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Two Matrix Model and Minimal Unitary Model *

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Abstract

We investigate the two matrix model with the $\mathbf{Z}_2$ symmetric even potential of degree $2v$. We determine the critical coupling constants, with which the two matrix model realizes the minimal unitary model coupled to two dimensional quantum gravity. By explicit calculation, the type $(p,q)=(m,m+1)$ string equation is obtained for $m=v+1\leq 6$. We also discuss the Douglas' P,Q operators.

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1. To find a true stable string vacuum from among a large number of classical vacua, we must study string theories nonperturbatively. Concerning the "space-time dimension" less than 1, the matrix model and the double scaling limit technique allow us the nonperturbative study of "string" theories[1]. One matrix model realizes $2d$ gravity coupled to $(p, q) = (2, 2m - 1)$ minimal conformal matter, and its string equation and correlation functions are closely related to the KdV hierarchy[1]. Douglas pointed out that more general (p, q) minimal conformal matter coupled to $2d$ gravity can be realized by $p - 1$ matrix model, and its string equation is related to the generalized KdV hierarchy (p-reduction of the KP hierarchy)[3]. In fact, $(p, q) = (3, 4)$ and $(4, 5)$ unitary minimal matters are realized by the two and three matrix models with quartic potentials respectively[4, 5]. Relations between $p - 1$ matrix model and W_p algebra, topological field theory were also pointed out[6].

Recently Tada and Yamaguchi studied the two matrix model with a sixth order potential and found that not only $(p, q) = (3, 8)$ minimal conformal model but also $(p, q) = (4, 5)$ minimal unitary model are realized[7]. They conjectured that the two matrix model with a higher order potential realizes all the minimal unitary models. Moreover Douglas discussed all (p, q) minimal conformal models can be realized by the two matrix model[8][1].

In this letter we will investigate the two matrix model with the $\mathbf{Z}_2$ symmetric even potential and try to show the Tada-Yamaguchi conjecture. By the method of [7], we present the general expression for the critical coupling constants that correspond to minimal unitary model, and express the Douglas' P, Q operators by the orthogonal polynomial method. Since the matrix model has a definite normalization, we will keep multiplicative constants carefully.

2. We will consider the two matrix model with the $\mathbf{Z}_2$ symmetric even potential of degree $2v$, whose partition function is

$$\mathcal{Z} = e^{\mathcal{F}} = \int dM_+ dM_- e^{-S}, \qquad S = \mathrm{tr}(V(M_+) + V(M_-) - cM_+M_-), \tag{1}$$

[1] After completion of our calculation, we received ref.[9], where Tada also conjectured the same statement and obtained $(p, q) = (3, 5)$ model by explicit calculation.

$$V(x) = \sum_{j=1}^{v} \frac{1}{N^{j-1}} \frac{g_{2j}}{(2j)!} x^{2j}, \qquad g_2 = 1, \tag{2}$$

where $M_\pm$ are $N \times N$ hermitian matrices. Introducing orthogonal monic polynomials $P_i(x)$ of degree i[2]

$$h_i \delta_{ij} = \int dx dy P_j(y) e^{-V(x)-V(y)+cxy} P_i(x), \quad F_i = \frac{h_i}{h_{i-1}}, \tag{3}$$

the partition function is expressed by F_i's. Polynomials P_i satisfy the following recursion relation:

$$x P_i(x) = \sum_j \alpha_{ij} P_j(x) = \sum_{k=0}^{v} R_i^{[k]} P_{i+1-2k}(x), \quad R_i^{[0]} = 1, \tag{4}$$

where $R^{[k]}$'s are unknown coefficients[2] and matrix α is defined by this equation. Depending on the potential, this recursion relation contains a number of adjustable coefficients. This is the reason why the two matrix model can realize the (p, q) minimal conformal model for any p. In the case of the one matrix model with even potential, the recursion relation contains only one coefficient. Therefore the one matrix model can realize $p = 2$ minimal conformal model only. In the case of the n-matrix model ($n \geq 3$), α is a general lower triangular matrix.

