the later discussions. As for the detail, the readers are referred to the original or review articles of the conformal field theory [1, 22]. The action for the Klein-Gordon field $\phi(x, \tau)$ is

$$S_{Eucl} = \int d\tau dx \frac{1}{2} \left((\partial_\tau \phi)^2 + (\partial_x \phi)^2 + m^2 \phi^2 \right).$$

If the system is massless $m = 0$ then it acquires the infinite dimensional conformal symmetry. We shall see how the generators of this conformal transformation are realized in terms of the Klein-Gordon field $\phi(x, \tau)$. Looking at the equation of motion

$$\partial_w \partial_{\bar{w}} \phi(w, \bar{w}) = 0, \qquad (w = x + i\tau), \tag{31}$$

we have the decoupling of ϕ into chiral and anti-chiral parts as

$$\phi(w, \bar{w}) = a(w) + \bar{a}(\bar{w}).$$

Therefore, we can study the chiral part and anti-chiral part separately. After the compactification of the space into the segment $0 \leq x \leq 2\pi$ with the periodic boundary condition and introducing the conformal mapping $z = e^{iw}$, we arrive at the expansion

$$a(z) = Q + a_0 \ln z - \sum_{n \neq 0} \frac{a_n}{n} z^{-n}. \tag{32}$$

The Poisson brackets for these modes are

$$\{a_n, a_m\}_{\text{P.B.}} = n\delta_{n+m,0} \qquad \{a_n, Q\}_{\text{P.B.}} = \delta_{n,0}. \tag{33}$$

The Virasoro current $L(z)$ is written as

$$L(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} = \frac{1}{2}\left(\partial a(z)\right)^2.$$

Using the formula

$$\{\partial a(z), \partial a(w)\}_{\text{P.B.}} = \frac{1}{z^2}\delta'(w/z),$$

we obtain

$$\{L_n, L_m\}_{\text{P.B.}} = (n-m)L_{n+m}. \tag{34}$$

To quantize the system, we replace the Poisson brackets by the commutators as

$$[a_n, a_m] = n\delta_{n+m} \qquad [a_n, Q] = \delta_{n,0}, \tag{35}$$

and define the Virasoro current with the normal ordered product as

$$L(z) = \frac{1}{2} : \left(\partial a(z)\right)^2 : .$$

The definition of this normal ordering is that we shift every annihilation operators (*i.e.*, a_n with $n \geq 0$) to the right of creation operators (*i.e.*, a_n with $n < 0$ and Q). For example $: a_{-1}a_2 := a_{-1}a_2$, $: a_3 a_{-2} := a_{-2}a_3$ and so on. This quantized Virasoro current obeys eq. (11) with $c = 1$. In general the central charge c depends on the model; $c = 1/2$ for real fermion, $c = 3k/(k+2)$ for $\widehat{su}(2)_k$ Kac-Moody algebra, for example. One more important construction of the Virasoro algebra which is relevant to our later discussion is the Feigin-Fuchs construction

$$L(z) = \frac{1}{2} : \left(\partial a(z)\right)^2 : + \alpha_0 \partial^2 a(z).$$

For this realization we have the central charge less than one

$$c = 1 - 12\alpha_0^2,$$

if α_0 is real. For the later discussion we will parameterize α_0 as

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(\sqrt{\beta} - \sqrt{1/\beta}\right).$$

We will see that this parameter β has the meaning of the coupling constant of the Calogero-Sutherland model. See Section 3.3.

3.2 singular vectors of the Virasoro algebra

What is very special in the representation space is the singular vectors, which correspond to decoupled states from the physical space. We review some of the explicit formulas of the singular vectors.

The highest weight state $|h\rangle$ is defined by $L_n|h\rangle = 0$ for $n > 0$ and $L_0|h\rangle = h|h\rangle$, and the Verma module $M(h)$ is spanned over the highest weight state $|h\rangle$ as $M(h) = \langle L_{-1}, L_{-2}, \cdots\rangle|h\rangle$. The singular vector $|\chi\rangle \in M(h)$ at level n is defined by $L_n|\chi\rangle = 0$ for $n > 0$ and $L_0|\chi\rangle = (h+n)|\chi\rangle$. By a standard argument, it has null norm with any states in the Verma module; $\langle *|\chi\rangle = 0$. The existence of such state depends crucially on the choice of parameter c and h. Celebrated Kac formula shows that if they are explicitly parameterized as eq. (12) and

$$h_{rs} = \frac{(\beta r - s)^2 - (\beta - 1)^2}{4\beta}, \tag{36}$$

for an arbitrary parameter $\beta(\neq 0) \in \mathbf{C}$ and integers r and s with $rs > 0$, there exists unique (up to normalizatin) null state of level rs. Some of the lower lying states can be explicitly obtained by solving the defining conditions. Let $|\chi_{rs}\rangle \in M(h_{rs})$ be the null state at level rs. For example, we obtain,

$$\begin{aligned}
|\chi_{11}\rangle &= L_{-1}|h_{11}\rangle, \\
|\chi_{12}\rangle &= (L_{-2} - \beta L_{-1}^2)|h_{12}\rangle, \\
|\chi_{22}\rangle &= \left(L_{-4} + \frac{2(\beta^2 - 3\beta + 1)}{3(\beta-1)^2} L_{-3}L_{-1} - \frac{(\beta+1)^2}{3\beta} L_{-2}^2 \right. \\
&\quad \left. + \frac{2(\beta^2+1)}{3(\beta-1)^2} L_{-2}L_{-1}^2 - \frac{\beta}{3(\beta-1)^2} L_{-1}^4 \right) |h_{22}\rangle,
\end{aligned} \tag{37}$$

and so on.

Since we have the Feigin-Fuchs realization of $L(z)$, the singular vectors are also written in the bosonic creation oscillators a_{-n} acting on the vacuum state $|\alpha\rangle$ (*i.e.*, $a_n|\alpha\rangle = 0$ for $n > 0$ and $a_0|\alpha\rangle = \alpha|\alpha\rangle$). It is quite remarkable that all these singular vectors can be regarded as the "Jack symmetric polynomials" when we replace a_{-n} by the power sum $\sum_{i=1}^N x_i^n$. As for the proof of this correspondence the reader is referred to [23].

In the Feigin-Fuchs construction, the Virasoro generators are

$$L_n = \frac{1}{2} \sum_{m \in \mathbf{Z}} : a_{n+m} a_{-m} : - \alpha_0 (n+1) a_n. \tag{38}$$

The highest weight state $|h_{rs}\rangle$ is realized as $|\alpha_{rs}\rangle \equiv e^{\alpha_{rs} Q}|0\rangle$, with

$$\alpha_{rs} = \frac{1}{\sqrt{2}} \left((1+r)\sqrt{\beta} - (1+s)\sqrt{1/\beta} \right). \tag{39}$$

In terms of this free boson oscillators, the null states (37) are written as,

$$
\begin{aligned}
|\chi_{11}\rangle &= a_{-1}|\alpha_{11}\rangle, \\
|\chi_{12}\rangle &= \left(a_{-2}+\sqrt{2\beta}a_{-1}^2\right)|\alpha_{12}\rangle, \\
|\chi_{22}\rangle &= \left(a_{-4}+\frac{4\sqrt{2\beta}}{1-\beta}a_{-3}a_{-1}-2\frac{1+\beta+\beta^2}{\sqrt{2\beta}(1-\beta)}a_{-2}^2-4a_{-2}a_{-1}^2-\frac{2\sqrt{2\beta}}{1-\beta}a_{-1}^4\right)|\alpha_{22}\rangle.
\end{aligned}
\tag{40}
$$

To translate these expressions into the Jack symmetric functions, one can apply the rule,

$$
a_{-n}\to\sqrt{\frac{\beta}{2}}\sum_{i=1}^{N}x_i^n, \qquad |\alpha_{rs}\rangle\to 1. \tag{41}
$$

Using this rule, we have the correspondence [23]

$$
|\chi_{rs}\rangle\sim J_{\{s^r\}}(x;\beta), \tag{42}
$$

where the R.H.S. is the Jack symmetric polynomial for the rectangular diagram $\{s^r\}$.

It was shown in [10] that the Jack polynomials for arbitrary Young diagrams are realized as the singular vector of the W_N algebra.

3.3 Calogero-Sutherland Hamiltonian, the Jack polynomials and the Virasoro algebra

In this section, we explain the "Calogero-Sutherland-Virasoro" correspondence. As for the details, the readers are referred to [10] and references therein.

The Jack symmetric polynomials arise in the Calogero-Sutherland (CS) model [24] as the wave functions of the excited states of this model. After a suitable coordinate transformation, the Hamiltonian and momentum of this system become

$$
\mathcal{H}=\sum_{i=1}^{N}D_i^2+\beta\sum_{i<j}\frac{x_i+x_j}{x_i-x_j}(D_i-D_j), \qquad \mathcal{P}=\sum_{i=1}^{N}D_i. \tag{43}
$$

where $D_i\equiv x_i\partial_{x_i}$. Their eigenfunctions are called as the Jack symmetric polynomials $J_\lambda(x;\beta)$ and the eigenvalues are

$$
\epsilon_\lambda=\sum_{i=1}^{M}\Big(\lambda_i^2+\beta(N+1-2i)\lambda_i\Big), \tag{44}
$$

with a Young diagram $\lambda=(\lambda_1,\cdots,\lambda_M)$, $\lambda_i\geq\lambda_{i+1}\in\mathbf{Z}_{\geq 0}$.

When we consider the limit of $N\to\infty$, $J_\lambda(x;\beta)$ must be regarded as a "symmetric function" [19]. It is the advantage of this limit that we are able to have the Heisenberg realization of the symmetric function $J_\lambda(x;\beta)$. In other words, the Jack symmetric functions are realized in the Fock space of the bosonic field [23, 10].

