EFI-96-45
DPSU-96-16
UT-762
November 1996

Quantum Deformation of the W_N Algebra[†]

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Abstract

We review the W_N algebra and its quantum deformation, based on free field realizations. The (quantum deformed) W_N algebra is defined through the (quantum deformed) Miura transformation, and its singular vectors realize the Jack (Macdonald) polynomials.

q-alg/9612001

[†] Talk presented by S.O. at the Nankai-CRM joint meeting on "Extended and Quantum Algebras and their Applications to Physics", Tianjin, China, August 19-24, 1996. To appear in the CRM series in mathematical physics, Springer Verlag.

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1 Introduction

The conformal field theory (CFT)[6], whose examples are string theories as a world sheet theory and statistical critical phenomena, has made remarkable progress contacting with various branches of mathematics. CFT is a massless theory and its infinitely many conserved quantities are controlled by the its symmetry algebra, the Virasoro algebra. By perturbing CFT, it becomes a massive theory and there does not exist the Virasoro algebra any longer. However, if we add a good perturbation, the theory has still infinitely many conserved quantities and they are called massive integrable theory (MIT). Behind the conserved quantities, there exist symmetries. So we would like to clarify

What symmetry ensures the integrability of MIT?

In some cases the quantum group, Yangian, degenerate affine Hecke algebra, etc. have an important role. For example[8], in spin one-half XXZ spin chain, its correlation functions were derived by using the quantum affine Lie algebra symmetry $U_q(\widehat{sl}_2)$. However these symmetries correspond to the current algebra (affine Lie algebra) symmetry in CFT, not to the Virasoro algebra. Naively, "quantum deformation" (q-deformation) of the Virasoro algebra has been expected. After the name of the quantum group had been known to physicists, several attempts to construct q-deformed Virasoro algebra have been made. But satisfactory q-deformation of the Virasoro algebra had not been obtained(at least to the talker)[1], because of the lack of definite guiding principle.

Last summer reasonable q-deformation of the Virasoro and W_N algebras was obtained[23, 3, 11] (see also important works [13, 19]). Here we will review these q-deformation. Putting aside the above physical motivation, we take the following point of view,

Algebra, Representation Theory, Free Field Realization.

Our guiding principle is the following. First we note the two facts:

1. In the free field realization, the singular vectors of the Virasoro and W_N algebras realize the Jack symmetric polynomials[22, 4].

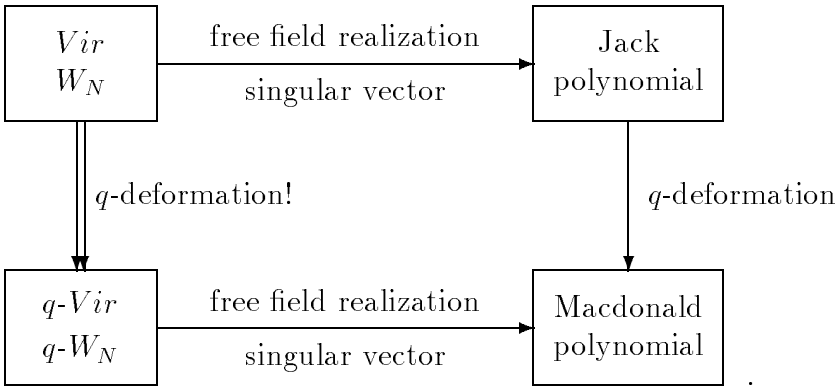
2. The Jack symmetric polynomials have the good q-deformation, the Macdonald symmetric polynomials[21].

Based on these, setting up the following question seems to be natural;

[1]Generally speaking, what is q-deformation? Although there is no precise definition of q-deformation, we would like to define q-deformation in the following way; (i) Theory deformed by adding one parameter q, which reduces to the original theory in the $q \to 1$ limit, (ii) (Some) Properties of the original theory remains in the q-world. The condition (ii) is obscure and arbitrary. So q-deformation is not unique. There exist "good" q-deformation, "bad" q-deformation and "usual" q-deformation.

- Construct the algebras whose singular vectors in the free field realization realize the Macdonald symmetric polynomials.

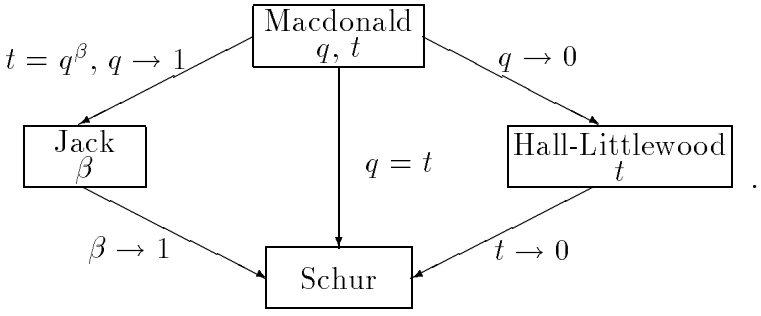
The resultant algebra are worth being called quantum deformation (q-deformation) of the Virasoro and W_N algebras in this sense. We call these as q-Virasoro and q-W_N algebras. We illustrate this scenario by a figure,



A priori these algebras have nothing to do with the symmetry of massive integrable models. But we believe that they are related very closely. In fact their relations have been revealed gradually[18, 10, 20, 16, 2, 1, 15, 14].

In this talk we would like to review the W_N and q-W_N algebras based on free field realizations. In section 2 the W_N algebra is defined by the Miura transformation, and singular vectors realize the Jack symmetric polynomials. In section 3 q-W_N algebra is introduced in a similar manner. We show that singular vectors realize the Macdonald symmetric polynomials and q-W_N reduces to W_N in the $q \to 1$ limit. Section 4 is devoted to the discussion. In appendix we present explicit examples.

Before going section 2, we recall the parameters of the Jack and Macdonald symmetric polynomials (for precise definitions of these polynomials, see other talks in this meeting). The Jack symmetric polynomials $J_\lambda(x;\beta)$ have one parameter β, and the Macdonald symmetric polynomials $P_\lambda(x;q,t)$ have two parameters q and t. The relation among the Macdonald, Jack, Hall-Littlewood and Schur symmetric polynomials is[21],



2 W_N Algebra

In this section we recapitulate the W_N algebra[12] from the free filed realization point of view. For comprehensive review of W algebras, see [7].

The W_N algebra (A_{N-1} type W algebra) is an associative algebra generated by spin k currents[2] $\bar{W}^k(z) = \sum_{n\in\mathbb{Z}} \bar{W}^k_n z^{-n-k}$ $(k = 2, \cdots, N)$ and the central charge c. To define its relations it is convenient to use a free field realization. Let us introduce free bosons $\bar{h}^i(z) = Q^i_h + h^i_0 \log z - \sum_{n\neq 0} \frac{1}{n} \bar{h}^i_n z^{-n}$ $(i = 1, \cdots, N)$, whose relations are given by

$$\begin{aligned} [\bar{h}^i_n, \bar{h}^j_m] &= (\delta^{ij} - \tfrac{1}{N}) n \delta_{n+m,0}, \quad \bar{h}^i_0 = h^i_0, \\ [\bar{h}^i_n, Q^j_h] &= (\delta^{ij} - \tfrac{1}{N}) \delta_{n0}, \quad [Q^i_h, Q^j_h] = 0, \quad \sum_{i=1}^N \bar{h}^i_n = 0, \quad \sum_{i=1}^N Q^i_h = 0. \end{aligned} \tag{1}$$

