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QUASI-HOPF TWISTORS FOR ELLIPTIC QUANTUM GROUPS

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Dedicated to Professor Mikio Sato on the occasion of his seventieth birthday

Abstract. The Yang–Baxter equation admits two classes of elliptic solutions, the vertex type and the face type. On the basis of these solutions, two types of elliptic quantum groups have been introduced (Foda et al. [FIJKMY1], Felder [Fe]). Frønsdal [Fr1, Fr2] made a penetrating observation that both of them are quasi-Hopf algebras, obtained by twisting the standard quantum affine algebra $U_q(\mathfrak{g})$. In this paper we present an explicit formula for the twistors in the form of an infinite product of the universal R matrix of $U_q(\mathfrak{g})$. We also prove the shifted cocycle condition for the twistors, thereby completing Frønsdal's findings.

This construction entails that, for generic values of the deformation parameters, the representation theory for $U_q(\mathfrak{g})$ carries over to the elliptic algebras, including such objects as evaluation modules, highest weight modules and vertex operators. In particular, we confirm the conjectures of Foda et al. concerning the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$.

1. Introduction

1. Elliptic algebras

Among the integrable models based on the Yang–Baxter equation (YBE), those related to elliptic solutions occupy a fundamental place. Elliptic algebras, or elliptic quantum groups, are certain algebraic structures introduced to account for these elliptic models. Nevertheless, the complexity of elliptic algebras has evaded their understanding for quite

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some time. The solutions of YBE (R-matrices) are classified into two types, vertex type and face type. Accordingly there are two types of elliptic algebras.

The vertex type elliptic algebras are associated with the R-matrix $R(u)$ of Baxter [Ba] and Belavin [Be]. The first example of this sort is the Sklyanin algebra [S], designed as an elliptic deformation of the Lie algebra $\mathfrak{sl}_2$. (An extension to $\mathfrak{sl}_n$ was discussed by Cherednik [C].) It is presented by the 'RLL'-relation

$$R^{(12)}(u_1-u_2)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2), \tag{1.1}$$

together with a specific choice of the form for $L(u)$. Here and after, the superscript $(1),(2),\cdots$ will refer to the tensor components. For further development concerning the Sklyanin algebra, see Feigin-Odesskii [FO] and references therein. An affine version of the Sklyanin algebra (deformation of $\widehat{\mathfrak{sl}}_2$) was proposed by Foda et al. [FIJKMY1]. The main point of [FIJKMY1] was to incorporate a central element c by modifying the RLL-relation to

$$R^{(12)}(u_1-u_2,r)L^{(1)}(u_1)L^{(2)}(u_2)=L^{(2)}(u_2)L^{(1)}(u_1)R^{(12)}(u_1-u_2,r-c), \tag{1.2}$$

where r denotes the elliptic modulus contained in $R(u)=R(u,r)$. In both these works, the coalgebra structure was missing.

The face type algebras are based on R-matrices of Andrews, Baxter, Forrester [ABF] and generalizations [DJMO, JMO1, JMO2]. In this case, besides the elliptic modulus, R and L depend also on extra parameter(s) λ. As Felder has shown [Fe], the RLL relation undergoes a 'dynamical' shift by elements h of the Cartan subalgebra $\mathfrak{h}$,

$$\begin{aligned}R^{(12)}(u_1-u_2,\lambda+h)L^{(1)}(u_1,\lambda)L^{(2)}(u_2,\lambda+h^{(1)})\\ =L^{(2)}(u_2,\lambda)L^{(1)}(u_1,\lambda+h^{(2)})R^{(12)}(u_1-u_2,\lambda).\end{aligned} \tag{1.3}$$

Likewise the YBE itself is modified to a dynamical one, see (1.7) below. As we shall see, a central extension of this algebra is obtained by introducing further a shift of the elliptic modulus analogous to (1.2) (see (4.5)–(4.6) and the remark following them).

The face type algebra has also been given an alternative formulation in terms of the Drinfeld currents. This is the approach adopted by Enriquez and Felder [EF] and one of the authors [K].[1] The Drinfeld currents are suited to deal with infinite dimensional representations. We plan to discuss this subject in a separate publication.

2. Quasi-Hopf twist

Babelon et al. [BBB] has pointed out that the natural framework for dealing with dynamical YBE is Drinfeld's theory of quasi-Hopf algebras [D2]. Since the work [BBB] is a prototype of our construction, let us recall the result. Consider the simplest quantum group $A=U_q(\mathfrak{sl}_2)$ with standard generators e,f,h. Given an arbitrary invertible element $F\in A^{\otimes 2}$, we can modify the coproduct Δ and the universal R matrix $\mathcal{R}$ by

$$\tilde{\Delta}(a)=F\Delta(a)F^{-1} \qquad (a\in A), \tag{1.4}$$

$$\tilde{\mathcal{R}}=F^{(21)}\mathcal{R}F^{-1}. \tag{1.5}$$

[1] The central extension in [K] is different from the one in [EF], and is closer to that of the present paper.

Here if $F = \sum a_i \otimes b_i$, then $F^{(21)} = \sum b_i \otimes a_i$. In general, the new coproduct $\tilde{\Delta}$ is no longer coassociative, and defines on A a quasi-Hopf algebra structure. The new R matrix $\tilde{\mathcal{R}}$ satisfies a YBE-type equation, which is somewhat complicated (see (A.10)). As Babelon et al. showed, this 'twisting' procedure leads to an interesting result when $F = F(\lambda)$ depends on a parameter λ in such a way that the shifted cocycle condition holds:

$$F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda). \tag{1.6}$$

If this is the case, then the YBE-type equation for $\tilde{\mathcal{R}} = \mathcal{R}(\lambda)$ simplifies to the dynamical YBE

$$\mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{1.7}$$

An explicit formula for such an $F(\lambda)$ was given in [BBB] as a formal power series in $q^{2\lambda}$. We shall refer to $F(\lambda)$ as a 'twistor'.

A key observation due to Frønsdal [Fr2] is that the RLL relations for the elliptic algebras of both types, (1.2) and (1.3), arise by the same mechanism as above. Namely, there exist two types of twistors which give rise to different comultiplications on the quantum affine algebras $U_q(\mathfrak{g})$, and the resultant quasi-Hopf algebras are nothing but the two types of elliptic quantum groups.

To substantiate this statement, we must find the corresponding twistors as elements in $U_q(\mathfrak{g})^{\otimes 2}$, satisfying the shifted cocycle condition (1.6). Frønsdal [Fr1, Fr2] launched a search for the twistor in the form of a formal series

$$F(\lambda) = 1 + \sum_{m \geq 1} \sum_{i_1,\dots,i_m} t_{i_1,\dots,i_m}(\lambda) e_{i_1} \dots e_{i_m} \otimes \tau^m(f_{i_1}) \dots \tau^m(f_{i_m}), \tag{1.8}$$

where e_i, f_i are the Chevalley generators, $t_{i_1,\dots,i_m}(\lambda)$ are certain functions of the 'Cartan' generators $h_i \otimes 1$ and $1 \otimes h_i$, and τ is a diagram automorphism.[2] Substituting (1.8) into (1.6), he obtained a recursion relation that determines the coefficients $t_{i_1,\dots,i_m}(\lambda)$ uniquely. Though a proof of the full cocycle condition (1.6) was left open, this construction was shown to reproduce correctly the classical limit [Fr2] and Baxter's R matrix [Fr1]. Another important observation presented in the work [Fr2] is that the twistor has an infinite product form $F(\lambda) = \cdots F^3(\lambda)F^2(\lambda)F^1(\lambda)$, and that the coefficients of each $F^m(\lambda)$ resemble those of the universal R matrix of $U_q(\mathfrak{g})$.

We remark that the quasi-Hopf structure of the face type algebra for $\widehat{\mathfrak{sl}}_2$ was studied in detail by Enriquez and Felder [EF] from a different point of view.

3. The present work

The aim of the present article is to complete the works of Babelon et al. and Frønsdal, by making explicit the aforementioned connection between the twistor and the universal R matrix, and supplying a proof of the shifted cocycle condition. We construct the two types of twistors in the form of an infinite product of the universal R matrix,

$$F(\lambda) = \cdots (\varphi_\lambda^3 \otimes \mathrm{id})(q^T \mathcal{R})^{-1} (\varphi_\lambda^2 \otimes \mathrm{id})(q^T \mathcal{R})^{-1} (\varphi_\lambda \otimes \mathrm{id})(q^T \mathcal{R})^{-1} \tag{1.9}$$

[2] For face type algebras $\tau = \mathrm{id}$, while for the vertex type $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ and τ is a cyclic diagram automorphism. See Section 2.

where φ_λ denotes a certain automorphism of $U_q(\mathfrak{g})^{\otimes 2}$ depending on λ, and T is an element of $\mathfrak{h}\otimes\mathfrak{h}$. For the face type algebras, λ is taken from the Cartan subalgebra, so that $F(\lambda)$ carries the same number of parameters as the rank of $\mathfrak{g}$. When $\mathfrak{g}$ is of affine type, the elliptic modulus appears as one of these parameters. For the vertex type algebras, λ is proportional to the central element. In this case the twistor, to be denoted $E(r)$, depends on only one parameter r, which is the elliptic modulus.

It is an old idea to construct the elliptic R matrices and L operators from the trigonometric ones by an 'averaging' procedure over the periods [Fa, RF, Tak]. Our formula, (1.9) along with (1.5), may be viewed as implementing this idea at the level of the universal R matrix.

Following [FIJKMY1], we shall denote the quasi-Hopf algebras associated with the vertex type twistor $E(r)$ by the symbol $\mathcal{A}_{q,p}(\mathfrak{g})$ (where $\mathfrak{g}=\widehat{\mathfrak{sl}}_n$), and the one associated with the face type twistor $F(\lambda)$ by $\mathcal{B}_{q,\lambda}(\mathfrak{g})$. As algebras, they are the same as the underlying $U_q(\mathfrak{g})$. Hence the representation theory for them should stay the same. Strictly speaking, the twistors are only formal power series with coefficients in $U_q(\mathfrak{g})$, but we expect they make sense in the 'good' category of representations and for generic values of the parameters. In such a case, the whole representation theory, including evaluation, highest weight modules and vertex operators, carry over to the elliptic algebras. We derive the commutation relations of vertex operators and the intertwining relations, regarding them as formal power series. In the special case of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ we recover the formulas conjectured in [FIJKMY1]. As Frønsdal has shown [Fr1, FG] for $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$, the formal series for the twistors in evaluation modules converge and can be computed explicitly. On the other hand, it is a nontrivial problem to compute their images in highest weight representations. We hope to come back to this issue in the future.

