Nonperturbative Analysis of Three Matrix Model

Hiroshi Kunitomo*†and Satoru Odake‡

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

The recently developed techniques are applied to give exact nonperturbative solutions of three matrix model. There exist three different critical points and the specific heats of the model corresponding to these points satisfy the different differential equations (string equations). A critical point expected to correspond to the tricritical Ising model is connected to the Ising model by a critical line. The string equations on each scaling limit coincide with the type $(p, q) = (4, 5)$, (2,7) and (3,8) equations of Douglas's general argument.

*kunitomo@tkyvax.hepnet, Fellow of the Japan Society for the Promotion of Science.
†Address after April 1990: Research Institute for Fundamental Physics, Kyoto University, Kyoto 606, Japan.
‡odake@tkyvax.hepnet, Fellow of the Japan Society for the Promotion of Science.

In spite of beautiful developments of string theory, we are still in the dark about its nonperturbative properties. Recently, some remarkable progress has been made in nonperturbative approach to two dimensional gravity or noncritical string theory.[1,2,3] They have given exact nonperturbative solution for the discretized two dimensional gravity by using the large-N matrix method with some double scaling limit for large-N and critical coupling. They also obtained the other solutions which exhibit multicritical behaviour and indexed by a positive integer m $(= 2, 3, \cdots)$.

These solutions are initially believed to correspond to minimal unitary conformal models coupled to quantum gravity. It is shown, however, that this conjecture is not true and $m = 3$ multicritical theory* corresponds to the Yang-Lee edge singularity[4] which is described by a non-unitary CFT with $c = -22/5$.[5]

Furthermore, it has been found that the correct exact specific heat of the Ising model, which is realized by two matrix model,[6,7,8] actually satisfies a different string equation from that of $(m = 3)$ multicritical model.[9,10,11] Therefore, we can easily imagine that the next candidate for the models describing the unitary CFT's are given by generalizing two matrix model[12,13] to n-matrix models.

The solvable n-matrix models are first introduced in Ref. [14] and described by n - $N\times N$ hermitian matrix M_i $(i = 1 \sim n)$ as†

$$\begin{aligned} Z_n &= \int \prod_{i=1}^{n} dM_i \exp\left(-\mathrm{tr}\sum_{i=1}^{n} V_i(M_i) + \mathrm{tr}\sum_{i=1}^{n-1} c_i M_i M_{i+1} \right), \\ V_i(x) &= \frac{1}{2}x^2 + \sum_{j\geq 3} \frac{1}{N^{\frac{j}{2}-1}} \frac{g_{i,j}}{j!} x^j. \end{aligned} \tag{1}$$

The nonperturbative property of these models is studied by Douglas[18] from point of view of the generalized KdV hierarchies (GKH). He claimed that these models are governed by some series of equations

$$\left[Q_+^{\frac{q}{p}}, Q \right] = 1, \tag{2}$$

*The parameter m is related to the general formula for the minimal conformal series as $c = 1 - \frac{6(p-q)^2}{pq}$ with $p = 2,\ q = 2m - 1$.

†In the limit $n \to \infty$ these models become one dimensional string theory (CFT with $c = 1$).[15,16,17] This fact also supports the above expectation.

with p-th order differential operator

$$Q = \partial^p + \sum_{i=2}^{p} a_i \partial^{p-i}. \tag{3}$$

The minimal unitary series is identified with $(p, q) = (m, m+1),\ m \geq 3$. However, we think that these Douglas's argument are still obscure and need further confirmation by solving the models concretely.

In this paper, we will solve three matrix model explicitly, and show that there is a critical point on which the system is actually controlled by the expected equation, eq. (2) with $(p, q) = (4, 5)$. Moreover, we will find another two critical points which are governed by the string equation, eq. (2) with $(p, q) = (2, 7)$ and (3,8).