These $\bar{h}^i(z)$'s correspond to the weights of the vector representation of A_{N-1} algebra, h_i. If we introduce an orthonormal set of free bosons $\varphi(z)^t \varphi(w) \sim \mathbf{1} \log(z-w)$, and project them on the root space, $\phi(z) = \varphi(z)\big|_{\text{root space}}$, then $\bar{h}^i$ is expressed as $\bar{h}^i(z) = h_i \cdot \phi(z)$. We will also use free bosons $\phi^a(z) = \alpha^a \cdot \phi(z)$, which correspond to the simple roots α^a $(a = 1, \cdots, N-1)$. $\phi^a(z)$ has the form $\phi^a(z) = Q^a_\alpha + \alpha^a_0 \log z - \sum_{n\neq 0} \frac{1}{n} \bar{\alpha}^a_n z^{-n}$ and $\alpha^a = h_a - h_{a+1}$ implies

$$[\bar{\alpha}^a_n, \bar{\alpha}^b_m] = A_{ab} n \delta_{n+m,0}, \quad \bar{\alpha}^a_0 = \alpha^a_0, \quad [\bar{\alpha}^a_n, Q^b_\alpha] = A_{ab} \delta_{n0}, \quad [Q^a_\alpha, Q^b_\alpha] = 0, \tag{2}$$

where $A = (A_{ab})$ is the Cartan matrix of A_{N-1} algebra. Note that $h_i \cdot h_j = \delta_{ij} - \frac{1}{N}$, $\alpha^a \cdot \alpha^b = A_{ab}$ and $\Lambda_a \cdot \Lambda_b = (A^{-1})_{ab}$, where $\Lambda_a = \sum_{i=1}^a h_i$ is the fundamental weight, $\alpha^a \cdot \Lambda_b = \delta_{ab}$.

The following Miura transformation gives the realization of the W_N algebra (this is just the definition of the W_N algebra by ref.[12]);

$$:\Big(\alpha_0 \partial_z + \partial \bar{h}^1(z)\Big)\Big(\alpha_0 \partial_z + \partial \bar{h}^2(z)\Big) \cdots \Big(\alpha_0 \partial_z + \partial \bar{h}^N(z)\Big): = \sum_{k=0}^N \bar{W}^k(z) \Big(\alpha_0 \partial_z\Big)^{N-k}. \tag{3}$$

Here $:*:$ stands for the normal ordering (i.e., non-negative mode oscillators are moved to the right for negative mode oscillators and Q), and α_0 is a parameter which determines the central charge. For later convenience we parameterize α_0 as

$$\alpha_0 = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}. \tag{4}$$

We have $W^0(z) = 1$, $W^1(z) = 0$, and

$$-W^2(z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \tfrac{1}{2} :\partial\phi(z) \cdot \partial\phi(z): + \alpha_0 \rho \cdot \partial^2 \phi(z). \tag{5}$$

[2] Bar ($\bar{}$) indicates non-q-deformed quantity.

This $L(z)$ generates the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{6}$$

with the central charge,

$$c = N - 1 - 12\alpha_0^2\rho^2, \tag{7}$$

where ρ is the half sum of the positive roots $\rho = \sum_{a=1}^{N-1} \Lambda_a$ and $\rho^2 = \frac{1}{12}N(N^2-1)$.

W_N generators satisfy quadratic relations. Symbolically it is

$$[\bar{W}_n^k, \bar{W}_m^\ell] = \sum((\bar{W}\bar{W}) + \bar{W} + 1). \tag{8}$$

Here quadratic terms should be normal-ordered; For operators $A(z) = \sum_{n\in\mathbb{Z}-h_A} A_n z^{-n-h_A}$ and $B(z) = \sum_{n\in\mathbb{Z}-h_B} B_n z^{-n-h_B}$, normal ordering $(AB)(z)$ is defined by

$$\begin{aligned}(AB)(w) &= \oint_w \frac{dz}{2\pi i}\frac{1}{z-w}A(z)B(w) \\ &= \oint_0 \frac{dz}{2\pi i z}\left(\frac{1}{1-\frac{w}{z}}A(z)B(w) + \frac{\frac{z}{w}}{1-\frac{z}{w}}B(w)A(z)\right) \\ &= \sum_{n\in\mathbb{Z}-h_A-h_B}\left(\sum_{m\le -h_A} A_m B_{n-m} + \sum_{m>-h_A} B_{n-m}A_m\right)\cdot w^{-n-h_A-h_B}.\end{aligned} \tag{9}$$

We consider the highest weight representations. The highest weight state $|\mathrm{hws}\bar{\rangle}$ is characterized by

$$\bar{W}_n^k|\mathrm{hws}\bar{\rangle} = 0\ (n>0), \quad \bar{W}_0^k|\mathrm{hws}\bar{\rangle} = \bar{w}_k|\mathrm{hws}\bar{\rangle}\ (\bar{w}_k \in \mathbb{C}). \tag{10}$$

The Verma module is obtained by successive action of $\bar{W}_{-n}^k$ $(n>0)$ on $|\mathrm{hws}\bar{\rangle}$. We remark that only finite number of terms in (WW) of (8) survives on $|\mathrm{hws}\bar{\rangle}$. If there exists a singular vector $|\bar{\chi}\bar{\rangle}$, which is characterized by

$$\bar{W}_n^k|\bar{\chi}\bar{\rangle} = 0\ (n>0), \quad \bar{W}_0^k|\bar{\chi}\bar{\rangle} = (\bar{w}_k + \bar{N}_k)|\bar{\chi}\bar{\rangle} = 0\ (\bar{N}_k \in \mathbb{C}), \tag{11}$$

then the Verma module is reducible. To obtain an irreducible module, the submodule on $|\bar{\chi}\bar{\rangle}$ has to be factored out.

In the following we consider the representations realized in the boson Fock space. The Fock vacuum $|\gamma\bar{\rangle}$ is characterized by

$$\bar{\alpha}_n^a|\gamma\bar{\rangle} = 0\ (n>0), \quad \alpha_0^a|\gamma\bar{\rangle} = \gamma^a|\gamma\bar{\rangle}, \tag{12}$$

where $\gamma = \sum_{a=1}^{N-1}\gamma^a\Lambda_a$ $(\gamma^a \in \mathbb{C})$. $|\gamma\bar{\rangle}$ can be obtained from $|0\bar{\rangle}$ $(\bar{\alpha}_n|0\bar{\rangle} = 0$ for $n \ge 0)$,

$$|\gamma\bar{\rangle} = \exp\Big(\sum_{a=1}^{N-1}\gamma^a Q_\Lambda^a\Big)\cdot|0\bar{\rangle}, \quad Q_\Lambda^a = \sum_{j=1}^{a} Q_h^j. \tag{13}$$

The Fock space $\bar{F}_\gamma$ is a linear span of $\bar{\alpha}^{a_1}_{-n_1}\bar{\alpha}^{a_2}_{-n_2}\cdots|\gamma\rangle\!\rangle$ $(n_1, n_2, \cdots > 0)$. Dual Fock space is defined similarly ($\langle\!\langle\gamma|\bar{\alpha}^a_n = \langle\!\langle\gamma|\gamma^a\delta_{n0}$ $(n \leq 0)$, with normalization $\langle\!\langle\gamma|\gamma'\rangle\!\rangle = \delta_{\gamma\gamma'}$). For studying representations in the Fock space, the most important tool is the screening current. Let us introduce the screening currents $\bar{S}^a_\pm(z)$ $(a = 1, \cdots N-1)$,

$$\bar{S}^a_\pm(z) = :e^{\alpha_\pm\phi^a(z)}:, \qquad \alpha_+ = \sqrt{\beta},\ \ \alpha_- = \frac{-1}{\sqrt{\beta}}. \tag{14}$$

The operator product expansion between W_N currents and the screening currents is

$$:\Big(\alpha_0\partial_z + \partial\bar{h}^1(z)\Big)\cdots\Big(\alpha_0\partial_z + \partial\bar{h}^N(z)\Big):\bar{S}^a_\pm(w) \sim \frac{\partial}{\partial w}\Big(\cdots\Big) + \text{reg.} \tag{15}$$

Therefore the screening charge $\oint dz\bar{S}^a_\pm(z)$ commutes with W_N algebra,

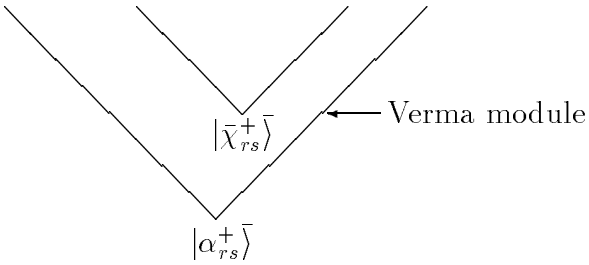
$$\Big[\ W_N, \oint dz\bar{S}^a_\pm(z)\ \Big] = 0. \tag{16}$$

We must comment that this proposition holds only on a suitable state on which the contour closes. To study representation in more detail, we have to construct the BRST charge and analyze the BRST cohomology[9]. Here we do not get into this direction any more.