The text is organized as follows. In Section 2, after preparing the notation, we present the formulas for the twistors. We then give a proof of the shifted cocycle condition. In Section 3, we discuss the examples $\mathcal{B}_{q,\lambda}(\mathfrak{sl}_2)$, $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$, and compute the images of the twistor and the universal R matrix in the two-dimensional evaluation representation. In Section 4, we define L operators and vertex operators for the elliptic algebras out of those of $U_q(\mathfrak{g})$, and derive various commutation relations among them. In particular, we derive a relation between the L^+ and L^- operators proposed earlier in [FIJKMY1] (see (4.8), (4.24)). In the Appendix we review the basics of quasi-Hopf algebras.

2. Quasi-Hopf twistors

In this section we construct the twistors which give rise to the elliptic algebras. For the face type algebras, the twistor $F=F(\lambda)\in U^{\otimes 2}$ depends on a parameter λ running over the Cartan subalgebra $\mathfrak{h}$. For the vertex type algebras, the twistor $E(r)$ depends on a single parameter $r\in\mathbb{C}$. Both of them are solutions of the shifted cocycle condition (see (2.22), (2.35) below).

1. Quantum groups

First let us fix the notation. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$. We fix an invariant inner product $(\ ,\ )$ on the Cartan subalgebra $\mathfrak{h}$ and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via $(\ ,\ )$. If $\{\alpha_i\}_{i\in I}$ denotes the set of simple roots, then $(\alpha_i,\alpha_j)=d_ia_{ij}$, where $d_i=\frac{1}{2}(\alpha_i,\alpha_i)$.

Consider the corresponding quantum group $U = U_q(\mathfrak{g})$. For simplicity of presentation, we choose to work over the ground ring $\mathbb{C}[[\hbar]]$ with $q = e^{\hbar}$. The algebra U has generators e_i, f_i $(i \in I)$ and h $(h \in \mathfrak{h})$, satisfying the standard relations

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h}), \tag{2.1}$$

$$[h, e_i] = (h, \alpha_i)e_i, \qquad [h, f_i] = -(h, \alpha_i)f_i \qquad (i \in I, h \in \mathfrak{h}), \tag{2.2}$$

$$[e_i, f_j] = \delta_{ij}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \qquad (i, j \in I), \tag{2.3}$$

and the Serre relations which we omit. In (2.3) we have set $q_i = q^{d_i}$, $t_i = q^{\alpha_i}$. We adopt the Hopf algebra structure given as follows.

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \tag{2.4}$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \tag{2.5}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(h) = 0, \tag{2.6}$$

$$S(e_i) = -t_i^{-1}e_i, \quad S(f_i) = -f_i t_i, \quad S(h) = -h, \tag{2.7}$$

where $i \in I$ and $h \in \mathfrak{h}$.

Let $\mathcal{R} \in U^{\otimes 2}$ denote the universal R matrix of U. It has the form

$$\mathcal{R} = q^{-T}\mathcal{C}, \tag{2.8}$$

$$\mathcal{C} = \sum_{\beta \in Q^+} q^{(\beta,\beta)}\left(q^{-\beta} \otimes q^{\beta}\right)\mathcal{C}_\beta = 1 - \sum_{i \in I}(q_i - q_i^{-1})e_i t_i^{-1} \otimes t_i f_i + \dots . \tag{2.9}$$

Here the notation is as follows. Take a basis $\{h_l\}$ of $\mathfrak{h}$, and its dual basis $\{h^l\}$. Then

$$T = \sum_l h_l \otimes h^l \tag{2.10}$$

denotes the canonical element of $\mathfrak{h} \otimes \mathfrak{h}$. The element $\mathcal{C}_\beta = \sum_j u_{\beta,j} \otimes u^j_{-\beta}$ is the canonical element of $U^+_\beta \otimes U^-_{-\beta}$ with respect to a certain Hopf pairing, where U^+ (resp. U^-) denotes the subalgebra of U generated by the e_i (resp. f_i), and $U^\pm_{\pm\beta}$ $(\beta \in Q^+)$ signifies the homogeneous components with respect to the natural gradation by $Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. (For details, the reader is referred, e.g., to [D1, Tan].) We shall need the following basic properties of the universal R matrix:

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \qquad \forall a \in U, \tag{2.11}$$

$$(\Delta \otimes \mathrm{id})\,\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(23)}, \tag{2.12}$$

$$(\mathrm{id} \otimes \Delta)\,\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(12)}, \tag{2.13}$$

$$(\varepsilon \otimes \mathrm{id})\,\mathcal{R} = (\mathrm{id} \otimes \varepsilon)\,\mathcal{R} = 1. \tag{2.14}$$

Here $\Delta' = \sigma \circ \Delta$ signifies the opposite coproduct, σ being the flip of the tensor components $\sigma(a \otimes b) = b \otimes a$. From (2.11)–(2.13) follows the Yang–Baxter equation

$$\mathcal{R}^{(12)}\mathcal{R}^{(13)}\mathcal{R}^{(23)} = \mathcal{R}^{(23)}\mathcal{R}^{(13)}\mathcal{R}^{(12)}. \tag{2.15}$$

2. Face type twistors

We are now in a position to describe the twistors for face type elliptic algebras. Let us prepare some notation.

Let $\rho \in \mathfrak{h}$ be an element such that $(\rho, \alpha_i) = d_i$ for all $i \in I$. Let ϕ be an automorphism of U given by

$$\phi = \mathrm{Ad}(q^{\frac{1}{2}\sum_l h_l h^l - \rho}), \tag{2.16}$$

where $\{h_l\}$, $\{h^l\}$ are as in (2.10). In other words, $\phi(e_i) = e_i t_i$, $\phi(f_i) = t_i^{-1} f_i$, $\phi(q^h) = q^h$. Since

$$\mathrm{Ad}(q^T) \circ (\phi \otimes \phi) = \mathrm{Ad}(q^{\frac{1}{2}\sum_l \Delta(h_l h^l) - \Delta(\rho)}), \tag{2.17}$$

we have

$$\mathrm{Ad}(q^T) \circ (\phi \otimes \phi) \circ \Delta = \Delta \circ \phi. \tag{2.18}$$

For $\lambda \in \mathfrak{h}$, introduce an automorphism

$$\varphi_\lambda = \mathrm{Ad}(q^{\sum_l h_l h^l + 2(\lambda - \rho)}) = \phi^2 \circ \mathrm{Ad}(q^{2\lambda}). \tag{2.19}$$

Then the expression

$$(\varphi_\lambda \otimes \mathrm{id})(q^T \mathcal{R}) \tag{2.20}$$

is a formal power series in the variables $x_i = q^{2(\lambda, \alpha_i)}$ $(i \in I)$ of the form

$$1 - \sum_i (q_i - q_i^{-1}) x_i e_i t_i \otimes t_i f_i + \dots .$$

We define the twistor $F(\lambda)$ as follows.

Definition 2.1. (Face type twistor).

$$F(\lambda) = \cdots (\varphi_\lambda^2 \otimes \mathrm{id})(q^T \mathcal{R})^{-1} (\varphi_\lambda \otimes \mathrm{id})(q^T \mathcal{R})^{-1} = \prod_{k \geq 1}^{\curvearrowleft} (\varphi_\lambda^k \otimes \mathrm{id})(q^T \mathcal{R})^{-1}. \tag{2.21}$$

Here and after, we use the ordered product symbol $\prod\limits_{k \geq 1}^{\curvearrowleft} A_k = \cdots A_3 A_2 A_1$. Note that the k-th factor in the product (2.21) is a formal power series in the x_i^k with leading term 1, and hence the infinite product makes sense. We shall refer to (2.21) as a *face type* twistor.

Our main result is the following.

Theorem 2.2. *The twistor* (2.21) *satisfies the shifted cocycle condition*

$$F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda). \tag{2.22}$$

We have in addition

$$(\varepsilon \otimes \mathrm{id})\, F(\lambda) = (\mathrm{id} \otimes \varepsilon)\, F(\lambda) = 1. \tag{2.23}$$

A proof of Theorem 2.2 will be given in subsection 4. In (2.22), if $\lambda = \sum_l \lambda_l h^l$, then $\lambda + h^{(1)}$ means $\sum_l (\lambda_l + h_l^{(1)}) h^l$. Hence we have, for example, $\mathrm{Ad}(q^{2lT^{(12)}}) F_k^{(23)}(\lambda) = F_k^{(23)}(\lambda + lh^{(1)})$, $\mathrm{Ad}(q^{2lT^{(13)}}) F_k^{(23)}(\lambda) = F_k^{(23)}(\lambda - lh^{(1)})$.

For convenience, let us give a name to the quasi-Hopf algebra associated with the twistor (2.21). As for the generalities on quasi-Hopf algebras, see Appendix A.

Definition 2.3. (Face type algebra). We define the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ of face type to be the set $(U_q(\mathfrak{g}), \Delta_\lambda, \varepsilon, \Phi(\lambda), \mathcal{R}(\lambda))$ together with $\alpha_\lambda = \sum_i S(d_i) e_i$, $\beta_\lambda = \sum_i f_i S(g_i)$ and the anti-automorphism S defined by (2.7). Here ε is defined by (2.6),

$$\Delta_\lambda(a) = F^{(12)}(\lambda)\, \Delta(a)\, F^{(12)}(\lambda)^{-1}, \tag{2.24}$$

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\, \mathcal{R}\, F^{(12)}(\lambda)^{-1}, \tag{2.25}$$

$$\Phi(\lambda) = F^{(23)}(\lambda) F^{(23)}(\lambda + h^{(1)})^{-1}, \tag{2.26}$$

and $\sum_i d_i \otimes e_i = F(\lambda)^{-1}$, $\sum_i f_i \otimes g_i = F(\lambda)$.

Let us consider the case where $\mathfrak{g}$ is of affine type, in which we are mainly interested. Let c be the canonical central element and d the scaling element. We set

$$\lambda - \rho = rd + s'c + \bar{\lambda} - \bar{\rho} \qquad (r, s' \in \mathbb{C}),$$

where $\bar{\lambda}$ stands for the classical part of $\lambda \in \mathfrak{h}$. Denote by $\{\bar{h}_j\}$, $\{\bar{h}^j\}$ the classical part of the dual basis of $\mathfrak{h}$. Since c is central, φ_λ is independent of s'. Writing $p = q^{2r}$, we have $\varphi_\lambda = \mathrm{Ad}(p^d q^{2cd}) \circ \bar{\varphi}_\lambda$, $\bar{\varphi}_\lambda = \mathrm{Ad}(q^{\sum \bar{h}_j \bar{h}^j + 2(\bar{\lambda} - \bar{\rho})})$. Set further[3]

$$\mathcal{R}(z) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}), \tag{2.27}$$

$$F(z, \lambda) = \mathrm{Ad}(z^d \otimes 1)(F(\lambda)), \tag{2.28}$$

$$\mathcal{R}(z, \lambda) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}(\lambda)) = \sigma(F(z^{-1}, \lambda)) \mathcal{R}(z) F(z, \lambda)^{-1}. \tag{2.29}$$

Here σ denotes the flip of the tensor components. (2.27), (2.28) are formal power series in z, whereas (2.29) contains both positive and negative powers of z. Note that $q^{c \otimes d + d \otimes c} \mathcal{R}(z)\big|_{z=0}$ reduces to the universal R matrix of $U_q(\bar{\mathfrak{g}})$ corresponding to the underlying finite dimensional Lie algebra $\bar{\mathfrak{g}}$. From the definition (2.21) of $F(\lambda)$, we have the difference equation

$$F(pq^{2c^{(1)}} z, \lambda) = (\bar{\varphi}_\lambda \otimes \mathrm{id})^{-1}\big(F(z, \lambda)\big) \cdot q^T \mathcal{R}(pq^{2c^{(1)}} z), \tag{2.30}$$

$$F(0, \lambda) = F_{\bar{\mathfrak{g}}}(\bar{\lambda}), \tag{2.31}$$

where $F_{\bar{\mathfrak{g}}}(\bar{\lambda})$ signifies the twistor corresponding to $\bar{\mathfrak{g}}$.

