QUASI-HOPF TWISTORS FOR ELLIPTIC QUANTUM GROUPS

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This talk is based on the collaboration with Jimbo, Konno and Shiraishi [1]. After reviewing a quasi-Hopf algebra and a twistor, we define elliptic quantum groups $\mathcal{A}_{q,p}(\widehat{sl}_n)$ and $\mathcal{B}_{q,\lambda}(\mathcal{G})$, which correspond to the solvable lattice models with elliptic solutions of the Yang-Baxter equation, the vertex type and the face type respectively.

1 Introduction

Conformal Field Theory (CFT) is a theory which is invariant under the conformal transformation. In two dimensional space (or 1+1 dimensional spacetime), the group of conformal transformations is infinite dimensional, whose algebra is known as the Virasoro algebra in the field theory realization. This symmetry is very powerful. By using its detailed representation theory one can determine the spectrum and even calculate correlation functions. CFT in 2 dimensions can be applied to the string theory (as a world sheet theory) and statistical critical phenomena in 2 dimensional space.

Quantum field theory and critical phenomena are the systems with infinite degrees of freedom. Hence they are difficult to treat. However one can sometimes solve some models, so-called solvable models. From symmetry point of view, "solvable" is stated as the following "equation":

$$\begin{aligned} &\frac{\text{System of } \textbf{infinite} \text{ degrees of freedom}}{\textbf{Infinite} \text{ dimensional symmetry}} \\ &= \text{System can be described by } \textbf{finite} \text{ degrees of freedom.} \end{aligned} \tag{1}$$

This is the reason why we are interested in infinite dimensional symmetries.

Since CFT is invariant under the scale transformation, CFT has no scale, in other words, it is a massless theory. If we add to CFT the perturbation which breaks the conformal symmetry, the theory becomes massive. General massive theories are very difficult. So we restrict ourselves to its subset, massive integrable models (MIM). If we perturb CFT in "good" manner, infinitely many conserved quantities survive. In the terminology of statistical mechanics, CFT corresponds to on-critical theory, and perturbation corresponds to off-critical procedure, and MIM corresponds to solvable lattice model. CFT is controlled by the Virasoro symmetry, but MIM is massive, therefore there is no Virasoro symmetry.

CFT	on-critical	massless	Virasoro
↓ good perturbation	↓ off-critical	↓	↓
massive integrable model	solvable lattice model	massive	no Virasoro

A natural question arises:

> What symmetry ensures
> the integrability or infinitely many conserved quantities of MIM?

We would like to answer this question. This is our main motivation.

In some cases the quantum group symmetry plays an important role. Kyoto group studied the XXZ spin chain and clarified its symmetry, the quantum affine Lie algebra $U_q(\widehat{sl}_2)$ [2]. But naively one expects some deformation of the Virasoro algebra. Such algebras, deformed Virasoro and deformed W_N algebras, were constructed in different point of view [3,4]. Later it was shown that the deformed Virasoro algebra appears in the Andrews-Baxter-Forester model as a symmetry [5].

Another possibility is elliptic quantum groups (elliptic algebras). Corresponding to the two types of elliptic solutions of the Yang-Baxter equation, there are two type of elliptic quantum groups [6,7]. These two elliptic quantum groups have a common structure [8]; They are quasi-Hopf algebras [9]. Along this line, we presented an explicit formula for the twistors and defined the vertex type algebra $\mathcal{A}_{q,p}(\widehat{sl}_n)$ and the face type algebra $\mathcal{B}_{q,\lambda}(\mathcal{G})$, see eq.(29) [1].

In section 2 we give a brief review of a quasi-Hopf algebra. Although several talkers already mention a quasi-Hopf algebra in this symposium, we also explain it because it is an important notion. In section 3 we present our result, an explicit formula for the twistors and the definition of elliptic quantum groups. Section 4 is devoted to discussion.