Let us modify the normalization of the bosonic oscillators as

$$\left[a_n, a_m\right] = \frac{1}{\beta} n \delta_{n+m,0}, \qquad [a_n, Q] = \frac{1}{\beta}\delta_{n,0}, \tag{45}$$

to make the "polynomial-boson" correspondence simple. Thus, the modification is ($n > 0$)

$$a_n = \sqrt{\frac{1}{2\beta}} a_n^{\rm old}, \qquad a_{-n} = \sqrt{\frac{2}{\beta}} a_{-n}^{\rm old}, \qquad a_0 = \sqrt{\frac{1}{2\beta}} a_0^{\rm old}, \qquad Q = \sqrt{\frac{2}{\beta}} Q^{\rm old}. \tag{46}$$

Correspondingly we change the notation for α_{rs} as

$$\alpha_{rs} = \frac{1}{2}\left((1+r)\beta - (1+s)\right),$$

and write $|h_{rs}\rangle = |\alpha_{rs}\rangle \equiv e^{\alpha_{rs} Q}|0\rangle$ as before.

One may derive bosonized Hamiltonian and momentum $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{P}}$ which satisfy $\mathcal{O}\pi_N\langle 0|\exp(\beta\sum_n \frac{a_n}{n}p_n) = \pi_N\langle 0|\exp(\beta\sum_n \frac{a_n}{n}p_n)\widehat{\mathcal{O}}$, where $\mathcal{O} = \mathcal{H}, \mathcal{P}$, and π_N denotes the projection to the N-particle space. They are given by,

$$\widehat{\mathcal{H}} = \beta \sum_{n>0} a_{-n} L_n + (\beta - 1 + \beta N - 2a_0)\,\widehat{\mathcal{P}}, \qquad \widehat{\mathcal{P}} = \beta \sum_{n=1}^{\infty} a_{-n} a_n. \tag{47}$$

Here L_n's are the annihilation operators of the Feigin-Fuchs construction of the Virasoro algebra with the center c in (12). Using these formulas, it is easily shown that the singular vector $|\chi_{rs}\rangle$ of the Virasoro algebra is proportional to the Jack polynomial for the Young diagram $\lambda = \{s^r\}$, since we have

$$\widehat{\mathcal{H}}|\chi_{rs}\rangle = \epsilon_{\{s^r\}}|\chi_{rs}\rangle. \tag{48}$$

with $\epsilon_{\{s^r\}} = (\beta(N-r)+s)\,rs$.

3.4 screening currents

In the Feigin-Fuchs construction we are able to have two weight-one primary fields $S_\pm(z)$, which are called screening currents. The condition of being weight-one primary gives us the equation

$$[L_n, S_\pm(z)] = \partial_z\left(z^{n+1} S_\pm(z)\right). \tag{49}$$

We have the solutions of this equation as follows

$$S_+(z) = \exp\left\{\sum_{n=1}^{\infty} \beta \frac{a_{-n}}{n} z^n\right\} \exp\left\{-\sum_{n=1}^{\infty} 2\beta \frac{a_n}{n} z^{-n}\right\} e^{\beta Q} z^{2\beta a_0}, \tag{50}$$

$$S_-(z) = \exp\left\{-\sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n\right\} \exp\left\{\sum_{n=1}^{\infty} 2\frac{a_n}{n} z^{-n}\right\} e^{-Q} z^{-2a_0}. \tag{51}$$

In the Fock module with the highest weight state $|\alpha_{r,s}\rangle$, we have a singular vector $|\chi_{r,s}\rangle$ at level rs. By using a screening current $S_+(z)$, $|\chi_{r,s}\rangle$ is given as follows [25, 26]:

$$|\chi_{r,s}\rangle = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i} \cdot \prod_{i=1}^{r} S_+(z_i)|\alpha_{-r,s}\rangle \tag{52}$$

$$= \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \cdot \prod_{\substack{i,j=1 \\ i<j}}^{r} (z_i - z_j)^{2\beta} \cdot \prod_{i=1}^{r} z_i^{(1-r)\beta - s} : \prod_{j=1}^{r} S_+(z_i) : |\alpha_{r,s}\rangle, \tag{53}$$

where the integration contour is the Felder's one [27]. Note that there is a similar formula for representing $|\chi_{r,s}\rangle$ by using $S_-(z)$.

This method of writing the singular vectors in terms of the screening currents thus gives us a systematic way to represent the Jack polynomials for rectangular diagrams.

3.5 Macdonald symmetric polynomials and $\mathcal{V}ir_{p,q}$

So far, we have explained the relations among the Calogero-Sutherland Hamiltonian, Jack polynomials and the Virasoro algebra. We want to study what will happen if we replace the Jack symmetric polynomial with the Macdonald symmetric polynomial. It will be shown that some of the Macdonald symmetric polynomials $P_\lambda(x;q,t)$ are related with the singular vectors of the quantum deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $p = qt^{-1}$.

The Macdonald symmetric polynomial $P_\lambda(x;q,t)$ is the eigenfunction of the Macdonald shift operator [19]

$$D_{q,t} = \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \tag{54}$$

where T_{q,x_i} is the q-shift operator,

$$T_{q,x_i} f(x_1, \cdots, x_N) = f(x_1, \cdots, qx_i, \cdots, x_N). \tag{55}$$

This shift operator plays the same role as the CS Hamiltonian $\mathcal{H}$. Here, a new complex deformation parameter q is introduced and t is related to β by $t = q^\beta$. In the limit of $q \to 1$ with fixed β, the Macdonald polynomials reduce to the Jack polynomials.

Amazingly, one may find a closed form of the bosonized Macdonald operator $\widehat{D}_{q,t}$ as

$$D_{q,t}\pi_N\langle 0| \exp\left\{\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n}\frac{a_n}{n}p_n\right\} = \pi_N\langle 0| \exp\left\{\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n}\frac{a_n}{n}p_n\right\} \widehat{D}_{q,t}, \tag{56}$$

$$\widehat{D}_{q,t} = \frac{t^N}{t-1}\oint \frac{dz}{2\pi i}\frac{1}{z} \exp\left\{\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^n\right\} \exp\left\{-\sum_{n=1}^{\infty} \frac{1-t^n}{n} a_n z^{-n}\right\} - \frac{1}{t-1}, \tag{57}$$

where the commutation relations for the bosonic oscillators are deformed as

$$[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}. \tag{58}$$

We can write the "natural criterion" for the quantum deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ associated with the Heisenberg realization of the Macdonald symmetric polynomials schematically in the following diagram.

$$\begin{array}{ccc}
\begin{array}{c}\text{Calogero-Sutherland Hamiltonian}\\ \widehat{\mathcal{H}} = \sum_{n=1}^{\infty} a_{-n} L_n + \cdots \\ L_n J_{\{s^r\}}(x;\beta) = 0 \quad \text{for } n > 0\end{array} & \xleftarrow{\text{singular vectors}} & L_n \text{ generates the Virasoro algebra} \\
q\text{-deformation} \Big\downarrow & & q\text{-deformation} \Big\downarrow \\
\begin{array}{c}\text{Macdonald difference operator}\\ \widehat{D}_{q,t} = \sum_{n=1}^{\infty} \psi_{-n} T_n + \cdots \\ T_n P_{\{s^r\}}(x;q,t) = 0 \quad \text{for } n > 0\end{array} & \xleftarrow{\text{singular vectors}} & T_n \text{ generates the } q\text{-Virasoro algebra } \mathcal{V}ir_{p,q}
\end{array}$$

Here, ψ_{-n} should be a suitable combination of the creation operators a_{-m}'s with degree n.

The problem is "to make this diagram commutative." However, we have many unknown operators ψ_{-n} and T_n. To have enough data to solve this problem, some knowledge of the q-deformed screening operators must be needed.

By studying the action of the bosonized Macdonald operator $\widehat{D}_{q,t}$, we are able to obtain bosonized realization for some of the Macdonald polynomials. Let,

$$\exp\left\{\sum_{n=1}^{\infty} \frac{1-q^{\gamma n}}{1-q^n} \frac{a_{-n}}{n} z^n\right\} = \sum_{n=0}^{\infty} \widehat{Q}_n^{(\gamma)} z^n \tag{59}$$

the states $\widehat{Q}_n^{(\gamma)}|0\rangle$ with $\gamma = \beta$ or -1 are the Macdonald polynomials $Q_\lambda(x;q,t)$ corresponding to the Young diagram with single row (n) or single column (1^n), respectively. As for the difference between $P_\lambda(x;q,t)$ and $Q_\lambda(x;q,t)$, see [19]. We obtained other examples; for the Young diagram with two rows $\lambda = (\lambda_1, \lambda_2)$ or two columns ${}^t\lambda = (\lambda_1, \lambda_2)$ we have

$$\begin{aligned}
\widehat{Q}^{(\gamma)}_{(\lambda_1,\lambda_2)}|0\rangle &= \sum_{\ell=0}^{\lambda_2} c^{(\gamma)}(\lambda_1-\lambda_2,\ell)\widehat{Q}^{(\gamma)}_{\lambda_1+\ell}\widehat{Q}^{(\gamma)}_{\lambda_2-\ell}|0\rangle, \\
c^{(\gamma)}(\lambda,\ell) &= \frac{1-q^{\frac{\beta}{\gamma}(\lambda+2\ell)}}{1-q^{\frac{\beta}{\gamma}(\lambda+\ell)}} \prod_{j=1}^{\ell} \frac{1-q^{\frac{\beta}{\gamma}(\lambda+j)}}{1-q^{\frac{\beta}{\gamma}j}} \cdot \prod_{i=1}^{\ell} \frac{q^{\gamma}-q^{\frac{\beta}{\gamma}(i-1)}}{1-q^{\gamma+\frac{\beta}{\gamma}(\lambda+i)}},
\end{aligned} \tag{60}$$

with $\gamma = \beta$ or -1, respectively. The Macdonald polynomials of single hook $(n, 1^m)$ are,

$$\widehat{Q}_{(n,1^m)}|0\rangle = \sum_{\ell=0}^{m} \frac{1-q^{n+\ell}t^{m-\ell}}{1-q} q^{m-\ell}\, \widehat{Q}^{(\beta)}_{n+\ell}\widehat{Q}^{(-1)}_{m-\ell}|0\rangle. \tag{61}$$

