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Two Matrix Model and Minimal Unitary Model

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Abstract

We investigate the two matrix model with the $\mathbf{Z}_2$ symmetric even potential of degree $2v$. We determine the critical coupling constants, with which the two matrix model realizes the minimal unitary model coupled to two dimensional quantum gravity. By explicit calculation, the type $(p,q) = (m, m+1)$ string equation is obtained for $m = v+1 \leq 6$. We also discuss the Douglas' P, Q operators and correlation functions.

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1. To find a true stable string vacuum from among a large number of classical vacua, we must study string theories nonperturbatively. Concerning the "space-time dimension" less than 1, the matrix model and the double scaling limit technique allow us the nonperturbative study of "string" theories[1]. Matrix models are solvable by the orthogonal polynomial method[2]. One matrix model with a quartic potential realizes a pure two dimensional gravity, i.e. $(p, q) = (2, 3)$, and its specific heat satisfies a nonlinear differential equation, so-called string equation[1]. One matrix model with higher order potential realizes $2d$ gravity coupled to $(p, q) = (2, 2m - 1)$ minimal conformal matter, and its string equation and correlation functions are closely related to the KdV hierarchy[1]. Douglas pointed out that more general (p, q) minimal conformal matter coupled to $2d$ gravity can be realized by $p - 1$ matrix model, and its string equation is related to the generalized KdV hierarchy (p-reduction of the KP hierarchy)[3]. In fact, $(p, q) = (3, 4)$ unitary minimal matter, i.e. Ising model, is realized by the two matrix model with quartic potential[4], and $(p, q) = (4, 5)$ minimal unitary matter, i.e. tricritical Ising model, is realized by the three matrix model with quartic potential[5]. Relations between $p - 1$ matrix model and W_p algebra, topological field theory were also pointed out[6].

Recently Tada and Yamaguchi studied the two matrix model with a sixth order potential and found that not only $(p, q) = (3, 8)$ minimal conformal model but also $(p, q) = (4, 5)$ minimal unitary model are realized[7]. They conjectured that the two matrix model with a higher order potential realizes all the minimal unitary models. Moreover Douglas discussed all (p, q) minimal conformal models can be realized by the two matrix model[8][1].

In this letter we will investigate the two matrix model with the $\mathbf{Z}_2$ symmetric even potential and try to show the Tada-Yamaguchi conjecture. By the method of [7], we present the general expression for the critical coupling constants that correspond to minimal unitary model, and express the Douglas' P, Q operators by the orthogonal polynomial method. In §2 we review the orthogonal polynomial method and determine the critical coupling constants. We show in §4 that these critical coupling constants really correspond to $(p, q) = (m, m + 1)$ minimal unitary model for $m \leq 6$, by explicit

[1]After completion of our calculation, we received ref.[12], where Tada also conjectured the same statement and obtained $(p, q) = (3, 5)$ model by explicit calculation.

calculation. We discuss the Douglas' P, Q operators in §3, correlation functions and critical lines in §5. In §6 we comment on the general (p, q) models. Since the matrix model has a definite normalization, we will keep multiplicative constants carefully.

2. We will consider the two matrix model with the $\mathbf{Z}_2$ symmetric even potential of degree $2v$, whose partition function is

$$\mathcal{Z} = e^{\mathcal{F}} = \int dM_+ dM_- e^{-S}, \qquad S = \mathrm{tr}(V(M_+) + V(M_-) - cM_+M_-), \tag{1}$$

$$V(x) = \sum_{j=1}^{v} \frac{1}{N^{j-1}} \frac{g_{2j}}{(2j)!} x^{2j}, \qquad g_2 = 1, \tag{2}$$

where $M_\pm$ are $N \times N$ hermitian matrices. We assume c is non-zero because, if $c = 0$, $\mathcal{Z}$ reduces to two copies of the one matrix model. We can diagonalize the matrices $M_\pm$ by using unitary matrices and integrate out the unitary matrices. Introducing orthogonal polynomials $P_i(x)$[2]

$$h_i \delta_{ij} = \int dx dy P_j(y) e^{-V(x)-V(y)+cxy} P_i(x), \quad F_i = \frac{h_i}{h_{i-1}}, \tag{3}$$

the partition function is rewritten as

$$\mathcal{Z} = e^{\mathcal{F}} = \pi^{N(N-1)} c^{-\frac{N(N-1)}{2}} N! h_0^N \prod_{i=1}^{N} \frac{1}{i!} \prod_{i=1}^{N} F_i^{N-i}. \tag{4}$$

