c=3d Algebra *

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Abstract

We define a superconformal algebra with the central charge $c = 3d$, which is the symmetry of the non-linear σ model on a complex d dimensional Calabi-Yau manifold. The $c = 3d$ algebra is an extended superconformal algebra obtained by adding the spectral flow generators to the $N = 2$ superconformal algebra. We study the representation theory and show that its representations are invariant under the integer-shift spectral flow. We present the character formulas and their modular transformation properties. We also discuss the relation to the $N = 4$ superconformal algebra.

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1 Introduction

In the search for four dimensional realistic string theories, space-time supersymmetry and world-sheet superconformal symmetry are the most important requirements and internal sector is restricted severely by them. If internal sector is considered as the non-linear σ model on certain manifold, from the study of the low energy supergravity, $N = 1$ space-time supersymmetry implies the manifold must be a Calabi-Yau manifold [1,2]. This manifold possesses a unique covariantly constant spinor and an (anti-)holomorphic d-form, where d is the complex dimension of the manifold and, in the case of four dimensional string theory, d is equal to 3. On the other hand, world-sheet $N = 2$ superconformal symmetry and $U(1)$ charge quantization are equivalent to space-time $N = 1$ supersymmetry[3,4,5,6,7,8]. The $N = 2$ superconformal algebra(SCA) has an automorphism (so-called spectral flow) due to the $U(1)$ Kac-Moody subalgebra[9], so that Neveu-Schwarz(NS) and Ramond(R) sectors are mapped onto each other by the spectral flow which are considered as space-time supersymmetry transformation. In previous papers[10,11], we studied the extension of the $N = 2$ SCA by adding the flow generators which generate the integer-shift spectral flow. Its representations are invariant under the integer-shift spectral flow because such flow corresponds to twice operation of space-time supersymmetry transformation. In this context the covariantly constant spinor corresponds to Ramond ground state and the (anti-)holomorphic d-forms correspond to the spectral flow generators.

In this talk we generalize the previous $d = 3$ result (the $c = 9$ algebra) to arbitrary d case (the $c = 3d$ algebra). Since space-time dimension of string theory is $10 - 2d$, d more than three case is not relevant to string compactification. However, study of $d > 3$ case is interesting because it is the symmetry of the non-linear σ model on a complex d dimensional Calabi-Yau manifold (i.e. manifold with $SU(d)$ holonomy, i.e. Ricci-flat Kähler manifold) and finding modular invariant partition functions will give one method to study the properties of Calabi-Yau manifold itself[6,8]. The $N = 2$ SCA, which is the symmetry of the non-linear σ model on a Kähler manifold, is invariant under the spectral flow but its representations are not so. We want to find the extended algebra which representations are invariant under the integer-shift spectral flow. For $c > 3$, its representation contains infinite many representations of the $N = 2$ SCA because representation of the $N = 2$ SCA never comes back to itself under the integer-shift spectral flow in contrast to the rational case $c < 3$. The $N = 4$ SCA, which is the symmetry of the non-linear σ model on a hyper-Kähler manifold, has this property but it is too large from this point of view and we want the smallest one, i.e. its representation contains only one highest weight state of the $N = 2$ SCA modulo the spectral flow of the $N = 2$ SCA. This is the $c = 3d$ algebra. The $c = 3d$ algebra is obtained from the $N = 2$ SCA by the addition of the spectra flow generators and by the requirement that the cental charge c is equal to $3d$. Characters of the $N = 4$ SCA [12] are decomposed into characters of the $c = 3d$ algebra and representations of the $N = 4$ SCA with $c > 6$ are infinitely reducible with respect to the $c = 3d$ algebra.

In this report we present a proof of the theorem about the structure of the representation space, which is omitted in [13]. For the details of the $c = 3d$ algebra (OPE, degeneracy conditions, subalgebras, realization, the spectral flow, irreducible unitary highest weight representations, character formulas, modular transformation properties, Witten index, decomposition of characters), see the original article [13]. We follow the notations of [13].