2 Quasi-Hopf Algebra (Introduction to Physicists)

In this section we illustrate an outline of a quasi-Hopf algebra. For precise definition we refer the readers to refs.[9,1]

In quantum mechanics, we know an addition of angular momentums very well. For two particles system the total angular momentum $\vec{J}$ is obtained simply by an addition of each angular momentum $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$,

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}. \tag{2}$$

In mathematics, this formula is written in the following way; $\vec{J}^{(1)}$ acts on the representation space V_1, $\vec{J}^{(2)}$ acts on V_2, and the total angular momentum $\vec{J}$ acts on $V_1 \otimes V_2$ by,

$$\vec{J} = \vec{J} \otimes 1 + 1 \otimes \vec{J}. \tag{3}$$

This is called the tensor product representation of Lie algebra $so(3)$. If a system has rotational symmetry, for example the Heisenberg spin chain (XXX spin chain), one can apply the representation theory of the rotational group $SO(3)$ (or its Lie algebra $so(3)$) to it. But if the system is perturbed and looses the rotational symmetry, then one can not apply $so(3)$ to it. Some models, however, have a good property. For example the XXZ spin chain has the same degeneracy of energy as the XXX spin chain. To treat such models we need some deformation of the Lie algebra or some deformation of the tensor product representation.

2.1 *Algebra and Coalgebra*

Let us begin with the definitions of an algebra and a coalgebra. For simplicity we take the complex field $\mathbf{C}$ as a base field. An algebra A is a vector space with two operations, product (multiplication) m and unit u, which satisfy

$$\begin{array}{ccccl} A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{id}} & A\otimes A & \text{product} & m : A\otimes A \to A \\ {\scriptstyle \mathrm{id}\otimes m}\downarrow & & \downarrow{\scriptstyle m} & \text{unit} & u : \mathbf{C} \to A \\ A\otimes A & \xrightarrow[m]{} & A & \text{associativity} & m\circ(m\otimes \mathrm{id}) = m\circ(\mathrm{id}\otimes m), \end{array} \tag{4}$$

and $m\circ(\mathrm{id}\otimes u) = \mathrm{id} = m\circ(u\otimes \mathrm{id})$ ($A\otimes\mathbf{C}$, A and $\mathbf{C}\otimes A$ are identified). If we write $m(a\otimes b) = ab$, the associativity becomes a usual form $(ab)c = a(bc)$.

A coalgebra is defined by reversing the arrows. A coalgebra A is a vector space with two operations, coproduct Δ and counit ε, which satisfy

$$\begin{array}{ccccl} A\otimes A\otimes A & \xleftarrow{\Delta\otimes \mathrm{id}} & A\otimes A & \text{coproduct} & \Delta : A \to A\otimes A \\ {\scriptstyle \mathrm{id}\otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} & \text{counit} & \varepsilon : A \to \mathbf{C} \\ A\otimes A & \xleftarrow[\Delta]{} & A & \text{coassociativity} & (\Delta\otimes \mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta, \end{array} \tag{5}$$

and $(\mathrm{id}\otimes\varepsilon)\circ\Delta = \mathrm{id} = (\varepsilon\otimes\mathrm{id})\circ\Delta$ ($A\otimes\mathbf{C}$, A and $\mathbf{C}\otimes A$ are identified).

Let us introduce σ ($\sigma : A\otimes A \to A\otimes A$, $\sigma(a\otimes b) = b\otimes a$) and define $m' = m\circ\sigma$ and $\Delta' = \sigma\circ\Delta$. An algebra is called commutative if $m' = m$, and a coalgebra is called cocommutative if $\Delta' = \Delta$.

2.2 *Hopf Algebra*

A Hopf algebra is a set $(A, m, u, \Delta, \varepsilon, S)$ satisfying the following conditions: A is an algebra and a coalgebra; m, u, Δ, ε are homomorphism; antipode $S : A \to A$ satisfies $m\circ(S\otimes\mathrm{id})\circ\Delta = u\circ\varepsilon = m\circ(\mathrm{id}\otimes S)\circ\Delta$. S is an anti-homomorphism.