3. Vertex type twistors

When $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$, it is possible to construct a different type of twistor. We call it *vertex type*. In this subsection, U will denote $U_q(\widehat{\mathfrak{sl}}_n)$.

Let us write $h_i = \alpha_i$ $(i = 0, \ldots, n-1)$. A basis of $\mathfrak{h}$ is $\{h_0, \ldots, h_{n-1}, d\}$. The element d gives the homogeneous grading, $[d, e_i] = \delta_{i0} e_i$, $[d, f_i] = -\delta_{i0} f_i$, for all $i =$

[3] The notation $\mathcal{R}(z)$ conflicts that of $\mathcal{R}(\lambda)$. Hopefully there is no confusion.

$0, \ldots, n-1$. Let the dual basis be $\{\Lambda_0, \ldots, \Lambda_{n-1}, c\}$. The Λ_i are the fundamental weights and c is the canonical central element. Let τ be the automorphism of U such that $\tau(e_i) = e_{i+1 \bmod n}$, $\tau(f_i) = f_{i+1 \bmod n}$, $\tau(h_i) = h_{i+1 \bmod n}$ and $\tau^n = \mathrm{id}$. Then we have $\tau(\Lambda_i) = \Lambda_{i+1 \bmod n} - \frac{n-1-2i}{2n}c$. The element $\rho = \sum_{i=0}^{n-1} \Lambda_i$ is invariant under τ. It gives the principal grading $[\rho, e_i] = e_i$, $[\rho, f_i] = -f_i$, for all $i = 0, \ldots, n-1$. Note also that $(\tau \otimes \tau) \circ \Delta = \Delta \circ \tau$, $(\tau \otimes \tau)(\mathcal{C}_\beta) = \mathcal{C}_{\tau(\beta)}$.

For $r \in \mathbb{C}$, we introduce an automorphism

$$\widetilde{\varphi}_r = \tau \circ \mathrm{Ad}\Big(q^{\frac{2(r+c)}{n}\rho}\Big). \tag{2.32}$$

Here and after, quantities related to the vertex type algebras will be denoted by the symbol $\widetilde{\ }$. Set

$$\widetilde{T} = \frac{1}{n}\Big(\rho \otimes c + c \otimes \rho - \frac{n^2-1}{12} c \otimes c\Big).$$

Then

$$(\widetilde{\varphi}_r \otimes \mathrm{id})(q^{\widetilde{T}}\mathcal{R})^{-1} \tag{2.33}$$

is a formal power series in $p^{\frac{1}{n}}$ where $p = q^{2r}$. Unlike the previous case of (2.20), (2.33) is a formal series with a nontrivial leading term $q^{T-\widetilde{T}}\,(1 + \cdots)$. Nevertheless, the n-fold product

$$\prod_{n \geq k \geq 1}^{\curvearrowleft} (\widetilde{\varphi}_r^k \otimes \mathrm{id})(q^{\widetilde{T}}\mathcal{R})^{-1}$$

takes the form $1 + \ldots$, because of the relation $\sum_{k=1}^{n} (\tau^k \otimes \mathrm{id})(T - \widetilde{T}) = 0$. We now define the vertex type twistor $E(r)$ as follows.

Definition 2.4. (Vertex type twistor).

$$E(r) \;\;=\;\; \prod_{k \geq 1}^{\curvearrowleft} (\widetilde{\varphi}_r^k \otimes \mathrm{id})(q^{\widetilde{T}}\mathcal{R})^{-1}. \tag{2.34}$$

The infinite product $\prod\limits_{k \geq 1}^{\curvearrowleft}$ is to be understood as $\lim\limits_{N \to \infty} \prod\limits_{nN \geq k \geq 1}^{\curvearrowleft}$. In view of the remark made above, $E(r)$ is a well defined formal series in $p^{\frac{1}{n}}$.

Theorem 2.5. *The twistor* (2.34) *satisfies the shifted cocycle condition*

$$E^{(12)}(r)(\Delta \otimes \mathrm{id})E(r) = E^{(23)}(r + c^{(1)})(\mathrm{id} \otimes \Delta)E(r). \tag{2.35}$$

We have in addition

$$(\varepsilon \otimes \mathrm{id})\, E(r) = (\mathrm{id} \otimes \varepsilon)\, E(r) = 1. \tag{2.36}$$

Definition 2.6. (Vertex type algebra). We define the quasi-Hopf algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ $(p = q^{2r})$ of vertex type to be the set $(U_q(\widehat{\mathfrak{sl}}_n), \Delta_r, \varepsilon, \Phi(r), \mathcal{R}(r))$ together with $\alpha_r =$

$\sum_i S(d_i)e_i$, $\beta_r = \sum_i f_i S(g_i)$ and the anti-automorphism S defined by (2.7). Here ε is defined by (2.6),

$$\Delta_r(a) = E^{(12)}(r)\,\Delta(a)\,E^{(12)}(r)^{-1}, \tag{2.37}$$
$$\mathcal{R}(r) = E^{(21)}(r)\,\mathcal{R}\,E^{(12)}(r)^{-1}, \tag{2.38}$$
$$\Phi(r) = E^{(23)}(r)E^{(23)}(r+c^{(1)})^{-1}, \tag{2.39}$$

and $\sum_i d_i \otimes e_i = E(r)^{-1}$, $\sum_i f_i \otimes g_i = E(r)$.

Let us set

$$\widetilde{\mathcal{R}}'(\zeta) = (\mathrm{Ad}(\zeta^{\rho}) \otimes \mathrm{id})\,(q^{\widetilde{T}}\mathcal{R}),$$
$$E(\zeta, r) = (\mathrm{Ad}(\zeta^{\rho}) \otimes \mathrm{id})\,E(r).$$

In just the same way as in the face type case, the definition (2.34) can be alternatively described as the unique solution of the difference equation

$$E(p^{1/n}q^{2c^{(1)}/n}\zeta, r) = (\tau \otimes \mathrm{id})^{-1}\big(E(\zeta, r)\big) \cdot \widetilde{\mathcal{R}}'(p^{1/n}q^{2c^{(1)}/n}\zeta),$$
$$E(0, r) = 1,$$

where $p = q^{2r}$.

4. Proof of the shifted cocycle condition

Let us prove the shifted cocycle condition (2.22) for the face type twistors. For $k = 0, 1, \cdots$ we set

$$F_k(\lambda) = (\phi^{2k} \otimes \mathrm{id})\mathcal{C}(\lambda)^{-1}, \tag{2.40}$$
$$\mathcal{C}(\lambda) = \mathrm{Ad}(q^{2\lambda} \otimes 1)(q^T\mathcal{R}) = \mathrm{Ad}(1 \otimes q^{-2\lambda})(q^T\mathcal{R}). \tag{2.41}$$

Then the twistor (2.21) can be written as

$$F(\lambda) = \prod_{k \geq 1}^{\curvearrowleft} F_k(k\lambda).$$

We have the invariance $[\Delta(h), F_k(\lambda)] = 0$, and in particular

$$\mathrm{Ad}(q^T) \circ (\phi \otimes \phi)\,(F_k(\lambda)) = F_k(\lambda). \tag{2.42}$$

From the properties (2.11)–(2.13) of the universal R matrix, we find

Lemma 2.7.

$$(\Delta \otimes \mathrm{id})\mathcal{C}(\lambda) = \mathcal{C}^{(13)}(\lambda - \frac{1}{2}h^{(2)})\mathcal{C}^{(23)}(\lambda), \tag{2.43}$$
$$(\mathrm{id} \otimes \Delta)\mathcal{C}(\lambda) = \mathcal{C}^{(13)}(\lambda + \frac{1}{2}h^{(2)})\mathcal{C}^{(12)}(\lambda), \tag{2.44}$$
$$\mathcal{C}^{(12)}(\lambda)\mathcal{C}^{(13)}(\lambda + \mu - \frac{1}{2}h^{(2)})\mathcal{C}^{(23)}(\mu)$$
$$= \mathcal{C}^{(23)}(\mu)\mathcal{C}^{(13)}(\lambda + \mu + \frac{1}{2}h^{(2)})\mathcal{C}^{(12)}(\lambda). \tag{2.45}$$

Lemma 2.8.

$$(\Delta\otimes\mathrm{id})F_k(\lambda)=F_k^{(23)}(\lambda+kh^{(1)})F_k^{(13)}(\lambda+(k-\frac{1}{2})h^{(2)}),\tag{2.46}$$

$$(\mathrm{id}\otimes\Delta)F_k(\lambda)=F_k^{(12)}(\lambda)F_k^{(13)}(\lambda+\frac{1}{2}h^{(2)}),\tag{2.47}$$

$$\begin{aligned}&F_k^{(12)}(\lambda)F_{k+l}^{(13)}(\lambda+\mu+(l+\frac{1}{2})h^{(2)})F_l^{(23)}(\mu+lh^{(1)})\\&\quad=F_l^{(23)}(\mu+lh^{(1)})F_{k+l}^{(13)}(\lambda+\mu+(l-\frac{1}{2})h^{(2)})F_k^{(12)}(\lambda).\end{aligned}\tag{2.48}$$

Proof. Using (2.43) we have LHS of (2.46) $=(\Delta\otimes\mathrm{id})(\phi^{2k}\otimes\mathrm{id})\mathcal{C}(\lambda)^{-1}=\mathrm{Ad}(q^{2kT^{(12)}})\cdot$ $\cdot(\phi^{2k}\otimes\phi^{2k}\otimes\mathrm{id})\left(\mathcal{C}^{(23)}(\lambda)^{-1}\mathcal{C}^{(13)}(\lambda-\frac{1}{2}h^{(2)})^{-1}\right)=F_k^{(23)}(\lambda+kh^{(1)})F_k^{(13)}(\lambda+(k-\frac{1}{2})h^{(2)})$. In the second equation we used (2.18). Eq.(2.47) can be verified in a similar way. Finally, (2.48) follows by applying $(\phi^{2(k+l)}\otimes\phi^{2l}\otimes\mathrm{id})\mathrm{Ad}(q^{2lT^{(12)}})$ to (2.45) and noting (2.42). □