$P_i(x)$ is a monic polynomial of degree i and has a definite parity $P_i(-x) = (-1)^i P_i(x)$, since our action is even. Polynomials P_i satisfy the following recursion relation:

$$xP_i(x) = \sum_j \alpha_{ij} P_j(x) = \sum_{k=0}^{v} R_i^{[k]} P_{i+1-2k}(x), \quad R_i^{[0]} = 1, \tag{5}$$

where $R^{[k]}$'s are unknown coefficients and matrix α is defined by this equation. Absence of the terms P_{i-2k} is a consequence of even potential and the upper bound[2] of sum over k is determined by the degree of the potential V. Depending on the potential, this recursion relation contains a number of adjustable coefficients. This is the reason why the two matrix model can realize the (p, q) minimal conformal model for any p. In the case of the one matrix model, the recursion relation contains only one coefficient.

[2]Strictly speaking, the upper bound of sum over k is $\min(v, [\frac{i+1}{2}])$.

Therefore the one matrix model can realize $p = 2$ minimal conformal model only. In the case of the n-matrix model ($n \geq 3$), α is a general lower triangular matrix.

All the necessary information for solving the two matrix model is contained in the following equations:

$$\begin{aligned}
cR_i^{[k]} &= F_i F_{i-1} \cdots F_{i+2-2k} V'(\alpha)_{i+1-2k,i}, \qquad (\frac{v}{2} < k \leq v), && (6)\\
cR_{i+k-1}^{[k]} &= F_{i+k-1} F_{i+k-2} \cdots F_{i+1-k} V'(\alpha)_{i-k,i+k-1}, \quad (1 \leq k \leq \frac{v}{2}), && (7)\\
i &= V'(\alpha)_{i,i-1} - cF_i. && (8)
\end{aligned}$$

We call the above first and second equations as constraint equations and the third one as a potential equation. In principle, by solving the constraint equations, we can express $R_i^{[k]}$'s in terms of F_i. We remark that $R_i^{[k]}$ ($\frac{v}{2} < k \leq v$), eq. (6), are solved in terms of F_i and $R_i^{[k]}$ ($1 \leq k \leq \frac{v}{2}$).

We first consider the naive large N limit. In this limit, F and $R^{[k]}$ are scaled as follows:

$$\frac{i}{N} \sim x, \quad \frac{1}{N} F_i \sim F(x), \quad \frac{1}{N^k} R_i^{[k]} \sim R^{[k]}(x), \quad (1 \leq k \leq v). \tag{9}$$

Let us denote the values of F and $R^{[k]}$ at $x = 1$ by

$$F(1) = f_0, \quad R^{[k]}(1) = f_0^k r_0^{[k]}, \quad (1 \leq k \leq v). \tag{10}$$

After eliminating $R^{[k]}$, we define a potential $W(F)$ by the right hand side of eq. (8) divided by N. Since the scaling laws arise from the singular behavior of $F(x)$ in the vicinity of $x = 1$, the potential W takes the form $W(F) - 1 \propto (F - f_0)^m$ near $x = 1$ at the m-th order critical point. So, the m-th order critical coupling constants c and g_{2j} are determined by the following requirement:

$$\begin{aligned}
W^{(k)}(f_0) &= 0, \quad (1 \leq k \leq m-1), && (11)\\
W(f_0) &= 1. && (12)
\end{aligned}$$

If we solve these equations completely for each m, we can draw the phase diagram of the two matrix model. In actual calculation it is convenient to differentiate $W(F)$ under the constraint equations instead of solving $R^{[k]}$ explicitly. However, this calculation is very hard for large v. So we give up finding out all the solutions and try to find some of the critical points.