We give two examples of a Hopf algebra, a group G and a Lie algebra $\mathcal{G}$, exactly speaking a function algebra of group $\mathrm{Fun}(G)$ and an enveloping algebra of Lie algebra $U(\mathcal{G})$ respectively. Their Hopf algebra structures are

$$\begin{array}{lll} & \mathrm{Fun}(G) = \mathrm{Map}(G,\mathbf{C}) & U(\mathcal{G}) \\ \text{product} & (f_1f_2)(x) = f_1(x)f_2(x) & XY \\ \text{unit} & (u(a))(x) = a & u(a) = a1 \\ \text{coproduct} & (\Delta(f))(x_1,x_2) = f(x_1x_2) & \Delta(X) = X\otimes 1 + 1\otimes X \\ \text{counit} & (\varepsilon(f))(x) = f(e) & \varepsilon(X) = 0 \\ \text{antipode} & (S(f))(x) = f(x^{-1}) & S(X) = -X, \end{array} \tag{6}$$

and $(m(g))(x) = g(x,x)$ for $g(x_1,x_2) \in \mathrm{Map}(G\times G,\mathbf{C})$, and $\Delta(1) = 1\otimes 1$, $\varepsilon(1) = 1$, $S(1) = 1$ for $U(\mathcal{G})$. Roughly speaking Δ, ε and S correspond to

$$\begin{array}{clll} & & \mathrm{Fun}(G) & U(\mathcal{G}) \\ \Delta & \leftrightarrow & \text{product of } G & \text{tensor product rep. of } U(\mathcal{G}) \\ \varepsilon & \leftrightarrow & \text{unit element of } G & \text{trivial rep. of } U(\mathcal{G}) \\ S & \leftrightarrow & \text{inverse element of } G & \text{contragredient rep. of } U(\mathcal{G}). \end{array} \tag{7}$$

Fun(G) is a commutative (and non-cocommutative) Hopf algebra and $U(\mathcal{G})$ is a cocommutative (and non-commutative) Hopf algebra. Non-commutative and non-cocommutative Hopf algebra may be regarded as an extension (deformation) of group or Lie algebra in this sense. This is the idea of quantum group (quantum algebra).

The quantum group (quantum algebra) is a Hopf algebra. We give an example of the quantum group, $U_q(sl_2)$, which is a deformation of $U(sl_2)$. $U_q(sl_2)$ is generated by $t = q^h$, e and f, which satisfy

$$\begin{array}{llll} [h,e]=2e & \Delta(h)=h\otimes 1+1\otimes h & \varepsilon(h)=0 & S(h)=-h \\ [h,f]=-2f & \Delta(e)=e\otimes 1+t\otimes e & \varepsilon(e)=0 & S(e)=-t^{-1}e \\ [e,f]=\frac{t-t^{-1}}{q-q^{-1}} & \Delta(f)=f\otimes t^{-1}+1\otimes f & \varepsilon(f)=0 & S(f)=-ft. \end{array} \quad (8)$$

This quantum algebra appears in the XXZ spin chain as a symmetry,

$$\begin{aligned} &[H_{\mathrm{XXZ}}, U_q(sl_2)] = 0, \\ &H_{\mathrm{XXZ}} = J\sum_{i=1}^{N-1}\Big(s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + \frac{q+q^{-1}}{2} s_i^z s_{i+1}^z\Big) + J\frac{q-q^{-1}}{4}(s_1^z - s_N^z), \\ &h = \sum_i 2s_i^z, \quad e = \sum_i q^{\sum_{j<i} 2s_j^z} s_i^+, \quad f = \sum_i s_i^- q^{-\sum_{j>i} 2s_j^z}, \end{aligned} \quad (9)$$

where $\vec{s} = \frac{1}{2}\vec{\sigma}$ and $s^{\pm} = s^1 \pm is^2$. In the $q \to 1$ limit, $U_q(sl_2)$ reduces to $U(sl_2)$.