Let us first summarize general formulae for n-matrix model, which are applicable to asymmetric coupling constants. The method to solve the n-matrix chain model described by eq. (1) with the restricted coupling constants, $c_i = c$ and $g_{i,j} = g_j$ for $^{\forall} i$, has been formulated in [14] by means of orthogonal polynomial method. We give here some formulae for more general coupling constants as in eq. (1). In these model, we can simultaneously diagonalize the matrices M_i as $M_i = U_i^{\dagger} X_i U_i$ by using unitary matrix U_i and the partition function eq. (1) is rewritten by diagonal matrices X_i as[12,13,14)]

$$\begin{aligned} Z_n &= K \int \prod_{i=1}^{n} dX_i \Delta(X_n) \Delta(X_1) \exp\left(-\mathrm{tr} \sum_{i=1}^{n} V_i(X_i) + \mathrm{tr} \sum_{i=1}^{n-1} c_i X_i X_{i+1}\right) &(4) \\ &= K N! \prod_{i=0}^{N-1} h_i, \qquad K = \left(\frac{\pi^n}{\prod_{i=1}^{n-1} c_i}\right)^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} i!^{-1}, &(5) \end{aligned}$$

after integrating out angular matrices U_i. Here $\Delta(X)$ is the Vandermonde's determinant and h_i's are defined by the following conditions of two orthogonal monic polynomials $P^+(x)$ and $P^-(y)$:

$$h_i \delta_{ij} = \int \prod_{k=1}^{n} dx_k P_j^-(x_n) P_i^+(x_1) \exp\left(-\sum_{k=1}^{n} V_k(x_k) + \sum_{k=1}^{n-1} c_k x_k x_{k+1}\right). \tag{6}$$

The free energy of the model, therefore, can be calculated from these constants h_i. In order to obtain these constants h_i, we can use the similar argument with [14]. If we introduce the following notation

$$\mathcal{M}_i^{\pm} f(x) = \int dy\ \exp\left(-V_i^{\pm}(y) + c_i^{\pm} xy\right) f(y), \tag{7}$$

$$\sum_j \left(x\delta_{ij} - \alpha^{\pm}_{(k)\ i,j} \right) \mathcal{M}^{\pm}_k \cdots \mathcal{M}^{\pm}_2 \mathcal{M}^{\pm}_1 P^{\pm}_j(x) = 0 \quad (0 \leq k \leq n-1), \tag{8}$$

$$\sum_j \left(\frac{d}{dx}\delta_{ij} - \beta^{\pm}_{i,j} \right) P^{\pm}_j(x) = 0, \tag{9}$$

where $c^-_{n-i} = c^+_i = c_i$, $V^-_{n+1-i} = V^+_i = V_i$, the expansion coefficients $\alpha^{\pm}_{(i)}$ can be computed from the formulae

$$c^{\pm}_i \alpha^{\pm}_{(i)} = V^{\pm\ \prime}_i(\alpha^{\pm}_{(i-1)}) - c^{\pm}_{i-1}\alpha^{\pm}_{(i-2)} \quad (1 \leq i \leq n-1), \tag{10}$$

with $c^{\pm}_0 \alpha^{\pm}_{(-1)} = \beta^{\pm}$, $[\alpha^{\pm}_{(0)}, \beta^{\pm}] = 1$, and

$$\alpha^-_{(k-1)\ i,j}\ h_j = \alpha^+_{(n-k)\ j,i}\ h_i \qquad (1 \leq k \leq n). \tag{11}$$

In particular, the last equation eq. (11) is most important and include all the informations we need.‡

In the case with Z_2-symmetric coupling constants such that $V_i = V_{n+1-i}$ and $c_i = c_{n-i}$, simplifications occur, because of the fact that we can choose the orthogonal polynomials as $P^+_i(x) = P^-_i(x)$ and then $\alpha^+_{(i)} = \alpha^-_{(i)}$, $\beta^+ = \beta^-$. Only these simpler formulae are needed for our purpose.