All the necessary information for solving the two matrix model is contained in the following equations:

$$cR_i^{[k]} = F_i F_{i-1} \cdots F_{i+2-2k} V'(\alpha)_{i+1-2k,i}, \qquad (\frac{v}{2} < k \leq v), \tag{5}$$

$$cR_{i+k-1}^{[k]} = F_{i+k-1} F_{i+k-2} \cdots F_{i+1-k} V'(\alpha)_{i-k,i+k-1}, \quad (1 \leq k \leq \frac{v}{2}), \tag{6}$$

$$i = V'(\alpha)_{i,i-1} - cF_i. \tag{7}$$

We first consider the naive large N limit, where F and $R^{[k]}$ are scaled as follows:

$$\frac{i}{N} \sim x, \quad \frac{1}{N} F_i \sim F(x), \quad \frac{1}{N^k} R_i^{[k]} \sim R^{[k]}(x), \quad (1 \leq k \leq v). \tag{8}$$

Let us denote the values of F and $R^{[k]}$ at $x = 1$ by

$$F(1) = f_0, \quad R^{[k]}(1) = f_0^k r_0^{[k]}, \quad (1 \leq k \leq v). \tag{9}$$

After eliminating $R^{[k]}$, we define a potential $W(F)$ by the right hand side of eq. (7) divided by N. Since the scaling laws arise from the singular behavior of $F(x)$ in the

[2]Strictly speaking, the upper bound of sum over k is $\min(v, [\frac{i+1}{2}])$.

vicinity of $x = 1$, the potential W takes the form $W(F) - 1 \propto (F - f_0)^m$ near $x = 1$ at the m-th order critical point. So, the m-th order critical coupling constants c and g_{2j} are determined by the following requirement:

$$W^{(k)}(f_0) = 0, \quad (1 \leq k \leq m - 1), \tag{10}$$

and $W(f_0) = 1$. If we solve these equations completely for each m, we can draw the phase diagram of the two matrix model. In actual calculation it is convenient to differentiate $W(F)$ under the constraints instead of solving $R^{[k]}$ explicitly. However, this calculation is very hard for large v. So we give up finding out all the solutions and try to find some of the critical points.

Instead of eq. (10), we require

$$d_n = 0, \quad (1 \leq n \leq p - 1), \tag{11}$$

where d_n's are defined by

$$d_n = \frac{1}{n!} \sum_{k=0}^{v} (2k - 1)^n r_0^{[k]}. \tag{12}$$

These equations with $p = m$ are sufficient conditions for eq. (10), and the meaning of them will be explained in the next section. Naive parameter counting shows that the maximum value of m is $v + 1$[3]. In the following we will concentrate our attention on this case, i.e. $m = p = v + 1$. Then eq. (11) is solved as follows:

$$r_0^{[k]} = \frac{(-1)^{k+1}}{2k - 1} \binom{v}{k}, \quad (0 \leq k \leq v). \tag{13}$$

Using these values, d_0, d_m, d_{m+1} are given by

$$d_0 = \frac{(2v)!!}{(2v - 1)!!}, \quad d_m = \frac{(-1)^{v+1}}{v + 1} 2^v, \quad d_{m+1} = \frac{(-1)^{v+1}}{v + 2} (v - 1) 2^v. \tag{14}$$

After some combinatorics, eqs. (5,6,7) become

$$r_0^{[k]} + \sum_{j=k}^{v} (-1)^j g'_{2j} \oint_0 \frac{dt}{2\pi i} \frac{1}{t^{j-k+1}} \left(\sum_{i=0}^{v} r_0^{[i]} t^i \right)^{2j-1} = 0, \quad (0 \leq k \leq v), \tag{15}$$

[3]In general there are exceptional critical coupling constants which give $m > v + 1$.

where $g'_{2j} = \frac{(-f_0)^{j-1}}{(2j-1)!c} g_{2j}$ $(2 \leq j \leq v)$, $g'_2 = \frac{1}{c}$ and $g'_0 = \frac{1}{cf_0}$. From this recursion relation, one can obtain the critical coupling constants g'_{2j} $(j = v, v-1, \cdots, 0)$ easily; for example,

$$
\begin{aligned}
g'_{2v} &= 1/(2v-1), \\
g'_{2(v-1)} &= 2v(v-1)/(2v-3), \\
g'_{2(v-2)} &= v(v-1)(6v^2-14v-1)/3(2v-5), \\
g'_{2(v-3)} &= 2v(v-1)(10v^4-60v^3+83v^2+16v+1)/15(2v-7), \\
g'_{2(v-4)} &= v(v-1)(420v^6-4620v^5+16184v^4-16180v^3 \\
&\qquad -5953v^2-831v-45)/630(2v-9), \\
g'_{2(v-5)} &= v(v-1)(252v^8-4368v^7+27468v^6-71808v^5 \\
&\qquad +53797v^4+33596v^3+7763v^2+886v+42)/945(2v-11).
\end{aligned}
\tag{16}
$$