Lemma 2.9. *For $l\in\mathbb{Z}_{\geq 0}$, we have the equality*

$$\begin{aligned}&\prod_{l\geq k\geq 1}^{\curvearrowleft}F_k^{(23)}(k\lambda+kh^{(1)})\cdot(\mathrm{id}\otimes\Delta)F(\lambda)\\&\quad=\prod_{k\geq 1}^{\curvearrowleft}F_k^{(12)}(k\lambda)F_{k+l}^{(13)}((k+l)\lambda+(l+\frac{1}{2})h^{(2)})\\&\qquad\times\prod_{l\geq k\geq 1}^{\curvearrowleft}F_k^{(23)}(k\lambda+kh^{(1)})F_k^{(13)}(k\lambda+(k-\frac{1}{2})h^{(2)}).\end{aligned}\tag{2.49}$$

Proof. We prove (2.49) by induction on l. The statement holds for $l=0$, since we have from (2.47) that

$$(\mathrm{id}\otimes\Delta)F(\lambda)=\prod_{k\geq 1}^{\curvearrowleft}F_k^{(12)}(k\lambda)F_k^{(13)}(k\lambda+\frac{1}{2}h^{(2)}).$$

Suppose the statement is correct for $l-1$. Then from (2.48) we obtain

$$\begin{aligned}&F_l^{(23)}(l\lambda+lh^{(1)})\cdot\prod_{k\geq 1}^{\curvearrowleft}F_k^{(12)}(k\lambda)F_{k+l-1}^{(13)}((k+l-1)\lambda+(l-\frac{1}{2})h^{(2)})\\&=F_l^{(23)}(l\lambda+lh^{(1)})\cdot\prod_{k\geq 1}^{\curvearrowleft}F_{k+l}^{(13)}((k+l)\lambda+(l-\frac{1}{2})h^{(2)})F_k^{(12)}(k\lambda)\cdot F_l^{(13)}(l\lambda+(l-\frac{1}{2})h^{(2)})\\&=\prod_{k\geq 1}^{\curvearrowleft}F_k^{(12)}(k\lambda)F_{k+l}^{(13)}((k+l)\lambda+(l+\frac{1}{2})h^{(2)})\cdot F_l^{(23)}(l\lambda+lh^{(1)})F_l^{(13)}(l\lambda+(l-\frac{1}{2})h^{(2)}).\end{aligned}$$

This means that the statement holds also for l. □

Proof of Theorem 2.2. Let $l \to \infty$ in (2.49). Then

$$\begin{aligned} & F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda) \\ &= \prod_{k\geq 1}^{\curvearrowleft} F_k^{(12)}(k\lambda) \cdot \prod_{k\geq 1}^{\curvearrowleft} F_k^{(23)}(k\lambda + kh^{(1)})F_k^{(13)}(k\lambda + (k-\frac{1}{2})h^{(2)}) \\ &= F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda). \end{aligned}$$

The last step is from (2.46). The statement (2.23) is evident from (2.14). □

The case of vertex type twistors (2.35) can be treated in an analogous manner. In place of the automorphism (2.16), we set $\widetilde{\phi} = \tau \circ \mathrm{Ad}(q^{\frac{2c}{n}\rho})$. Then we have $\mathrm{Ad}(q^{2\widetilde{T}}) \circ (\widetilde{\phi} \otimes \widetilde{\phi}) \circ \Delta = \Delta \circ \widetilde{\phi}$. The twistor can be written as

$$E(r) = \prod_{k\geq 1}^{\curvearrowleft} E_k(kr),$$

with the definition

$$E_k(r) = (\widetilde{\phi}^k \otimes \mathrm{id})\widetilde{\mathcal{C}}(r)^{-1}, \tag{2.50}$$

$$\widetilde{\mathcal{C}}(r) = \mathrm{Ad}(q^{\frac{2r}{n}\rho} \otimes 1)(q^{\widetilde{T}}\mathcal{R}) = \mathrm{Ad}(1 \otimes q^{-\frac{2r}{n}\rho})(q^{\widetilde{T}}\mathcal{R}). \tag{2.51}$$

We have also that $\mathrm{Ad}(q^{2\widetilde{T}}) \circ (\widetilde{\phi} \otimes \widetilde{\phi})\,(E_k(r)) = E_k(r)$. The rest of the proof is much the same with that of the face type, so we omit the details.

3. Examples

1. The case $\mathcal{B}_{q,\lambda}(\mathfrak{sl}_2)$

Let $\mathfrak{g} = \mathfrak{sl}_2$, with the generators e, f, h as in (2.2),(2.3). In this case the universal R matrix is given by [D1]

$$\mathcal{R} = q^{-T} \exp_{q^2}\left(-(q-q^{-1})et^{-1} \otimes tf\right), \qquad T = \frac{1}{2}h \otimes h. \tag{3.1}$$

Here the q-exponential symbol is defined by

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \qquad \exp_q(x)\exp_{q^{-1}}(-x) = 1,$$

$$(n)_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

Let us set $\lambda = (s+1)\frac{1}{2}h,\ w = q^{2s}$, and write $F(w)$ for $F(\lambda)$. Since $\varphi_\lambda = \mathrm{Ad}(q^{\frac{1}{2}h^2}w^{\frac{1}{2}h})$ and $\varphi_\lambda^k(e) = (q^2w)^k et^{2k}$, the formula for the twistor (2.21) becomes

$$F(w) = \prod_{k\geq 1}^{\curvearrowleft} \exp_{q^{-2}}\left((q-q^{-1})(q^2w)^k \cdot et^{2k-1} \otimes tf\right). \tag{3.2}$$

Using the formula

$$\sum_{n=0}^{\infty} \frac{1}{(n)_q!(a;q)_n} b^n = \prod_{k\geq 0}^{\curvearrowleft} \exp_q(a^k b) \qquad \text{if} \quad ba = q^2 ab,$$

we find

$$F(w) = \sum_{n=0}^{\infty} \frac{(q^2 w)^n (q-q^{-1})^n}{(n)_{q^{-2}}!(q^{-2}w(t^2\otimes 1);q^{-2})_n} (et)^n \otimes (tf)^n. \tag{3.3}$$

The formulas (3.2)–(3.3) are due to [BBB, Fr1, Fr2]. In the two-dimensional representation $(\pi, \mathbb{C}^2)$

$$\pi(e) = E_{12}, \quad \pi(f) = E_{21}, \quad \pi(h) = E_{11} - E_{22}, \tag{3.4}$$

with E_{ij} denoting the matrix with 1 at the (i,j)-th place and 0 elsewhere, we have

$$(\pi\otimes\pi)F(w) = 1 + (q-q^{-1})\frac{w}{1-w} E_{12}\otimes E_{21}. \tag{3.5}$$

2. The case $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$

Next consider the case of the affine Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. Taking a basis $\{c, d, h_1\}$ of $\mathfrak{h}$, we write $\lambda = (r+2)d + s'c + (s+1)\frac{1}{2}h_1$ $(r, s', s \in \mathbb{C})$. φ_λ is independent of s'. Writing

$$p = q^{2r}, \quad w = q^{2s}, \tag{3.6}$$

we set

$$\mathcal{R}(z) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}), \tag{3.7}$$

$$F(z;p,w) = \mathrm{Ad}(z^d\otimes 1)(F(\lambda)), \tag{3.8}$$

$$\mathcal{R}(z;p,w) = \mathrm{Ad}(z^d\otimes 1)(\mathcal{R}(\lambda)) = \sigma(F(z^{-1};p,w))\mathcal{R}(z)F(z;p,w)^{-1}. \tag{3.9}$$

In particular, for $z=0$, $q^{c\otimes d + d\otimes c}\mathcal{R}(0)$ reduces to the universal R matrix (3.1) of $U_q(\mathfrak{sl}_2)$. From (2.30),(2.31) we have

$$F(pq^{2c^{(1)}}z;p,w) = (\bar{\varphi}_w^{-1}\otimes \mathrm{id})\,(F(z;p,w)) \times q^T \mathcal{R}(pq^{2c^{(1)}}z), \tag{3.10}$$

$$F(0;p,w) = F_{\mathfrak{sl}_2}(w), \tag{3.11}$$

where $\bar{\varphi}_w = \mathrm{Ad}(q^{h_1^2/2} w^{h_1/2})$. In (3.11), the right hand side means the twistor (3.2) in the previous example.

Let us calculate the image of (3.8),(3.9) in the two-dimensional representation (π, V), $V = \mathbb{C}^2$, where e_1, f_1, h_1 are mapped as in (3.4) and $\pi(e_0) = \pi(f_1)$, $\pi(f_0) = \pi(e_1)$, $\pi(h_0) = -\pi(h_1)$. We set $F_{VV}(z;p,w) = (\pi\otimes\pi)F(z;p,w)$, $R_{VV}(z;p,w) = (\pi\otimes\pi)\mathcal{R}(z;p,w)$. The image $R_{VV}(z) = (\pi \otimes \pi)\mathcal{R}(z)$ is known to be given as follows (see e.g. [IIJMNT]).

$$R_{VV}(z) = \rho(z)\overline{R}_{VV}(z),$$

$$\rho(z) = q^{-\frac{1}{2}} \frac{(q^2 z; q^4)_\infty^2}{(z; q^4)_\infty (q^4 z; q^4)_\infty}, \tag{3.12}$$

$$\overline{R}_{VV}(z) = \begin{bmatrix} 1 & & & \\ & b(z) & c(z) & \\ & zc(z) & b(z) & \\ & & & 1 \end{bmatrix}, \tag{3.13}$$

$$b(z) = \frac{(1-z)q}{1-q^2 z}, \qquad c(z) = \frac{1-q^2}{1-q^2 z}. \tag{3.14}$$

Here and after, we use the infinite product symbol

$$(z; t_1, t_2, \cdots, t_n)_\infty = \prod_{i_1, i_2, \cdots, i_n = 0}^{\infty} (1 - z t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}).$$