Instead of eq. (11), we require

$$d_n = 0, \quad (1 \le n \le p-1), \tag{13}$$

where d_n's are defined by

$$d_n = \frac{1}{n!} \sum_{k=0}^{v} (2k-1)^n r_0^{[k]}. \tag{14}$$

These equations with $p = m$ are sufficient conditions for eq. (11), and the meaning of them will be explained in the next section. Naive parameter counting shows that the maximum value of m is $v+1$[3]. In the following we will concentrate our attention on this case, i.e. $m = p = v+1$. Then eq. (13) is solved as follows:

$$r_0^{[k]} = \frac{(-1)^{k+1}}{2k-1} \binom{v}{k}, \quad (0 \le k \le v). \tag{15}$$

Using these values, d_0, d_m, d_{m+1} are given by

$$d_0 = \frac{(2v)!!}{(2v-1)!!}, \quad d_m = \frac{(-1)^{v+1}}{v+1} 2^v, \quad d_{m+1} = \frac{(-1)^{v+1}}{v+2} (v-1) 2^v. \tag{16}$$

Critical coupling constants are determined by the constraint and potential equations. After some combinatorics, eqs. (6,7,8) become

$$r_0^{[k]} + \sum_{j=k}^{v} (-1)^j g'_{2j} \oint_0 \frac{dt}{2\pi i} \frac{1}{t^{j-k+1}} \left(\sum_{i=0}^{v} r_0^{[i]} t^i \right)^{2j-1} = 0, \quad (0 \le k \le v), \tag{17}$$

where $g'_{2j} = \frac{(-f_0)^{j-1}}{(2j-1)!c} g_{2j}$ $(2 \le j \le v)$, $g'_2 = \frac{1}{c}$ and $g'_0 = \frac{1}{cf_0}$. From this recursion relation, one can obtain the critical coupling constants g'_{2j} $(j = v, v-1, \cdots, 0)$ easily; for example,

$$\begin{aligned}
(2v-1) g'_{2v} &= 1, \\
(2v-3) g'_{2(v-1)} &= 2v(v-1), \\
3(2v-5) g'_{2(v-2)} &= v(v-1)(6v^2 - 14v - 1), \\
15(2v-7) g'_{2(v-3)} &= 2v(v-1)(10v^4 - 60v^3 + 83v^2 + 16v + 1), \\
630(2v-9) g'_{2(v-4)} &= v(v-1)(420v^6 - 4620v^5 + 16184v^4 - 16180v^3 \\
&\qquad -5953v^2 - 831v - 45), \\
945(2v-11) g'_{2(v-5)} &= v(v-1)(252v^8 - 4368v^7 + 27468v^6 - 71808v^5 \\
&\qquad +53797v^4 + 33596v^3 + 7763v^2 + 886v + 42).
\end{aligned} \tag{18}$$

[3]In general there are exceptional critical coupling constants which give $m > v+1$.

Next we consider the double scaling limit in order to include the contributions from the higher genus Riemann surfaces. As refs.[1], we introduce a lattice spacing constant a such that the continuum limit is $a \to 0$. The renormalized cosmological constant is $\mu_R = (\mu_B - \mu_{cri})/a^2$, i.e. $\mu_R = \frac{g_{2j}^{cri} - g_{2j}}{(j-1) g_{2j}^{cri} a^2}$. Introducing the scaled variable z,

$$\frac{i}{N} \sim 1 - a^2 z, \quad \frac{1}{N} \sim \tilde{a}^{2m+1}, \quad a = \tilde{a}^m, \tag{19}$$

F and $R^{[k]}$ have the following expansions:

$$\frac{1}{N} F_{i+l} \sim f_0 (1 + \sum_{n \geq 1} \tilde{a}^{2n} f_{2n}(z - l\tilde{a})) \tag{20}$$

$$\frac{1}{N^k} R^{[k]}_{i+k-1+l} \sim f_0^k (r_0^{[k]} + \sum_{n \geq 1} \tilde{a}^{2n} r^{[k]}_{2n}(z - l\tilde{a})), \quad (1 \leq k \leq \frac{v}{2}). \tag{21}$$

