UT-569
October 1990

$W_{1+\infty}$ and Super W_∞ Algebras with $SU(N)$ Symmetry

Satoru Odake* and Takashi Sano†

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

We extend the $W_{1+\infty}$ algebra proposed by Pope, Roman and Shen, by adding the affine $\widehat{su}(N)$ algebra with level k. The central charge c of this algebra, denoted by $\widehat{su}(N)_k$-$W_{1+\infty}$, turns out to be $c = Nk$. We also obtain the supersymmetric extension whose bosonic sector is $W_\infty \oplus \widehat{su}(N)_k$-$W_{1+\infty}$. These algebras can be realized in terms of bilinears of free fermions and bosons.

*odake@tkyvax.hepnet, Fellow of the Japan Society for the Promotion of Science.
†sano@tkyvax.hepnet.

The Virasoro algebra is the infinite dimensional Lie algebra which plays a central role in two-dimensional conformal field theories. For some years several extensions of the Virasoro algebra have been attempted. The typical one is Zamolodchikov's W_N algebra, which contains fields of conformal spin $2, \cdots, N$ [1]. But, for $N \geq 3$, W_N doesn't have the structure of Lie algebra, on account of appearance of non-linear terms. Recently new extensions of the Virasoro algebra have been considered, which can be regarded as large N limit of W_N algebra. These extended algebras have fields of arbitrary conformal spin l, $l \geq 2$, and are Lie algebras. One example is the w_∞ algebra[2], which is defined in the following way:

$$[w_m^{(i)}, w_n^{(j)}] = ((j-1)m - (i-1)n) w_{m+n}^{(i+j-2)}. \tag{1}$$

The generator $w_m^{(i)}$ has conformal spin i, and $w_m^{(2)}$ corresponds to the Virasoro generator L_m. The w_∞ algebra can be naturally interpreted as the algebra of area-preserving diffeomorphisms of 2-surfaces. But unfortunately this algebra admits central extensions only in the Virasoro sector, so all the higher-spin generators act on the physical states trivially. This situation spoils physical interest. In some recent papers, Pope, Romans, Shen etc. have constructed and studied the new algebra W_∞[3, 6], which admits central extensions for all sectors of arbitrary conformal spin. In addition, they constructed the $W_{1+\infty}$ algebra[4], which contains the conformal spin 1 field, and super W_∞, and their realizations[5, 7, 8].

In this letter we extend the $W_{1+\infty}$ algebra by adding the $SU(N)$ Kac-Moody algebra. Additional generators transform according to the adjoint representation of $su(N)$. We also construct its supersymmetric extension, whose fermionic generators transform according to the $\mathbf{N}$ and $\bar{\mathbf{N}}$ representations of $su(N)$. These algebras admit free field realizations and spectral flow invariance.

In ref. [4], Pope, Romans and Shen constructed the $W_{1+\infty}$ algebra, which contains conformal spin $i+2$ fields $V^i(z)$ ($i \geq -1$) (not primary in general). We first consider adding $SU(N)$ symmetry to this algebra. We take t^a ($a = 1, \cdots, N^2-1$) as basis of $su(N)$, which are traceless hermitian $N \times N$ matrices with normalization $\mathrm{tr} t^a t^b = \delta^{ab}$.

We define the structure constants f^{abc} and the d^{abc} symbol as follows:

$$[t^a, t^b] = if^{abc}t^c, \quad \mathrm{tr}\{t^a, t^b\}t^c = 2d^{abc}. \tag{2}$$

The t^a's satisfy the completeness condition $t^a_{\alpha\beta}t^a_{\gamma\delta} + \frac{1}{N}\delta_{\alpha\beta}\delta_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma}$. In addition to $V^i(z)$, we introduce other fields $W^{i,a}(z)$ $(i \geq -1)$, which have conformal spin $i+2$. As usual, mode expansion of a field $A(z)$ with conformal spin h is given by $A(z) = \sum A_n z^{-n-h}$.