These explicit formulas in terms of (59) strongly suggest that the screening currents for $\mathcal{V}ir_{p,q}$ should be

$$S_+(z) = \exp\left\{\sum_{n=1}^{\infty}\frac{1-t^n}{1-q^n}\frac{a_{-n}}{n}z^n\right\}\exp\left\{\text{annihilation part for} S_+\right\}e^{\beta Q}z^{2\beta a_0}, \tag{62}$$

$$S_-(z) = \exp\left\{-\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right\}\exp\left\{\text{annihilation part for} S_-\right\}e^{-Q}z^{-2a_0}. \tag{63}$$

Note that this can be regarded as a good deformation of the screening currents (50) and (51). It is assumed that the zero-mode parts are not deformed.

In [12], it is shown that the problem of finding all these unknown parts for $\widehat{D}_{q,t}$, ψ_n and $S^{\pm}(z)$ is "uniquely solved" if we start from a nice ansatz for the operators. The reason or mechanism behind the process of solving the problem have not been well investigated yet and seem somehow mysterious. What is clear so far is that the "locality" for the operators seems quite important, i.e., the behaviors of the δ-functions in the OPE factors must be well controlled in some sence. The mathematical structure of this "locality" should be understood in the future. Therefore, we will not show the processes of solving this problem here. Let us just summarize the results.

Theorem 1. *[12] The quantum deformed Virasoro current $T(z)$ is realized as*

$$T(z) = \Lambda^+(z) + \Lambda^-(z), \tag{64}$$

where

$$\Lambda^+(z) = p^{1/2}\exp\left\{-\sum_{n=1}^{\infty}\frac{1-t^n}{1+p^n}\frac{a_{-n}}{n}z^n t^{-n}p^{-n/2}\right\}\exp\left\{-\sum_{n=1}^{\infty}(1-t^n)\frac{a_n}{n}z^{-n}p^{n/2}\right\}q^{\beta a_0},$$

$$\Lambda^-(z) = p^{-1/2}\exp\left\{\sum_{n=1}^{\infty}\frac{1-t^n}{1+p^n}\frac{a_{-n}}{n}z^n t^{-n}p^{n/2}\right\}\exp\left\{\sum_{n=1}^{\infty}(1-t^n)\frac{a_n}{n}z^{-n}p^{-n/2}\right\}q^{-\beta a_0}. \tag{65}$$

By studying the operator product expansions, it is easy to see that this $T(z)$ satisfies the relation of $\mathcal{V}ir_{p,q}$ (4). We can observe that this formula has strong resemblance to the dressed vacuum form (DVF) in the algebraic Bethe ansatz. This profound q-Virasoro-DVF correspondence ($T(z) = \Lambda^+(z) + \Lambda^-(z)$) was discovered by Frenkel and Reshetikhin [8].

Theorem 2. *[12] The Macdonald operator is written as*

$$\begin{aligned}\widehat{D}_{q,t} &= \frac{t^N}{t-1}\left[\oint\frac{dz}{2\pi\sqrt{-1}}\frac{1}{z}\psi(z)T(z) - p^{-1}q^{-2\beta a_0}\right] - \frac{1}{t-1}\\ &= \frac{t^N}{t-1}\left[\sum_{n=0}^{\infty}\psi_{-n}T_n - p^{-1}q^{-2\beta a_0}\right] - \frac{1}{t-1},\end{aligned} \tag{66}$$

where the field $\psi(z)$ is given by

$$\psi(z) = \sum_{n=0}^{\infty}\psi_{-n}z^n = p^{-1/2}\exp\left\{-\sum_{n=1}^{\infty}\frac{1-t^n}{1+p^n}\frac{a_{-n}}{n}z^n p^{n/2}t^{-n}\right\}q^{-\beta a_0}. \tag{67}$$

Theorem 3. *[12] The screening currents for $\mathcal{V}ir_{p,q}$*

$$S_+(z) = \exp\left\{\sum_{n=1}^{\infty}\frac{1-t^n}{1-q^n}\frac{a_{-n}}{n}z^n\right\}\exp\left\{-\sum_{n=1}^{\infty}(1+p^n)\frac{1-t^n}{1-q^n}\frac{a_n}{n}z^{-n}\right\}e^{\beta Q}z^{2\beta a_0}, \quad (68)$$

$$S_-(z) = \exp\left\{-\sum_{n=1}^{\infty}\frac{a_{-n}}{n}z^n\right\}\exp\left\{\sum_{n=1}^{\infty}(1+p^n)\frac{a_n}{n}z^{-n}p^{-n}\right\}e^{-Q}z^{-2a_0}, \quad (69)$$

satisfies the commutation relation:

$$\left[T_n, S_+(w)\right] = -(1-q)(1-t^{-1})\frac{d_q}{d_q w}\left((p^{-\frac{1}{2}}w)^{n+1}p^{\frac{1}{2}} : \Lambda^-(p^{-\frac{1}{2}}w)S_+(w) :\right), \quad (70)$$

$$\left[T_n, S_-(w)\right] = -(1-q^{-1})(1-t)\frac{d_t}{d_t w}\left((p^{\frac{1}{2}}w)^{n+1}p^{-\frac{1}{2}} : \Lambda^+(p^{\frac{1}{2}}w)S_-(w) :\right), \quad (71)$$

where the difference operator with one parameter is defined by

$$\frac{d_\xi}{d_\xi z}g(z) = \frac{g(z)-g(\xi z)}{(1-\xi)z}. \quad (72)$$

Since the Kac determinant has the same structure as the ordinary Virasoro algebra has, the structure of the singular vectors are not changed in an essential manner. As for the case of the Jack polynomials, we can write down all the singular vectors at least formally in the following way

$$|\chi_{r,s}\rangle = \oint\prod_{j=1}^{r}\frac{dz_j}{2\pi i}\cdot\prod_{i=1}^{r}S_+(z_i)|\alpha_{-r,s}\rangle, \quad (73)$$

however, we have to be very careful about the integration cycle. Recently, very nice construction of the integration cycles together with some modification of the integration kernel is achieved in [6, 28]. A q-deformation of the Felder complex is constructed in these works, and mathematical treatment becomes much "easier" than the original case.

Finally, we have as the Jack case,

Theorem 4. *There exists a one to one correspondence between the singular vectors $|\chi_{r,s}\rangle$ of the q-Virasoro algebra $\mathcal{V}ir_{p,q}$ and the Macdonald functions $P_{\{s^r\}}(x;q,t)$ with the rectangular Young diagram $\{s^r\}$ up to normalization constants. It is simply given by*

$$P_{\{s^r\}}(x;q,t) \propto \langle\alpha_{r,s}|\exp\left\{\sum_{n=1}^{\infty}\frac{1-t^n}{1-q^n}\frac{a_n}{n}p_n\right\}|\chi_{r,s}\rangle, \quad (74)$$

where $\langle\alpha_{r,s}|\alpha_{r,s}\rangle = 1$.

For the general Young diagram, the Macdonald polynomials correspond to the singular vectors of q-$\mathcal{W}$ algebras [14].

3.6 $\mathcal{V}ir_{p,q}$ and the Hall-Littlewood polynomial

If we take the limit $q \to 0$, the Macdonald polynomial $P_\lambda(x;q,t)$ reduces to the Hall-Littlewood polynomial $P_\lambda(x;t)$ [19]. Let us study the limit of $\mathcal{V}ir_{p,q}$ in $q \to 0$, and the connection between $\mathcal{V}ir_{p,q}$ and the Hall-Littlewood polynomials $P_\lambda(x;t)$. The commutation relations are already given in (22). The Kac determinants at lower levels are calculated as

$$\begin{aligned}
\det{}_1 &= \langle\lambda|\tilde{T}_1\tilde{T}_{-1}|\lambda\rangle = 1-t^{-1}, \\
\det{}_2 &= \begin{vmatrix} \langle\lambda|\tilde{T}_2\tilde{T}_{-2}|\lambda\rangle & \langle\lambda|\tilde{T}_2\tilde{T}_{-1}\tilde{T}_{-1}|\lambda\rangle \\ \langle\lambda|\tilde{T}_1\tilde{T}_1\tilde{T}_{-2}|\lambda\rangle & \langle\lambda|\tilde{T}_1\tilde{T}_1\tilde{T}_{-1}\tilde{T}_{-1}|\lambda\rangle \end{vmatrix} \\
&= (1-t^{-1})^2(1-t^{-2}).
\end{aligned} \tag{75}$$