Let us apply the general formulae summarized above to the three matrix model with Z_2-symmetric coupling constants,

$$\begin{aligned} Z_3 &= \int dAdBdC\ e^{-S(A,B,C)}, & (12)\\ &= \int dXdYdZ\Delta(X)\Delta(Z)\ e^{-S(X,Y,Z)}, & (13)\\ S(A,B,C) &= \frac{1}{2}\mathrm{tr}(A^2+B^2+C^2) \\ &\qquad + \frac{1}{N}\frac{1}{4!}\mathrm{tr}(gA^4+g'B^4+gC^4) - c\,\mathrm{tr}(AB+BC). & (14) \end{aligned}$$

It should be noted here that two coupling constants g and g' can be different in general. If we set $g' = 0$, B-integration becomes Gaussian and we get the two matrix model after integrating it out. The relation of the coupling constants between two models is $c_{(two)} = c^2/(1-c^2)$ and $g_{(two)} = g/(1-c^2)^2$ in our notation (1). Three matrix model,

‡In these formulae, we assume $n \geq 2$. For the one matrix model case, only one relation $V'(\alpha) = \beta$ is enough to solve it.

therefore, include the two matrix model, *i.e.* the Ising model, for a special case. For another coupling constants, we assume $g \neq 0,\ c \neq 0$ hereafter.

We use a familiar notation for $\alpha_{(0)}$ in the general formulae:

$$xP_i(x) = P_{i+1}(x) + R_i P_{i-1}(x) + S_i P_{i-3}(x) + \cdots . \tag{15}$$

(Note that the absence of terms $P_i,\ P_{i-2}\cdots$ in this recursion relation is a consequence of the fact that $P_i(x)$ has a definite parity $P_i(-x) = (-)^i P_i(x)$ since our action S is quartic (even).) The next coefficients $\alpha_{(1)}$ are calculated from this recursion relation as[14)]

$$c\alpha_{(1)i,j} = \begin{cases} 0 & \text{for } |i-j|>3 \\ \frac{g}{6N} & \text{for } j=i+3 \\ \frac{g}{6N} f_i f_{i-1} f_{i-2} & \text{for } j=i-3 \\ 1+\frac{g}{6N}\left(R_i+R_{i+1}+R_{i+2}\right) & \text{for } j=i+1 \\ f_i\left(1+\frac{g}{6N}\left(R_{i-1}+R_i+R_{i+1}\right)\right) & \text{for } j=i-1, \end{cases} \tag{16}$$

where $f_i = h_i/h_{i-1}$. From the relation (11), we obtain

$$cS_i = \left(\alpha_{(1)} + \frac{g'}{6N}\alpha_{(1)}^3\right)_{i,i-3}, \tag{17}$$

$$c\left(f_i + R_i\right) = \left(\alpha_{(1)} + \frac{g'}{6N}\alpha_{(1)}^3\right)_{i.i-1}, \tag{18}$$

$$R_i\left(1+\frac{g}{6N}\left(R_{i-1}+R_i+R_{i+1}\right)\right) + \frac{g}{6N}\left(S_i+S_{i+1}+S_{i+2}\right) - c\alpha_{(1)i,i-1} = i. \tag{19}$$

These three equations (17)-(19) are sufficient to get the necessary information about the nonperturbative solution.

We now try to find critical points of the model by considering the naive large-N limit. In the naive large-N limit, we have

$$\frac{i}{N} \sim x, \qquad \frac{1}{N}f_i \sim \frac{f(x)}{g}, \qquad \frac{1}{N}R_i \sim \frac{R(x)}{g}, \tag{20}$$

which give the possibility to derive from (18) and (19) the algebraic equations

$$c^4\left(f+R\right) = f\left(1+\frac{R}{2}\right)\left[c^2 + \frac{g'}{g}\left\{\frac{1}{36}f^3 + \frac{1}{12}f^2\left(1+\frac{R}{2}\right) + \frac{1}{2}f\left(1+\frac{R}{2}\right)^2\right\}\right], \tag{21}$$

$$(R-f)\left(1+\frac{R}{2}\right) + \frac{f^3}{12c^4}\left[c^2 + \frac{g'}{g}\left\{\frac{1}{72}f^3 + f\left(1+\frac{R}{2}\right)^2 + \left(1+\frac{R}{2}\right)^3\right\}\right] = gx. \tag{22}$$

Here we have eliminated S_i by using eq. (17).[§] In principle, we can further eliminate R from the lhs of eq. (22) by solving eq. (21). Then we define a "potential" $W(f)$ by the lhs of eq. (22) as the function of f. Since the scaling laws arise from the singular behaviour of $f(x)$ near $x = 1$, the form of $W(f)$ around there control the criticality. Therefore, in order to tune all the couplings g, g' and c on critical, we must adjust it such that the first three derivatives of $W(f)$ vanish

$$W^{'}(f_0) = W^{''}(f_0) = W^{'''}(f_0) = 0. \tag{23}$$

The coupling g is related to W as $g = W(f_0)$. From the practical stand point, however, it is not useful to solve eq. (21) and to treat $W(f)$ as the function of only f. We solve these criticality conditions under the constraint eq. (21) without solving it in the actual calculation.