Next we consider the double scaling limit in order to include the contributions from the higher genus Riemann surfaces. As refs.[1], we introduce a lattice spacing constant a such that the continuum limit is $a \to 0$. The renormalized cosmological constant is $\mu_R = (\mu_B - \mu_{cri})/a^2$, i.e. $\mu_R = \frac{g_{2j}^{cri} - g_{2j}}{(j-1) g_{2j}^{cri} a^2}$. Introducing the scaled variable z,

$$\frac{i}{N} \sim 1 - a^2 z, \quad \frac{1}{N} \sim \tilde{a}^{2m+1}, \quad a = \tilde{a}^m, \tag{17}$$

F and $R^{[k]}$ have the following expansions:

$$\frac{1}{N} F_{i+l} \sim f_0 (1 + \sum_{n \geq 1} \tilde{a}^{2n} f_{2n}(z - l\tilde{a})) \tag{18}$$

$$\frac{1}{N^k} R^{[k]}_{i+k-1+l} \sim f_0^k (r_0^{[k]} + \sum_{n \geq 1} \tilde{a}^{2n} r_{2n}^{[k]}(z - l\tilde{a})), \quad (1 \leq k \leq \frac{v}{2}). \tag{19}$$

$R_i^{[k]}$ $(\frac{v}{2} < k \leq v)$ have been exactly eliminated in terms of F_i and $R_i^{[k]}$ $(1 \leq k \leq \frac{v}{2})$ by using eq. (5). The function $f_2(z)$ is related to the free energy as $\frac{d^2}{d\mu_R^2} \mathcal{F} = f_2(z)|_{z=\mu_R}$. Substituting these expansions into eqs. (6,7) and solving them order by order in $\tilde{a}$, we obtain a coupled nonlinear differential equation, so-called string equation. This calculation is tedious but straightforward. We expect that this string equation agrees with the type $(p, q) = (m, m+1)$ equation of the KP hierarchy (q-th flow of p-reduction of the KP hierarchy). In fact we will check this conjecture for $m \leq 6$ in section 4.

3. To compare the string equation obtained in the previous section and the string equation in Douglas' form[3], we introduce the normalized polynomials $\mathcal{P}_i(x) = \frac{1}{\sqrt{h_i}} P_i(x)$.

We define matrices $\tilde{Q}$ and $\tilde{P}$ as follows:

$$x\mathcal{P}_i(x) = \sum_j \tilde{Q}_{ij}\mathcal{P}_j(x), \quad \frac{d}{dx}\mathcal{P}_i(x) = \sum_j \tilde{P}_{ij}\mathcal{P}_j(x). \tag{20}$$

$\tilde{Q}$ and $\tilde{P}$ satisfy the commutation relation

$$[\tilde{Q}, \tilde{P}] = 1, \tag{21}$$

and this equation reduces to $[{}^t\tilde{Q}, \tilde{Q}] = \frac{1}{c}$, because $\tilde{P}$ is expressed as $\tilde{P} = -c\,{}^t\tilde{Q} + V'(\tilde{Q})$, where ${}^t\tilde{Q}$ is a transposed matrix of $\tilde{Q}$. In the double scaling limit, the matrix $\tilde{Q}$ is scaled as follows[7]:

$$\tilde{Q}_{ij} = \sqrt{F_{i+1}}\delta_{i+1,j} + \sum_{k=1}^{v} R_i^{[k]}(F_iF_{i-1}\cdots F_{i+2-2k})^{-\frac{1}{2}}\delta_{i+1-2k,j}, \tag{22}$$

$$\frac{1}{\sqrt{N}}\tilde{Q}_{ij} \sim \sqrt{\frac{F_{i+1}}{N}}e^{-\tilde{a}\partial} + \sum_{k=1}^{v}\frac{R_i^{[k]}}{N^k}\left(\frac{F_i}{N}\frac{F_{i-1}}{N}\cdots\frac{F_{i+2-2k}}{N}\right)^{-\frac{1}{2}}e^{(2k-1)\tilde{a}\partial} \tag{23}$$

$$\stackrel{\text{def}}{=} \sqrt{f_0}\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n, \tag{24}$$