$R^{[k]}_i$ ($\frac{v}{2} < k \leq v$) have been exactly eliminated by using eq. (6). The function $f_2(z)$ is related to the free energy as $\frac{d^2}{d\mu_R^2} \mathcal{F} = f_2(z)|_{z=\mu_R}$. Substituting these expansions into eqs. (7,8) and solving the constraint equations order by order in $\tilde{a}$, we obtain a coupled nonlinear differential equation, so-called string equation. This calculation is tedious but straightforward. We expect that this string equation agrees with the type $(p, q) = (m, m+1)$ equation of the KP hierarchy (q-th flow of p-reduction of the KP hierarchy). In fact we will check this conjecture for $m \leq 6$ in section 4.

3. To compare the string equation obtained in the previous section and the string equation in Douglas' form[3], we introduce the normalized polynomials $\mathcal{P}_i(x) = \frac{1}{\sqrt{h_i}} P_i(x)$. We define matrices $\tilde{Q}$ and $\tilde{P}$ as follows:

$$x \mathcal{P}_i(x) = \sum_j \tilde{Q}_{ij} \mathcal{P}_j(x), \quad \frac{d}{dx} \mathcal{P}_i(x) = \sum_j \tilde{P}_{ij} \mathcal{P}_j(x). \tag{22}$$

$\tilde{Q}$ and $\tilde{P}$ satisfy the commutation relation

$$[\tilde{Q}, \tilde{P}] = 1, \tag{23}$$

and this equation reduces to $[{}^t\tilde{Q}, \tilde{Q}] = \frac{1}{c}$, because $\tilde{P}$ is expressed as $\tilde{P} = -c\, {}^t\tilde{Q} + V'(\tilde{Q})$, where ${}^t\tilde{Q}$ is a transposed matrix of $\tilde{Q}$. In the double scaling limit, the matrix $\tilde{Q}$ is scaled as follows[7]:

$$\tilde{Q}_{ij} = \sqrt{F_{i+1}} \delta_{i+1,j} + \sum_{k=1}^{v} R^{[k]}_i (F_i F_{i-1} \cdots F_{i+2-2k})^{-\frac{1}{2}} \delta_{i+1-2k,j}, \tag{24}$$

$$\frac{1}{\sqrt{N}}\tilde{Q}_{ij} \sim \sqrt{\frac{F_{i+1}}{N}}e^{-\tilde{a}\partial} + \sum_{k=1}^{v}\frac{R_i^{[k]}}{N^k}\left(\frac{F_i}{N}\frac{F_{i-1}}{N}\cdots\frac{F_{i+2-2k}}{N}\right)^{-\frac{1}{2}}e^{(2k-1)\tilde{a}\partial} \tag{25}$$

$$\stackrel{\text{def}}{=} \sqrt{f_0}\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n, \tag{26}$$

where $\frac{F}{N}$ and $\frac{R^{[k]}}{N^k}$ have the expansions eqs. (20,21) and ∂ represents $\frac{\partial}{\partial z}$. $\tilde{Q}_n$ is a differential operator of order n, and its coefficient of ∂^n is d_n. Therefore eq. (13) is the necessary condition for the vanishing of $\tilde{Q}_n = 0$ $(1 \leq n \leq p-1)$. In the next section we will see that it is also sufficient. For later use, we rewrite $\tilde{Q}_n$ as $\tilde{Q}_n = d_n Q_n$. Expansion of the transposed matrix ${}^t\tilde{Q}$ becomes $\frac{1}{\sqrt{N}}({}^t\tilde{Q})_{ij} \sim \sqrt{f_0}\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n^\dagger$, where $\dagger$ is defined by the following properties: $\partial^\dagger = -\partial$, $\varphi(z)^\dagger = \varphi(z)$ for any function φ, $(AB)^\dagger = B^\dagger A^\dagger$ for any pseudodifferential operators A and B, and linearity. On the other hand ${}^t\tilde{Q}$ has the following expansion:

$$\frac{1}{\sqrt{N}}({}^t\tilde{Q})_{ij} \sim \sqrt{\frac{F_i}{N}}e^{\tilde{a}\partial} + \sum_{k=1}^{v}\frac{R_{i+2k-1}^{[k]}}{N^k}\left(\frac{F_{i+2k-1}}{N}\frac{F_{i+2k-2}}{N}\cdots\frac{F_{i+1}}{N}\right)^{-\frac{1}{2}}e^{-(2k-1)\tilde{a}\partial}. \tag{27}$$