2 Structure of the Representation Space

We will present a proof of the following structure theorem:
Representation space of the $c = 3d$ algebra is a direct sum of representation spaces of the $N = 2$ SCA, which are mapped from the highest weight state of the $c = 3d$ algebra by the integer shift spectral flow of the $N = 2$ SCA.
From this theorem, one obtains character formulas of the $c = 3d$ algebra as a sum of character formulas of the $N = 2$ SCA. We will prove following lemmas, propositions and theorems by induction and using the operator relations coming from the associativity of the $c = 3d$ algebra:

$$(IX)(z) - \partial X(z) = 0 \tag{2.1}$$

$$(IY)(z) - \partial Y(z) + \frac{1}{2}(\bar{G}X)(z) = 0 \tag{2.2}$$

$$(GY)(z) + \frac{1}{2}(\partial IX)(z) - (TX)(z) = 0 \tag{2.3}$$

$$(\partial^j XX)(z) = 0 \quad (j = 0, 1, \cdots, d-1) \tag{2.4}$$

$$(\partial^j XY)(z) = (\partial^j YY)(z) = 0 \quad (j = 0, 1, \cdots, d-2) \tag{2.5}$$

and their hermitian conjugates. In the following $N = 2$ means $N = 2$ SCA with $c = 3d$ and $(N = 2)$ stands for creation operators of the $N = 2$ SCA and we assume $d \geq 2$.

First we show the following lemma.

Lemma 1 $A_{-r}\bar{B}_{-s}|h, Q\rangle = (N = 2)|h, Q\rangle \quad (A, B = X, Y)$.

(proof) Using eq. (2.1) etc., we can show the following operator relations:

$$(A\partial^l \bar{B})(z) = (N = 2 \text{ generators})(z) \quad (l \geq 0). \tag{2.6}$$

For the case $A = B = X$, this is easily checked because, in bosonized form of $U(1)$ current $I(z) = \sqrt{d}i\partial\phi(z)$, $X(z)$ is $: e^{i\sqrt{d}\phi(z)} :$, so $(X\partial^l \bar{X})(z)$ is expressed as a differential polynomial of $I(z)$. Eq. (2.6) says

$$(A\partial^l \bar{B})_{-r-s}|h, Q\rangle = (N = 2)|h, Q\rangle \quad (l \geq 0). \tag{2.7}$$

Combining these relations with various l proves this lemma.□

From this, the next proposition holds.

Proposition 1

$$\overbrace{X \cdots XY \cdots Y}^{\frac{n+m}{2}} \overbrace{\bar{X} \cdots \bar{X}\bar{Y} \cdots \bar{Y}}^{\frac{n-m}{2}} |h, Q\rangle$$

$$= \begin{cases} \sum (N = 2) \overbrace{X \cdots XY \cdots Y}^{m} |h, Q\rangle & m \geq 0 \\ \sum (N = 2) \underbrace{\bar{X} \cdots \bar{X}\bar{Y} \cdots \bar{Y}}_{-m} |h, Q\rangle & m < 0. \end{cases}$$

Here we abbreviate $X_{-n_1} \cdots X_{-n_k}$ to $X \cdots X$ etc. Roughly speaking, these lemma and proposition say that X and Y have "flow charge" $+1$, $\bar{X}$ and $\bar{Y}$ have flow charge -1,

T, I, G and $\bar{G}$ have flow charge 0 and this flow charge is compatible with the $c = 3d$ algebra and its representation.

1. massive representations.

We define the states $|m\rangle$ as

$$|m\rangle = \begin{cases} \prod_{j=1}^{m} Y_{-k(j-\frac{1}{2})-Q}|h, Q\rangle & m \geq 0 \\ \prod_{j=1}^{-m} \bar{Y}_{-k(j-\frac{1}{2})+Q}|h, Q\rangle & m < 0, \end{cases} \tag{2.8}$$

where $\prod_{j=n}^{n-1} * = 1$.

Lemma 2 $X_{-n}, Y_{-n}|m\rangle = 0 \quad (n < k(m + \frac{1}{2}) + Q, \quad m \geq 0)$.