Eq. (3.10) implies a difference equation for $F_{VV}(z; p, w)$. Noting $\pi(c) = 0$ and $\pi \circ \bar{\varphi}_w = \mathrm{Ad}(D_w)^{-1} \circ \pi$ where $D_w = \mathrm{diag}(1, w)$, we find

$$F_{VV}(pz; p, w)^t = R_{VV}(pz)^t \, K (D_w \otimes 1)^{-1} \cdot F_{VV}(z; p, w)^t \, (D_w \otimes 1),$$

where X^t means the transpose of X, and we have set $K = \mathrm{diag}(q^{1/2}, q^{-1/2}, q^{-1/2}, q^{1/2})$. This means that each column of $F_{VV}(z; p, w)^t$ satisfies a difference equation of the same sort as the q-KZ equation. Solving this with the initial condition which follows from (3.11), we obtain the result

$$F_{VV}(z; p, w) = \varphi(z; p) \begin{bmatrix} 1 & & & \\ & X_{11}(z) & X_{12}(z) & \\ & X_{21}(z) & X_{22}(z) & \\ & & & 1 \end{bmatrix},$$

where

$$\varphi(z; p) = \frac{(pz; q^4, p)_\infty (pq^4 z; q^4, p)_\infty}{(pq^2 z; q^4, p)_\infty^2}, \tag{3.15}$$

and

$$X_{11}(z) = {}_2\phi_1 \left(\begin{matrix} wq^2 \;\; q^2 \\ w \end{matrix} ; p, pq^{-2} z \right),$$

$$X_{12}(z) = \frac{w(q - q^{-1})}{1 - w} {}_2\phi_1 \left(\begin{matrix} wq^2 \;\; pq^2 \\ pw \end{matrix} ; p, pq^{-2} z \right),$$

$$X_{21}(z) = \frac{pw^{-1}(q - q^{-1})}{1 - pw^{-1}} z \, {}_2\phi_1 \left(\begin{matrix} pw^{-1} q^2 \;\; pq^2 \\ p^2 w^{-1} \end{matrix} ; p, pq^{-2} z \right),$$

$$X_{22}(z) = {}_2\phi_1 \left(\begin{matrix} pw^{-1} q^2 \;\; q^2 \\ pw^{-1} \end{matrix} ; p, pq^{-2} z \right).$$

Here ${}_2\phi_1\left(\begin{matrix} q^a\ q^b \\ q^c \end{matrix};q,z\right)$ denotes the basic hypergeometric series

$$
{}_2\phi_1\left(\begin{matrix} q^a & q^b \\ & q^c \end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(q^a;q)_n(q^b;q)_n}{(q;q)_n(q^c;q)_n} z^n.
$$

The image of the R matrix is determined from (3.9) and the connection formula for the basic hypergeometric series

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} q^a & q^b \\ & q^c \end{matrix};q,\frac{1}{z}\right) \\
&= \frac{\Gamma_q(c)\Gamma_q(b-a)\Theta_q(q^{1-a}z)}{\Gamma_q(b)\Gamma_q(c-a)\Theta_q(qz)} {}_2\phi_1\left(\begin{matrix} q^a & q^{a-c+1} \\ & q^{a-b+1} \end{matrix};q,q^{c-a-b+1}z\right) \\
&\quad + \frac{\Gamma_q(c)\Gamma_q(a-b)\Theta_q(q^{1-b}z)}{\Gamma_q(a)\Gamma_q(c-b)\Theta_q(qz)} {}_2\phi_1\left(\begin{matrix} q^b & q^{b-c+1} \\ & q^{b-a+1} \end{matrix};q,q^{c-a-b+1}z\right),
\end{aligned}
$$

where

$$
\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad \Theta_q(z) = (z;q)_\infty(qz^{-1};q)_\infty(q;q)_\infty.
$$

We find

$$
R_{VV}(z;p,w) = \rho(z;p)\begin{bmatrix} 1 & & & \\ & b(z;p,w) & c(z;p,w) & \\ & \bar{c}(z;p,w) & \bar{b}(z;p,w) & \\ & & & 1 \end{bmatrix},
$$

with the coefficients given by

$$
\rho(z;p) = q^{-1/2}\frac{(q^2z;p,q^4)_\infty^2}{(z;p,q^4)_\infty(q^4z;p,q^4)_\infty}\frac{(pz^{-1};p,q^4)_\infty(pq^4z^{-1};p,q^4)_\infty}{(pq^2z^{-1};p,q^4)_\infty^2}, \tag{3.16}
$$

$$
b(z;p,w) = q\frac{(pw^{-1}q^2;p)_\infty(pw^{-1}q^{-2};p)_\infty}{(pw^{-1};p)_\infty^2}\frac{\Theta_p(z)}{\Theta_p(q^2z)}, \tag{3.17}
$$

$$
\bar{b}(z;p,w) = q\frac{(wq^2;p)_\infty(wq^{-2};p)_\infty}{(w;p)_\infty^2}\frac{\Theta_p(z)}{\Theta_p(q^2z)}, \tag{3.18}
$$

$$
c(z;p,w) = \frac{\Theta_p(q^2)}{\Theta_p(w)}\frac{\Theta_p(wz)}{\Theta_p(q^2z)}, \tag{3.19}
$$

$$
\bar{c}(z;p,w) = z\frac{\Theta_p(q^2)}{\Theta_p(pw^{-1})}\frac{\Theta_p(pw^{-1}z)}{\Theta_p(q^2z)}. \tag{3.20}
$$

As expected, these are (up to a gauge) the Boltzmann weights of the Andrews-Baxter-Forrester model [ABF].

3. The case $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$

The case of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ can be treated similarly. Let

$$\widetilde{\mathcal{R}}(\zeta) = \mathrm{Ad}(\zeta^\rho \otimes 1)(\mathcal{R}), \tag{3.21}$$

$$E(\zeta;p) = \mathrm{Ad}(\zeta^\rho \otimes 1)(E(r)), \tag{3.22}$$

$$\widetilde{\mathcal{R}}(\zeta;p) = \mathrm{Ad}(\zeta^\rho \otimes 1)(\mathcal{R}(r)) = \sigma\left(E(\zeta^{-1};p)\right)\widetilde{\mathcal{R}}(\zeta)E(\zeta;p)^{-1}, \tag{3.23}$$

where $p = q^{2r}$. In this case we have simply $q^T\widetilde{\mathcal{R}}(0) = 1$. Thus $E(\zeta;p)$ is characterized by $E(p^{1/2}q^{c^{(1)}}\zeta;p) = (\tau^{-1} \otimes \mathrm{id})\,(E(\zeta;p)) \times q^T\widetilde{\mathcal{R}}(p^{1/2}q^{c^{(1)}}\zeta)$, $E(0;p) = 1$.

The calculation of the image in the two-dimensional representation can be done directly. Since $\pi \circ \tau = \mathrm{Ad}(\sigma^x) \circ \pi$ with $\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$(\pi \otimes \pi)(\tau^{2k} \otimes \mathrm{id})\left(\widetilde{\mathcal{R}}(p^k\zeta)\right) = q^{-1/2}\rho(p^{2k}\zeta^2)\begin{bmatrix} 1 & & & \\ & b_{2k} & c_{2k} & \\ & c_{2k} & b_{2k} & \\ & & & 1 \end{bmatrix},$$

$$(\pi \otimes \pi)(\tau^{2k-1} \otimes \mathrm{id})\left[\widetilde{\mathcal{R}}(p^{k-\frac{1}{2}}\zeta)\right] = q^{-1/2}\rho(p^{2k-1}\zeta^2)\begin{bmatrix} b_{2k-1} & & & c_{2k-1} \\ & 1 & & \\ & & 1 & \\ c_{2k-1} & & & b_{2k-1} \end{bmatrix},$$

where b_l, c_l are given in terms of (3.14) by $b_l = b(p^l\zeta^2)$, $c_l = p^{\frac{l}{2}}\zeta\, c(p^l\zeta^2)$. The infinite product can be readily calculated, yielding the result

$$(\pi \otimes \pi)(E(\zeta;p)) = \varphi(\zeta^2;p)\begin{bmatrix} a_E(\zeta) & & & d_E(\zeta) \\ & b_E(\zeta) & c_E(\zeta) & \\ & c_E(\zeta) & b_E(\zeta) & \\ d_E(\zeta) & & & a_E(\zeta) \end{bmatrix}, \tag{3.24}$$

where $\varphi(z;p)$ is given by (3.15), and

$$a_E(\zeta) \pm d_E(\zeta) = \frac{(\mp p^{1/2}q\zeta;p)_\infty}{(\mp p^{1/2}q^{-1}\zeta;p)_\infty}, \tag{3.25}$$

$$b_E(\zeta) \pm c_E(\zeta) = \frac{(\mp pq\zeta;p)_\infty}{(\mp pq^{-1}\zeta;p)_\infty}. \tag{3.26}$$

Finally the image of the R matrix (3.23) is given by

$$(\pi \otimes \pi)(\widetilde{\mathcal{R}}(\zeta;p)) = \rho(\zeta^2,p)\begin{bmatrix} a^+(\zeta) & & & d^+(\zeta) \\ & b^+(\zeta) & c^+(\zeta) & \\ & c^+(\zeta) & b^+(\zeta) & \\ d^+(\zeta) & & & a^+(\zeta) \end{bmatrix}, \tag{3.27}$$

with

$$a^+(\zeta) \pm d^+(\zeta) = \frac{(\mp p^{1/2}q^{-1}\zeta;p)_\infty}{(\mp p^{1/2}q\zeta;p)_\infty}\frac{(\mp p^{1/2}q\zeta^{-1};p)_\infty}{(\mp p^{1/2}q^{-1}\zeta^{-1};p)_\infty}, \tag{3.28}$$

$$b^+(\zeta) \pm c^+(\zeta) = q\frac{1 \pm q^{-1}\zeta}{1 \pm q\zeta}\frac{(\mp pq^{-1}\zeta;p)_\infty}{(\mp pq\zeta;p)_\infty}\frac{(\mp pq\zeta^{-1};p)_\infty}{(\mp pq^{-1}\zeta^{-1};p)_\infty}. \tag{3.29}$$

This agrees with the R matrix of the eight vertex model (cf. eqs.(2.5)-(2.9) in [FIJKMY2] wherein the sign of $p^{1/2}$ is changed). The results (3.24)-(3.29) are due to Frønsdal [Fr1, FG].

4. Dynamical RLL-relations and vertex operators

The L-operators and vertex operators for the elliptic algebras can be constructed from those of $U_q(\mathfrak{g})$ by 'dressing' the latter with the twistors. In this section, we examine various commutation relations among these operators. We shall mainly discuss the case of the face type algebra $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ where $\mathfrak{g}$ is of affine type. We touch upon the vertex type algebras $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$ briefly at the end.

1. RLL-relation for $\mathcal{B}_{q,\lambda}(\mathfrak{g})$

Hereafter we write $U = U_q(\mathfrak{g})$, $\mathcal{B} = \mathcal{B}_{q,\lambda}(\mathfrak{g})$. By a representation of the quasi-Hopf algebra $\mathcal{B}$, we mean that of the underlying associative algebra U. Let (π_V, V) be a finite dimensional module over U, and let $(\pi_{V,z}, V_z)$ be the evaluation representation associated with it where $\pi_{V,z} = \pi_V \circ \mathrm{Ad}(z^d)$.

Definition 4.1. We define L-operators for $\mathcal{B}$ by

$$L^\pm_V(z,\lambda) = (\pi_{V,z} \otimes \mathrm{id})\,\mathcal{R}'^{\pm}(\lambda), \tag{4.1}$$

$$\mathcal{R}'^{+}(\lambda) = q^{c\otimes d + d\otimes c}\mathcal{R}(\lambda), \tag{4.2}$$

$$\mathcal{R}'^{-}(\lambda) = \mathcal{R}^{(21)}(\lambda)^{-1}q^{-c\otimes d - d\otimes c}. \tag{4.3}$$

Likewise we set $R^\pm_{VW}(z_1/z_2,\lambda) = (\pi_{V,z_1} \otimes \pi_{W,z_2})\,\mathcal{R}'^{\pm}(\lambda)$.