2.3 Quasi-Triangular Hopf Algebra

Using the coproduct, a tensor product representation of two representations (π_i, V_i) $(i = 1,2)$ of the Hopf algebra can be defined in the following way,

$$\Big((\pi_1 \otimes \pi_2)\circ\Delta, V_1\otimes V_2\Big). \quad (10)$$

Coassociativity implies the isomorphism,

$$(V_1\otimes V_2)\otimes V_3 \cong V_1\otimes(V_2\otimes V_3) \quad (\text{as } A \text{ module}). \quad (11)$$

But cocommutativity does not hold in general, so the following isomorphism depends on the detail of the Hopf algebra:

$$V_1\otimes V_2 \overset{?}{\cong} V_2\otimes V_1 \quad (\text{as } A \text{ module}). \quad (12)$$

Of course we have $V_1\otimes V_2 \cong V_2\otimes V_1$ as vector space, by $P_{V_1V_2} : V_1\otimes V_2 \overset{\cong}{\to} V_2\otimes V_1$, $P_{V_1V_2}(v_1\otimes v_2) = v_2\otimes v_1$. But the problem is the commutativity of the action of A and $P_{V_1V_2}$.

Drinfeld considered the situation that the isomorphism (12) does hold. A quasi-triangular Hopf algebra $(A, m, u, \Delta, \varepsilon, S, \mathcal{R})$ (we abbreviate it as $(A, \Delta, \mathcal{R})$)

is a Hopf algebra with a universal R matrix $\mathcal{R}$, which satisfies

$$\begin{aligned} &\mathcal{R} \in A \otimes A : \underline{\text{universal R matrix}} \\ &\quad \Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad (\forall a \in A), \\ &\quad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(23)}, \quad (\varepsilon \otimes \mathrm{id})\mathcal{R} = 1, \\ &\quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(12)}, \quad (\mathrm{id} \otimes \varepsilon)\mathcal{R} = 1. \end{aligned} \tag{13}$$

Then we have an intertwiner

$$R_{V_1V_2} = P_{V_1V_2} \circ (\pi_1 \otimes \pi_2)(\mathcal{R}) : V_1 \otimes V_2 \xrightarrow{\cong} V_2 \otimes V_1 \quad (\text{as } A \text{ module}), \tag{14}$$

and $\mathcal{R}$ satisfies the Yang-Baxter equation,

$$\mathcal{R}^{(12)}\mathcal{R}^{(13)}\mathcal{R}^{(23)} = \mathcal{R}^{(23)}\mathcal{R}^{(13)}\mathcal{R}^{(12)}. \tag{15}$$

2.4 Quasi-Triangular Quasi-Hopf Algebra

As presented in subsection 2.2, the quantum group is obtained from the Lie algebra by relaxing one condition, cocommutativity. Here we relax one more condition, coassociativity. Coassociativity (5) is modified by a coassociator Φ in the following way,

$$\begin{aligned} &\Phi \in A \otimes A \otimes A : \underline{\text{coassociator}} \\ &\quad (\mathrm{id} \otimes \Delta)\Delta(a) = \Phi(\Delta \otimes \mathrm{id})\Delta(a)\Phi^{-1} \quad (\forall a \in A), \text{ etc.} \end{aligned} \tag{16}$$