(proof)

(i) $m = 0$

Operating $-n$ mode of $(IX)(z) - \partial X(z)(= 0)$ to $|h, Q\rangle$, we obtain

$$(n - \frac{d}{2} - Q)X_{-n}|h, Q\rangle = \sum_{-n<p\leq -1} I_p X_{-n-p}|h, Q\rangle. \tag{2.9}$$

From this, $X_{-n}|h, Q\rangle = 0$ for $n < \frac{d}{2} + Q$. Operating $-n$ mode of $(IY)(z) - \partial Y(z) + \frac{1}{2}(\bar{G}X)(z)(= 0)$ to $|h, Q\rangle$, we obtain

$$(n - \frac{k}{2} - Q)Y_{-n}|h, Q\rangle = (\sum_{-n\leq p\leq -1} I_p Y_{-n-p} + \frac{1}{2} \sum_{\frac{k}{2}+Q-n\leq p\leq -\frac{1}{2}} \bar{G}_p X_{-n-p})|h, Q\rangle. \tag{2.10}$$

From this, $Y_{-n}|h, Q\rangle = 0$ for $n < \frac{k}{2} + Q$. Thus $m = 0$ case is OK.

(ii) Assume claim is true for m.

We have only to show $X_{-n}, Y_{-n}|m + 1\rangle = 0$ for $n_0 \leq n < n_0 + k$, where $n_0 = k(m + \frac{1}{2}) + Q$. Operating $-(n + n_0)$ mode of $(\partial^l XY)(z)$ $(= 0, \quad 0 \leq l \leq k - 1)$ to $|m\rangle$,

$$\begin{aligned} 0 &= (\partial^l XY)_{-n-n_0}|m\rangle \quad (l = 0, 1, \cdots, k - 1) \\ &= \sum_{n_0\leq p\leq n} \prod_{r=0}^{l-1} (p - \frac{d}{2} - r) X_{-p} Y_{p-n-n_0}|m\rangle. \end{aligned} \tag{2.11}$$

From this we get

$$\sum_{j=0}^{l} a_{ij} X_{-n_0-j-\frac{1}{2}} Y_{j-l-n_0}|m\rangle = 0, \tag{2.12}$$

where $a_{ij} = \prod_{r=0}^{i-1}(j+n_0-\frac{k}{2}-r)$ $(0 \leq i,j \leq l)$ is a regular matrix because, in general, a determinant of a matrix $A = (a_{ij})_{0\leq i,j\leq n-1}$ with $a_{ij} = \prod_{r=0}^{i-1}(x+j-r)$ is $\det A = \prod_{r=0}^{n-1} r!$. So we obtain

$$X_{-n_0-j-\frac{1}{2}} Y_{j-l-n_0}|m\rangle = 0 \quad (0 \leq j \leq l). \tag{2.13}$$

Let $j = l$, we get desired results $X_{-n_0-l-\frac{1}{2}}|m+1\rangle = 0$ $(0 \leq l \leq k-1)$. Similarly, operating $-(n+n_0)$ mode of $(\partial^l YY)(z)$ $(=0, \ \ 0 \leq l \leq k-1)$ to $|m\rangle$,

$$\begin{aligned} 0 &= (\partial^l YY)_{-n-n_0}|m\rangle \quad (0 \leq l \leq k-1) \\ &= \sum_{n_0 \leq p \leq n} \prod_{r=0}^{l-1}(p - \frac{d+1}{2} - r) Y_{-p} Y_{p-n-n_0}|m\rangle. \end{aligned} \tag{2.14}$$

From this we get

$$\sum_{j=0}^{l} b_{ij} Y_{-n_0-j} Y_{j-l-n_0}|m\rangle = 0, \tag{2.15}$$

where $b_{ij} = \prod_{r=0}^{i-1}(j+n_0-\frac{d+1}{2}-r)$ $(0 \leq i,j \leq l)$. Since b_{ij} is an invertible matrix, we obtain

$$Y_{-n_0-j} Y_{j-l-n_0}|m\rangle = 0 \quad (0 \leq j \leq l) \tag{2.16}$$

and for $j = l$ we obtain $Y_{-n_0-l}|m+1\rangle = 0$ $(0 \leq l \leq k-1)$. $m+1$ case is also OK.