Setting further

$$\mathcal{R}'^{\pm}(z,\lambda) = \mathrm{Ad}(z^d \otimes 1)\mathcal{R}'^{\pm}(\lambda), \tag{4.4}$$

we find from the dynamical YBE (A.18) that

$$\begin{aligned}
&\mathcal{R}'^{\pm(12)}(z_1/z_2,\lambda+h^{(3)})\mathcal{R}'^{\pm(13)}(q^{\mp c^{(2)}}z_1/z_3,\lambda)\mathcal{R}'^{\pm(23)}(z_2/z_3,\lambda+h^{(1)})\\
&\quad = \mathcal{R}'^{\pm(23)}(z_2/z_3,\lambda)\mathcal{R}'^{\pm(13)}(q^{\pm c^{(2)}}z_1/z_3,\lambda+h^{(2)})\mathcal{R}'^{\pm(12)}(z_1/z_2,\lambda),\\
&\mathcal{R}'^{+(12)}(q^{c^{(3)}}z_1/z_2,\lambda+h^{(3)})\mathcal{R}'^{+(13)}(z_1/z_3,\lambda)\mathcal{R}'^{-(23)}(z_2/z_3,\lambda+h^{(1)})\\
&\quad = \mathcal{R}'^{-(23)}(z_2/z_3,\lambda)\mathcal{R}'^{+(13)}(z_1/z_3,\lambda+h^{(2)})\mathcal{R}'^{+(12)}(q^{-c^{(3)}}z_1/z_2,\lambda).
\end{aligned}$$

Applying $\pi_V \otimes \pi_W \otimes \mathrm{id}$, we obtain the dynamical RLL relation.

Proposition 4.2.

$$
\begin{aligned}
&R^{\pm(12)}_{VW}(z_1/z_2,\lambda+h)L^{\pm(1)}_V(z_1,\lambda)L^{\pm(2)}_W(z_2,\lambda+h^{(1)})\\
&\quad= L^{\pm(2)}_W(z_2,\lambda)L^{\pm(1)}_V(z_1,\lambda+h^{(2)})R^{\pm(12)}_{VW}(z_1/z_2,\lambda), &(4.5)\\
&R^{+(12)}_{VW}(q^c z_1/z_2,\lambda+h)L^{+(1)}_V(z_1,\lambda)L^{-(2)}_W(z_2,\lambda+h^{(1)})\\
&\quad= L^{-(2)}_W(z_2,\lambda)L^{+(1)}_V(z_1,\lambda+h^{(2)})R^{+(12)}_{VW}(q^{-c}z_1/z_2,\lambda). &(4.6)
\end{aligned}
$$

Here the index (1) (resp. (2)) refers to V (resp. W), and h, c (without superfix) are elements of $\mathfrak{h}\subset\mathcal{B}$. If we write $\lambda-\rho=rd+s'c+\bar\lambda-\bar\rho$ ($r,s'\in\mathbb{C},\bar\lambda\in\bar{\mathfrak{h}}$), then

$$
\lambda+h^{(1)}=(r+h^\vee+c^{(1)})d+(s'+d^{(1)})c+(\bar\lambda+\bar{h^{(1)}}) \tag{4.7}
$$

where $h^\vee$ is the dual Coxeter number. The parameter r plays the role of the elliptic modulus. Note that, in (4.5)–(4.6), r also undergoes a shift depending on the central element c.

Actually the two L-operators (4.1) are not independent.

Proposition 4.3. *We have*

$$
L^+_V(pq^c z,\lambda)=q^{-2\bar T_{V,\bullet}}(Ad(\bar X_\lambda)\otimes\mathrm{id})^{-1}L^-_V(z,\lambda), \tag{4.8}
$$

where $\bar T_{V,\bullet}=\sum\pi(\bar h_j)\otimes\bar h^j$, $\bar X_\lambda=\pi(q^{\sum\bar h_j\bar h^j+2(\bar\lambda-\bar\rho)})$.

Proof. In the notation of (4.4), we have

$$
\begin{aligned}
\mathcal{R}'^{+}(z,\lambda)&=\sigma\left(F(z^{-1},\lambda)\right)\Big|_{z\mapsto q^{c^{(2)}-c^{(1)}}z}\mathcal{R}'^{+}(z)F(z,\lambda)^{-1},\\
\mathcal{R}'^{-}(z,\lambda)&=\sigma\left(F(z^{-1},\lambda)\right)\mathcal{R}'^{-}(z)F(q^{c^{(2)}-c^{(1)}}z,\lambda)^{-1}.
\end{aligned}
$$

Now the difference equation (2.30) implies

$$
\begin{aligned}
&q^{\bar T}\mathcal{R}'^{+}(pz)F(pz,\lambda)^{-1}=(\bar\varphi_\lambda\otimes\mathrm{id})^{-1}F(q^{-2c^{(1)}}z,\lambda),\\
&\sigma\left(F(p^{-1}z^{-1},\lambda)\right)\Big|_{z\mapsto q^{-2c^{(1)}}z}=(\mathrm{id}\otimes\bar\varphi_\lambda)\left(\sigma(F(z^{-1},\lambda))\mathcal{R}'^{-}(z)q^{-\bar T}\right).
\end{aligned}
$$

Noting that $\mathrm{Ad}(q^{2\bar T})(\bar\varphi_\lambda\otimes\bar\varphi_\lambda)\mathcal{R}'=\mathcal{R}'$, we find

$$
\mathcal{R}'^{+}(pq^{c^{(1)}+c^{(2)}}z,\lambda)=q^{-2\bar T}(\bar\varphi_\lambda\otimes\mathrm{id})^{-1}\mathcal{R}'^{-}(z,\lambda).
$$

Taking the image in V we obtain the assertion. □

2. Vertex operators for $\mathcal{B}_{q,\lambda}(\mathfrak{g})$

Let $(\pi_{V,z},V_z)$ be as before, and let $V(\mu)$ be a highest weight module with highest weight μ. Consider intertwiners of U-modules of the form $\Phi^{(\nu,\mu)}_V(z):V(\mu)\longrightarrow V(\nu)\otimes V_z$,

${\Psi^*}_V^{(\nu,\mu)}(z) : V_z \otimes V(\mu) \longrightarrow V(\nu)$. We call them vertex operators (VO's) of type I and type II, respectively.[4] Define the corresponding VO's for $\mathcal{B}$ as follows [FG]:

$$\Phi_V^{(\nu,\mu)}(z,\lambda) = (\mathrm{id} \otimes \pi_z) F(\lambda) \circ \Phi_V^{(\nu,\mu)}(z), \tag{4.9}$$
$$\Psi^{*}{}_V^{(\nu,\mu)}(z,\lambda) = \Psi^{*}{}_V^{(\nu,\mu)}(z) \circ (\pi_z \otimes \mathrm{id}) F(\lambda)^{-1}. \tag{4.10}$$

When there is no fear of confusion, we often drop the sub(super)scripts V or (ν,μ). It is clear that (4.9),(4.10) satisfy the intertwining relations relative to the coproduct Δ_λ (2.24), $\Delta_\lambda(a)\Phi(z,\lambda) = \Phi(z,\lambda)a$, $a\Psi^*(z,\lambda) = \Psi^*(z,\lambda)\Delta_\lambda(a)$ ($\forall a \in \mathcal{B}$). These intertwining relations can be encapsulated to commutation relations with the L-operators.

Proposition 4.4. *The 'dressed' VO's* (4.9),(4.10) *satisfy the following dynamical intertwining relations* (*see the diagram below*)*:*

$$\Phi_W(z_2,\lambda) L_V^+(z_1,\lambda) = R_{VW}^+(q^c z_1/z_2, \lambda+h) L_V^+(z_1,\lambda) \Phi_W(z_2, \lambda + h^{(1)}), \tag{4.11}$$
$$\Phi_W(z_2,\lambda) L_V^-(z_1,\lambda) = R_{VW}^-(z_1/z_2, \lambda+h) L_V^-(z_1,\lambda) \Phi_W(z_2, \lambda + h^{(1)}), \tag{4.12}$$
$$L_V^+(z_1,\lambda) \Psi_W^*(z_2, \lambda + h^{(1)}) = \Psi_W^*(z_2,\lambda) L_V^+(z_1, \lambda + h^{(2)}) R_{VW}^+(z_1/z_2, \lambda), \tag{4.13}$$
$$L_V^-(z_1,\lambda) \Psi_W^*(z_2, \lambda + h^{(1)}) = \Psi_W^*(z_2,\lambda) L_V^-(z_1, \lambda + h^{(2)}) R_{VW}^-(q^c z_1/z_2, \lambda). \tag{4.14}$$

$$\begin{tikzcd}
V_{z_1} \otimes V(\mu) \arrow[r, "\Phi_W"] \arrow[d, "L_V^\pm"'] & V_{z_1} \otimes V(\nu) \otimes W_{z_2} \arrow[r, "L_V^\pm"] & V_{z_1} \otimes V(\nu) \otimes W_{z_2} \arrow[d, "R_{VW}^\pm"] \\
V_{z_1} \otimes V(\mu) \arrow[rr, "\Phi_W"'] & & V_{z_1} \otimes V(\nu) \otimes W_{z_2}
\end{tikzcd}$$

$$\begin{tikzcd}
V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[r, "R_{VW}^\pm"] \arrow[d, "\Psi_W^*"'] & V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[r, "L_V^\pm"] & V_{z_1} \otimes W_{z_2} \otimes V(\mu) \arrow[d, "\Psi_W^*"] \\
V_{z_1} \otimes V(\nu) \arrow[rr, "L_V^\pm"'] & & V_{z_1} \otimes V(\nu)
\end{tikzcd}$$

Proof. Let Δ be the original coproduct (2.5) for U. The properties (A.16), (A.17) are equivalently rewritten as

$$(\Delta \otimes \mathrm{id})\mathcal{R}(\lambda) = F^{(12)}(\lambda + h^{(3)})^{-1} \mathcal{R}^{(13)}(\lambda) \mathcal{R}^{(23)}(\lambda + h^{(1)}) F^{(12)}(\lambda),$$
$$(\mathrm{id} \otimes \Delta)\mathcal{R}(\lambda) = F^{(23)}(\lambda)^{-1} \mathcal{R}^{(13)}(\lambda + h^{(2)}) \mathcal{R}^{(12)}(\lambda) F^{(23)}(\lambda + h^{(1)}).$$

[4] In this paper we treat VO's which have Fourier expansion in integral powers of z. In the notation of [IIJMNT], they are denoted with the symbol $\widetilde{\ }$; e.g. $\Phi_V^{(\nu,\mu)}(z)$ here is written as $\widetilde{\Phi}_\mu^{\nu,V}(z)$ in [IIJMNT].