where $\frac{F}{N}$ and $\frac{R^{[k]}}{N^k}$ have the expansions eqs. (18,19) and ∂ represents $\frac{\partial}{\partial z}$. $\tilde{Q}_n$ is a differential operator of order n, and its coefficient of ∂^n is d_n. Therefore eq. (11) is the necessary condition for the vanishing of $\tilde{Q}_n$ $(1 \leq n \leq p-1)$. In the next section we will see that it is also sufficient. For later use, we rewrite $\tilde{Q}_n$ as $\tilde{Q}_n = d_nQ_n$. Expansion of the transposed matrix ${}^t\tilde{Q}$ becomes $\frac{1}{\sqrt{N}}({}^t\tilde{Q})_{ij} \sim \sqrt{f_0}\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n^\dagger$, where $\dagger$ stands for a formal adjoint of a (pseudo)differential operator. On the other hand ${}^t\tilde{Q}$ has the following expansion:

$$\frac{1}{\sqrt{N}}({}^t\tilde{Q})_{ij} \sim \sqrt{\frac{F_i}{N}}e^{\tilde{a}\partial} + \sum_{k=1}^{v}\frac{R_{i+2k-1}^{[k]}}{N^k}\left(\frac{F_{i+2k-1}}{N}\frac{F_{i+2k-2}}{N}\cdots\frac{F_{i+1}}{N}\right)^{-\frac{1}{2}}e^{-(2k-1)\tilde{a}\partial}. \tag{25}$$

Comparing eq. (23) and eq. (25), we obtain

$$\sum_{n\geq 0}\tilde{a}^n\tilde{Q}_n^\dagger(z) = \sum_{n\geq 0}(-\tilde{a})^n\tilde{Q}_n(z-\tilde{a}), \tag{26}$$

where we write the z-dependence of the coefficients of ∂ explicitly. This is a consequence of the $\mathbf{Z}_2$ symmetry. If $\tilde{Q}_n = 0$ $(1 \leq n \leq m-1)$, then, by using the above equation, we can show that

$$\tilde{Q}_m^\dagger = (-1)^m\tilde{Q}_m, \quad \tilde{Q}_{m+1}^\dagger = (-1)^{m+1}(\tilde{Q}_{m+1} + \tilde{Q}_m'), \tag{27}$$

where $'$ represents a differentiation, i.e. $A' = \sum_n a'_n \partial^n$ for $A = \sum_n a_n \partial^n$. Under the condition $\tilde{Q}_n = 0$ $(1 \leq n \leq m-1)$, by substituting these expansions into eq. (21), i.e. $[\frac{1}{\sqrt{N}}\tilde{Q}, \frac{1}{\sqrt{N}}\tilde{P}] = \frac{1}{N} \sim \tilde{a}^{2m+1}$, we obtain

$$[\tfrac{1}{2}(Q_{m+1} + (-1)^{m+1}Q^\dagger_{m+1}),\ Q_m] = \frac{(-1)^{m+1}g'_0}{2d_m d_{m+1}}. \tag{28}$$

This commutation relation suggests the following identification of the Douglas' P, Q operators with Q_n's which are obtained from the orthogonal polynomial method,

$$Q = Q_m, \quad P = \frac{1}{2}(Q_{m+1} + (-1)^{m+1}Q^\dagger_{m+1}) = Q_{m+1} + \frac{d_m}{2d_{m+1}}Q'_m. \tag{29}$$

We remark that Q is the order $\tilde{a}^m$ term of $\tilde{Q}$ but P is not the order $\tilde{a}^{m+1}$ term of $\tilde{P}$. This form of P was conjectured in [7]. Q and P have definite parity $Q^\dagger = (-1)^m Q$, $P^\dagger = (-1)^{m+1}P$ due to the $\mathbf{Z}_2$ symmetry.

Let us recall the type (p,q) string equation in Douglas form[3]. Q is a differential operator of order p and $P = Q^{\frac{q}{p}}{}_+$:

$$Q = \partial^p + \sum_{n=2}^{p-1}\{\partial^{p-n}, c_n\} + c_p, \quad Q^{\frac{q}{p}} = \partial^q + \sum_{n\geq 2} b_n \partial^{q-n}. \tag{30}$$

In the case of the $\mathbf{Z}_2$ symmetric potential, c_{odd} vanishes, and Q and P have definite parity. Setting all the integration constants equal to zero, the string equation in Douglas form ($[P,Q] = \text{const}$) is equivalent to[5]

$$b_n = 0, \quad (q+1 \leq n \leq p+q-2), \quad b_{p+q-1} = \frac{\text{const}}{p} z. \tag{31}$$

In our case, $p = m = v+1$, $q = m+1$, const $= \frac{(-1)^{m+1}g'_0}{2d_m d_{m+1}}$. In the case of the $\mathbf{Z}_2$ symmetric potential, $b_{odd} = 0$ can be derived from other equations, and unknown functions are c_{2n} $(1 \leq n \leq [\frac{p}{2}])$[4].