From (i)(ii), the claim holds by induction. □

Proposition 2 *$|m\rangle$ are the highest weight states of the $N=2$ SCA with squared norm* $= \prod_{j=1}^{m} \frac{1}{2}(h - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{Q}{k})^2) > 0 \quad (m \geq 0)$.

(proof)

(i) $m = 0$ OK.

(ii) Assume that m case is OK.

Let $n_0 = k(m+\frac{1}{2}) + Q$. By assumption,

$$A_n|m+1\rangle = [A_n, Y_{-n_0}\}|m\rangle \quad (n > 0, \quad A = T, I, G, \bar{G}), \tag{2.17}$$

so we will consider only $0 < n < n_0$. $A = \bar{G}$ case is trivial. $A_n|m+1\rangle$ is proportional to $Y_{n-n_0}|m\rangle$ for $A = T, I$, and $X_{n-n_0}|m\rangle$ for G. These states are null states because of Lemma2. So $|m+1\rangle$ is also the highest weight state of the $N = 2$ SCA.

Squared norm of $|m\rangle$ is given by recursion formula

$$\begin{aligned}\langle m+1|m+1\rangle &= (-1)^d \langle m|[Y_{-n_0}, \bar{Y}_{n_0}\}|m\rangle \\ &= \frac{1}{2}(h + \frac{Q}{2} + \frac{k}{2}m^2 + (\frac{k}{2} + Q)m)\langle m|m\rangle. \end{aligned} \tag{2.18}$$

In the first line of this equation, we have used $\bar{X}_n, \bar{Y}_n|m\rangle = 0 \ \ (n \geq n_0, \ m \geq 0)$, which is proved by induction, and in the second line we have used the formula eq.(2.9) of [13]. □

Lemma 3 $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \quad (n \geq k(m+\frac{1}{2}) + Q, \quad m \geq 0)$.

(proof) Operating $-n$ mode of $(GY)(z) + \frac{1}{2}(\partial IX)(z) - (TX)(z)$ $(= 0)$ and $(IY)(z) - \partial Y(z) + \frac{1}{2}(\bar{G}X)(z)$ $(= 0)$ to $|m\rangle$, we obtain

$$\begin{aligned}(h_m + \frac{1}{2}Q_m)X_{-n}|m\rangle &= (\sum_{p\leq -\frac{1}{2}} G_p Y_{-n-p} - \sum_{p\leq -1} L_p X_{-n-p} \\ &\quad -\frac{1}{2}\sum_{p\leq -2}(p+1)I_p X_{-n-p})|m\rangle \\ (n - Q_m - \frac{k}{2})Y_{-n}|m\rangle &= (\sum_{p\leq -1} I_p Y_{-n-p} + \frac{1}{2}\sum_{p\leq -\frac{1}{2}} \bar{G}_p X_{-n-p})|m\rangle. \end{aligned} \tag{2.19}$$

Let $n_0 = k(m+\frac{1}{2}) + Q$. By definition, $Y_{-n_0}|m\rangle = |m+1\rangle$.

From eq. (2.19), $X_{-n_0-\frac{1}{2}}|m\rangle$, $Y_{-n_0-1}|m\rangle$, $X_{-n_0-\frac{3}{2}}|m\rangle$, $Y_{-n_0-2}|m\rangle$, $\cdots$ are expressed as $(N=2)|m+1\rangle$ in turn. □

From these the next proposition holds.

Proposition 3 $\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m \geq 0)$.

We can show the similar results for $|m\rangle$ $(m < 0)$. Therefore we obtain the theorem:

Theorem 1 $V_{h,Q} = \bigoplus_{m\in\mathbf{Z}} V^{N=2}_{h_m,Q_m}$.

Here V denotes the representation space.

For massless representations, we only mention the statements.

2. massless representation with $Q=0$.

We define the states

$$|m\rangle = \begin{cases} \prod_{j=2}^{m} Y_{-k(j-\frac{1}{2})-1} X_{-\frac{d}{2}}|0,0\rangle & m>0 \\ |0,0\rangle & m=0 \\ \prod_{j=2}^{-m} \bar{Y}_{-k(j-\frac{1}{2})-1} \bar{X}_{-\frac{d}{2}}|0,0\rangle & m<0. \end{cases} \tag{2.20}$$

Lemma 4 $X_{-n}, Y_{-n}|m\rangle = 0 \quad (n < k(m+\frac{1}{2})+1,\ m>0)\ ;\ (n<\frac{d}{2},\ m=0)$.