The Fock vacuum $|\gamma\rangle\!\rangle$ satisfies the highest weight state condition of the W_N algebra. If we choose γ as special values, the Verma module contains singular vectors. Let us define $\alpha^\pm_{rs}$ and $\tilde{\alpha}^\pm_{rs}$ as

$$\begin{aligned}
\alpha^+_{rs} &= \sum_{a=1}^{N-1}\Big(\sqrt{\beta}(1 + r_a - r_{a-1}) + \frac{-1}{\sqrt{\beta}}(1 + s_a)\Big)\Lambda_a, \quad \tilde{\alpha}^+_{rs} = \alpha^+_{rs} - \sqrt{\beta}\sum_{a=1}^{N-1} r_a\alpha^a, \\
\alpha^-_{rs} &= \sum_{a=1}^{N-1}\Big(\frac{-1}{\sqrt{\beta}}(1 + r_a - r_{a-1}) + \sqrt{\beta}(1 + s_a)\Big)\Lambda_a, \quad \tilde{\alpha}^-_{rs} = \alpha^-_{rs} - \frac{-1}{\sqrt{\beta}}\sum_{a=1}^{N-1} r_a\alpha^a,
\end{aligned} \tag{17}$$

where r_a, s_a are positive integers such that $r_1 > r_2 > \cdots > r_{N-1}$ and $r_0 = 0$. The Verma module on $|\alpha^+_{rs}\rangle\!\rangle$ contains a singular vector $|\bar{\chi}^+_{rs}\rangle\!\rangle$,



$$\begin{aligned}
|\bar{\chi}^+_{rs}\rangle\!\rangle &= \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i}\cdot\bar{S}^1_+(x^1_1)\cdots\bar{S}^1_+(x^1_{r_1})\cdots\bar{S}^{N-1}_+(x^{N-1}_1)\cdots\bar{S}^{N-1}_+(x^{N-1}_{r_{N-1}})|\tilde{\alpha}^+_{rs}\rangle\!\rangle \\
&= \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i x^a_j}\cdot\prod_{a=1}^{N-1}\bar{\pi}\Big(\frac{1}{x^a}, x^{a+1}\Big)\bar{\Delta}(x^a)\prod_{j=1}^{r_a}\Big(x^a_j\Big)^{-s_a}\Big[\bar{S}^a_+(x^a_j)\Big]_-\cdot|\alpha^+_{rs}\rangle\!\rangle,
\end{aligned} \tag{18}$$

where $\bar{\Delta}(x)$ and $\bar{\pi}(x,y)$ are

$$\bar{\Delta}(x) = \prod_{i\neq j}\Big(1-\frac{x_i}{x_j}\Big)^{\beta}, \quad \bar{\pi}(x,y) = \prod_{i,j}(1-x_iy_j)^{-\beta}, \tag{19}$$

and $[*]_-$ stands for the negative mode oscillator part.

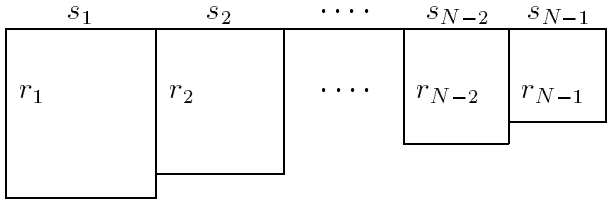
This singular vector is related to the Jack symmetric polynomial. To see this, we consider a map from the Fock space to the ring of symmetric function;

$$\bar{F}_\gamma \ni |f\bar{\rangle} \mapsto f(x) = \bar{\langle}\gamma|\exp\Big(\sqrt{\beta}\sum_{n>0}\tfrac{1}{n}\bar{h}^1_n p_n\Big)|f\bar{\rangle}, \tag{20}$$

where p_n is a power sum symmetric polynomial $p_n = \sum_i (x_i)^n$. By this map, $\bar{\alpha}^a_{-n}$ is replaced by $\delta^{a1}\sqrt{\beta}p_n$. We remark that $\exp(\sqrt{\beta}\sum_{n>0}\frac{1}{n}\bar{h}^1_n z^{-n})$ is the positive oscillator part of the vertex operator corresponding to the vector representation. Then the singular vector $|\bar{\chi}^+_{rs}\bar{\rangle}$ is mapped to the Jack symmetric polynomial,

$$\begin{aligned}
&\bar{\langle}\alpha^+_{rs}|\exp\Big(\sqrt{\beta}\sum_{n>0}\tfrac{1}{n}\bar{h}^1_n p_n\Big)|\bar{\chi}^+_{rs}\bar{\rangle} \\
&= \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i x^a_j}\cdot\bar{\pi}(x,x^1)\prod_{a=1}^{N-1}\bar{\pi}\Big(\frac{1}{x^a},x^{a+1}\Big)\bar{\Delta}(x^a)\prod_{j=1}^{r_a}\Big(x^a_j\Big)^{-s_a} \\
&\propto J_\lambda(x;\beta).
\end{aligned} \tag{21}$$

In the last equation we have changed the integration variable $x^a \to \frac{1}{x^a}$ and used the integral representation of the Jack symmetric polynomial[4]. Here the partition λ is $\lambda' = ((r_1)^{s_1},(r_2)^{s_2},\cdots,(r_{N-1})^{s_{N-1}})$, namely, corresponds to the following Young diagram

$\lambda =$ 

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Therefore, in the free field realization, the singular vector of the W_N algebra realizes the Jack symmetric polynomial with the Young diagram composed of $N-1$ rectangles. Similarly we have singular vectors $|\bar{\chi}^-_{rs}\bar{\rangle} = \oint \bar{S}_-\cdots\bar{S}_-|\bar{\alpha}^-_{rs}\bar{\rangle}$, and another type of integral representation of the Jack polynomial[4].

3 Quantum Deformed W_N Algebra

In this section we define and explain the quantum deformed W_N algebra[3, 11] along the line of W_N in section 2.