Comparing eq. (25) and eq. (27), we obtain

$$\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n^\dagger(z) = \sum_{n\geq 0}(-\tilde{a})^n\tilde{Q}_n(z-\tilde{a}), \tag{28}$$

where we write the z-dependence of the coefficients of ∂ explicitly. This is a consequence of the $\mathbf{Z}_2$ symmetry. If $\tilde{Q}_n = 0$ $(1 \leq n \leq m-1)$, then, by using the above equation, we can show that

$$\tilde{Q}_m^\dagger = (-1)^m\tilde{Q}_m, \quad \tilde{Q}_{m+1}^\dagger = (-1)^{m+1}(\tilde{Q}_{m+1} + \tilde{Q}_m'), \tag{29}$$

where $'$ represents a differentiation, i.e. $A' = \sum_n a_n'\partial^n$ for $A = \sum_n a_n\partial^n$. Under the condition $\tilde{Q}_n = 0$ $(1 \leq n \leq m-1)$, by substituting these expansions into eq. (23), i.e. $[\frac{1}{\sqrt{N}}\tilde{Q}, \frac{1}{\sqrt{N}}\tilde{P}] = \frac{1}{N} \sim \tilde{a}^{2m+1}$, we obtain

$$[\tfrac{1}{2}(Q_{m+1} + (-1)^{m+1}Q_{m+1}^\dagger),\ Q_m] = \frac{(-1)^{m+1}g_0'}{2d_m d_{m+1}}. \tag{30}$$

This commutation relation suggests the following identification of the Douglas' P, Q operators with Q_n's which are obtained from the orthogonal polynomial method,

$$Q = Q_m, \quad P = \frac{1}{2}(Q_{m+1} + (-1)^{m+1}Q_{m+1}^\dagger) = Q_{m+1} + \frac{d_m}{2d_{m+1}}Q_m'. \tag{31}$$

We remark that Q is the order $\tilde{a}^m$ term of $\tilde{Q}$ but P is not the order $\tilde{a}^{m+1}$ term of $\tilde{P}$. This form of P was conjectured in [7]. Q and P have definite parity $Q^\dagger = (-1)^m Q$, $P^\dagger = (-1)^{m+1} P$ due to the $\mathbf{Z}_2$ symmetry.

Let us recall the type (p, q) string equation in Douglas form[3]. Q is a differential operator of order p and $P = Q^{\frac{q}{p}}{}_+$:

$$Q = \partial^p + \sum_{n=2}^{p-1} \{\partial^{p-n}, c_n\} + c_p, \quad Q^{\frac{q}{p}} = \partial^q + \sum_{n \geq 2} b_n \partial^{q-n}. \tag{32}$$

In the case of the $\mathbf{Z}_2$ symmetric potential, c_{odd} vanishes, and Q and P have definite parity. In the language of KP hierarchy, $Q = L^p = L^p{}_+$ (p-reduction) and $P = L^q{}_+$, where L is a pseudodifferential operator $L = \partial + \sum_{n \geq 2} u_n \partial^{1-n}$. Setting all the integration constants equal to zero, the string equation in Douglas form ($[P, Q] = \text{const}$) is equivalent to[5]

$$b_n = 0, \quad (q + 1 \leq n \leq p + q - 2), \quad b_{p+q-1} = \frac{\text{const}}{p} z. \tag{33}$$

In our case, $p = m = v+1$, $q = m+1$, const $= \frac{(-1)^{m+1} g_0'}{2 d_m d_{m+1}}$. In the case of the $\mathbf{Z}_2$ symmetric potential, $b_{odd} = 0$ can be derived from other equations, and unknown functions are c_{2n} $(1 \leq n \leq [\frac{p}{2}])$[4].