Proposition 4 *$|m\rangle$ are the highest weight states of the $N=2$ SCA with squared norm* $= \prod_{j=2}^{m} \frac{1}{2}(-\frac{1}{2}-\frac{k}{8}-\frac{1}{2k}+\frac{k}{2}(j-\frac{1}{2}+\frac{1}{k})^2) > 0 \quad (m\geq 0)$.

Lemma 5 $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle$
$(n\geq k(m+\frac{1}{2})+1,\ m>0)\ ;\ (n\geq\frac{d}{2},\ m=0)$.

Proposition 5 $\underbrace{X\cdots XY\cdots Y}_{m}|0\rangle = (N=2)|m\rangle \qquad (m\geq 0)$.

Theorem 2 $V_{0,0} = \bigoplus_{m<0} V^{N=2}_{h_m,Q_m} \oplus V^{N=2}_{0,0} \bigoplus_{m>0} V^{N=2}_{h_m,Q_m}$.

3. massless representations with $Q>0$.

We define the states

$$|m\rangle = \begin{cases} \prod_{j=1}^{m} Y_{-k(j-\frac{1}{2})-Q}|\frac{Q}{2},Q\rangle & m\geq 0 \\ \prod_{j=2}^{-m} \bar{Y}_{-k(j-\frac{1}{2})+Q-1} \bar{X}_{-\frac{d}{2}+Q}|\frac{Q}{2},Q\rangle & m<0. \end{cases} \tag{2.21}$$

Lemma 6 $X_{-n}, Y_{-n}|m\rangle = 0 \quad (n < k(m+\frac{1}{2}) + Q,\ m \geq 0).$

Proposition 6 *$|m\rangle$ are the highest weight states of the $N=2$ SCA with squared norm* $= \prod_{j=1}^{m} \frac{1}{2}(\frac{Q}{2} - \frac{k}{8} - \frac{Q^2}{2k} + \frac{k}{2}(j - \frac{1}{2} + \frac{Q}{k})^2) > 0 \quad (m \geq 0).$

Lemma 7 $X_{-n}, Y_{-n}|m\rangle = (N=2)|m+1\rangle \quad (n \geq k(m+\frac{1}{2}) + Q,\ m \geq 0).$

Proposition 7 $\underbrace{X \cdots X Y \cdots Y}_{m} |0\rangle = (N=2)|m\rangle \qquad (m \geq 0).$

Lemma 8 $\bar{X}_{-n}, \bar{Y}_{-n}|m\rangle = 0 \quad (n < k(-m+\frac{1}{2}) + 1 - Q,\ m < 0)\ ;\ (n < \frac{d}{2} - Q,\ m = 0).$

Proposition 8 *$|m\rangle$ are the highest weight states of the $N=2$ SCA with squared norm* $= \prod_{j=2}^{-m} \frac{1}{2}(\frac{Q-1}{2} - \frac{k}{8} - \frac{(Q-1)^2}{2k} + \frac{k}{2}(j - \frac{1}{2} - \frac{Q-1}{k})^2) > 0 \quad (m < 0).$

Lemma 9 $\bar{X}_{-n}, \bar{Y}_{-n}|m\rangle = (N=2)|m-1\rangle$

$$(n \geq k(-m+\frac{1}{2}) + 1 - Q,\ m < 0)\ ;\ (n \geq \frac{d}{2} - Q,\ m = 0).$$

Proposition 9 $\underbrace{\bar{X} \cdots \bar{X} \bar{Y} \cdots \bar{Y}}_{-m} |0\rangle = (N=2)|m\rangle \qquad (m < 0).$

Theorem 3 $V_{\frac{Q}{2},Q} = \bigoplus_{m<0} V^{N=2}_{h_m,Q_m} \bigoplus_{m\geq 0} V^{N=2}_{h_m,Q_m}.$

Similar results hold for massless representations with $Q < 0$.

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