We require that $J^a(z) \equiv 4qW^{-1,a}(z)$ [1] should generate the affine $\widehat{su}(N)$ algebra with level k and $W^{i,a}_m$ should transform as members of the adjoint representation of $su(N)$:

$$\begin{aligned} [J^a_m, J^b_n] &= if^{abc}J^c_{m+n} + \delta^{ab}\delta_{m+n,0}km, &(3)\\ [J^a_0, W^{j,b}_n] &= if^{abc}W^{j,c}_n. &(4) \end{aligned}$$

Then the commutators between V^i_m and $W^{i,a}_m$ are all determined by the Jacobi identity. We call this Lie algebra as $\widehat{su}(N)_k$-$W_{1+\infty}$ algebra and its commutation relations are as follows:

$$\begin{aligned} [V^i_m, V^j_n] &= \sum_{r\geq 0, even} q^r g^{ij}_r(m,n)V^{i+j-r}_{m+n} + \delta^{ij}\delta_{m+n,0}q^{2i}c_i(m), &(5)\\ [V^i_m, W^{j,a}_n] &= \sum_{r\geq 0, even} q^r g^{ij}_r(m,n)W^{i+j-r,a}_{m+n}, &(6)\\ [W^{i,a}_m, W^{j,b}_n] &= \tfrac{1}{2}if^{abc}\sum_{r\geq -1, odd} q^r g^{ij}_r(m,n)W^{i+j-r,c}_{m+n} + \delta^{ij}\delta^{ab}\delta_{m+n,0}q^{2i}k_i(m) \\ &\quad + \sum_{r\geq 0, even} q^r g^{ij}_r(m,n)(d^{abc}W^{i+j-r,c}_{m+n} + \tfrac{1}{N}\delta^{ab}V^{i+j-r}_{m+n}). &(7) \end{aligned}$$

The structure constants are given by[7]

$$\begin{aligned} g^{ij}_r(m,n) &= \tfrac{1}{2(r+1)!}\phi^{ij}_r(0,-\tfrac{1}{2})N^{i,j}_r(m,n), &(8)\\ N^{x,y}_r(m,n) &= \sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k} \\ &\quad \times[x+1+m]_{r+1-k}[x+1-m]_k[y+1+n]_k[y+1-n]_{r+1-k}, &(9)\\ \phi^{ij}_r(x,y) &= {}_4F_3\Big[\begin{matrix} -\frac{1}{2}-x-2y, \frac{3}{2}-x+2y, -\frac{r+1}{2}+x, -\frac{r}{2}+x \\ -i-\frac{1}{2}, -j-\frac{1}{2}, i+j-r+\frac{5}{2}\end{matrix};1\Big], &(10) \end{aligned}$$

[1] q is a parameter and we will fix $q = \frac{1}{4}$ in this letter.

$$
{}_4F_3\Big[\begin{array}{c} a_1,a_2,a_3,a_4 \\ b_1,b_2,b_3 \end{array};z\Big]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n}\frac{z^n}{n!}\ , \tag{11}
$$

where $[x]_n = x(x-1)\cdots(x-n+1)$ and $(x)_n = x(x+1)\cdots(x+n-1)$. The central terms are given by

$$
c_i(m) = c_i \prod_{j=-i-1}^{i+1}(m+j) \quad , \qquad k_i(m) = k_i \prod_{j=-i-1}^{i+1}(m+j), \tag{12}
$$

$$
c_i = \frac{2^{2i-2}((i+1)!)^2}{(2i+1)!!(2i+3)!!}c \quad , \qquad k_i = \frac{2^{2i-2}((i+1)!)^2}{(2i+1)!!(2i+3)!!}k. \tag{13}
$$

The relation between c (central charge of the Virasoro generator $V^0(z)$) and k (level of $J^a(z)$) is also determined by the Jacobi identity:

$$
c = Nk, \tag{14}
$$

so c takes integer value for any unitary representation. In the case of $N=1$, $\widehat{su}(N)_k$-$W_{1+\infty}$ reduces to $W_{1+\infty}$. Instead of $V^i(z)$ and $W^{i,a}(z)$, we can take another basis $W^{i,(\alpha\beta)}(z)$ $(i \geq -1)$, which are defined by

$$
W^{i,(\alpha\beta)}(z) = \tfrac{1}{N}\delta^{\alpha\beta}V^i(z) + t^a_{\beta\alpha}W^{i,a}(z). \tag{15}
$$