4. In §2,3, we have argued that the two matrix model with the critical coupling constants eqs. (17,18) and eq. (15), gives the type $(p, q) = (m, m + 1)$ string equation. However there are some unproved facts. They are

(i) $\tilde{Q}_n = 0, \quad (1 \leq n \leq m - 1)$.

(ii) $\frac{1}{2}(Q_{m+1} + (-1)^{m+1} Q^\dagger_{m+1}) = (Q_m)^{\frac{m+1}{m}}{}_+$.

(iii) the string equation derived from eqs. (7,8) agrees with the string equation in Douglas form eq. (33) with $p = m = v + 1 = q - 1$, const $= \dfrac{(-1)^{m+1} g_0'}{2 d_m d_{m+1}}$.

$v = 1$ case is trivial because the action is Gaussian. We only consider the case $v = m - 1 \geq 2$. For $m \leq 6$, we have checked (i)(ii)(iii) by explicit calculation, and we are convinced that (i)(ii)(iii) hold for all m. Here we give only the relations between f_{2n},

[4] In our notation, suffixes of $f_n, r_n^{[k]}, \tilde{Q}_n, Q_n, c_n, b_n, u_n$ represent "weight" n. ∂ has weight 1.

$r_{2n}^{[k]}$ and c_{2n}, because the string equation has a lengthy expression for large v, and the critical coupling constants can be easily obtained from eq. (18). They are

$$v = 2: \quad c_2 = \tfrac{3}{4} f_2, \tag{34}$$

$$v = 3: \quad c_2 = f_2, \quad c_4 = (r_4^{[1]} - f_4) + f_2^2, \tag{35}$$

$$v = 4: \quad c_2 = \tfrac{5}{4} f_2, \quad c_4 = \tfrac{5}{8}((r_4^{[1]} - f_4) + \tfrac{3}{2} f_2^2 - \tfrac{1}{2} f_2''), \tag{36}$$

$$v = 5: \quad c_2 = \tfrac{3}{2} f_2, \quad c_4 = \tfrac{3}{4}((r_4^{[1]} - f_4) + 2 f_2^2 - \tfrac{4}{3} f_2''), \tag{37}$$

$$c_6 = \tfrac{3}{8}((r_6^{[1]} + r_6^{[2]} + 3 f_6) + \tfrac{7}{3}(r_4^{[1]} - f_4)'' + 5(r_4^{[1]} - f_4) f_2 + \tfrac{4}{3} f_2'''' - \tfrac{8}{3} f_2'^2 + \tfrac{8}{3} f_2^3). \tag{38}$$

c_2 is related to f_2 as $c_2 = \frac{m}{4} f_2$ for all m.

5. One point functions of $\mathrm{tr} M_\pm^{2n}$ and $\mathrm{tr} M_+ M_-$ are given by

$$\frac{1}{N^{n+1}} \langle \mathrm{tr} M_\pm^{2n} \rangle = \frac{1}{N} \sum_{i=0}^{N-1} \frac{1}{N^n} (\alpha^{2n})_{ii}, \tag{39}$$

$$\frac{1}{N^2} \langle \mathrm{tr} M_+ M_- \rangle = \frac{1}{c} \frac{1}{N} \sum_{i=0}^{N-1} (\frac{1}{N} (\alpha V'(\alpha))_{ii} - \frac{2i+1}{N}), \tag{40}$$

where $\langle \mathcal{O} \rangle = \int dM_+ dM_- \mathcal{O} e^{-S} / \mathcal{Z}$. Using the Euler-Maclaurin formula, the first derivative of the sum $\frac{1}{N} \sum_{i=1}^{N} \varphi(\frac{i}{N})$ with respect to the renormalized cosmological constant can be written

$$-\frac{1}{a^2} \frac{d}{d\mu_R} \frac{1}{N} \sum_{i=1}^{N} \varphi(\frac{i}{N}) \sim \sum_{r=0}^{M} \frac{B_{2r}}{(2r)!} \tilde{a}^{2r} \tilde{\varphi}^{(2r)}(z) \Big|_{z=\mu_R} + \cdots, \tag{41}$$