Although it is a little lengthy, it presents no difficulties to solve the criticality conditions eq. (23) and eq. (21). We find just three different solutions:[¶]

$$\text{i)}: \quad f_0 = -\frac{7}{22},\ R_0 = -\frac{23}{22},\ c^2 = \frac{441}{7216},\ \frac{g'}{g} = \frac{9801}{11767},\ g = -\frac{245}{594}, \tag{24}$$

$$\begin{aligned}\text{ii)}: \quad & f_0 = \frac{1 \pm \sqrt{7}}{4},\ R_0 = \mp\sqrt{7} f_0,\ c^2 = \frac{5 \pm \sqrt{7}}{32}, \\ & \frac{g'}{g} = \frac{32 \pm 10\sqrt{7}}{7},\ g = \frac{50 \pm 35\sqrt{7}}{108},\end{aligned} \tag{25}$$

$$\text{iii)}: \quad f_0 = \frac{1}{2},\ R_0 = -\frac{3}{2},\ c^2 = -\frac{9}{16},\ \frac{g'}{g} = -81,\ g = -\frac{35}{54}. \tag{26}$$

These solutions would yield three different critical points of the model on the sphere. We investigate the critical point i) in detail since it leads to a string equation expected to represent the tricriticl Ising model (the unitary CFT with $c = \frac{7}{10}$).

To keep the higher genus contributions, we must expand eq. (18) and eq. (19) in $1/N$. As in Refs. [1-3], it is convenient to introduce a lattice spacing a so that the renormalized "cosmological" constant is given by $\mu_R = (g_c - g)/g_c a^2$. Then we have

---

[§]It is worthwhile to note that,by using eq. (17), S_i can be eliminated without any approximation.

[¶]Strictly speaking, the solution (24) gives $\frac{0}{0}$ if we put it into eq. (23). This is interpreted as the limit $t \to -7/22$ on a line explained later.

the following expansions,

$$\frac{1}{N}f_{i+l} \sim \frac{f_0}{g}\left(1+\sum_{k\geq 1} a^{\frac{2k}{m}} f_k(z-la^{\frac{1}{m}})\right), \tag{27}$$

$$\frac{1}{N}R_{i+l} \sim \frac{f_0}{g}\left(\frac{R_0}{f_0}+\sum_{k\geq 1} a^{\frac{2k}{m}} r_k(z-la^{\frac{1}{m}})\right), \tag{28}$$

with the scaled variable z

$$\frac{i}{N} \sim 1-a^2 z, \tag{29}$$

and the critical values (24) for couplings. The parameter m is fixed by the order of criticality[2] and $m=4$ in our case. The coefficient function f_1 related to the free energy F as $f_1(z)=F''(z)$. Substituting in eqs.(18) and (19) these expansions, we can find differential equations for f_i and r_i. These differential equations reduce to a single equation for f_1 in the case of all the nonperturbatively solved models so far.[1,2,3,9,19]

In our case, however, situation is a little different. String equation satisfied by f_1 does not become a single equation but two coupled differential equations

$$\begin{aligned} f'''' + 10ff'' + 5f'^2 + 10f^3 + 10r'' + 20fr &= 0, \\ f^{(6)} + 13ff'''' + 24f'f''' + 23f''^2 + 60f^2f'' + 45ff'^2 + 15f^4 & \\ -14r'''' - 70fr'' - 20f'r' + 20f''r + 60r^2 &= \frac{770}{3}z, \end{aligned} \tag{30}$$

where $f^{(i)} = d^i f/dz^i$, $f=f_1$ and $r=r_2-f_2$. The constant 770/3 in front of z can be taken to 1 by appropriate rescaling. We will see next these string equations can be also derived from the GKH stand point.