In this basis, $V^i(z)$ and $W^{i,a}(z)$ are expressed as $V^i(z) = W^{i,(\alpha\alpha)}(z)$ and $W^{i,a}(z) = t^a_{\alpha\beta}W^{i,(\alpha\beta)}(z)$, and eqs. (5)-(7) become

$$
\begin{aligned}
[W^{i,(\alpha\beta)}_m, W^{j,(\gamma\delta)}_n] \ &= \ \tfrac{1}{2}\sum_{r\geq -1} q^r g^{ij}_r(m,n)(\delta^{\beta\gamma}W^{i+j-r,(\alpha\delta)}_{m+n} + (-1)^r\delta^{\alpha\delta}W^{i+j-r,(\gamma\beta)}_{m+n}) \\
&\quad +\delta^{ij}\delta^{\alpha\delta}\delta^{\beta\gamma}\delta_{m+n,0}q^{2i}k_i(m).
\end{aligned} \tag{16}
$$

Calculations are sometimes easier when we take this basis. We remark that above commutation relations are compatible with the hermiticity properties of the generators: $V^{i\dagger}_n = V^i_{-n}$, $W^{i,a\dagger}_n = W^{i,a}_{-n}$ $(W^{i,(\alpha\beta)\dagger}_n = W^{i,(\beta\alpha)}_{-n})$.

Next we consider the super extension of $\widehat{su}(N)$-$W_{1+\infty}$. As in ref. [7], we must introduce the W_∞ algebra, which is generated by conformal spin $i+2$ fields $\tilde{V}^i(z)$ $(i \geq 0)$, and consequently the super $\widehat{su}(N)_k$-W_∞ algebra contains the $\widehat{su}(N)_k$-$W_{1+\infty}$ and the W_∞ algebras as bosonic parts. In addition we introduce the fermionic generators

$G^{i,\alpha}(z)$ and $\bar{G}^{i,\alpha}(z)$ $(i \geq 0, \alpha = 1, \cdots, N)$ whose conformal spin is $i + \frac{3}{2}$, and we require that $G^{i,\alpha}_m$ and $\bar{G}^{i,\alpha}_m$ should transform as members of the $\mathbf{N}$ and $\bar{\mathbf{N}}$ representations of $su(N)$:

$$[J^a_0, G^{j,\alpha}_n] = -t^a_{\alpha\beta} G^{j,\beta}_n, \tag{17}$$

$$[J^a_0, \bar{G}^{j,\alpha}_n] = t^a_{\beta\alpha} \bar{G}^{j,\beta}_n. \tag{18}$$

$G^{i,\alpha}_m$ and $\bar{G}^{i,\alpha}_m$ have $U(1)$ charges [2] 1 and -1 respectively. This situation is a generalization of the N-extended superconformal algebras which were simultaneously discovered by Bershadsky and Knizhnik [9]. The (anti-)commutators are all determined by the Jacobi identity, and the super $\widehat{su}(N)_k$-W_∞ algebra is defined by eqs. (5)-(7) and

$$[\tilde{V}^i_m, \tilde{V}^j_n] = \sum_{r \geq 0, even} q^r \tilde{g}^{ij}_r(m,n) \tilde{V}^{i+j-r}_{m+n} + \delta^{ij} \delta_{m+n,0} q^{2i} \tilde{c}_i(m), \tag{19}$$

$$[V^i_m, G^{j,\alpha}_n] = \sum_{r \geq -1} q^r a^{ij}_r(m,n) G^{i+j-r,\alpha}_{m+n}, \tag{20}$$

$$[V^i_m, \bar{G}^{j,\alpha}_n] = \sum_{r \geq -1} q^r (-1)^r a^{ij}_r(m,n) \bar{G}^{i+j-r,\alpha}_{m+n}, \tag{21}$$

$$[W^{i,a}_m, G^{j,\alpha}_n] = t^a_{\alpha\beta} \sum_{r \geq -1} q^r a^{ij}_r(m,n) G^{i+j-r,\beta}_{m+n}, \tag{22}$$