where $\varphi(x) = \tilde{\varphi}(z)$ and Bernoulli numbers are defined by $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$. We abbreviate $-\frac{1}{a^2} \frac{d}{d\mu_R} \langle \mathcal{O} \rangle$ as $\langle\langle \mathcal{O} \rangle\rangle$[5]. In the case of (p, q) minimal conformal model, the correlation function of the scaling operator $\mathcal{O}_{(k)\alpha}$, which is the gravitational descendant of the primary scaling operator $\mathcal{O}_\alpha$ $(0 \le \alpha \le p-2)$, is given by[9]

$$\langle\langle \mathcal{O}_{(k)\alpha} \rangle\rangle \propto \mathrm{Res}\ Q^{k + \frac{\alpha+1}{p}}. \tag{42}$$

Since we are dealing with only the $\mathbf{Z}_2$ symmetric potential, $\mathbf{Z}_2$ odd operators, such as a spin operator of the Ising model, cannot be handled. The $\mathbf{Z}_2$ even operator can be

[5] Usually this is written as $\langle \mathcal{P} \mathcal{O} \rangle$, where $\mathcal{P}$ is the puncture operator.

expressed as a linear combination of $\mathrm{tr}M_\pm^{2n}$ and $\mathrm{tr}M_+M_-$. For example,

$$v=2: \ \langle\langle \tfrac{1}{N^3}\mathrm{tr}M_+^4\rangle\rangle - \tfrac{34}{5}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+^2\rangle\rangle \propto \mathrm{Res}\ Q^{\frac{1}{3}}, \tag{43}$$

$$\langle\langle \tfrac{1}{N^4}\mathrm{tr}M_+^6\rangle\rangle - \tfrac{144}{7}\langle\langle \tfrac{1}{N^3}\mathrm{tr}M_+^4\rangle\rangle + \tfrac{4356}{35}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+^2\rangle\rangle - \tfrac{29376}{175}\langle\langle \tfrac{1}{N}\mathrm{tr}1\rangle\rangle \propto \mathrm{Res}\ Q^{\frac{5}{3}}, \tag{44}$$

$$v=3: \ \langle\langle \tfrac{1}{N^3}\mathrm{tr}M_+^4\rangle\rangle - \tfrac{1375}{1008}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+^2\rangle\rangle \propto \mathrm{Res}\ Q^{\frac{1}{4}}, \tag{45}$$

$$\langle\langle \tfrac{1}{N^4}\mathrm{tr}M_+^6\rangle\rangle - \tfrac{638}{595}\langle\langle \tfrac{1}{N^3}\mathrm{tr}M_+^4\rangle\rangle + \tfrac{3751}{5712}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+^2\rangle\rangle \propto \mathrm{Res}\ Q^{\frac{3}{4}}, \tag{46}$$

$$\begin{aligned}&\langle\langle \tfrac{1}{N^5}\mathrm{tr}M_+^8\rangle\rangle - \tfrac{979}{672}\langle\langle \tfrac{1}{N^4}\mathrm{tr}M_+^6\rangle\rangle + \tfrac{75625}{56448}\langle\langle \tfrac{1}{N^3}\mathrm{tr}M_+^4\rangle\rangle - \tfrac{22234355}{18966528}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+^2\rangle\rangle \\ &\quad - \tfrac{33275}{1806336}\langle\langle \tfrac{1}{N^2}\mathrm{tr}M_+M_-\rangle\rangle + \tfrac{1590378625}{531062784}\langle\langle \tfrac{1}{N}\mathrm{tr}1\rangle\rangle \propto \mathrm{Res}\ Q^{\frac{7}{4}}.\end{aligned} \tag{47}$$

In the last equation we have used the string equation, and this correlation function was firstly obtained in analysis of the three matrix model with the $\mathbf{Z}_2$ symmetric quartic potential[11]. These operators correspond to the $\mathbf{Z}_2$ even relevant operator ϕ^{2n} in the notation of the Landau-Ginzburg description of the usual minimal unitary model. There are many ways of expressing $\mathcal{O}_{(k)\alpha}$ in terms of $\mathrm{tr}M_\pm^{2n}$ and $\mathrm{tr}M_+M_-$, but we do not know the general expressions.