The quantum deformed W_N algebra, q-W_N, is an associative algebra generated by $W^i(z) = \sum_{n\in\mathbb{Z}} W^i_n z^{-n}$ $(i = 1, \cdots, N-1)$ and contains two parameters q and t $(q, t \in \mathbb{C})$. We will often use the following notation,

$$p = qt^{-1}, \quad q = e^{\hbar} = e^{\frac{1}{\sqrt{\beta}}\hbar'}, \quad t = q^{\beta} = e^{\sqrt{\beta}\hbar'}. \tag{22}$$

To define the relations of q-W_N generators, we first introduce fundamental bosons h^i_n $(n \in \mathbb{Z})$ and Q^i_h $(i = 1, \cdots, N)$,

$$\begin{aligned}
[h^i_n, h^j_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\frac{p^{\frac{n}{2}N(\delta^{ij}-\frac{1}{N})} - p^{-\frac{n}{2}N(\delta^{ij}-\frac{1}{N})}}{p^{\frac{n}{2}N} - p^{-\frac{n}{2}N}} p^{\frac{n}{2}N\mathrm{sgn}(j-i)}\delta_{n+m,0}, \\
[h^i_n, Q^j_h] &= (\delta^{ij} - \tfrac{1}{N})\delta_{n0}, \quad [Q^i_h, Q^j_h] = 0, \quad \sum_{i=1}^{N} p^{in} h^i_n = 0, \quad \sum_{i=1}^{N} Q^i_h = 0,
\end{aligned} \tag{23}$$

where $\mathrm{sgn}(x)$ is $\mathrm{sgn}(x) = 1(\text{for } x > 0), 0(\text{for } x = 0), -1(\text{for } x < 0)$. We remark that the fractional part containing p is a p-analogue of $\delta^{ij} - \frac{1}{N}$. Let us define transformations θ, ω, and $\omega' = \theta\omega$ as follows:

	θ	ω	ω'
q	q^{-1}	t	t^{-1}
t	t^{-1}	q	q^{-1}
h^i_n $(n \neq 0)$	h^{N+1-i}_n	h^{N+1-i}_n	h^i_n
h^i_0	$-h^{N+1-i}_0$	h^{N+1-i}_0	$-h^i_0$
Q^i_h	$-Q^{N+1-i}_h$	Q^{N+1-i}_h	$-Q^i_h$

(24)

For example $\theta \cdot q = q^{-1}$, $\omega \cdot h^i_0 = h^{N+1-i}_0$. These are involutions, $\theta^2 = \omega^2 = \omega'^2 = 1$. Then (23) is invariant under θ, ω and ω'. For later convenience we list up transformation rules for various quantities (for their definitions see below):

	θ	ω	ω'
$\hbar'$	$-\hbar'$	$\hbar'$	$-\hbar'$
β	β	β^{-1}	β^{-1}
p	p^{-1}	p^{-1}	p
α^a_n $(n \neq 0)$	$-\alpha^{N-a}_n$	$-\alpha^{N-a}_n$	α^a_n
α^a_0	α^{N-a}_0	$-\alpha^{N-a}_0$	$-\alpha^a_0$
Q^a_α	Q^{N-a}_α	$-Q^{N-a}_\alpha$	$-Q^a_\alpha$
$\Lambda_i(z)$	$\Lambda_{N+1-i}(z)$	$\Lambda_{N+1-i}(z)$	$\Lambda_i(z)$
$W^i(z)$	$W^i(z)$	$W^i(z)$	$W^i(z)$
$f^{ij}(x)$	$f^{ij}(x)$	$f^{ij}(x)$	$f^{ij}(x)$
$S^a_\pm(z)$	$S^{N-a}_\pm(z)$	$S^{N-a}_\mp(z)$	$S^a_\mp(z)$

(25)

In the case of W_N algebra, the building block of W_N currents is a boson $\bar{h}^i(z)$, and the Miura transformation is an equation of differential operator. On the other hand the building block of q-W_N currents is an exponentiated boson $\Lambda_i(z)$ $(i=1,\cdots,N)$,

$$\Lambda_i(z) = :\exp\Big(\sum_{n\neq 0} h^i_n z^{-n}\Big): q^{\sqrt{\beta}h^i_0} p^{\frac{N+1}{2}-i}, \tag{26}$$

and the differential operator (shift operator) ∂_z is replaced by p-difference operator (p-shift operator) p^{D_z} $(D_z = z\partial_z)$; $p^{D_z} f(z) = f(pz)$. q-deformed Miura transformation is given by

$$:\Big(p^{D_z}-\Lambda_1(z)\Big)\Big(p^{D_z}-\Lambda_2(p^{-1}z)\Big)\cdots\Big(p^{D_z}-\Lambda_N(p^{1-N}z)\Big): = \sum_{i=0}^{N}(-1)^i W^i(p^{\frac{1-i}{2}}z)p^{(N-i)D_z}. \tag{27}$$

From this, $W^i(z)$ is expressed as

$$W^i(z) = \sum_{1\leq j_1<j_2<\cdots<j_i\leq N} :\Lambda_{j_1}(p^{\frac{i-1}{2}}z)\Lambda_{j_2}(p^{\frac{i-3}{2}}z)\cdots\Lambda_{j_i}(p^{-\frac{i-1}{2}}z):, \tag{28}$$

and $W^0(z) = W^N(z) = 1$. $W^i(z)$ is invariant under θ, ω and ω'. Eq.(27) is equivalent to

$$:\Big(p^{-D_z}-\Lambda_N(z)\Big)\Big(p^{-D_z}-\Lambda_{N-1}(pz)\Big)\cdots\Big(p^{-D_z}-\Lambda_1(p^{N-1}z)\Big): = \sum_{i=0}^{N}(-1)^i W^i(p^{\frac{i-1}{2}}z)p^{-(N-i)D_z},$$

$$:\Big(1-\Lambda_1(z)p^{-D_z}\Big)\Big(1-\Lambda_2(z)p^{-D_z}\Big)\cdots\Big(1-\Lambda_N(z)p^{-D_z}\Big): = \sum_{i=0}^{N}(-1)^i W^i(p^{\frac{1-i}{2}}z)p^{-iD_z}, \tag{29}$$

$$:\Big(1-\Lambda_N(z)p^{D_z}\Big)\Big(1-\Lambda_{N-1}(z)p^{D_z}\Big)\cdots\Big(1-\Lambda_1(z)p^{D_z}\Big): = \sum_{i=0}^{N}(-1)^i W^i(p^{\frac{i-1}{2}}z)p^{iD_z}.$$

The relation of q-W_N algebra is quadratic. Symbolically it is

$$f^{ij}(\tfrac{w}{z})W^i(z)W^j(w) - W^j(w)W^i(z)f^{ji}(\tfrac{z}{w}) = \sum\Big(\mathop{\circ}_{\circ} W W \mathop{\circ}_{\circ} + W + 1\Big). \tag{30}$$

We remark that this relation is invariant under θ, ω and ω'. The structure function $f^{ij}(x) = \sum_{\ell=0}^{\infty} f^{ij}_\ell x^\ell$ is defined by

$$f^{ij}(x) = \exp\Big(-\sum_{n>0}\frac{1}{n}(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}}) \times\frac{p^{\frac{n}{2}\min(i,j)}-p^{-\frac{n}{2}\min(i,j)}}{p^{\frac{n}{2}}-p^{-\frac{n}{2}}}\frac{p^{\frac{n}{2}(N-\max(i,j))}-p^{-\frac{n}{2}(N-\max(i,j))}}{p^{\frac{n}{2}N}-p^{-\frac{n}{2}N}}x^n\Big). \tag{31}$$

Note that the fractional part containing p is a p-analogue of $(A^{-1})_{ij}$ and $f^{ij}(x) = f^{ji}(x) = f^{N-i,N-j}(x)$. Quadratic terms have to be normal-ordered,

$$\begin{aligned}
&\mathop{\circ}_{\circ} W^i(rw)W^j(w)\mathop{\circ}_{\circ} \\
&= \oint\frac{dz}{2\pi i z}\left(\frac{1}{1-\frac{rw}{z}}f^{ij}(\tfrac{w}{z})W^i(z)W^j(w) + \frac{\frac{z}{rw}}{1-\frac{z}{rw}}W^j(w)W^i(z)f^{ji}(\tfrac{z}{w})\right) \\
&= \sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty}\sum_{\ell=0}^{m} f^{ij}_\ell\Big(r^{m-\ell}W^i_{-m}W^j_{n+m} + r^{\ell-m-1}W^j_{n-m-1}W^i_{m+1}\Big)\cdot w^{-n},
\end{aligned} \tag{32}$$

where $(1-x)^{-1}$ stands for $\sum_{n\geq 0} x^n$. This normal ordering $^\circ_\circ * ^\circ_\circ$ is a generalization of (9).