$$[W^{i,a}_m, \bar{G}^{j,\alpha}_n] = t^a_{\beta\alpha} \sum_{r \geq -1} q^r (-1)^r a^{ij}_r(m,n) \bar{G}^{i+j-r,\beta}_{m+n}, \tag{23}$$

$$[\tilde{V}^i_m, G^{j,\alpha}_n] = \sum_{r \geq -1} q^r \tilde{a}^{ij}_r(m,n) G^{i+j-r,\alpha}_{m+n}, \tag{24}$$

$$[\tilde{V}^i_m, \bar{G}^{j,\alpha}_n] = \sum_{r \geq -1} q^r (-1)^r \tilde{a}^{ij}_r(m,n) \bar{G}^{i+j-r,\alpha}_{m+n}, \tag{25}$$

$$\begin{aligned} \{G^{i,\alpha}_m, \bar{G}^{j,\beta}_n\} &= \delta^{\alpha\beta} \sum_{r \geq 0} q^r \left(\tfrac{1}{N} b^{ij}_r(m,n) V^{i+j-r}_{m+n} + \tilde{b}^{ij}_r(m,n) \tilde{V}^{i+j-r}_{m+n}\right) \\ &\quad + t^a_{\alpha\beta} \sum_{r \geq 0} q^r b^{ij}_r(m,n) W^{i+j-r,a}_{m+n} + \delta^{ij} \delta^{\alpha\beta} \delta_{m+n,0} q^{2i} \check{c}_i(m), \end{aligned} \tag{26}$$

$$[V^i_m, \tilde{V}^j_n] = [W^{i,a}_m, \tilde{V}^j_n] = \{G^{i,\alpha}_m, G^{j,\beta}_n\} = \{\bar{G}^{i,\alpha}_m, \bar{G}^{j,\beta}_n\} = 0. \tag{27}$$

The structure constants are essentially the same as those of super W_∞ of [7]:

$$\tilde{g}^{ij}_r(m,n) = \tfrac{1}{2(r+1)!} \phi^{ij}_r(0,0) N^{i,j}_r(m,n), \tag{28}$$

$$a^{ij}_r(m,n) = \tfrac{(-1)^r}{4(r+2)!} ((i+1)\phi^{ij}_{r+1}(0,0) - (i-r-1)\phi^{ij}_{r+1}(0,-\tfrac{1}{2})) N^{i,j-\frac{1}{2}}_r(m,n), \tag{29}$$

[2] $U(1)$ current is defined by $J(z) \equiv -4qV^{-1}(z)$.

$$\tilde{a}_r^{ij}(m,n) = \tfrac{-1}{4(r+2)!}((i-r)\phi_{r+1}^{ij}(0,0)-(i+2)\phi_{r+1}^{ij}(0,-\tfrac{1}{2}))N_r^{i,j-\frac{1}{2}}(m,n), \tag{30}$$

$$\begin{aligned} b_r^{ij}(m,n) &= \tfrac{(-1)^r 4}{r!}((i+j+2-r)\phi_r^{ij}(\tfrac{1}{2},-\tfrac{1}{4}) \\ &\quad -(i+j+\tfrac{3}{2}-r)\phi_{r+1}^{ij}(\tfrac{1}{2},-\tfrac{1}{4}))N_{r-1}^{i-\frac{1}{2},j-\frac{1}{2}}(m,n), \end{aligned} \tag{31}$$

$$\begin{aligned} \tilde{b}_r^{ij}(m,n) &= -\tfrac{4}{r!}((i+j+1-r)\phi_r^{ij}(\tfrac{1}{2},-\tfrac{1}{4}) \\ &\quad -(i+j+\tfrac{3}{2}-r)\phi_{r+1}^{ij}(\tfrac{1}{2},-\tfrac{1}{4}))N_{r-1}^{i-\frac{1}{2},j-\frac{1}{2}}(m,n). \end{aligned} \tag{32}$$