From this it follows that

$$(\mathrm{id}\otimes\Delta)\mathcal{R}'^{+}(z,\lambda) = F^{(23)}(\lambda)^{-1}\mathcal{R}'^{+(13)}(q^{c^{(2)}}z,\lambda+h^{(2)})\mathcal{R}'^{+(12)}(z,\lambda)F^{(23)}(\lambda+h^{(1)}),$$
$$(\mathrm{id}\otimes\Delta)\mathcal{R}'^{-}(z,\lambda) = F^{(23)}(\lambda)^{-1}\mathcal{R}'^{-(13)}(z,\lambda+h^{(2)})\mathcal{R}'^{-(12)}(q^{c^{(3)}}z,\lambda)F^{(23)}(\lambda+h^{(1)}).$$

Using the intertwining relation $\Phi_W(z)a = \Delta(a)\Phi_W(z)$, we obtain

$$\begin{aligned}
&\Phi_W(z_2,\lambda)L_V^+(z_1,\lambda)\\
&= (\pi_{V,z_1}\otimes\mathrm{id}\otimes\pi_{W,z_2})(F^{(23)}(\lambda))\Phi_W(z_2)(\pi_{V,z_1}\otimes\mathrm{id})(\mathcal{R}'^{+}(\lambda))\\
&= (\pi_{V,z_1}\otimes\mathrm{id}\otimes\pi_{W,z_2})(F^{(23)}(\lambda)\cdot(\mathrm{id}\otimes\Delta)\mathcal{R}'^{+}(\lambda))\Phi_W(z_2)\\
&= (\pi_{V,z_1}\otimes\mathrm{id}\otimes\pi_{W,z_2})(\mathcal{R}'^{+(13)}(q^{c^{(2)}},\lambda+h^{(2)})\mathcal{R}'^{+(12)}(\lambda)F^{(23)}(\lambda+h^{(1)}))\Phi_W(z_2)\\
&= R^+_{VW}(q^c z_1/z_2,\lambda+h)L_V^+(z_1,\lambda)\Phi_W(z_2,\lambda+h^{(1)}).
\end{aligned}$$

The other cases are similar. □

From the theory of qKZ-equation[FR], we know the VO's for U satisfy the commutation relations of the form

$$\check{R}_{VV}(z_1/z_2)\Phi_V^{(\nu,\mu)}(z_1)\Phi_V^{(\mu,\kappa)}(z_2) = \sum_{\mu'}\Phi_V^{(\nu,\mu')}(z_2)\Phi_V^{(\mu',\kappa)}(z_1)W_I\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right), \tag{4.15}$$

$$\Psi_V^{*(\nu,\mu)}(z_1)\Psi_V^{*(\mu,\kappa)}(z_2)\check{R}_{VV}(z_1/z_2)^{-1} = \sum_{\mu'}W_{II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi_V^{*(\nu,\mu')}(z_2)\Psi_V^{*(\mu',\kappa)}(z_1), \tag{4.16}$$

$$\Phi_V^{(\nu,\mu)}(z_1)\Psi_V^{*(\mu,\kappa)}(z_2) = \sum_{\mu'}W_{I,II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi_V^{*(\nu,\mu')}(z_2)\Phi_V^{(\mu',\kappa)}(z_1). \tag{4.17}$$

Here [5] $\check{R}_{VV}(z) = PR_{VV}(z)$, $P(v\otimes v') = v'\otimes v$, $R_{VV}(z_1/z_2) = (\pi_{V,z_1}\otimes\pi_{V,z_2})\mathcal{R}$ is the 'trigonometric' R matrix. In (4.15)–(4.17) we used a slightly abbreviated notation. For example, the left hand side of (4.15) means the composition

$$V(\kappa)\xrightarrow{\Phi(z_2)}V(\mu)\otimes V_{z_2}\xrightarrow{\Phi(z_1)\otimes\mathrm{id}}V(\nu)\otimes V_{z_1}\otimes V_{z_2}\xrightarrow{\mathrm{id}\otimes\check{R}(z_1/z_2)}V(\nu)\otimes V_{z_2}\otimes V_{z_1}.$$

Similarly (4.16), (4.17) are maps $V_{z_2}\otimes V_{z_1}\otimes V(\kappa)\longrightarrow V(\nu)$, $V_{z_2}\otimes V(\kappa)\longrightarrow V(\nu)\otimes V_{z_1}$, respectively. For $U_q(\widehat{\mathfrak{sl}}_2)$, the formulas for the W-factors in the simplest case can be found e.g. in [IIJMNT].

The 'dressed' VO's satisfy similar relations with appropriate dynamical shift. Setting $\check{R}_{VV}(z,\lambda) = PR_{VV}(z,\lambda)$, we have

[5] The R matrix used here is the image of the universal R matrix. It differs by a scalar factor from the one used, e.g., in [JM, IIJMNT] in the commutation relations of VO's.

Proposition 4.5.

$$\begin{aligned}
&\check{R}_{VV}(z_1/z_2,\lambda+h^{(1)})\Phi_V^{(\nu,\mu)}(z_1,\lambda)\Phi_V^{(\mu,\kappa)}(z_2,\lambda)\\
&\quad=\sum_{\mu'}\Phi_V^{(\nu,\mu')}(z_2,\lambda)\Phi_V^{(\mu',\kappa)}(z_1,\lambda)W_I\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right), \qquad (4.18)\\
&\Psi_V^{*(\nu,\mu)}(z_1,\lambda)\Psi_V^{*(\mu,\kappa)}(z_2,\lambda+h^{(1)})\check{R}_{VV}(z_1/z_2,\lambda)^{-1}\\
&\quad=\sum_{\mu'}W_{II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi_V^{*(\nu,\mu')}(z_2,\lambda)\Psi_V^{*(\mu',\kappa)}(z_1,\lambda+h^{(1)}), \qquad (4.19)\\
&\Phi_V^{(\nu,\mu)}(z_1,\lambda)\Psi_V^{*(\mu,\kappa)}(z_2,\lambda)\\
&\quad=\sum_{\mu'}W_{I,II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi_V^{*(\nu,\mu')}(z_2,\lambda)\Phi_V^{(\mu',\kappa)}(z_1,\lambda+h^{(1)}). \qquad (4.20)
\end{aligned}$$

Notice that the W-factors stay the same with the trigonometric case, and are not affected by a dynamical shift.

Proof. Let us verify (4.19) as an example. We drop the suffix V. From the intertwining relation $a\Psi^*(z)=\Psi^*(z)\Delta(a)$, we have

$$\begin{aligned}
&\text{LHS of (4.19)}\\
&=\Psi^{*(\nu,\mu)}(z_1)(\pi_z\otimes\mathrm{id})(F(\lambda)^{-1})(\mathrm{id}\otimes\Psi^{*(\mu,\kappa)}(z_2))\\
&\quad\times(\pi_{z_1}\otimes\pi_{z_2}\otimes\mathrm{id})(F^{(23)}(\lambda+h^{(1)})^{-1})\\
&=\Psi^{*(\nu,\mu)}(z_1)(\mathrm{id}\otimes\Psi^{*(\mu,\kappa)}(z_2))\\
&\quad\times(\pi_{z_1}\otimes\pi_{z_2}\otimes\mathrm{id})((\mathrm{id}\otimes\Delta)F(\lambda)^{-1}\cdot F^{(23)}(\lambda+h^{(1)})^{-1}). \qquad (4.21)
\end{aligned}$$

In the right hand side of (4.21), the first factor equals

$$\sum_{\mu'}W_{II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi^{*(\nu,\mu')}(z_2)\Psi^{*(\mu',\kappa)}(z_1)P^{(12)}\left(\pi_{z_1}\otimes\pi_{z_2}\otimes\mathrm{id}\right)(\mathcal{R}^{(12)}),$$

while the second is $(\pi_{z_1}\otimes\pi_{z_2}\otimes\mathrm{id})((\Delta\otimes\mathrm{id})F(\lambda)^{-1}F^{(12)}(\lambda)^{-1})$ by the shifted cocycle condition. Since $\mathcal{R}^{(12)}(\Delta\otimes\mathrm{id})F(\lambda)^{-1}F^{(12)}(\lambda)^{-1}=(\Delta'\otimes\mathrm{id})F(\lambda)^{-1}F^{(21)}(\lambda)^{-1}\mathcal{R}^{(12)}(\lambda)$, (4.21) becomes

$$\begin{aligned}
&\sum_{\mu'}W_{II}\left(\begin{array}{cc|c}\kappa & \mu & \frac{z_1}{z_2}\\ \mu' & \nu & \end{array}\right)\Psi^{*(\nu,\mu')}(z_2)\Psi^{*(\mu',\kappa)}(z_1)\\
&\times\left(\pi_{z_2}\otimes\pi_{z_1}\otimes\mathrm{id}\right)\left((\Delta\otimes\mathrm{id})F(\lambda)^{-1}F^{(12)}(\lambda)^{-1}\right)\cdot P^{(12)}R(z_1/z_2,\lambda).
\end{aligned}$$

Using again the shifted cocycle condition we arrive at the right hand side of (4.19). □

3. The case of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_n)$

The case of vertex type algebras can be treated in a parallel way. Let $(\widetilde{\pi}_{V,\zeta}, V)$, $\widetilde{\pi}_{V,\zeta} = \pi \circ \mathrm{Ad}(\zeta^{\rho})$ stand for the evaluation module defined via the principal gradation operator ρ. In place of d we use ρ/n to define the R-matrix and L-operators as follows:

$$
\begin{aligned}
&\widetilde{L}_V^{\pm}(\zeta, r) = (\widetilde{\pi}_{V,\zeta} \otimes \mathrm{id})\,\widetilde{\mathcal{R}}'^{\pm}(r),\\
&\widetilde{\mathcal{R}}'^{+}(r) = q^{\widetilde{T}}\widetilde{\mathcal{R}}(r),\\
&\widetilde{\mathcal{R}}'^{-}(r) = \widetilde{\mathcal{R}}^{(21)}(r)^{-1}q^{-\widetilde{T}},\\
&\widetilde{R}^{\pm}_{VW}(\zeta_1/\zeta_2, r) = (\pi_{V,\zeta_1} \otimes \pi_{W,\zeta_2})\,\widetilde{\mathcal{R}}'^{\pm}(r).
\end{aligned}
$$

Then the RLL relations (4.5)–(4.6) remain valid if we replace the shift $\lambda + h$ by $r + c$ and read $q^c z$ as $q^{c/n}\zeta$. Since c is mapped to 0 in $(\pi_{V,\zeta}, V)$, the relations somewhat simplify. The result reads as follows.