Before considering about concrete case, let us rewrite eq. (2) more explicit and convenient form. If we write

$$Q = \partial^p + \sum_{i=2}^{p} a_i \partial^{p-i}, \tag{31}$$

$$Q^{\frac{q}{p}} = \partial^q + \sum_{i=2}^{\infty} b_i \partial^{q-i}, \tag{32}$$

and use the fact that $[Q, Q_+^{\frac{q}{p}}] = [Q_-^{\frac{q}{p}}, Q]$, eq. (2) impose[‖]

$$(b_i + \cdots)' = 0 \qquad \text{for} \qquad (q+1 \leq i \leq p+q-2),$$

‖The authors would like to thank T. Eguchi for pointing out this fact.

$$p(b_{p+q-1}+\cdots)' \;=\; 1, \tag{33}$$

where "$\cdots$" represents some differential polynomials of a_j which disappear if all the integration constants are equal to zero.

Since the coefficient functions b_i's are uniquely determined as the differential polynomial of a_i, eqs. (33) become the differential equations of a_i. In particular, we note that the last equation in eqs. (33) is always integrated out to the form of the familiar string equation.

Eqs. (33) are the most general equations without any restriction. If we wish it to have Z_2-symmetry it must be restricted that Q is a self-adjoint operator. This is equivalent to impose Q has a definite parity with respect to $\partial \to -\partial$ when we write it a symmetric form

$$Q = \partial^p + \sum_{i=2}^{p}\left(c_i\partial^{p-i} + \partial^{p-i}c_i\right). \tag{34}$$

Now consider the case $(p,q)=(4,5)$. Q is a 4-th differential operator in this case

$$Q = \partial^4 + (u_2\partial^2 + \partial^2 u_2) + (u_3\partial + \partial u_3) + u_4, \tag{35}$$

and we should take $u_3 = 0$ for Z_2-symmetry. The commutation relation is calculated as

$$\left[Q, Q_+^{\frac{5}{4}}\right] = -4b_6'\partial^2 - (4b_7' + 6b_6'')\partial - (4b_8' + 6b_7'' + 4b_6''' + 2a_2b_6' + a_2'b_6), \tag{36}$$

with

$$32b_6 \;=\; u_2'''' - 10u_2u_2'' - 15u_2'^2 - 10u_2^3 + 10u_4'' + 20u_2u_4, \tag{37}$$

$$\begin{aligned} 128b_8 \;=\;& u_2^{(6)} + 6u_2u_2'''' + 12u_2'u_2''' + 9u_2''^2 + 50u_2^2u_2'' + 50u_2u_2'^2 + 35u_2^4 \\ &+2u_4'''' - 20u_2u_4'' + 20u_2'u_4' + 20u_2''u_4 + 20u_4^2 - 60u_2^2u_4, \end{aligned} \tag{38}$$

and $64b_7 = -32b_6'$. Then we get two coupled differential equations

$$b_6' = 0, \tag{39}$$

$$4b_8' = 1. \tag{40}$$

This coincides with eq. (30) up to integration constants with the identification $f = u_2,\; r = u_4 - u_2^2$.

Let us now study a few properties of string equation eq. (30). If we assume the solution of eq. (30) for $f(z)$ has at most second order moving pole singularity as in the case of Painlevé type equations[9,19)] $f(z)$ and $r(z)$ can be expanded as

$$
\begin{aligned}
f(z) &= \sum_{i=0}^{\infty} c_i (z - z_0)^{i-2}, &(41)\\
r(z) &= \sum_{i=0}^{\infty} d_i (z - z_0)^{i-4}, &(42)
\end{aligned}
$$

and these expansions converge in the vicinity of z_0. The coefficients c_i and d_i are determined such that $f(z)$ and $r(z)$ are the solution of eq. (30). It is known that if we can arrange the number of free parameters in expansions (41) is equal to the order of differential equation, the solution of such a equation has no moving cut singularity in general.[20)]** In our case of eq. (30), expansions (41) must have eight free parameters.[††] The explicit calculation leads, if we choose $c_0 = -2$, eight parameters $z_0,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6,\ c_7$ and c_{10} remain unfixed and the simple pole term vanish, $c_1 = 0$. Therefore our string equation eq. (30) does not admit the solution with any moving branch point.[‡‡]

The facts that $c_0 = -2$ and $c_1 = 0$ are important. Since f is the second derivative of free energy F, this double pole singularity corresponds exactly to logarithmic singularities of the free energy. Thus the partition function $Z = e^F$ is entire in z if and only if the residue c_0 is equal to negative integer and there is no simple pole.