The central terms are given by

$$\tilde{c}_i(m) = \tilde{c}_i \prod_{j=-i-1}^{i+1}(m+j) \quad , \qquad \check{c}_i(m) = \check{c}_i \prod_{j=-i-1}^{i}(m+j+\frac{1}{2}), \tag{33}$$

$$\tilde{c}_i = \frac{2^{2i-3} i!(i+2)!}{(2i+1)!!(2i+3)!!}\tilde{c} \quad , \qquad \check{c}_i = \frac{2^{2i} i!(i+1)!}{3((2i+1)!!)^2}\check{c}, \tag{34}$$

where $\tilde{c}$ (central charge of the Virasoro generator $\tilde{V}^0(z)$) and $\check{c}$ are related to the level k :

$$\tilde{c} = 2k, \quad \check{c} = 3k, \tag{35}$$

so $\tilde{c}$ must be an even integer for unitary representations. In the case of $N=1$, super $\widehat{su}(N)_k$-W_∞ reduces to super W_∞ of [7]. We remark that the above (anti-)commutators are compatible with the hermiticity properties of the generators :

$$V_n^{i\dagger} = V_{-n}^i, \; W_n^{i,a\dagger} = W_{-n}^{i,a}, \; \tilde{V}_n^{i\dagger} = \tilde{V}_{-n}^i, \; G_n^{i,\alpha\dagger} = \bar{G}_{-n}^{i,\alpha}. \tag{36}$$

The $\widehat{su}(N)_k$-$W_{1+\infty}$ and the super $\widehat{su}(N)_k$-W_∞ algebras admit free field realizations. Their generators can be represented in terms of bilinears of free fermion and free boson fields. By introducing complex free fermions $\psi^\alpha(z) = \sum \psi_n^\alpha z^{-n-\frac{1}{2}}$ $(\alpha = 1, \cdots, N)$ and a complex free boson $i\partial\varphi(z) = \sum \alpha_n z^{-n-1}$ having the following operator product expansions

$$\bar{\psi}^\alpha(z)\psi^\beta(w) \sim \frac{\delta^{\alpha\beta}}{z-w}, \quad i\partial\bar{\varphi}(z) i\partial\varphi(w) \sim \frac{1}{(z-w)^2}, \tag{37}$$

the level $k=1$ realization of the super $\widehat{su}(N)$-W_∞ algebra is given by

$$V^j(z) = \frac{2^{j-1}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j+1} (-1)^r \binom{j+1}{r}^2 : \partial^{j+1-r}\bar{\psi}^\alpha(z)\partial^r\psi^\alpha(z) :, \tag{38}$$

$$
\begin{aligned}
W^{j,a}(z) &= \frac{2^{j-1}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j+1} (-1)^r \binom{j+1}{r}^2 : \partial^{j+1-r} \bar{\psi}^\alpha(z) t^a_{\alpha\beta} \partial^r \psi^\beta(z) :, \quad (39) \\
\tilde{V}^j(z) &= \frac{2^{j-1}(j+2)!}{(2j+1)!!} q^j \sum_{r=0}^{j} \frac{(-1)^r}{j+1} \binom{j+1}{r} \binom{j+1}{r+1} : \partial^{j-r} i\partial\bar{\varphi}(z) \partial^r i\partial\varphi(z) :, \quad (40) \\
G^{j,\alpha}(z) &= \frac{2^{j+\frac{1}{2}}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j} (-1)^r \binom{j+1}{r} \binom{j}{r} \partial^{j-r} i\partial\bar{\varphi}(z) \partial^r \psi^\alpha(z), \quad (41) \\
\bar{G}^{j,\alpha}(z) &= \frac{2^{j+\frac{1}{2}}(j+1)!}{(2j+1)!!} q^j \sum_{r=0}^{j} (-1)^{j+r} \binom{j+1}{r} \binom{j}{r} \partial^{j-r} i\partial\varphi(z) \partial^r \bar{\psi}^\alpha(z). \quad (42)
\end{aligned}
$$

The coefficients in the above realization are essentially the same as those of super W_∞ algebra of [7]. $V^0(z)$ and $\tilde{V}^0(z)$ are the Virasoro generators for N complex free fermions and one complex boson, and the central charges are $c = N$ and $\tilde{c} = 2$ respectively. Using the hermiticity properties of free fields ($\psi_n^{\alpha\dagger} = \bar{\psi}^\alpha_{-n}$, $\alpha_n^\dagger = \bar{\alpha}_{-n}$), we can check the hermiticity properties of the generators, eq. (36), which is a good exercise of combinatorics.