Here we present some examples of them. The relation of $W^1(z)$ and $W^j(z)$ for $j \geq 1$ is

$$\begin{aligned} & f^{1j}(\tfrac{w}{z})W^1(z)W^j(w) - W^j(w)W^1(z)f^{j1}(\tfrac{z}{w}) \\ = & -\frac{(1-q)(1-t^{-1})}{1-p}\Big(\delta(p^{\frac{j+1}{2}}\tfrac{w}{z})W^{j+1}(p^{\frac{1}{2}}w) - \delta(p^{-\frac{j+1}{2}}\tfrac{w}{z})W^{j+1}(p^{-\frac{1}{2}}w)\Big), \end{aligned} \tag{33}$$

with $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$; and that of $W^2(z)$ and $W^j(z)$ for $j \geq 2$ is

$$\begin{aligned} & f^{2j}(\tfrac{w}{z})W^2(z)W^j(w) - W^j(w)W^2(z)f^{j2}(\tfrac{z}{w}) \\ = & -\frac{(1-q)(1-t^{-1})}{1-p}\frac{(1-qp)(1-t^{-1}p)}{(1-p)(1-p^2)}\Big(\delta(p^{\frac{j}{2}+1}\tfrac{w}{z})W^{j+2}(pw) - \delta(p^{-\frac{j}{2}-1}\tfrac{w}{z})W^{j+2}(p^{-1}w)\Big) \\ & -\frac{(1-q)(1-t^{-1})}{1-p}\Big(\delta(p^{\frac{j}{2}}\tfrac{w}{z})^\circ_\circ W^1(p^{-\frac{1}{2}}z)W^{j+1}(p^{\frac{1}{2}}w)^\circ_\circ - \delta(p^{-\frac{j}{2}}\tfrac{w}{z})^\circ_\circ W^1(p^{\frac{1}{2}}z)W^{j+1}(p^{-\frac{1}{2}}w)^\circ_\circ\Big) \\ & +\frac{(1-q)^2(1-t^{-1})^2}{(1-p)^2}\Big(\delta(p^{\frac{j}{2}}\tfrac{w}{z})\Big(\frac{p^2}{1-p^2}W^{j+2}(pw) + \frac{1}{1-p^j}W^{j+2}(w)\Big) \\ & \qquad -\delta(p^{-\frac{j}{2}}\tfrac{w}{z})\Big(\frac{p^j}{1-p^j}W^{j+2}(w) + \frac{1}{1-p^2}W^{j+2}(p^{-1}w)\Big)\Big), \end{aligned} \tag{34}$$

with $W^i(z) = 0$ for $i > N$. The main term of

$$f^{ij}(\tfrac{w}{z})W^i(z)W^j(w) - W^j(w)W^i(z)f^{ji}(\tfrac{z}{w}) \qquad (i \leq j)$$

is

$$\begin{aligned} & -\frac{(1-q)(1-t^{-1})}{1-p}\sum_{k=1}^{\min(i,N-j)}\prod_{\ell=1}^{k-1}\frac{(1-qp^\ell)(1-t^{-1}p^\ell)}{(1-p^\ell)(1-p^{\ell+1})} \\ & \times\Big(\delta(p^{\frac{j-i}{2}+k}\tfrac{w}{z})^\circ_\circ W^{i-k}(p^{-\frac{k}{2}}z)W^{j+k}(p^{\frac{k}{2}}w)^\circ_\circ - \delta(p^{-\frac{j-i}{2}-k}\tfrac{w}{z})^\circ_\circ W^{i-k}(p^{\frac{k}{2}}z)W^{j+k}(p^{-\frac{k}{2}}w)^\circ_\circ\Big). \end{aligned} \tag{35}$$

By taking $\beta \to 0$ limit (q fixed), this main term reduces to q-deformed W_N Poisson bracket algebra in [13].

The highest weight state $|\mathrm{hws}\rangle$ is characterized by

$$W^i_n|\mathrm{hws}\rangle = 0 \ (n > 0), \quad W^i_0|\mathrm{hws}\rangle = w_i|\mathrm{hws}\rangle \ (w_i \in \mathbb{C}). \tag{36}$$

We remark that only finite number of terms in $^\circ_\circ W W^\circ_\circ$ of (30) are non-vanishing on $|\mathrm{hws}\rangle$. The boson Fock space F_γ is defined like as in section 2. To construct screening currents we define root bosons α^a_n ($n \in \mathbb{Z}$) and Q^a_α ($a = 1, \cdots, N-1$) as $\alpha^a_n = h^a_n - h^{a+1}_n$, $Q^a_\alpha = Q^a_h - Q^{a+1}_h$. They satisfy

$$\begin{aligned} [\alpha^a_n, \alpha^b_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\frac{p^{\frac{n}{2}A_{ab}} - p^{-\frac{n}{2}A_{ab}}}{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}p^{\frac{n}{2}\mathrm{sgn}(b-a)}\delta_{n+m,0}, \\ [\alpha^a_n, Q^b_\alpha] &= A_{ab}\delta_{n0}, \quad [Q^a_\alpha, Q^b_\alpha] = 0. \end{aligned} \tag{37}$$

The fractional part containing p is a p-analogue of A_{ab}. Screening currents are defined by

$$S^a_+(z) = :\exp\Big(-\sum_{n\neq 0}\frac{\alpha^a_n}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}z^{-n}\Big): e^{\sqrt{\beta}Q^a_\alpha} z^{\sqrt{\beta}\alpha^a_0}, \tag{38}$$

$$S^a_-(z) = :\exp\Big(\sum_{n\neq 0}\frac{\alpha^a_n}{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}z^{-n}\Big): e^{\frac{-1}{\sqrt{\beta}}Q^a_\alpha} z^{\frac{-1}{\sqrt{\beta}}\alpha^a_0}. \tag{39}$$

Commutation relation of $W^i(z)$ and $S^a_\pm(w)$ can be expressed as a total difference:

$$\begin{aligned}
&\Big[:\Big(p^{D_z}-\Lambda_1(z)\Big)\Big(p^{D_z}-\Lambda_2(p^{-1}z)\Big)\cdots\Big(p^{D_z}-\Lambda_N(p^{1-N}z)\Big): , S^a_\pm(w)\Big] \\
&= (q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})\frac{d}{d^q_t w}\Big(:\Big(p^{D_z}-\Lambda_1(z)\Big)\cdots\Big(p^{D_z}-\Lambda_{a-1}(p^{2-a}z)\Big) \\
&\qquad \times w\delta(p^{a-1}\tfrac{w}{z})A^a_\pm(w)p^{D_z}\times\Big(p^{D_z}-\Lambda_{a+2}(p^{-1-a}z)\Big)\Big(p^{D_z}-\Lambda_N(p^{1-N}z)\Big):\Big),
\end{aligned} \tag{40}$$

where $A^a_\pm(w)$ and $\frac{d}{d_\xi w}$ are given by[3]

$$A^a_+(w) = :\exp\Big(\sum_{n\neq 0}\frac{q^{\frac{n}{2}}h^{a+1}_n - q^{-\frac{n}{2}}h^a_n}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}w^{-n}\Big): e^{\sqrt{\beta}Q^a_\alpha} w^{\sqrt{\beta}\alpha^a_0} q^{\frac{1}{2}\sqrt{\beta}(h^a_0+h^{a+1}_0)} p^{\frac{N}{2}-a},$$

$$A^a_-(w) = \omega'\cdot A^a_+(w), \tag{41}$$

$$\frac{d}{d_\xi w}f(w) = \frac{f(\xi^{\frac{1}{2}}w)-f(\xi^{-\frac{1}{2}}w)}{(\xi^{\frac{1}{2}}-\xi^{-\frac{1}{2}})w} = \frac{d}{d_{\xi^{-1}}w}f(w). \tag{42}$$

Then screening charge $\oint dz S^a_\pm(z)$ commutes with q-W_N algebra,

$$\Big[\, q\text{-}W_N, \oint dz\, S^a_\pm(z)\,\Big] = 0. \tag{43}$$

Exactly speaking we have to include the zero mode factor[20, 1, 15] and study BRST cohomology.