Until now we have analyzed a critical point which is expected to correspond the tricritical Ising model coupled to gravity. We comment here on the behaviour of another two (25) and (26).

**We call Painlevé test to check a nonlinear differential equation has enough number of free parameters.

††This counting comes from the following fact. If we allow the the nonpolynomial differential equation it is possible to eliminate $r(z)$ from eq. (30) by solving the third order algebraic equation. Then the resultant differential equation of f becomes eighth order.

‡‡We also find that this situation is common for all the known string equations. The location of second order pole z_0 is always a free parameter and another ones c_j appear at $j = 6$, for $m = 2$, $j = 2, 5, 8$ for $m = 3$, $j = 2, 4 \leq j \leq 2m-3, 2m-1, 2m+2$ for $m \geq 4$ for the multicritical one matrix model and at $j = 3, 4, 8$ for the Ising model with the choice $c_0 = -2$. In all the cases, we have no simple pole singularity, $c_1 = 0$.

First, consider the case (25). The scaling property on this point governed by the differential equations indexed by $(p, q) = (2, 7)$ of GKH classification. Since it is known that this equation is derived from $m = 4$ multicritical one matrix model, this critical point may be expected to correspond some Yang-Lee type edge singularity described by some non-unitary CFT.

The case (26) needs a little discussion. Although we find this point as third-order critical point (highest order we can tune with three couplings in general), the fourth-order derivative of potential W accidentally vanish on this point so criticality jumps to fourth-order.

From this reason, we must change the lattice spacing expansions to (27) and (28) with $m = 5$ rather than $m = 4$. With this change one can derive the string equation as follows:

$$\begin{aligned}
&4f^{(8)} + 60ff^{(6)} + 180f'f^{(5)} + 402f''f'''' + 306f^2f'''' + 252f'''^2 \\
&+1224ff'f''' + 918ff''^2 + 891f'^2f'' + 630f^3f'' + 945f^2f'^2 + 108f^5 = -420z,
\end{aligned} \tag{43}$$

where $f = f_1$. This differential equation eq. (43) was first found in the analysis of two matrix model with higher order criticality.[21)]

Eq. (43) can be also derived from the GKH with $(p, q) = (3, 8)$. The operator Q in this case is

$$Q = \partial^3 + \frac{3}{4}(u_2\partial + \partial u_2) + u_3, \tag{44}$$

and we set $u_3 = 0$ for Z_2-symmetry. The commutation relation becomes

$$\left[Q, Q_+^{\frac{8}{3}}\right] = -3b_9'\partial - 3(b_{10}' + b_9'') \tag{45}$$

with

$$\begin{aligned}
b_9 &= 0, \\
1296b_{10} &= 4u_2^{(8)} + 60u_2u_2^{(6)} + 180u_2'u_2^{(5)} + 402u_2''u_2'''' + 306u_2^2u_2'''' + 252u_2'''^2 \\
&\quad +1224u_2u_2'u_2''' + 918u_2u_2''^2 + 891u_2'^2u_2'' + 630u_2^3u_2'' + 945u_2^2u_2'^2 + 108u_2^5.
\end{aligned} \tag{46}$$

Therefore GKH give the same equation with eq. (43) up to integration constant. The fact that the criticality jumps by two steps at the point (26) is in accord with the description by GKH. Since there is no GKH equation for $(p,q)=(3,6)$, $([Q,Q_{+}^{\frac{6}{3}}]=0)$, the system has to jump from (3,4) to (3,8).[21]

The Painlevé test for this equation gives a result similar to the other cases. Free parameters are $z_0,\ c_3,\ c_4,\ c_5,\ c_6,\ c_7,\ c_8,\ c_{12}$ with $c_0=-2$ and $c_1=0$.