Here, we observe that the Kac determinants do not depend on λ. Therefore, if t is generic, we have no singular vectors for any λ. To study the degeneration of the boson realization, we have to restrict the zero-mode charge of the vacuum $|\alpha_{r,s}\rangle$ to $s=0$, otherwise, we are not able to obtain nontrivial algebra. It is easy to see that the operators $\tilde{\Lambda}^\pm(z) \equiv \lim_{q\to 0}\Lambda^\pm(p^{\pm 1/2}z)$ are well behaving ones. Introducing the renormalized boson $b_n = -t^n a_n$, $b_{-n} = a_{-n}$ for $n>0$, we have in $q \to 0$,

$$[b_n, b_m] = n\frac{1}{1-t^{-|n|}}\delta_{n+m,0}, \tag{76}$$

$$\begin{aligned}
\tilde{\Lambda}^+(z) &= t^{r/2}\exp\left\{\sum_{n=1}^\infty(1-t^{-n})\frac{b_{-n}}{n}z^n\right\}\exp\left\{-\sum_{n=1}^\infty(1-t^{-n})\frac{b_n}{n}z^{-n}\right\}, \\
\tilde{\Lambda}^-(z) &= t^{-r/2}\exp\left\{-\sum_{n=1}^\infty(1-t^{-n})\frac{b_{-n}}{n}z^n\right\}\exp\left\{\sum_{n=1}^\infty(1-t^{-n})\frac{b_n}{n}z^{-n}\right\},
\end{aligned} \tag{77}$$

on the Fock space spanned over $|\alpha_{r,0}\rangle$. Note that these are essentially the same as Jing's operators $H(z)$ and $H^*(z)$ for the Hall-Littlewood polynomial $P_\lambda(x;t^{-1})$ [20]. Using this notation, the rescaled $\mathcal{V}ir_{p,q}$ generator $\tilde{T}_n$ is expressed as

$$\tilde{T}_n = \oint\frac{dz}{2\pi i z}\left(\,\theta[\,n\le 0\,]\tilde{\Lambda}^+(z) + \theta[\,n\ge 0\,]\tilde{\Lambda}^-(z)\,\right)z^n, \tag{78}$$

where $\theta[P]=1$ or 0 if the proposition P is true or false, respectively. This formula and the coincidence of our $\tilde{\Lambda}^+(z)$ with Jing's $H(z)$ means that in $q\to 0$ limit, "all the vectors in the Fock module" are written in terms of the Hall-Littlewood polynomials.

The screening currents will disappear in this limit in the sense that $S_-(z)$ and $\left[\tilde{T}_n, S_+(w)\right]$ become singular. The disappearance of the singular vectors in the Fock module which is derived from the study of the Kac determinants is explained by this singular behavior of the screening currents.

It seems interesting to study the relation between $\mathcal{V}ir_{p,q}$ and the Hall algebra [19] which is related with the analysis over the local fields. Is it possible to have a geometric interpretation of $\mathcal{V}ir_{p,q}$ for $q=0$?

4 Further aspects of $\mathcal{V}ir_{p,q}$ and relations with other models

Here we discuss *i)* the representation theories, *ii)* application to the ABF model and *iii)* a hidden elliptic algebra generated by screening current. We will also study the limits of $\beta = 1$, $3/2$ and 2, and find some connections with the Kac-Moody algebras when $\beta = 1$, for $\beta = 3/2$, the q-Virasoro generator is given by a BRST exact form and construct a topological model, and when $\beta = 2$, the q-Virasoro algebra relates with $c = 1$ $\mathcal{W}_{1+\infty}$ algebra.

4.1 symmetric realization of $\mathcal{V}ir_{p,q}$ and vertex operators

When we introduced the Feigin-Fuchs realization, the creation operators and the annihilation operators had nice symmetry. However we destroyed this symmetry to make many formulas for the Jack and Macdonald symmetric polynomials become simple. We will study some applications of $\mathcal{V}ir_{p,q}$ to other solvable systems. So, it helps us very much to have a "symmetric expressions" of $T(z)$ and $S_\pm(z)$ [13, 14, 6, 29, 30].

Let us introduce the fundamental Heisenberg algebra h_n ($n \in \mathbf{Z}$), Q_h having the commutation relations

$$[h_n, h_m] = \frac{1}{n}\frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}\delta_{n+m,0}, \qquad [h_n, Q_h] = \frac{1}{2}\delta_{n,0}. \tag{79}$$

The correspondence to the bosonic oscillators in the last two subsections is ($n > 0$)

$$h_n = \frac{t^n - 1}{n}a_n, \qquad h_{-n} = \frac{1}{n}\frac{1 - t^{-n}}{1 + p^n}a_{-n}, \qquad h_0 = \sqrt{\beta}a_0, \qquad Q_h = \frac{\sqrt{\beta}}{2}Q. \tag{80}$$

By these, the Virasoro current $T(z)$ and the screening current $S_\pm(z)$ (with some modification) are written as

$$T(z) = \Lambda^+(z) + \Lambda^-(z), \tag{81}$$

$$\Lambda^\pm(z) = :\exp\left\{\pm\sum_{n\neq 0} h_n p^{\pm\frac{n}{2}} z^{-n}\right\}: q^{\pm\sqrt{\beta}h_0} p^{\pm\frac{1}{2}}, \tag{82}$$

$$S_+(z) = :\exp\left\{-\sum_{n\neq 0}\frac{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}h_n z^{-n}\right\}: e^{2\sqrt{\beta}Q_h} z^{2\sqrt{\beta}h_0}, \tag{83}$$

$$S_-(z) = :\exp\left\{\sum_{n\neq 0}\frac{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}h_n z^{-n}\right\}: e^{-\frac{2}{\sqrt{\beta}}Q_h} z^{-\frac{2}{\sqrt{\beta}}h_0}. \tag{84}$$

If we introduce the isomorphisms θ and ω of the Heisenberg algebra related with (7), (8):

$$(\mathrm{II}') \quad \theta : (q,t) \mapsto (q^{-1}, t^{-1}), \quad h_n \mapsto -h_n \ (n \neq 0), \quad h_0 \mapsto h_0, \quad Q_h \mapsto Q_h,$$

$$\text{(III}'\text{)} \quad \omega : \quad q \leftrightarrow t \quad , \quad h_n \mapsto -h_n, \quad Q_h \mapsto -Q_h, \tag{85}$$

then $\Lambda^-(z) = \theta \cdot \Lambda^+(z) = \omega \cdot \Lambda^+(z)$, $S_-(z) = \omega \cdot S_+(z)$ and $\theta \cdot S_\pm(z) = S_\pm(z)$. Here $\omega \cdot \beta$ should be understood as $1/\beta$. Under the isomorphism σ such that:

$$\text{(IV)} \quad \sigma : q \leftrightarrow 1/t, \qquad \sqrt{\beta} \leftrightarrow -\sqrt{1/\beta}, \tag{86}$$

$\sigma \cdot \Lambda^\pm(z) = \Lambda^\pm(z)$ and $\sigma \cdot S_\pm(z) = S_\mp(z)$.

The free boson realization for $T(z)$ is expressed as the following deformed Miura transformation [8]

$$:\left(p^D - \Lambda^+(z)\right)\left(p^D - \Lambda^-(z)\right): = p^{2D} - T(z)p^D + 1, \tag{87}$$

which has been generalized to define the q-deformed $\mathcal{W}$ algebra [13, 14]. By using this transformation, Frenkel-Reshetikhin [8] proposed a generalization of their quasi-classical q-Virasoro algebra to $ABCD$-type cases. An analogy to the Baxter's dressed vacuum form Q defined by $:\left(p^D - \Lambda^-(z)\right) Q(z): = 0$, so $\Lambda^\pm(z) = :Q(zp^{\mp 1})\, Q(z)^{-1}:$, seems to be of some interest.

The vertex operator defined by

$$V_{2,1}(z) \equiv :\exp\left\{\sum_{n\neq 0} \frac{h_n}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} z^{-n}\right\}: e^{-\sqrt{\beta}Q_h} z^{-\sqrt{\beta}h_0}, \tag{88}$$

satisfies $\theta \cdot V_{2,1}(z) = V_{2,1}(z)$ and

$$g\left(\frac{w}{z}p^{\pm\frac{1}{2}}\right) T(z)V_{2,1}(w) - V_{2,1}(w)T(z)g^{-1}\left(\frac{z}{w}p^{\mp\frac{1}{2}}\right) \tag{89}$$

$$= t^{\pm\frac{1}{4}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\,\delta\left(t^{\pm\frac{1}{2}}\frac{w}{z}\right) V_{2,1}(q^{\pm\frac{1}{2}}w)p^{\mp\frac{1}{2}}, \tag{90}$$

with

$$g(x) = t^{\pm\frac{1}{4}} \exp\left\{\pm \sum_{n>0} \frac{1}{n}\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{p^{\frac{n}{2}} - p^{-\frac{n}{2}}} x^n\right\}. \tag{91}$$

Note that $V_{2,1}(q^{\pm\frac{1}{2}}w)p^{\mp\frac{1}{2}} = :\Lambda^\mp(t^{\mp\frac{1}{2}}w)V_{2,1}(w):$. If we let $V_{1,2}(z) \equiv \sigma \cdot V_{2,1}(z)$ and their fusion as

$$V_{\ell+1,k+1}(z) \equiv :\prod_{i=1}^{\ell} V_{2,1}(q^{\frac{\ell+1-2i}{2\ell}}z) \prod_{j=1}^{k} V_{1,2}(t^{\frac{k+1-2j}{2k}}z):, \tag{92}$$

then they also obey a similar commutation relation as (90), which reduces to the usual defining relation for the Virasoro primary field of the conformal weight $h_{\ell+1,k+1}$, in the limit $q \to 1$ [30]. The adjont action of the $\mathcal{V}ir_{p,q}$ generator $T(z)$ on this fused vertex operator $V_{\ell+1,k+1}$ may be closely connected with a coproduct of the algebra $\mathcal{V}ir_{p,q}$. Similar but slightly different definition for fused vertex operators

$$:\prod_{i=1}^{\ell} V_{2,1}(q^{-\frac{k}{2}}t^{\frac{\ell+1}{2}-i}z) \prod_{j=1}^{k} V_{1,2}(t^{-\frac{\ell}{2}}q^{\frac{k+1}{2}-j}z):, \tag{93}$$

was proposed in [29]. The meaning of fused operators given by (92) or (93) has not been made clear yet.