A quasi-triangular quasi-Hopf algebra $(A, m, u, \Delta, \varepsilon, \Phi, S, \alpha, \beta, \mathcal{R})$ (we abbreviate it as $(A, \Delta, \Phi, \mathcal{R})$) satisfies (16) and

$$\begin{aligned} &\mathcal{R} \in A \otimes A \\ &\quad \Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad (\forall a \in A), \\ &\quad (\Delta \otimes \mathrm{id})\mathcal{R} = \Phi^{(312)}\mathcal{R}^{(13)}{\Phi^{(132)}}^{-1}\mathcal{R}^{(23)}\Phi^{(123)}, \\ &\quad (\mathrm{id} \otimes \Delta)\mathcal{R} = {\Phi^{(231)}}^{-1}\mathcal{R}^{(13)}\Phi^{(213)}\mathcal{R}^{(12)}{\Phi^{(123)}}^{-1}, \end{aligned} \tag{17}$$

where $\Phi^{(312)}$ means $\Phi^{(312)} = \sum_i Z_i \otimes X_i \otimes Y_i$ for $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$. Then $\mathcal{R}$ enjoys the Yang-Baxter type equation,

$$\mathcal{R}^{(12)}\Phi^{(312)}\mathcal{R}^{(13)}{\Phi^{(132)}}^{-1}\mathcal{R}^{(23)}\Phi^{(123)} = \Phi^{(321)}\mathcal{R}^{(23)}{\Phi^{(231)}}^{-1}\mathcal{R}^{(13)}\Phi^{(213)}\mathcal{R}^{(12)}. \tag{18}$$

A quasi-Hopf algebra with $\Phi = 1$ is nothing but a Hopf algebra.

2.5 Twist

Quasi-Hopf algebras admit an important operation, twist. For any invertible element $F \in A \otimes A$ ($(\varepsilon \otimes \mathrm{id})F = (\mathrm{id} \otimes \varepsilon)F = 1$), which is called as a twistor, there is a map from quasi-Hopf algebras to quasi-Hopf algebras:

$$\begin{aligned} \text{quasi-Hopf algebra} &\longrightarrow \text{quasi-Hopf algebra} \\ (A, \Delta, \Phi, \mathcal{R}) &\xrightarrow{F} (A, \widetilde{\Delta}, \widetilde{\Phi}, \widetilde{\mathcal{R}}). \end{aligned} \tag{19}$$

New coproduct, coassociator, R matrix etc. are given by

$$
\begin{aligned}
& F \in A \otimes A : \underline{\text{twistor}} \\
& \quad \widetilde{\Delta} = F\Delta(a)F^{-1} \quad (\forall a \in A), \\
& \quad \widetilde{\Phi} = \Big(F^{(23)}(\mathrm{id}\otimes\Delta)F\Big)\Phi\Big(F^{(12)}(\Delta\otimes\mathrm{id})F\Big)^{-1}, \\
& \quad \widetilde{\mathcal{R}} = F^{(21)}\mathcal{R}F^{-1}, \\
& \quad \text{etc.}
\end{aligned} \tag{20}
$$

We remark that an algebra A itself is unchanged. If a twistor F satisfies the cocycle condition, this twist operation maps a Hopf algebra to a Hopf algebra. For a general twistor F, however, a Hopf algebra is mapped to a quasi-Hopf algebra:

$$
\begin{aligned}
\text{Hopf algebra} &\longrightarrow \text{quasi-Hopf algebra} \\
(A, \Delta, \mathcal{R}) &\xrightarrow{F} (A, \widetilde{\Delta}, \widetilde{\Phi}, \widetilde{\mathcal{R}}).
\end{aligned} \tag{21}
$$

Let H be an Abelian subalgebra of A, with the product written additively. A twistor $F(\lambda) \in A \otimes A$ depending on $\lambda \in H$ is a shifted cocycle if it satisfies the relation (shifted cocycle condition),

$$
\begin{aligned}
& F(\lambda) : \underline{\text{shifted cocycle}} \\
\Leftrightarrow\ & F^{(12)}(\lambda)(\Delta\otimes\mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id}\otimes\Delta)F(\lambda),
\end{aligned} \tag{22}
$$

for some $h \in H$.