$$
\begin{aligned}
&\widetilde{R}^{\pm(12)}_{VW}(\zeta_1/\zeta_2, r+c)\widetilde{L}^{\pm(1)}_V(\zeta_1, r)\widetilde{L}^{\pm(2)}_W(\zeta_2, r)\\
&\quad = \widetilde{L}^{\pm(2)}_W(\zeta_2, r)\widetilde{L}^{\pm(1)}_V(\zeta_1, r)\widetilde{R}^{\pm(12)}_{VW}(\zeta_1/\zeta_2, r), \qquad (4.22)\\
&\widetilde{R}^{+(12)}_{VW}(q^{c/n}\zeta_1/\zeta_2, r+c)\widetilde{L}^{+(1)}_V(\zeta_1, r)\widetilde{L}^{-(2)}_W(\zeta_2, r)\\
&\quad = \widetilde{L}^{-(2)}_W(\zeta_2, r)\widetilde{L}^{+(1)}_V(\zeta_1, r)\widetilde{R}^{+(12)}_{VW}(q^{-c/n}\zeta_1/\zeta_2, r). \qquad (4.23)
\end{aligned}
$$

By the same method as in the face type case, we find also $\widetilde{\mathcal{R}}'^{+}(p^{1/n}q^{(c^{(1)}+c^{(2)})/n}\zeta, r) = (\tau \otimes \mathrm{id})^{-1}\widetilde{\mathcal{R}}'^{-}(\zeta, r)$. Taking the image in the vector representation $V = \mathbb{C}^n = \mathbb{C}v_1 \oplus \ldots \oplus \mathbb{C}v_n$ and noting that τ is implemented by a conjugation $\pi \circ \tau = \mathrm{Ad}(h) \circ \pi$, $hv_j = v_{j+1 \bmod n}$, we obtain

$$
\widetilde{L}^{+}_V(p^{1/n}q^{c/n}\zeta, r) = (\mathrm{Ad}(h) \otimes \mathrm{id})^{-1}\widetilde{L}^{-}_V(\zeta, r). \qquad (4.24)
$$

The relations (4.22),(4.23) and (4.24) first appeared (for $n = 2$) in [FIJKMY1].

Similarly we define the VO's by

$$
\begin{aligned}
&\widetilde{\Phi}^{(\nu,\mu)}_V(\zeta, r) = (\mathrm{id} \otimes \pi_{V,\zeta})(E(r)) \circ \widetilde{\Phi}^{(\nu,\mu)}_V(\zeta),\\
&\widetilde{\Psi}^{*(\nu,\mu)}_V(\zeta, r) = \widetilde{\Psi}^{*(\nu,\mu)}_V(\zeta) \circ (\pi_{V,\zeta} \otimes \mathrm{id})(E(r))^{-1}.
\end{aligned}
$$

Here $\widetilde{\Phi}^{(\nu,\mu)}_V(\zeta)$, $\widetilde{\Psi}^{*(\nu,\mu)}_V(\zeta)$ are the VO's of $U_q(\widehat{\mathfrak{sl}}_2)$ in the principal gradation. The intertwining relations can be obtained from (4.11)–(4.14) by a simple replacement as explained above:

$$
\begin{aligned}
&\widetilde{\Phi}_W(\zeta_2, r)\widetilde{L}^{+}_V(\zeta_1, r) = \widetilde{R}^{+}_{VW}(q^{c/n}\zeta_1/\zeta_2, r+c)\widetilde{L}^{+}_V(\zeta_1, r)\widetilde{\Phi}_W(\zeta_2, r), \qquad (4.25)\\
&\widetilde{\Phi}_W(\zeta_2, r)\widetilde{L}^{-}_V(\zeta_1, r) = \widetilde{R}^{-}_{VW}(\zeta_1/\zeta_2, r+c)\widetilde{L}^{-}_V(\zeta_1, r)\widetilde{\Phi}_W(\zeta_2, r), \qquad (4.26)\\
&\widetilde{L}^{+}_V(\zeta_1, r)\widetilde{\Psi}^{*}_W(\zeta_2, r) = \widetilde{\Psi}^{*}_W(\zeta_2, r)\widetilde{L}^{+}_V(\zeta_1, r)\widetilde{R}^{+}_{VW}(\zeta_1/\zeta_2, r), \qquad (4.27)\\
&\widetilde{L}^{-}_V(\zeta_1, r)\widetilde{\Psi}^{*}_W(\zeta_2, r) = \widetilde{\Psi}^{*}_W(\zeta_2, r)\widetilde{L}^{-}_V(\zeta_1, r)\widetilde{R}^{-}_{VW}(q^{c/n}\zeta_1/\zeta_2, r). \qquad (4.28)
\end{aligned}
$$

These formulas agree with those conjectured in [FIJKMY1, FIJKMY2], if we identify $q^{2(r+c)}$ with p there.

The same can be done about the commutation relations of VO (4.18)–(4.20). We do not repeat the formulas.

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A. Quasi-Hopf algebras

We summarize here some basic notions concerning quasi-Hopf algebras [D2, D3]. Let k be a commutative ring.

Definition A.1. A quasi-bialgebra is a set $(A, \Delta, \varepsilon, \Phi)$ consisting of a unital associative k-algebra A, homomorphisms $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ and an invertible element $\Phi \in A \otimes A \otimes A$, satisfying the following axioms.

$$(\mathrm{id} \otimes \Delta)\Delta(a) = \Phi(\Delta \otimes \mathrm{id})\Delta(a)\Phi^{-1} \quad \forall a \in A, \tag{A.1}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})\Phi = (1 \otimes \Phi) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\Phi \cdot (\Phi \otimes 1), \tag{A.2}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{A.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi = 1. \tag{A.4}$$

Definition A.2. A quasi-Hopf algebra is a quasi-bialgebra $(A, \Delta, \varepsilon, \Phi)$ together with elements $\alpha, \beta \in A$ and an anti-automorphism S, satisfying the following conditions.

$$\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha, \qquad \sum_i b_i \beta S(c_i) = \varepsilon(a)\beta, \tag{A.5}$$

for $a \in A$, $\Delta(a) = \sum_i b_i \otimes c_i$, and

$$\sum_i X_i \beta S(Y_i) \alpha Z_i = 1, \tag{A.6}$$

where $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$.

Definition A.3. A quasi-triangular quasi-Hopf algebra is a set $(A, \Delta, \varepsilon, \Phi, R)$, where $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra and $R \in A \otimes A$ is an invertible element such that

$$\Delta'(a) = R\Delta(a)R^{-1}, \tag{A.7}$$

$$(\Delta \otimes \mathrm{id})R = \Phi^{(312)} R^{(13)} \Phi^{(132)^{-1}} R^{(23)} \Phi^{(123)}, \tag{A.8}$$

$$(\mathrm{id} \otimes \Delta)R = \Phi^{(231)^{-1}} R^{(13)} \Phi^{(213)} R^{(12)} \Phi^{(123)^{-1}}. \tag{A.9}$$

Here $\Delta' = \sigma \circ \Delta$ ($\sigma(a \otimes b) = b \otimes a$) is the opposite comultiplication. In (A.8)–(A.9), if $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$, then we write $\Phi^{(312)} = \sum_i Z_i \otimes X_i \otimes Y_i$, $\Phi^{(213)} = \sum_i Y_i \otimes X_i \otimes Z_i$, and so forth. Similar notations will be used throughout.

The properties (A.7)–(A.9) imply in particular the Yang–Baxter type equation

$$R^{(12)} \Phi^{(312)} R^{(13)} \Phi^{(132)^{-1}} R^{(23)} \Phi^{(123)} = \Phi^{(321)} R^{(23)} \Phi^{(231)^{-1}} R^{(13)} \Phi^{(213)} R^{(12)}. \tag{A.10}$$

There is an important operation called *twist*, which associates a new quasi-bialgebra with a given one. Let $(A, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra, and let $F \in A \otimes A$ be an invertible element such that $(\mathrm{id} \otimes \varepsilon)F = 1 = (\varepsilon \otimes \mathrm{id})F$. Set

$$\tilde{\Delta}(a) = F\Delta(a)F^{-1} \qquad (\forall a \in A), \tag{A.11}$$

$$\tilde{\Phi} = \left(F^{(23)}(\mathrm{id} \otimes \Delta)F\right) \Phi \left(F^{(12)}(\Delta \otimes \mathrm{id})F\right)^{-1}. \tag{A.12}$$

Then $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ is also a quasi-bialgebra. We refer to the element F as a *twistor*. If in addition $(A, \Delta, \varepsilon, \Phi)$ is a quasi-Hopf algebra, with α, β, S satisfying (A.5) and (A.6), then $(A, \tilde{\Delta}, \varepsilon, \tilde{\Phi})$ defined by (A.11) and (A.12) together with

$$\tilde{S} = S, \qquad \tilde{\alpha} = \sum_i S(d_i)\alpha e_i, \qquad \tilde{\beta} = \sum_i f_i \beta S(g_i),$$

is also a quasi-Hopf algebra. Here we have set $\sum_i d_i \otimes e_i = F^{-1}$ and $\sum_i f_i \otimes g_i = F$. Finally, a twist of a quasi-triangular quasi-Hopf algebra is again quasi-triangular, with the choice of new R given by

$$\tilde{R} = F^{(21)} R F^{(12)^{-1}}. \tag{A.13}$$

An important special case is a twist of a quasi-triangular *Hopf* algebra $(A, \Delta, \varepsilon, R)$ (i.e. a quasi-triangular quasi-Hopf algebra with $\Phi = 1$) by a shifted cocycle. Let H be an abelian subalgebra of A, with the product written additively.

Definition A.4. A twistor $F(\lambda)$ depending on $\lambda \in H$ is a shifted cocycle if it satisfies the relation

$$F^{(12)}(\lambda)\,(\Delta \otimes \mathrm{id})\,F(\lambda) = F^{(23)}(\lambda + h^{(1)})\,(\mathrm{id} \otimes \Delta)\,F(\lambda) \tag{A.14}$$

for some $h \in H$.

Let $(A, \Delta_\lambda, \varepsilon, \Phi(\lambda), R(\lambda))$ be the quasi-triangular quasi-Hopf algebra obtained by a twist by $F(\lambda)$. The shifted cocycle condition (A.14) simplifies the properties of $\Phi(\lambda)$ and $R(\lambda)$ as follows.

Proposition A.5. *We have*

$$\Phi(\lambda) = F^{(23)}(\lambda) F^{(23)}(\lambda + h^{(1)})^{-1}, \tag{A.15}$$

$$(\Delta_\lambda \otimes \mathrm{id})R(\lambda) = \Phi^{(312)}(\lambda) R^{(13)}(\lambda) R^{(23)}(\lambda + h^{(1)}), \tag{A.16}$$

$$(\mathrm{id} \otimes \Delta_\lambda)R(\lambda) = R^{(13)}(\lambda + h^{(2)}) R^{(12)}(\lambda) \Phi^{(123)}(\lambda)^{-1}. \tag{A.17}$$

As a corollary the dynamical Yang–Baxter relation holds:

$$R^{(12)}(\lambda + h^{(3)}) R^{(13)}(\lambda) R^{(23)}(\lambda + h^{(1)}) = R^{(23)}(\lambda) R^{(13)}(\lambda + h^{(2)}) R^{(12)}(\lambda). \tag{A.18}$$

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