The fundamental vertex operators $V_{2,1}(z)$ and $V_{1,2}(z)$, that satisfy fermion like anti-commutation relation, are especially important. Because the q-Virasoro generator and screening currents are expressed by them as follows;

$$\Lambda^+(zp^{\frac{1}{2}}) = :V_{2,1}^+(zq^{-\frac{1}{2}})V_{2,1}^-(zq^{\frac{1}{2}}):p^{\frac{1}{2}}, \qquad S^+(z) = :V_{2,1}^-(zp^{\frac{1}{2}})V_{2,1}^-(zp^{-\frac{1}{2}}):, \tag{94}$$

and the relations obtained by ω. Here $V_{\ell+1,k+1}^{\pm}(z) \equiv V_{\ell+1,k+1}^{\pm 1}(z)$. Moreover, the boson and power-sum correspondence operator in eq. (56) is also realized as $:\prod_{i=1}^N V_{2,1}(q^{\frac{1}{2}}x_i^{-1}):$. Hence, they must play more important role in the q-Virasoro algebra.

4.2 ABF model and $\mathcal{V}ir_{p,q}$

In the papers [6], the explicit formula for the multipoint correlation functions is successfully obtained. We review their method and the relation to $\mathcal{V}ir_{p,q}$.

The q-Virasoro algebra can be applied to the off-critical phenomena, especially to the ABF model in the regime III [35]. Let $z \equiv p^v$ and the vertex operators $\Phi_\pm(z)$ be

$$\Phi_+(z) \equiv V_{2,1}(z), \qquad \Phi_-(z') \equiv \oint \frac{dz}{2\pi i z} V_{2,1}(z')S_+(z)z^\beta f(v-v',\pi), \tag{95}$$

where $[v] \equiv p^{\frac{1}{2}((1-\beta)v^2-v)}(z;q)_\infty(qz^{-1};q)_\infty(q;q)_\infty$ and

$$f(v,w) \equiv \frac{[v+\frac{1}{2}-w]}{[v-\frac{1}{2}]}, \qquad \pi \equiv -\frac{2h_0}{\sqrt{\beta}-\sqrt{1/\beta}}. \tag{96}$$

The integration contour is a closed curve around the origin satisfying $p|z'| < |z| < p^{-1}|z'|$.

Then the vertex operators satisfy the following commutation relation;

Theorem 5. *[6]* [8]

$$\Phi_{\ell_3-\ell_2}(z_1)\Phi_{\ell_2-\ell_1}(z_2) = \sum_{\ell_4} W\left(\begin{matrix}\ell_3 & \ell_4\\ \ell_2 & \ell_1\end{matrix}\middle|\frac{z_1}{z_2}\right)\Phi_{\ell_3-\ell_4}(z_2)\Phi_{\ell_4-\ell_1}(z_1), \tag{97}$$

where $\ell_{i+1}-\ell_i = \pm 1$ with $\ell_5 \equiv \ell_1$. Here $W(\ell|z)$ is the Boltzmann weight of the ABF model such that

$$\begin{aligned}
W\left(\begin{matrix}\ell\pm 2 & \ell\pm 1\\ \ell\pm 1 & \ell\end{matrix}\middle|z\right) &= R(z), & R(z) &= z^{\frac{\beta}{2}}\frac{g(z^{-1})}{g(z)},\\
W\left(\begin{matrix}\ell & \ell\pm 1\\ \ell\pm 1 & \ell\end{matrix}\middle|z\right) &= R(z)\frac{[\ell\pm v][1]}{[\ell][1-v]}, & g(z) &= \frac{(pz;q,p^2)_\infty(pqz;q,p^2)_\infty}{(qz;q,p^2)_\infty(p^2z;q,p^2)_\infty},\\
W\left(\begin{matrix}\ell & \ell\mp 1\\ \ell\pm 1 & \ell\end{matrix}\middle|z\right) &= -R(z)\frac{[\ell\pm 1][v]}{[\ell][1-v]},
\end{aligned} \tag{98}$$

[8] The notations in [6] are $x = p^{\frac{1}{2}}$ and $r = 1/(1-\beta)$.

with $(z;q,p)_\infty \equiv \prod_{n,m\geq 0}(1-zq^np^m)$.

Besides this commutation relation these vertex operators also satisfy the other defining relations, homogeneity and normalization condition [36], of those of the ABF model in the regime III; $0<p<z<1$. Here $p=0$ and 1 correspond to the zero temperature and the critical point, respectively. Thus these vertex operators of the q-Virasoro algebra can be regarded as those of the ABF model.

The exchange relation (97) was generalized to general vertex operators by [29, 30], and to $sl(n)$ RSOS model by [7].

4.3 Felder resolution and the space of ABF model

Not only the vertex operators themselves but also the Hilbert space on which the physical operators act are able to be identified with that of the ABF model. The Hilbert space of the ABF model can be constructed through a deformed Felder-type BRST resolution.

Let $\mathcal{F}_{r,s}$ be the Fock module generated by the highest weight state $|r,s\rangle$ such that

$$h_{n>0}|r,s\rangle = 0, \qquad h_0|r,s\rangle = -\frac{1}{2}\left(r\sqrt{\beta}-s\sqrt{1/\beta}\right)|r,s\rangle. \tag{99}$$

Suppose $\beta = P_-/P_+$ with coprime integers $P_+ > P_- \in \mathbf{N}$ and let the screening charge $X_+ : \mathcal{F}_{r,s} \to \mathcal{F}_{r-2,s}$ be

$$X_+ = \oint \frac{dz}{2\pi i z} S_+(z) z^\beta f(v,\pi), \tag{100}$$

and define the BRST charges Q_j^+ $(j\in\mathbf{Z})$ as

$$\begin{aligned} Q_{2j}^+ &= X_+^r \quad : \quad \mathcal{F}_{r-2jP_+,s} \to \mathcal{F}_{-r-2jP_+,s}, \\ Q_{2j+1}^+ &= X_+^{P_+-r} : \quad \mathcal{F}_{-r-2jP_+,s} \to \mathcal{F}_{r-2(j+1)P_+,s}. \end{aligned} \tag{101}$$

We also define the dual screening charge $X_- : \mathcal{F}_{r,s} \to \mathcal{F}_{r,s-2}$ and the dual BRST charges Q_j^- by the replacement $\sqrt{\beta} \leftrightarrow -\sqrt{1/\beta}$, $q \leftrightarrow 1/t$ and $r \leftrightarrow s$.

Proposition 4. *[6, 28] The screening charges $X_\pm$ commute with each other and with q-Virasoro generators*

$$\begin{aligned} [X_+, X_-] &= 0, \\ [T(z), X_\pm^{n_\pm}] &= 0, \quad \text{on} \quad \mathcal{F}_{r_+,r_-}, \quad \text{with} \quad n_\pm \equiv r_\pm \bmod P_\pm, \end{aligned} \tag{102}$$

and are also nilpotent

$$Q_j^\pm Q_{j-1}^\pm = X_\pm^{P_\pm} = 0, \quad P_\pm > 1. \tag{103}$$

Hence we can construct Felder type BRST complexes, for example, by X_+

$$\cdots \xrightarrow{X_+^r} \mathcal{F}_{-r+2P_+,s} \xrightarrow{X_+^{P_+-r}} \mathcal{F}_{r,s} \xrightarrow{X_+^r} \mathcal{F}_{-r,s} \xrightarrow{X_+^{P_+-r}} \mathcal{F}_{r-2P_+,s} \xrightarrow{X_+^r} \cdots. \tag{104}$$

From the Kac determinant in (30), the Fock module $\mathcal{F}_{r,s}$ with $r,\ s \in \mathbf{N}$ is reducible. To obtain an irreducible one $\mathcal{L}_{r,s}$, we have to factor out the submodules by the Felder

resolution. In a special case, this irreducible module coincides with the space of the ABF model.

To see this, we have to introduce a grading operator, which plays the role of the corner Hamiltonian in the ABF model,

$$H_c = \sum_{n>0} n^2 \frac{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}{\left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)\left(t^{\frac{n}{2}} - t^{-\frac{n}{2}}\right)} h_{-n} h_n + h_0^2 - \frac{1}{24}. \tag{105}$$

This commutes with screening currents up to a total divergence

$$[H_c, S_\pm(z)] z^{\beta^{\pm 1}} = \frac{\partial}{\partial z}\left(S_\pm(z) z^{\beta^{\pm 1}}\right), \tag{106}$$

and its eigenvalues $\varepsilon_{r,s}$ on the Fock module $\mathcal{F}_{r,s}$ are

$$\varepsilon_{r,s} = h_{r,s} - \frac{c}{24} + n, \qquad n \in \mathbf{Z}_{n\geq 0}, \tag{107}$$

with $h_{r,s}$ in eq. (36). When $P_- = P_+ - 1$, these values coincide with the eigenvalues of the corner Hamiltonian of the ABF model corresponding to the 1-d configurations given by the rule: *i*) each height takes an integer value between 1 and $P_+ - 1$, *ii*) the allowed values of difference in any neighboring heights are ± 1, *iii*) the height at the origin is r, *iv*) the asymptotic configuration is $\cdots, s, s+1, s, s+1, \cdots$. However, we should note that the multiplicities of the bosonic Fock space and that of ABF model are different.