4. In §2,3, we have argued that the two matrix model with the critical coupling constants eqs. (15,16) and eq. (13), gives the type $(p,q) = (m, m+1)$ string equation. However there are some unproved facts. They are

(i) $\tilde{Q}_n = 0, \quad (1 \leq n \leq m-1)$.

[4]In our notation, suffixes of $f_n, r^{[k]}_n, \tilde{Q}_n, Q_n, c_n, b_n, u_n$ represent "weight" n. ∂ has weight 1.

(ii) $\frac{1}{2}(Q_{m+1}+(-1)^{m+1}Q^\dagger_{m+1}) = (Q_m)^{\frac{m+1}{m}}{}_+$.

(iii) the string equation derived from eqs. (6,7) agrees with the string equation in Douglas form eq. (31) with $p = m = v+1 = q-1$, const $= \dfrac{(-1)^{m+1}g'_0}{2d_m d_{m+1}}$.

We only consider the case $v = m-1 \geq 2$, because $v = 1$ case is trivial. For $m \leq 6$, we have checked (i)(ii)(iii) by explicit calculation, and we are convinced that (i)(ii)(iii) hold for all m. Here we give only the relations between f_{2n}, $r^{[k]}_{2n}$ and c_{2n}, because the string equation has a lengthy expression for large v, and the critical coupling constants can be easily obtained from eq. (16). They are

$$v=2: \qquad c_2 = \tfrac{3}{4}f_2, \tag{32}$$

$$v=3: \qquad c_2 = f_2, \quad c_4 = (r^{[1]}_4 - f_4) + f_2^2, \tag{33}$$

$$v=4: \qquad c_2 = \tfrac{5}{4}f_2, \quad c_4 = \tfrac{5}{8}((r^{[1]}_4 - f_4) + \tfrac{3}{2}f_2^2 - \tfrac{1}{2}f''_2), \tag{34}$$

$$v=5: \qquad c_2 = \tfrac{3}{2}f_2, \quad c_4 = \tfrac{3}{4}((r^{[1]}_4 - f_4) + 2f_2^2 - \tfrac{4}{3}f''_2), \tag{35}$$

$$c_6 = \tfrac{3}{8}((r^{[1]}_6 + r^{[2]}_6 + 3f_6) + \tfrac{7}{3}(r^{[1]}_4 - f_4)'' + 5(r^{[1]}_4 - f_4)f_2 + \tfrac{4}{3}f''''_2 - \tfrac{8}{3}f'^2_2 + \tfrac{8}{3}f_2^3). \tag{36}$$

c_2 is related to f_2 as $c_2 = \frac{m}{4}f_2$ for all m.

5. In order to realize the general (p,q) minimal conformal matter, we must consider the $\mathbf{Z}_2$ asymmetric action[8, 9]. In this case $P^\pm_i(x)$, $\alpha^\pm$, $R^{[k]\pm}_i$, $d^\pm_n$, etc. are introduced. Corresponding to eq. (11), it is expected that the critical coupling constants of type (p,q) are obtained by the requirement that d^+_n should vanish for $1 \leq n \leq p-1$ and d^-_n should vanish for $1 \leq n \leq q-1$, and the string equation is derived from $[{}^t\tilde{Q}^-, \tilde{Q}^+] = \frac{1}{c}$. One can change p by adjusting c and V^-, and q by V^+. From the parameter counting, minimal values of v^- and v^+ are $p-1$ and q respectively, but in contrast to the unitary case, the coupling constants are not uniquely determined.

One point functions of $\mathrm{tr}M^{2n}_\pm$ and $\mathrm{tr}M_+M_-$ are also calculable by the orthogonal polynomial method. If we take appropriate linear combinations, the first derivative of them with respect to μ_R show scaling behavior. We can find such linear combinations by brute force for small v, but we have not succeeded in finding the general expressions.

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