When a twistor $F(\lambda)$ satisfies the shifted cocycle condition, we obtain a quasi-triangular quasi-Hopf algebra from a quasi-triangular Hopf algebra by twisting,

$$
\begin{aligned}
\text{Hopf algebra} &\longrightarrow \text{quasi-Hopf algebra} \\
(A, \Delta, \mathcal{R}) &\xrightarrow{F(\lambda)} (A, \Delta_\lambda, \Phi(\lambda), \mathcal{R}(\lambda)),
\end{aligned} \tag{23}
$$

and we have

$$
\begin{aligned}
& \Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}, \\
& (\Delta_\lambda \otimes \mathrm{id})\mathcal{R}(\lambda) = \Phi^{(312)}(\lambda)\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}), \\
& (\mathrm{id}\otimes\Delta_\lambda)\mathcal{R}(\lambda) = \mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda)\Phi^{(123)}(\lambda)^{-1},
\end{aligned} \tag{24}
$$

and R matrix satisfies the dynamical Yang-Baxter equation,

$$
\mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{25}
$$

3 Qausi-Hopf Twistors and Elliptic Quantum Groups

It is well known that a statistical lattice model in two dimensional space is solvable if its Boltzmann weights satisfy the Yang-Baxter equation. The Yang-Baxter equation admits two types of elliptic solutions, the vertex-type and the face-type.

Corresponding to these two there are two types of elliptic quantum groups (algebras).

The first example of the vertex-type elliptic algebras is the Sklyanin algebra [10], designed as an elliptic deformation of the Lie algebra sl_2. It is presented by the RLL relation,

$$R^{(12)}(u_1-u_2)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2). \tag{26}$$

R and L depend on an elliptic modulus r. Foda et al.[6] proposed its affine version, $\mathcal{A}_{q,p}(\widehat{sl}_2)$,

$$R^{(12)}(u_1-u_2,r)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2,r-c), \tag{27}$$

whose main point of is the shift of r by a central element c. On the other hand, in the case of the face type algebras, R and L depend on the elliptic modulus r and other extra parameters λ. Felder [7] showed that the RLL relation undergoes a dynamical shift by elements h of the Cartan subalgebra,

$$\begin{aligned}R^{(12)}(u_1-u_2;\lambda+h)L^{(1)}(u_1,\lambda)L^{(2)}(u_2,\lambda+h^{(1)})\\ =L^{(2)}(u_2,\lambda)L^{(1)}(u_1,\lambda+h^{(2)})R^{(12)}(u_1-u_2;\lambda),\end{aligned} \tag{28}$$

and $R(u,\lambda)$ satisfies the dynamical Yang-Baxter equation(25).

These two algebras seemed to be different but Frønsdal [8] pointed out that they have a common structure; they are quasi-Hopf algebras obtained by twisting quantum affine algebras. We constructed an explicit formula for the twistors satisfying the shifted cocycle condition and defined two types of elliptic quantum groups, $\mathcal{A}_{q,p}(\widehat{sl}_n)$ and $\mathcal{B}_{q,\lambda}(\mathcal{G})$: [1]

$$\begin{array}{ccccc}\text{type} & \text{twistor} & \text{Hopf algebra} & & \text{quasi-Hopf algebra}\\ \text{face} & F(\lambda) & U_q(\mathcal{G}) & \xrightarrow{F(\lambda)} & \mathcal{B}_{q,\lambda}(\mathcal{G})\\ \text{vertex} & E(r) & U_q(\widehat{sl}_n) & \xrightarrow{E(r)} & \mathcal{A}_{q,p}(\widehat{sl}_n),\end{array} \tag{29}$$

where $\mathcal{G}$ is the Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix.