Lukyanov and Pugai [5, 6] showed that, after the Felder-type BRST resolution by the dual screening current $S_-(z)$, the multiplicities in the irreducible Fock module $\mathcal{L}_{r,s}$ coincide those of the ABF model. Therefore, ABF model is completely described by the representation of the q-Virasoro algebra and the multi-point local height probabilities of ABF model [36] are realized as correlation functions of the vertex operators. For example, the probability that the heights at the same vertical column sites have the values $1 \leq r_1, r_2, \cdots, r_n \leq P_+ - 1$ is proportional to

$$Tr_{\mathcal{L}_{r_1,s}}\left[p^{2H_c}\Phi_{-\sigma_1}(z_1/p)\cdots\Phi_{-\sigma_{n-1}}(z_{n-1}/p)\Phi_{\sigma_{n-1}}(z_{n-1})\cdots\Phi_{\sigma_1}(z_1)\right], \tag{108}$$

where $\sigma_s = r_{s+1} - r_s$.

4.4 elliptic algebra generated by the screening currents and $k = 1$ affine Lie algebra

The properties of screening currents are quite important in the representation theory of the infinite-dimensional algebra; they govern the irreducibility and the physical states as mentioned above subsections. Moreover they relate with hidden quantum symmetries.

4.4.1 an elliptic algebra generated by $S^{\pm}(z)$

Here, we show that the screening currents generate an elliptic hidden symmetry, which reduces to the (quantum) affine Lie algebra with a special center when c tends to 1. Let us introduce a new current $\Psi(z) \equiv :S_+(q^{\pm\frac{1}{2}}z)\, S_-(t^{\pm\frac{1}{2}}z):$, *i.e.*,

$$\Psi(z) = \exp\left\{\sum_{n\neq 0} \frac{p^n - p^{-n}}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} h_n z^{-n}\right\} e^{2\alpha Q} z^{2\alpha h_0}, \tag{109}$$

with $\alpha = \sqrt{\beta} - 1/\sqrt{\beta}$, then we have

Proposition 5. *Screening Currents $S_{\pm}(z)$ and $\Psi(z)$ generate the following elliptic two-parameter algebra;*

$$f_{00}\left(\frac{w}{z}\right)\Psi(z)\Psi(w) = \Psi(w)\Psi(z) f_{00}\left(\frac{z}{w}\right), \tag{110}$$

$$f_{0\pm}\left(\frac{w}{z}\right)\Psi(z)S_{\pm}(w) = S_{\pm}(w)\Psi(z) f_{\pm 0}\left(\frac{z}{w}\right), \tag{111}$$

$$f_{\pm\pm}\left(\frac{w}{z}\right)S_{\pm}(z)S_{\pm}(w) = S_{\pm}(w)S_{\pm}(z) f_{\pm\pm}\left(\frac{z}{w}\right), \tag{112}$$

$$[\, S_+(z), S_-(w)\,] = \frac{1}{(p-1)w}\left[\delta\left(p^{\frac{1}{2}}\frac{w}{z}\right)\Psi(t^{-\frac{1}{2}}w) - \delta\left(p^{-\frac{1}{2}}\frac{w}{z}\right)\Psi(q^{-\frac{1}{2}}w)\right], \tag{113}$$

where $f_{00}(x) = f_{++}(x) f_{+-}^2(xp^{\frac{1}{2}}) f_{--}(x)$ and $f_{0\pm}(x) = f_{\pm 0}(x) = f_{+\pm}(xq^{\frac{1}{2}}) f_{-\pm}(xt^{\frac{1}{2}})$ with

$$f_{+-}(x) = f_{-+}(x) = \exp\left\{-\sum_{n>0}\frac{1}{n}(p^{\frac{n}{2}} + p^{-\frac{n}{2}})x^n\right\} x^{-1}, \tag{114}$$

$$f_{++}(x) = \exp\left\{-\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}(p^{\frac{n}{2}} + p^{-\frac{n}{2}})x^n\right\} x^{\beta}, \tag{115}$$

and $f_{--}(x) = \omega \cdot f_{++}(x)$.

The relation between $\Psi(z)$ and the q-Virasoro generators $\Lambda^{\pm}(z)$ is simply written as

$$[\,\Lambda^{\pm}(z), \Psi(w)\,] = \mp p^{\mp\frac{1}{2}}(p^{\frac{1}{2}} - p^{-\frac{1}{2}})\delta\left(\frac{w}{z}\right) :\Lambda^{\pm}(w)\Psi(w):. \tag{116}$$

As we shall see explicitly in the next subsection, in the limit of q and t tend to 0 with p and $t^{-\frac{|n|}{2}}h_n$ fixed, the relations (110)–(113) reduce to those of $k = 1$ $U_q(\widehat{sl}_2)$. Therefore, the algebra generated by $S^{\pm}(z)$ and $\Psi(z)$ can be regarded as an elliptic generalization of $U_q(\widehat{sl}_2)$ with level-one. We can regard p, q and t as three independent parameters. Even in this case, screening currents (83) and (84) and new currents $\Psi_{\pm}(z) \equiv :S_+(p^{\pm\frac{1}{4}}z)\, S_-(p^{\mp\frac{1}{4}}z):$ generate an elliptic algebra. These extended algebras may help us to investigate elliptic-type integrable models.

In the sense of analytic continuation, these relations are also rewritten by using elliptic theta functions[13],

$$S_{\pm}(z)S_{\pm}(w) = U_{\pm}\left(\frac{w}{z}\right) S_{\pm}(w)S_{\pm}(z), \tag{117}$$

with

$$U_\pm(x) = -x^{1-2\beta} \exp\left\{ \sum_{n\neq 0} \frac{1}{n} \frac{q^{\frac{n}{2}} t^{-n} - q^{-\frac{n}{2}} t^n}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} x^n \right\} = -x^{2(1-\beta)} \frac{\vartheta_1(px;q)}{\vartheta_1(px^{-1};q)}, \tag{118}$$

and $U_-(x) = \omega \cdot U_+(x)$. Note that $U_\pm(x)$ are quasi-periodic functions, namely for $U_+(x)$, we have

$$U_+(qx) = U_+(x), \qquad U_+(e^{2\pi i}x) = e^{-4\pi i\beta} U_+(x). \tag{119}$$

It should be noted that the screening currents of q-$\mathcal{W}$ algebra [13, 14] and $U_q(\widehat{sl}_N)$ [11] also obey similar elliptic relations.

4.4.2 two $c = 1$ limits

It is possible to consider two different $c = 1$ limits of $\mathcal{V}ir_{p,q}$. They are related with the (quantum) affine algebra of $A_1^{(1)}$-type.

(A) Let us consider the limit, $q \to 0$, $t \to 0$, p is fixed. In this limit we must have $\beta \to 1$. thus, this is a "$c = 1$" limit (see (12)). We will see that the screening currents $S_\pm(z)$ reduces to the Frenkel-Jing realization of $U_q(\widehat{sl}_2)$ at level-one [31].

Introducing rescaled bosons as

$$a_n = -h_n q^{|n|/2} p^{-|n|/4} [2n] \qquad (n \neq 0), \qquad a_0 = -2h_0, \qquad Q = -2Q_h, \tag{120}$$

we obtain

$$[a_n, a_m] = \frac{[2n][n]}{n} \delta_{n+m,0}, \quad [a_n, Q] = 2\delta_{n,0}, \tag{121}$$

$$\begin{aligned} S_-(z) &\to \exp\left\{ \sum_{n=1}^\infty \frac{1}{[n]} a_{-n} z^n p^{-\frac{n}{4}} \right\} \exp\left\{ -\sum_{n=1}^\infty \frac{1}{[n]} a_n z^{-n} p^{-\frac{n}{4}} \right\} e^Q z^{a_0}, \\ S_+(z) &\to \exp\left\{ -\sum_{n=1}^\infty \frac{1}{[n]} a_{-n} z^n p^{\frac{n}{4}} \right\} \exp\left\{ \sum_{n=1}^\infty \frac{1}{[n]} a_n z^{-n} p^{\frac{n}{4}} \right\} e^{-Q} z^{-a_0}, \end{aligned} \tag{122}$$

where $[n] = (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$. After replacing $p^{\frac{1}{2}} \to q$, we can identify screening currents $S_\pm$ with the Frenkel-Jing realization of the Drinfeld currents of $U_q(\widehat{sl}_2)$.

It is quite unfortunate that at this limit, the $\mathcal{V}ir_{p,q}$ current becomes singular. Namely, it seems difficult to extract nontrivial object from $T(z)$ at this limit. Thus it is still a challenging problem to find a Sugawara construction for $U_q(\widehat{sl}_2)$. The limit discussed in the next paragraph (B) may be relevant to this problem.

(B) Next, we consider the limit of $\beta \to 1$ with q fixed. This is another "$c = 1$" limit. In this limit, the screening currents degenerate to the Frenkel-Kac realization of the level-one $\widehat{sl}_2$ [32].

Introducing rescaled bosons as

$$a_n = \frac{2n}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} h_n \quad (n \neq 0), \quad a_0 = 2h_0, \quad Q = 2Q_h, \tag{123}$$

we obtain

$$[a_n, a_m] = 2n\delta_{n+m,0}, \quad [a_n, Q] = 2\delta_{n,0}, \tag{124}$$

$$S_\pm(z) \to \exp\left\{\pm\sum_{n\neq 0}\frac{a_n}{n}z^{-n}\right\} e^{\pm Q} z^{\pm a_0}. \tag{125}$$

From these, it can be seen that the screening currents have reduced to the Frenkel-Kac realization of $\widehat{sl}_2$.