So far we require eq. (13) with $p = v+1$, so that critical coupling constants are uniquely determined. If we relax the condition, we obtain the critical line (or surface, bulk, etc.). We take $v=3$ case as an example. Requiring $d_1 = 0$ only, we obtain a critical surface parameterized by g_4' and g_6' ($g_6' \neq -\frac{1}{15}$). At a general point on this surface, we obtain a $(p,q)=(2,3)$ string equation. At a general point on a line ($g_4' = 50{g_6'}^2 + 10g_6'$) in this surface, we get a $(p,q)=(2,5)$ equation. If we require $d_2 = 0$ moreover, then g_4' is related to g_6', and we obtain a critical line:

$$g_4' = \tfrac{1}{3}(75{g_6'}^2 + 40g_6' + 1), \qquad \tfrac{1}{c} = \tfrac{1}{3}(375{g_6'}^3 + 500{g_6'}^2 + 155g_6' + 12), \tag{48}$$

$$\tfrac{1}{cf_0} = \tfrac{10}{3}(19{g_6'}^2 + 8g_6' + 1), \qquad r_0^{[1]} = 2 + 5g_6'. \tag{49}$$

On this critical line except for $g_6' = \frac{1}{5}, \frac{1}{15}$, we obtain a $(p,q)=(3,4)$ equation. The point $g_6' = \frac{1}{15}$ realizes a $(p,q)=(3,8)$ equation[10,7], and in this case, $m = 5 > v+1$. In the case of $g_6' = \frac{1}{5}$, d_3 vanishes. So we obtain a $(p,q)=(4,5)$ equation[7]. We remark that this point is an intersection point of two critical lines: $(p,q)=(2,5)$ line ($g_4' = 50{g_6'}^2 + 10g_6'$) and $(3,4)$ line ($g_4' = \frac{1}{3}(75{g_6'}^2 + 40g_6' + 1)$). We observed the same phenomena in analysis of the three matrix model with the $\mathbf{Z}_2$ symmetric quartic potential[11]. In $(p,q)=(4,5)$ model realized by the three matrix model, the gravitationally dressed vacancy operator of the tricritical Ising model is expressed as a linear combination of $\mathrm{tr}(A^{2n} + C^{2n})$, $\mathrm{tr}B^{2n}$ $(n = 1, 2)$, and $\mathrm{tr}(AB + BC)$, and its coefficients are determined by the critical line (eq. (48) of the first ref. of [5]). In the case of the two matrix model, we have not succeeded in deriving such coefficients from critical lines.

6. In this letter we have presented the critical coupling constants, with which two matrix model realize $(p, q) = (m, m + 1)$ minimal unitary conformal matter coupled to two dimensional quantum gravity, and we have explicitly checked them for $m \leq 6$ by using the orthogonal polynomial method. Douglas' P, Q operators are also described by the same method.

It has been conjectured that general (p, q) minimal conformal models coupled to $2d$ gravity can be realized by the two matrix model with the $\mathbf{Z}_2$ asymmetric action $S = \mathrm{tr}(V^+(M_+) + V^-(M_-) - cM_+M_-)$[8,12]. In fact Tada obtained $(p, q) = (3, 5)$[12]. In the case of the $\mathbf{Z}_2$ asymmetric potential, orthogonal polynomials are defined by eq. (3) with $P_i(x)$ and $P_j(y)$ replaced by $P_i^+(x)$ and $P_j^-(y)$ respectively, and $\alpha^\pm$, $R_i^{[k]\pm}$, $d_n^\pm$, etc. are introduced. Corresponding to eq. (13), it is expected that the critical coupling constants of type (p, q) are obtained by the requirement that d_n^+ should vanish for $1 \leq n \leq p - 1$ and d_n^- should vanish for $1 \leq n \leq q - 1$, and the string equation is derived from $[{}^t\tilde{Q}^-, \tilde{Q}^+] = \frac{1}{c}$. One can change p by adjusting c and V^-, and q by V^+. From the parameter counting, minimal values of v^- and v^+ are $p-1$ and q respectively, but in contrast to the unitary case, the coupling constants are not uniquely determined.

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