The Fock vacuum $|\gamma\rangle$ satisfies the highest weight condition of q-W_N. In the Verma module generated by $|\alpha^+_{rs}\rangle$ (α^+_{rs} is given in (17)) we have a singular vector $|\chi^+_{rs}\rangle$,

$$\begin{aligned}
|\chi^+_{rs}\rangle &= \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i}\cdot S^1_+(x^1_1)\cdots S^1_+(x^1_{r_1})\cdots S^{N-1}_+(x^{N-1}_1)\cdots S^{N-1}_+(x^{N-1}_{r_{N-1}})|\tilde{\alpha}^+_{rs}\rangle \\
&= \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i x^a_j}\cdot\prod_{a=1}^{N-1}\pi\Big(\frac{1}{x^a},px^{a+1}\Big)\Delta(x^a)C(x^a)\prod_{j=1}^{r_a}\Big(x^a_j\Big)^{-s_a}\Big[S^a_+(x^a_j)\Big]_-\cdot|\alpha^+_{rs}\rangle,
\end{aligned} \tag{44}$$

where $\Delta(x)$, $\pi(x,y)$ and $C(x)$ are given by

$$\Delta(x)=\prod_{i\neq j}\exp\Big(-\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}p^{-\frac{n}{2}}\frac{x^n_j}{x^n_i}\Big),\quad \pi(x,y)=\prod_{i,j}\exp\Big(\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}p^{-\frac{n}{2}}x^n_i y^n_j\Big),$$

[3] We have used the slightly different notation from that of [3] (say 'old');
$S^a_+(w) = S^{a,\mathrm{old}}_+(q^{-\frac{1}{2}}w)q^{\frac{1}{2}\sqrt{\beta}\alpha^a_0}$, $S^a_-(w) = S^{a,\mathrm{old}}_-(t^{-\frac{1}{2}}w)q^{-\frac{1}{2}\sqrt{\beta}\alpha^a_0}$, $A^a_+(w) = A^{a,\mathrm{old}}_+(w)q^{\frac{1}{2}\sqrt{\beta}\alpha^a_0}p^{\frac{1}{2}}$,
$A^a_-(w) = A^{a,\mathrm{old}}_-(w)q^{-\frac{1}{2}\sqrt{\beta}\alpha^a_0}p^{-\frac{1}{2}}$, $\frac{d}{d_\xi w}f(w) = \xi^{\frac{1}{2}}\Big(\frac{d}{d_\xi w}\Big)^{\mathrm{old}}f(\xi^{-\frac{1}{2}}w)$

$$C(x)=\prod_{i<j}^{r}\exp\Big(\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}\Big(p^{-\frac{n}{2}}\frac{x_i^n}{x_j^n}-p^{\frac{n}{2}}\frac{x_j^n}{x_i^n}\Big)\Big)\cdot\prod_{i=1}^{r}x_i^{(r+1-2i)\beta}. \tag{45}$$

We remark that $C(x)$ is a pseudo-constant under q-shift; $q^{D_{x_i}}C(x)=C(x)$ $(\forall i)$.

What we want to show is that this singular vector $|\chi^+_{rs}\rangle$ is related to the Macdonald symmetric polynomial. To establish this relation, we consider a map from the Fock space to the ring of symmetric function;

$$F_\gamma\ni|f\rangle\mapsto f(x)=\langle\gamma|\exp\Big(\sum_{n>0}\frac{h^1_n}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}p_n\Big)|f\rangle,\quad p_n=\sum_i(x_i)^n. \tag{46}$$

This map replaces α^a_{-n} with $\delta^{a1}\frac{1}{n}(t^{\frac{n}{2}}-t^{-\frac{n}{2}})p^{\frac{n}{2}}p_n$. We remark that $\exp(\sum_{n>0}\frac{h^1_n}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}z^{-n})$ is the positive oscillator part of the vertex operator corresponding to the vector representation. The singular vector $|\chi^+_{rs}\rangle$ is mapped to the Macdonald symmetric polynomial,

$$\begin{aligned}&\langle\alpha^+_{rs}|\exp\Big(\sum_{n>0}\frac{h^1_n}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}p_n\Big)|\chi^+_{rs}\rangle\\ =\ &\oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dx^a_j}{2\pi i x^a_j}\cdot\pi(x,px^1)\prod_{a=1}^{N-1}\pi\Big(\frac{1}{x^a},px^{a+1}\Big)\Delta(x^a)C(x^a)\prod_{j=1}^{r_a}\Big(x^a_j\Big)^{-s_a}\\ \propto\ &P_\lambda(x;q,t).\end{aligned} \tag{47}$$

In the last equation we have changed the integration variable $x^a\to(p^ax^a)^{-1}$ and used the integral representation of the Macdonald symmetric polynomial[4, 5]. Here the partition λ is same as in section 2, $\lambda'=((r_1)^{s_1},(r_2)^{s_2},\cdots,(r_{N-1})^{s_{N-1}})$. Therefore, in the free field realization, the singular vector of the q-W_N algebra realizes the Macdonald symmetric polynomial with the Young diagram composed of $N-1$ rectangles. Similarly we have singular vectors $|\chi^-_{rs}\rangle=\omega'\cdot|\chi^+_{rs}\rangle=\oint S_-\cdots S_-|\tilde\alpha^-_{rs}\rangle$, and another type of integral representation of the Macdonald polynomial[3].

We have shown the condition (ii) of q-deformation (see footnote 1). Next let us check the condition (i); the classical limit, i.e., $q\to1$ limit ($\hbar'\to0$, β fixed). The fundamental boson h^i_n can be expressed in a linear combination of $\bar h^i_n$,

$$h^i_n=\hbar'\sum_{j=1}^{N}d^{ij}_n\bar h^j_n=\hbar'\bar h^i_n+O(\hbar'^2)\quad(n\neq0), \tag{48}$$

where $d^{ij}_n\in\mathbb{C}$. Then $\Lambda_i(z)$ and p^{D_z} have the following $\hbar'$ expansion,

$$\begin{aligned}\Lambda_i(z)&=1+\hbar'\Big(D\bar h^i(z)-(\tfrac{N+1}{2}-i)\alpha_0\Big)+O(\hbar'^2),\\ p^{D_z}&=1-\hbar'\alpha_0D_z+O(\hbar'^2),\\ p^{D_z}-\Lambda_i(p^{1-i}z)&=-\hbar'z^{\frac{N+1}{2}-i+1}\Big(\alpha_0\partial_z+\partial\bar h^i(z)\Big)z^{-(\frac{N+1}{2}-i)}+O(\hbar'^2).\end{aligned} \tag{49}$$

Thus L.H.S. of (27) is

$$(-\hbar')^N z^{\frac{N+1}{2}} : \Big(\alpha_0\partial_z + \partial\bar{h}^1(z)\Big)\Big(\alpha_0\partial_z + \partial\bar{h}^2(z)\Big)\cdots\Big(\alpha_0\partial_z + \partial\bar{h}^N(z)\Big) : \ z^{\frac{N-1}{2}} \times \Big(1 + O(\hbar')\Big). \tag{50}$$

This is nothing but (3). So q-W_N algebra reduces to W_N algebra with central charge (7). However relation between $W^i(z)$ and $\bar{W}^k(z)$ is nontrivial. R.H.S. of (27) contains $\hbar'^{\ell}$ ($\ell < N$) terms which must vanish, and $\hbar'^{N}$ term yields the W_N algebra. We will demonstrate these for explicit examples in the Appendix.