In this limit, $\mathcal{V}ir_{p,q}$ survives and satisfies the relation

$$\begin{aligned} &\frac{\sqrt{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})}}{1-\frac{w}{z}}T(z)T(w) - T(w)T(z)\frac{\sqrt{(1-q\frac{z}{w})(1-q^{-1}\frac{z}{w})}}{1-\frac{z}{w}} \\ = \;& 2(1-q)(1-q^{-1})\frac{w}{z}\delta'\left(\frac{w}{z}\right). \end{aligned} \tag{126}$$

It is shown that all the vectors in the Fock space spanned over the vacuum $|0\rangle$ are singular vectors of this algebra by studying the Kac determinant at level-one. This fact and the normalization of the bosonic oscillators in this limit mean that the the Fock space is spanned by the Schur symmetric polynomials. This degeneration is not accidental because the Macdonald polynomial $P_\lambda(x;q,t)$ reduces to the Schur polynomial in the limit $t \to q$.

In this "$c = 1$" limit, $\mathcal{V}ir_{p,q}$ survives and acts on the Fock space on which the bosonized currents of $U_q(\widehat{sl}_2)$ can also act. The relationship between these algebras has not been made clear yet.

4.5 $c = 0$ topological q-superconformal model

The $c = 0$ Virasoro algebra describes a topological conformal model. We shall call one of the screening currents of this model $G^+(z)$, and set $G^-(z) =: (G^+(z))^{-1} :$. The screening charge $\oint dz G^+(z)$ plays the role of the BRST operator. Since the model is topological, the energy-momentum tensor $L_{c=0}(z)$ should be BRST exact. Actually, we have $\{\oint dz' G^+(z'), G^-(z)\} = 2L_{c=0}(z)$.

There exists a similar structure in the q-Virasoro case[9]. Let us consider the case when $c = 0$, *i.e.*, $\beta = 3/2$ ($q = p^{-2}$). Denote one of the screening current $S_+(z)$ and the normal ordering of its inverse as $G^+(z)$ and $G^-(z)$, respectively

$$G^\pm(z) =: \exp\left\{\pm\sum_{n\neq 0}\frac{h_n}{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}z^{-n}\right\} : \; e^{\pm 2\sqrt{\beta}Q} z^{\pm 2\sqrt{\beta}h_0}. \tag{127}$$

[9]This was inspired by the discussion with T. Kawai

Proposition 6. *The fields $T(z)$ and $G^{\pm}(z)$ satisfy the relations*

$$
\begin{aligned}
f\left(\frac{w}{z}\right)T(z)T(w)-T(w)T(z)f\left(\frac{z}{w}\right) &= -(p^{\frac{1}{2}}+p^{-\frac{1}{2}})(p^{\frac{3}{2}}-p^{-\frac{3}{2}})\left(\delta\left(\frac{wp}{z}\right)-\delta\left(\frac{w}{zp}\right)\right),\\
f\left(\frac{w}{z}\right)T(z)G^{-}(w)-G^{-}(w)T(z)f\left(\frac{z}{w}\right) &= G^{-}(z)(p^{\frac{3}{2}}-p^{-\frac{3}{2}})\left(p^{2}\delta\left(\frac{wp}{z}\right)-p^{-2}\delta\left(\frac{w}{zp}\right)\right),\\
\left\{\oint dzG^{+}(z),G^{-}(w)\right\} &= \frac{p^{-1}}{w^{2}(p-p^{-1})(p^{\frac{1}{2}}-p^{-\frac{1}{2}})}\left(T(w)-(p^{\frac{1}{2}}+p^{-\frac{1}{2}})\right),\\
\left[T(z),\oint dwG^{+}(w)\right] &= 0, \qquad \left\{G^{\pm}(z),G^{\pm}(w)\right\}=0.
\end{aligned}
\tag{128}
$$

where $f(x)$ is given by (3) with $q=p^{-2}$.

Note that $G^{-}(z)$ is a primary field and its commutation relation with $T(z)$ is given by the same function $f(x)$ in the defining relation of the q-Virasoro algebra.

We can regard these relations as a $c=0$ topological q-Virasoro algebra. The screening charge $\oint dzG_{+}(z)$ may play the role of BRST operator which reduces the bosonic Fock space to irreducible representation space of the q-Virasoro algebra. At the value of the coupling constant $\beta=3/2$, the central charge vanishes and the entire Fock space contains only BRST trivial states, except for the vacuum state. The q-Virasoro generator itself (up to a constant) is given by a BRST exact form. Thus the $\beta=3/2$ q-Virasoro algebra is a topological field theory same as $q=1$ case.

The relations between the currents $G^{\pm}(z)$ and $\Lambda^{\pm}(z)$ are written as

$$
\begin{aligned}
f\left(\frac{w}{z}\right)\Lambda^{\pm}(zp^{\pm1})G^{+}(w)-G^{+}(w)\Lambda^{\pm}(zp^{\pm1})f\left(\frac{z}{w}\right) &= \mp p^{\mp1}(p^{\frac{3}{2}}-p^{-\frac{3}{2}})\delta\left(\frac{w}{z}p^{\mp1}\right)G^{+}(z),\\
f\left(\frac{w}{z}\right)\Lambda^{\pm}(z)G^{-}(w)-G^{-}(w)\Lambda^{\pm}(z)f\left(\frac{z}{w}\right) &= \pm p^{\pm2}(p^{\frac{3}{2}}-p^{-\frac{3}{2}})\delta\left(\frac{w}{z}p^{\pm1}\right)G^{-}(z),\\
\left\{G^{+}(z),G^{-}(w)\right\} &= \frac{p^{-3}}{zw^{2}(p-p^{-1})(p^{\frac{1}{2}}-p^{-\frac{1}{2}})}\\
&\times\left(\Lambda^{+}(w)\,\delta\left(\frac{w}{zp}\right)+\Lambda^{-}(w)\,\delta\left(\frac{wp}{z}\right)-\left(p^{\frac{1}{2}}+p^{-\frac{1}{2}}\right)\delta\left(\frac{w}{z}\right)\right).
\end{aligned}
\tag{129}
$$

Their Fourier modes given by $G^{\pm}(z)=\sum_{n}G_{n}^{\pm}z^{-n}$ and $\Lambda^{\pm}(z)=\sum_{n}\Lambda_{n}^{\pm}z^{-n}$ satisfy

$$
\left\{G_{n+1}^{+},G_{m+1}^{-}\right\}=\frac{p^{-3}}{(p-p^{-1})(p^{\frac{1}{2}}-p^{-\frac{1}{2}})}\left(\Lambda_{n+m-1}^{+}p^{-n}+\Lambda_{n+m-1}^{-}p^{n}-\left(p^{\frac{1}{2}}+p^{-\frac{1}{2}}\right)\delta_{n+m-1,0}\right).
$$

Note that, the relation between G^{+} and G^{-} can be expressed in other ways. For example, use the fact that $G^{+}(z)G^{-}(w)z/w+G^{-}(w)G^{+}(z)w/z$ and $\{G^{+}(z),G^{-}(w)\}\,z^{-r}z^{2}w^{2}/(z+w)$ are also written by $\Lambda^{\pm}$ for any $r\in\mathbf{C}$.

In the $q=1$ case, $c=0$ topological algebra can be constructed from $N=2$ superconformal algebra by the operation so-called "twisting"[33]. What we have obtained here is a deformation of this twisted superconformal algebra. So far, we have not been able to find a mechanism of "untwisting" in the deformed case. We expect that our topological q-Virasoro algebra helps us to find a supersymmetric generalization of the q-Virasoro algebra $\mathcal{V}ir_{p,q}$ and a deformed twisting operation.

4.6 $c=-2$ $\mathcal{V}ir_{p,q}$ and $c=1$ $\mathcal{W}_{1+\infty}$ algebra

Since $c=0$ Virasoro algebra is realized by the differential operator $L_n=-z^nD$ with $D=z\partial_z$, one can expect that q-Virasoro algebra $\mathcal{V}ir_{p,q}$ has a similar representation by the difference or shift operator as $T_n\sim z^nq^D$. However, this shift operator is nothing but the generating function of the $c=0$ $\mathcal{W}_{1+\infty}$ generators. Thus we expect some relations between the q-Virasoro and the $\mathcal{W}_{1+\infty}$ algebra. Indeed this is the case when $\beta=2$, we show a relation with $c=1$ $\mathcal{W}_{1+\infty}$ algebra.