Finally, we give some speculation about a model identification. We may expect from our result, a critical point (24) realize the tricritical Ising model coupled to gravity. As a fact which support this expectation, we find a critical line connecting between the point (24) and the one of the Ising model.

In the above investigation we have considered the case of all the coupling constants tuned to critical values by imposing the condition (23). If we displace one of these constants off critical, by relaxing the condition (23) to,

$$W'(f_0)=W''(f_0)=0, \tag{47}$$

a line which represents lower criticality can be found:

$$\begin{aligned} &f_0=t,\ R_0=-3t-2,\ c^2=\frac{81t^2}{4(212t+101)},\\ &\frac{g'}{g}=-\frac{729(5t+2)}{t(212t+101)^2},\ g=\frac{10t(187t+91)}{243}. \end{aligned} \tag{48}$$

One can easily see this line connect a critical point (24) $(t=-7/22)$ to that of the Ising model $(t=-2/5)$.

The scaling behaviour of the model on this line is governed by the differential equation

$$f''''+9ff''+\frac{9}{2}f'^2+6f^3=\frac{5(187t+91)}{3(22t+7)}z, \tag{49}$$

except for a point $t=-7/22$. This equation is a string equation for the Ising model derived in Refs. [9,19] if we rescale f and z appropriately. On a point (24) the critical behaviour of the system jumps to one controlled by eq. (30). This criticality structure corresponds to the tricritical Ising model, in which the chemical potential of the lattice vacancy parameterizes the critical line. The detailed analysis of phase structure will be reported in a separate paper.[22]

We also check the positivity of partition function[19] by means of concrete calculation. It is found that each term of genus expansion is positive up to 60-th order.

The correspondence between three matrix model and the tricritical Ising model should be further tested by alternate construction such as that of Kostov.[23]

Acknowledgments

The authors would like to thank T. Eguchi for valuable discussions and reading the manuscript. The authors also would like to acknowledge useful discussions with M. Fukuma, A. Kato, H. Kawai, T. Kawai and K. Ogawa. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan No.63790157 and No.01790191.

References

[1] E. Brézin and V. Kazakov, ENS preprint LPS-ENS-89-175 (1989),

[2] M. Douglas and S. Shenker, Rutgers preprint RU-89-34 (1989),

[3] D. Gross and A. Migdal, Princeton preprint PUTP-1148 (1989).

[4] M. Staudacher, Illinois preprint ILL-(TH)-89-54 (1989).

[5] J. Cardy, *Phys. Rev. Lett.* **54**, 1354 (1985).

[6] M. Bershadsky and A. Migdal, *Phys. Lett.* **B174**, 393 (1986).

[7] V. Kazakov, *Phys. Lett.* **A119**, 140 (1986).

[8] D. Boulatov and V. Kazakov, *Phys. Lett.* **B186**, 369 (1987).

[9] E. Brézin, M. Douglas, V. Kazakov and S. Shenker, Rutgers preprint RU-89-47 (1989).

[10] D. Gross and A. Migdal, *Phys. Rev. Lett.* **64**, 717 (1990).

[11] C. Crnković, P. Ginsparg and G. Moore, Harvard Yale preprint YCTP-P20-89, HUTP-89/A058 (1989).

[12] C. Itzykson and J. Zuber, *J. Math. Phys.* **21**, 411 (1980).

[13] M. Mehta, *Commun. Math. Phys.* **79**, 327 (1981).

[14] S. Chadha, G. Mahoux and M. Mehta, *J. Phys.* **A14**, 579 (1981).

[15] E. Brézin, V. Kazakov and Al.B. Zamolodchkov, ENS preprint LPS-ENS-89-182 (1989).

[16] D. Gross and N. Miljković, Princeton preprint PUPT-1160 (1990).

[17] G. Parisi, Roma Tor Vergata preprint ROM2F-90/2 (1990).

[18] M. Douglas, Rutgers preprint RU-89-51 (1989).

[19] D. Gross and A. Migdal, Princeton preprint PUTP-1159 (1989).

[20] T. Eguchi, private communication.

[21] A. Kato and T. Eguchi, unpublished.

[22] T. Eguchi, A. Kato, H. Kunitomo and S. Odake, in preparation.

[23] I. Kostov, *Nucl. Phys.* **B326**, 583 (1989).