For other quantities, $q \to 1$ limit is straightforward,

$$\begin{aligned}
&\alpha_n^a = \hbar'\bar{\alpha}_n^a \times \Big(1 + O(\hbar)\Big) \quad (n \neq 0), \quad |\gamma\rangle = |\bar{\gamma}\rangle, \\
&S_{\pm}^a(z) = \bar{S}_{\pm}^a(z) \times \Big(1 + O(\hbar)\Big), \quad |\chi_{rs}^{\pm}\rangle = |\bar{\chi}_{rs}^{\pm}\rangle \times \Big(1 + O(\hbar)\Big), \\
&\Delta(x) = \bar{\Delta}(x) \times \Big(1 + O(\hbar)\Big), \quad \pi(x,y) = \bar{\pi}(x,y) \times \Big(1 + O(\hbar)\Big), \text{ etc.}
\end{aligned} \tag{51}$$

4 Discussion

We have defined a quantum deformed W_N algebra, q-W_N. There are many interesting points to be clarified in the future study; representation theory and applications to physics. Here we write down some of them.

(i) Explicit form of the defining relation. Even for the W_N algebra, the explicit form of the defining relation in terms of $\bar{W}_n^k$ has not been known for general N. It may be easier for q-W_N. We remark that when we study representation theory by free field realization and BRST cohomology technique, the explicit form of the defining relation is not necessary and what we need are q-Miura transformation, screening currents and vertex operators just like as W_N case.

(ii) Various limits. (a) $q \to 1$: $\hbar'^{N}$ term of (27) gives (3), but explicit relation between $W^i(z)$ and $\bar{W}^k(z)$ are unknown for general N. What is the meaning of $\hbar'^{\ell}$ ($\ell > N$) terms? Are they related to conserved quantities in CFT? (b) $q \to 0$: For quantum (affine) Lie algebras, $q \to 0$ limit yielded an important notion, crystal base[17]. For q-Vir, see our another talk in this meeting.

(iii) Kac determinant. For $N = 2$ case (q-Vir), we calculated and conjectured the Kac determinant of q-Vir[23]. It has same structure as that of the Virasoro algebra and extra zeros when q (or t) is a root of unity. We guess that the Kac determinant of q-W_N is also so.

(iv) Tensor product representation. Tensor product representation of the Virasoro algebra is trivial because the Virasoro algebra is a Lie algebra. But q-Vir (q-W_N, W_N) satisfies quadratic relation. So its tensor product representation is nontrivial.

(v) Relation to a quantum affine Lie algebra. Since the Virasoro algebra is obtained from affine Lie algebra $\hat{\mathcal{G}}$ by the Sugawara construction, we naively expect that q-Vir is obtained from quantum affine Lie algebra $U_q(\hat{\mathcal{G}})$. In the critical level, the q-Virasoro Poisson algebra was constructed through q-Sugawara form[13]. An interesting relation between screening currents of q-W_N and $U_q(\hat{\mathcal{G}})$ was pointed out in [11].
(vi) Various type of W algebras. q-W_N is A_{N-1}-type. The Hamiltonian reduction technique gives us various type of W algebra from affine Lie algebras[7]. Can we extend this technique to the q-world? $W_{1+\infty}$ algebra is obtained from W_N by taking a suitable $N \to \infty$ limit. What algebra is obtained from q-W_N by $N \to \infty$ limit? We remark that for special value of β, q-Vir is related to $W_{1+\infty}$. Supersymmetric extension is also important (see our another talk in this meeting).
(vii) Macdonald operators. The Macdonald operator can be bosonized and expressed in terms of q-Vir generators[23]. Bosonized form of the higher Macdonald operators may be expressed by q-W_N generators.
(viii) Vertex operators. Vertex operators(primary fields) are indispensable for applications to physics. There are several proposals for vertex operators[3, 20, 16, 2, 1] but at present general definition has not been obtained. The condition (i) of q-deformation (see footnote 1) is of course satisfied but the problem exists in the condition (ii); what property should we impose on vertex operators?
(ix) BRST cohomology. By using above vertex operators and screening currents, the BRST property of the free field representation of q-W_N can be studied. To construct BRST charge and screened vertex operators, we should include zero mode factor[20, 16, 2, 1, 15].
(x) Correlation functions. By using vertex operators and screening currents, correlation functions can be calculated[20, 2, 1]. In CFT correlation functions for the minimal model are characterized by the differential equations. It is expected that correlation functions for q-W_N satisfy some q-difference equations[16, 2].
(xi) Application to solvable models. In the work[20], multi-point local hight probabilities for the ABF model in the regime III were calculated, where the vertex operator of the ABF model is identified with that of q-Vir. Its higher rank generalization was partially studied in [1]. They constructed (type I) vertex operators of the RSOS model. To calculate its correlation functions, however, the knowledge of BRST cohomology will be needed.
(xii) Applications to other physics. In the work[18], the q-Vir generator $T(z)$ is identified with the Zamolodchikov-Faddeev operator for the basic scalar particle in the XYZ model. In the scaling limit XYZ model is described by the sine-Gordon model, and the two-particle S-matrix of the basic scalar particle in the sine-Gordon model can be obtained from the defining relation of q-Vir. We remark that in CFT the meaning of z in $L(z)$ is the local coordinate of the Riemann surface, but in this case z of $T(z)$ corresponds to the rapidity of particle.

Our original motivation is to find and study the symmetry of massive integrable models. We hope that q-W_N will be found out to be a useful symmetry.

Acknowledgments
We would like to thank members of Nankai Institute for their kind hospitality. S.O. would like to thank the organizers of the meeting for giving him the opportunity to present our results. This work is supported in part by Grant-in-Aid for Scientific Research from Ministry of Science and Culture.

Appendix: Explicit examples

We present explicit defining relations of q-W_N and $q \to 1$ limit for $N = 2, 3$ cases. To check consistency of $q \to 1$ limit (see below (50)), we need higher order terms of (48). We set

$$h_n^i = \hbar' \sum_{j=1}^{N} d_n^{ij} \bar{h}_n^j \quad (n \neq 0), \qquad d_n^{ij} = \tilde{d}_n^{ij} \sqrt{\frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{n\hbar} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n\hbar\beta}}. \tag{A.1}$$

We remark that d_n^{ij} is not uniquely determined because of $\sum_{i=1}^N \bar{h}_n^i = 0$, and $\tilde{d}_n^{ij}$ can be chosen as a function of p only and $\tilde{d}_n^{N+1-i,N+1-j} = \tilde{d}_n^{ij}\Big|_{p\to p^{-1}}$.

$N = 2$ case (q-Vir)[4]. q-Miura transformation is

$$:\Big(p^{D_z} - \Lambda_1(z)\Big)\Big(p^{D_z} - \Lambda_2(p^{-1}z)\Big): = p^{2D_z} - W^1(z)p^{D_z} + 1, \quad W^1(z) = \Lambda_1(z) + \Lambda_2(z), \tag{A.2}$$

and the relation is

$$f(\tfrac{w}{z})T(z)T(w) - T(w)T(z)f(\tfrac{z}{w}) = -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\Big(\delta(p\tfrac{w}{z}) - \delta(p^{-1}\tfrac{w}{z})\Big), \tag{A.3}$$

where $T(z) = W^1(z)$ and $f(x) = f^{11}(x)$.