Due to space limitation, here we give the results only and refer the readers to ref.[1] for the details. The face type twistor $F(\lambda)\in U_q(\mathcal{G})^{\otimes 2}$ (λ is an element of the Cartan subalgebra of $\mathcal{G}$) and the vertex type twistor $E(r)\in U_q(\widehat{sl}_n)^{\otimes 2}$ ($r\in\mathbf{C}$) are represented in the form of an infinite product of the universal R matrix,

$$\underline{\text{face type twistor}}\quad F(\lambda)=\prod_{k\geq 1}^{\leftarrow}\Big(\varphi_\lambda^k\otimes\mathrm{id}\Big)\Big(q^T\mathcal{R}\Big)^{-1}, \tag{30}$$

$$\underline{\text{vertex type twistor}}\quad E(r)=\prod_{k\geq 1}^{\leftarrow}\Big(\widetilde{\varphi}_r^k\otimes\mathrm{id}\Big)\Big(q^{\widetilde{T}}\mathcal{R}\Big)^{-1}, \tag{31}$$

where φ_λ, $\widetilde{\varphi}_r$ are automorphisms of (algebra)$^{\otimes 2}$, and T, $\widetilde{T}$ are canonical elements in (Cartan subalgebra)$^{\otimes 2}$. These two twistor satisfy the shifted cocycle condition,

$$F^{(12)}(\lambda)(\Delta\otimes\mathrm{id})F(\lambda)=F^{(23)}(\lambda+h^{(1)})(\mathrm{id}\otimes\Delta)F(\lambda), \tag{32}$$

$$E^{(12)}(r)(\Delta\otimes\mathrm{id})E(r)=E^{(23)}(r+c^{(1)})(\mathrm{id}\otimes\Delta)E(r). \tag{33}$$

Twisting the quantum algebra $U_q(\mathcal{G})$ ($U_q(\widehat{sl}_n)$) by the twistor $F(\lambda)$ ($E(r)$), we define the elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathcal{G})$ ($\mathcal{A}_{q,p}(\widehat{sl}_n)$) respectively,

face type algebra $\mathcal{B}_{q,\lambda}(\mathcal{G})$

$$\begin{array}{ccccl} \text{Hopf algebra} & & \text{quasi-Hopf algebra} & & \\ U_q(\mathcal{G}) & \xrightarrow{F(\lambda)} & \mathcal{B}_{q,\lambda}(\mathcal{G}) & & \\ \Delta & & \Delta_\lambda & & \Delta_\lambda(a) = F(\lambda)\Delta(a)F(\lambda)^{-1} \\ \mathcal{R} & & \mathcal{R}(\lambda) & & \mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F(\lambda)^{-1} \\ \vdots & & \Phi(\lambda) & & \Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}, \\ & & \vdots & & \end{array} \tag{34}$$

vertex type algebra $\mathcal{A}_{q,p}(\widehat{sl}_n)$

$$\begin{array}{ccccl} \text{Hopf algebra} & & \text{quasi-Hopf algebra} & & \\ U_q(\widehat{sl}_n) & \xrightarrow{E(r)} & \mathcal{A}_{q,p}(\widehat{sl}_n) & & p = q^{2r} \\ \Delta & & \Delta_r & & \Delta_r(a) = E(r)\Delta(a)E(r)^{-1} \\ \mathcal{R} & & \mathcal{R}(r) & & \mathcal{R}(r) = E^{(21)}(r)\mathcal{R}E(r)^{-1} \\ \vdots & & \Phi(r) & & \Phi(r) = E^{(23)}(r)E^{(23)}(r + c^{(1)})^{-1}. \\ & & \vdots & & \end{array} \tag{35}$$

4 Discussion

We have presented an explicit formula for the twistors which satisfy the shifted cocycle condition, and defined elliptic quantum groups by twisting quantum groups with the twistors (29). We also studied the RLL relation ($RLL = LLR^*$), the vertex operators in ref.[1] and $U_{q,p}(\widehat{sl}_2)$ in ref.[11] Applications of these elliptic quantum groups to the solvable lattice models are to be studied in future.

Finally we mention other related works; Relation to the deformed Virasoro and W algebras is studied in [12,13], and an extension to the Lie superalgebra is studied in [14,15].

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