First, recall that the q-Virasoro generator is expressed by the fundamental vertex operator and its dual $V^{\pm}_{2,1}(z)$ as eq. (94). When $\beta=2$, *i.e.*, $c=-2$, $(q=1/p)$, they reduce to the fermions such that

$$\{V^+_{2,1}(z),V^-_{2,1}(w)\}=\frac{1}{z}\delta\left(\frac{w}{z}\right),\qquad \{V^{\pm}_{2,1}(z),V^{\pm}_{2,1}(w)\}=0. \tag{130}$$

On the other hand, the generating function of $c=1$ $\mathcal{W}_{1+\infty}$ algebra is known to be also realized by a complex fermion. Therefore, we have found;

Proposition 7. *The $\beta=2$ $(c=-2)$ q-Virasoro algebra $T(z)=\Lambda^+(z)+\Lambda^-(z)$ generates the $c=1$ $\mathcal{W}_{1+\infty}$ algebra and it is realized by fermions $V^{\pm}_{2,1}(z)$ as follows;*

$$\Lambda^{\pm}(zq^{\frac{1}{2}\mp 1})=:V^-_{2,1}(z)q^{\mp D}V^+_{2,1}(z):\ q^{\mp\frac{1}{2}}. \tag{131}$$

Next we show the relation between q-Virasoro and $\mathcal{W}_{1+\infty}$ algebras more explicitly comparing their commutation relations. Let

$$X^k(z)\equiv:\exp\left\{\sum_{n\neq 0}\frac{1-q^{kn}}{1-q^n}h_nz^{-n}\right\}:\ q^{k\sqrt{2}h_0}\equiv\sum_{n\in\mathbf{Z}}X^k_nz^{-n}. \tag{132}$$

Note that the q-Virasoro generator is now $T(z)=X^1(z)+X^{-1}(z)$. Then

$$\begin{aligned}\left[X^k(z),X^\ell(w)\right]&=\frac{(q^k-1)(q^\ell-1)}{q^{k+\ell}-1}\left(X^{k+\ell}(z)\delta\left(q^k\frac{w}{z}\right)-X^{k+\ell}(w)\delta\left(q^\ell\frac{z}{w}\right)\right),\\ \left[X^k(z),X^{-k}(w)\right]&=(q^{\frac{k}{2}}-q^{-\frac{k}{2}})^2\left(\left(1+\sum_{n\neq 0}\frac{1-q^{kn}}{1-q^n}nh_nz^{-n}\right)\delta\left(q^k\frac{w}{z}\right)-\delta'\left(q^k\frac{w}{z}\right)\right),\end{aligned}$$

for $k,\ell,n,m\in\mathbf{Z}$. Their modes

$$W^k_n=\frac{X^k_n}{q^k-1}-\frac{1}{1-q^{-k}}\delta_{n,0},\qquad W^0_n=\frac{n}{q^n-1}h_n, \tag{133}$$

for $k\neq 0$, satisfy

$$\begin{aligned}\left[W^k_n,W^\ell_m\right]&=(q^{-km}-q^{-\ell n})W^{k+\ell}_{n+m}+\frac{q^{-km}-q^{-\ell n}}{1-q^{-k-\ell}}\delta_{n+m,0},\\ \left[W^k_n,W^{-k}_m\right]&=(q^{-km}-q^{kn})W^0_{n+m}+nq^{kn}\delta_{n+m,0}.\end{aligned} \tag{134}$$

This is nothing but the algebra of the generating functions for the $c = 1$ $\mathcal{W}_{1+\infty}$ generators $W_n^k = W\left(z^n q^{-kD}\right)$ in the notation of [34].

The $c = 0$ $\mathcal{W}_{1+\infty}$ algebra has a meaning of an area-preserving diffeomorphism and relates with classical membrane. We expect this relation between the q-Virasoro algebra with the $\mathcal{W}_{1+\infty}$ algebra is a key for a geometrical interpretation of $\mathcal{V}ir_{p,q}$ and the quantization of the membrane.

5 Summary and further issues

Our presentation has aimed to show that the new Virasoro-type elliptic algebra $\mathcal{V}ir_{p,q}$ defined by eq. (4) can be regarded as a universal symmetry of the massive integrable models.

The algebra $\mathcal{V}ir_{p,q}$ has two parameters p and q ($qp^{-1} = q^{\beta}$) and it can be regarded as a generating function for several different Virasoro-type symmetry algebras appearing in solvable models. At the limit of $q \to 1$, the algebra $\mathcal{V}ir_{p,q}$ reduces to the ordinary Virasoro algebra with the central charge c related to β (11). When we consider the limit of $q \to p$, it reduces to the q-Virasoro algebra of Frenkel-Reshetikhin (13). When $q \to 0$, we obtain Jing's generating operators for the Hall-Littlewood polynomials (77). The topological algebra (128) and the $c = 1$ $\mathcal{W}_{1+\infty}$ algebra (134) are constructed from the special cases $p = q^{-1/2}$ and $p = 1/q$ respectively.

One of the peculiar features of the algebra $\mathcal{V}ir_{p,q}$ is its non-linearity. Since it is a quadratic algebra like the Yang-Baxter relation for the transfer matrix, the associativity is quite non-trivial (1), and the Yang-Baxter equation determines the structure function uniquely, *i.e.*, fixes the algebra itself! Moreover it turns out to be the Zamolodchikov-Faddeev algebra of the particle-creation operators for the XYZ and the sine-Gordon models (16).

The next essential nature is its infinite-dimensionality, which connects with the integrability of massive models. The representation space of the algebra $\mathcal{V}ir_{p,q}$ possesses rich structure enough to describe the physical space of massive models. Despite its non-linearity, the Kac determinant is very similar to that of the Virasoro case (30).

The algebra $\mathcal{V}ir_{p,q}$ is realized by the free fields (64) and (81) in a quite simple way. It shows us that the integrability of the model due to $\mathcal{V}ir_{p,q}$ symmetry can be investigated in a natural manner in terms of the bosonic field. Furthermore this free field realization is described by a deformed Miura transformation (87), and the deformed Miura transformation brings about an interesting analogy with the dressed vacuum form. A generalization of this transformation gives the q-deformed $\mathcal{W}$ algebra [13, 14].

To study infinite dimensional algebras, *e.g.*, not only Virasoro and Kac-Moody algebras but also $\mathcal{V}ir_{p,q}$, one of the most essential oubjects in the representation theory is the screening current. Using the screening charge, one can define the physical states by the BRST method (101), write down the null states (73), and study the nontrivial monodromic property of the screened vertex operators. The null states relate with the wave functions of the Ruijsenaars model (54) which is a relativistic generalization of the Calogero-Sutherland model. The Hamiltonian of this model is realized by the positive

modes of the $\mathcal{V}ir_{p,q}$ generators (66) and causes the correspondence between the null states and excited states. The monodromy matrix is connected to the ABF model. The exchange relation of the vertex operators possesses a quantum group structure characterized by the solution of the face type Yang-Baxter equation(97) and leads to the identification of the deformed Virasoro vertex operators with that of the ABF model. Moreover the screening currents show us an hidden symmetry; that is an elliptic generalizaton of $k = 1$ $U_q(\widehat{sl}_2)$ algebra (Prop. 5).

Finally, we mention some comments on further issues coming from mathematical or physical points of view.

Mathematically, there are many things to be clarified. To obtain a fusion of the ABF model or to find a suitable primary fields, one needs a tensor product representation or suitable adjoint action of the algebra $\mathcal{V}ir_{p,q}$. In other words, a co-product structure must be discovered. Since the algebra, however, is quadratic, it seems a highly non-trivial task. Studying the limit $q \to 0$ may help us to reveal it.

Dispense with a help of free fields, the correlation function must be determined by a difference equation coming from the Ward-Takahashi identity. To find this equation, we need to recognize a geometrical meaning of the algebra $\mathcal{V}ir_{p,q}$; how is a difference operator realized in $\mathcal{V}ir_{p,q}$? Restricting the q-$\mathcal{W}_{1+\infty}$ algebra [37], which is defined as an algebra of higher pseud-difference operators, to the first order difference one, we might obtain the algebra $\mathcal{V}ir_{p,q}$ and its difference operator realization. Seeking for a realization on the infinite q-wedge [38] might be able to connect both problems mentioned above; a co-product and a geometrical interpretation.

The relation with the quantum affine Lie algebras also seems to be of some interest. Frenkel-Reshetikhin's q-Virasoro algebra was constructed by a q-Sugawara method [8] from the $U_q(\widehat{sl}_n)$ at the critical level. Are there any such constructions for the algebra $\mathcal{V}ir_{p,q}$ from $U_q(\widehat{sl}_n)$ or, more hopefully, from their elliptic generalization $\mathcal{A}_{q,p}$?

Physically, we anticipate many applications. There are two approaches to investigate the 2-dimensional integrable models, one is an Abelian method based on the algebraic Bethe-ansatz and the other is a non-Abelian one based on the Virasoro algebra and its generalizations. The latter approach describes the model more in detail, however its applicability is restricted only to critical phenomena and some off-critical trigonometric-type models. The deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ should be a synthesis of the massive integrable models including elliptic-type models.

In the dual resonance model, which was a precursor of string theory, the Veneziano amplitude has its generalizations to non-linearly rising Regge trajectories (see for example, [39]). However the absence of their operator representations has disturbed their development, including to prove a no-ghost theorem. In a special case, the amplitude reduces to q-beta function, which is very similar to a four-point function of our algebra $\mathcal{V}ir_{p,q}$. We hope that $\mathcal{V}ir_{p,q}$ gives an operator representation of a generalized Veneziano amplitude and opens further avenues to the exploration of new string theories.

The $c = 0$ $\mathcal{W}_{1+\infty}$ algebra, an area-preserving diffeomorphism, is a symmetry of the classical membrane. The relation between the $c = -2$ $\mathcal{V}ir_{p,q}$ algebra and $c = 1$ $\mathcal{W}_{1+\infty}$

algebra may be a key for a geometrical interpretation of the algebra $\mathcal{V}ir_{p,q}$ and the quantization of the membrane as the basic object of the 11(12)-dimensional M(F) theory. On the other hand, quantum membrane can be represented by a large N matrix model [40], and the partition functions of more general conformal matrix models are described by eigenstates of the Calogero-Sutherland models [10]. Do the algebra $\mathcal{V}ir_{p,q}$ or eigenstates of the Ruijsenaars model relate with a relativistic generalization of the quantum membrane?

What is a field theoretical interpretation of the algebra $\mathcal{V}ir_{p,q}$? The low energy 4-dimensional $N = 2$ super YM theories are described by some integrable models, periodic Toda chain or elliptic CS model [41]. If we generalize them to 5D's one by Kaluza-Klein method with an extra dimension compactified to a circle, then they are described by the relativistic generalization of 4D's one [42]. It may suggest that the Kaluza-Klein with a radius R can lead to a relativistic generalization with the speed of light $1/R$ or a q-deformation with $q = e^R$ of a original theory. Is the algebra $\mathcal{V}ir_{p,q}$ understood as a Kaluza-Klein from CFT?

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