By multiplying $z^{\frac{1}{2}}$ from the left and $z^{-\frac{1}{2}}$ from the right, q-Miura transformation (27) becomes (see (50))

$$(-\hbar')^2 z^2 :\Big(\alpha_0\partial_z + \partial\bar{h}^1(z)\Big)\Big(\alpha_0\partial_z + \partial\bar{h}^2(z)\Big): + O(\hbar'^3) = p^{2(D_z - \frac{1}{2})} - T(z)p^{D_z - \frac{1}{2}} + 1. \tag{A.4}$$

To check the consistency of this equation, we need $\tilde{d}_n^{ij}$, which is chosen as

$$\tilde{d}_n^{11} = \frac{1}{2}p^{\frac{n}{2}}\sqrt{2\frac{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}{p^n - p^{-n}}}, \quad \tilde{d}_n^{12} = -\tilde{d}_n^{11}, \quad \tilde{d}_n^{3-i,3-j} = \tilde{d}_n^{ij}\Big|_{p\to p^{-1}}. \tag{A.5}$$

[4]Since $h_n^1 + p^n h_n^2 = 0$, it is convenient to set $h_n = p^{-\frac{n}{2}} h_n^1$.

Then $T(z)$ has the following $\hbar'$ expansion,

$$T(z) = 2 + \hbar'^2\Big(z^2 L(z) + \tfrac{1}{4}\alpha_0^2\Big) + O(\hbar'^4), \tag{A.6}$$

where $L(z)$ is the Virasoro generator (5). We remark that $T(z)$ is an even function of $\hbar'$. Using these, $O(\hbar'^\ell)$ terms ($\ell < 2$) of R.H.S. of (A.4) actually vanish and $O(\hbar'^2)$ term gives Miura transformation of $L(z)$.

$N = 3$ case (q-W_3). q-Miura transformation is

$$:\Big(p^{D_z} - \Lambda_1(z)\Big)\Big(p^{D_z} - \Lambda_2(p^{-1}z)\Big)\Big(p^{D_z} - \Lambda_3(p^{-2}z)\Big): = p^{3D_z} - W^1(z)p^{2D_z} + W^2(p^{-\frac{1}{2}}z)p^{D_z} - 1,$$

$$\begin{aligned} W^1(z) &= \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z), \\ W^2(z) &= :\Lambda_1(p^{\frac{1}{2}}z)\Lambda_2(p^{-\frac{1}{2}}z): + :\Lambda_1(p^{\frac{1}{2}}z)\Lambda_3(p^{-\frac{1}{2}}z): + :\Lambda_2(p^{\frac{1}{2}}z)\Lambda_3(p^{-\frac{1}{2}}z):, \end{aligned} \tag{A.7}$$

and the relations are

$$\begin{aligned} & f^{11}(\tfrac{w}{z})W^1(z)W^1(w) - W^1(w)W^1(z)f^{11}(\tfrac{z}{w}) \\ =\ & -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\Big(\delta(p\tfrac{w}{z})W^2(p^{\frac{1}{2}}w) - \delta(p^{-1}\tfrac{w}{z})W^2(p^{-\frac{1}{2}}w)\Big), \\ & f^{22}(\tfrac{w}{z})W^2(z)W^2(w) - W^2(w)W^2(z)f^{22}(\tfrac{z}{w}) \qquad \Big(f^{22}(x) = f^{11}(x)\Big) \\ =\ & -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\Big(\delta(p\tfrac{w}{z})W^1(p^{\frac{1}{2}}w) - \delta(p^{-1}\tfrac{w}{z})W^1(p^{-\frac{1}{2}}w)\Big), \\ & f^{12}(\tfrac{w}{z})W^1(z)W^2(w) - W^2(w)W^1(z)f^{21}(\tfrac{z}{w}) \qquad \Big(f^{21}(x) = f^{12}(x)\Big) \\ =\ & -\frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\Big(\delta(p^{\frac{3}{2}}\tfrac{w}{z}) - \delta(p^{-\frac{3}{2}}\tfrac{w}{z})\Big). \end{aligned} \tag{A.8}$$

We remark that there is no distinction between $W^1(z)$ and $W^2(z)$ in algebraically.

By multiplying z from the left and z^{-1} from the right, q-Miura transformation (27) becomes (see (50))

$$\begin{aligned} & (-\hbar')^3 z^3 :\Big(\alpha_0\partial_z + \partial\bar{h}^1(z)\Big)\Big(\alpha_0\partial_z + \partial\bar{h}^2(z)\Big)\Big(\alpha_0\partial_z + \partial\bar{h}^3(z)\Big): + O(\hbar'^4) \\ =\ & p^{3(D_z-1)} - W^1(z)p^{2(D_z-1)} + W^2(p^{-\frac{1}{2}}z)p^{D_z-1} - 1. \end{aligned} \tag{A.9}$$

We choose $\tilde{d}_n^{ij}$ as

$$\begin{aligned} & \tilde{d}_n^{11} = -\tilde{d}_n^{13} = \frac{1}{2}p^n\sqrt{2\frac{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}{p^n - p^{-n}}}, \quad \tilde{d}_n^{22} = -\tilde{d}_n^{21} = \frac{1}{2}\sqrt{\frac{3}{2}\frac{p^n - p^{-n}}{p^{\frac{3}{2}n} - p^{-\frac{3}{2}n}}}, \\ & \tilde{d}_n^{12} = -\frac{1}{2}p^{\frac{3}{2}n}\sqrt{2\frac{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}{p^n - p^{-n}}}\sqrt{3\frac{p^{\frac{n}{2}} - p^{-\frac{n}{2}}}{p^{\frac{3}{2}n} - p^{-\frac{3}{2}n}}}, \qquad \tilde{d}_n^{4-i,4-j} = \tilde{d}_n^{ij}\Big|_{p\to p^{-1}}. \end{aligned} \tag{A.10}$$

Then $W^1(z)$ and $W^2(z)$ have the following $\hbar'$ expansion,

$$\begin{aligned} W^1(z) &= 3+\hbar'^2\Big(z^2L(z)+\alpha_0^2\Big) \\ &\quad +\hbar'^3\Big(\tfrac{1}{2}z^3\Big(W(z)+\tfrac{1}{2}\alpha_0\partial L(z)\Big)+\tfrac{1}{4}\alpha_0 z^2\Big(2X(z)+DX(z)\Big)\Big)+O(\hbar'^4), \\ W^2(z) &= 3+\hbar'^2\Big(z^2L(z)+\alpha_0^2\Big) \\ &\quad +\hbar'^3\Big(-\tfrac{1}{2}z^3\Big(W(z)+\tfrac{1}{2}\alpha_0\partial L(z)\Big)+\tfrac{1}{4}\alpha_0 z^2\Big(2X(z)+DX(z)\Big)\Big)+O(\hbar'^4), \end{aligned} \tag{A.11}$$

where $L(z)=-\bar{W}^2(z)$, $W(z)=\bar{W}^3(z)$, $D=z\frac{d}{dz}$ and $X(z)$ is

$$X(z)=\tfrac{1}{2}:\Big(\partial\phi^1(z)\Big)^2:+\alpha_0\partial^2\phi^1(z)-\tfrac{1}{2}:\Big(\partial\phi^2(z)\Big)^2:-\alpha_0\partial^2\phi^2(z). \tag{A.12}$$

We remark that the combination $W(z)+\frac{1}{2}\alpha_0\partial L(z)$ is an primary field of the Virasoro algebra. Using these, $O(\hbar'^\ell)$ terms ($\ell<3$) of R.H.S. of (A.9) actually vanish and $O(\hbar'^3)$ term gives the Miura transformation of the W_3 algebra.

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