Due to the Kac-Moody symmetry, the $\widehat{su}(N)$-$W_{1+\infty}$ and the super $\widehat{su}(N)$-W_∞ algebras have the automorphism, so called spectral flow [10]. Namely (anti-)commutators of these algebras are invariant under the transformations of the generators. Lower-order examples are

$$
\begin{aligned}
W_m^{-1,(\alpha\beta)\prime} &= W^{-1,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} - \delta^{\alpha\beta}\delta_{m0}\tfrac{1}{4q}\eta_\alpha k, \quad (43) \\
W_m^{0,(\alpha\beta)\prime} &= W^{0,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} - 2q(\eta_\alpha+\eta_\beta) W^{-1,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} + \delta^{\alpha\beta}\delta_{m0}\tfrac{1}{2}\eta_\alpha^2 k, \quad (44) \\
W_m^{1,(\alpha\beta)\prime} &= W^{1,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} - 4q(\eta_\alpha+\eta_\beta) W^{0,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} \\
&\quad + \tfrac{16}{3}q^2\left(\tfrac{1}{2}(\eta_\beta-\eta_\alpha)m + \eta_\alpha^2 + \eta_\alpha\eta_\beta + \eta_\beta^2\right) W^{-1,(\alpha\beta)}_{m+\eta_\beta-\eta_\alpha} - \delta^{\alpha\beta}\delta_{m0}\tfrac{4}{3}q\eta_\alpha^3 k, \quad (45) \\
G_m^{0,\alpha\prime} &= G^{0,\alpha}_{m+\eta_\alpha}, \quad (46) \\
G_m^{1,\alpha\prime} &= G^{1,\alpha}_{m+\eta_\alpha} - \tfrac{8}{3}q\eta_\alpha G^{0,\alpha}_{m+\eta_\alpha}, \quad (47) \\
G_m^{2,\alpha\prime} &= G^{2,\alpha}_{m+\eta_\alpha} - \tfrac{24}{5}q\eta_\alpha G^{1,\alpha}_{m+\eta_\alpha} + \tfrac{8}{5}q^2\eta_\alpha(2m+5\eta_\alpha) G^{0,\alpha}_{m+\eta_\alpha}, \quad (48)
\end{aligned}
$$

where η_α's are arbitrary real parameters. The transformation of $\bar{G}^{i,\alpha}_m$ is that of $G^{i,\alpha}_m$ with (η_α, q) replaced by $(-\eta_\alpha, -q)$, and $\tilde{V}^i_m$ are all left invariant. It can be understood most easily by the free field realization eqs. (38)-(42). The transformations of the generators are derived from those of the free fields:

$$
\psi^{\alpha\prime}(z) = z^{\eta_\alpha}\psi^\alpha(z),\ \bar{\psi}^{\alpha\prime}(z) = z^{-\eta_\alpha}\bar{\psi}^\alpha(z),\ \varphi'(z) = \varphi(z),\ \bar{\varphi}'(z) = \bar{\varphi}(z), \quad (49)
$$

where the normal ordering : : is understood as : $A(z)B(z) :=\oint_z \frac{dx}{2\pi i}\frac{1}{x-z}A(x)B(z)$.

In this letter we have constructed the new higher spin algebra $\widehat{su}(N)$-$W_{1+\infty}$, its supersymmetric extension and their free field realizations. Although we do not have complete analytic proofs for some of the results, we can check lower-order results by explicit calculations. For example, we have checked that the free field realization eqs. (38)-(42) give the correct (anti-)commutation relations: $[X^i_m, Y^j_n\}$ eqs. (5)-(7),(19)-(27) for $i, j \leq 10$, so that the Jacobi identities are satisfied to this order. As is in [6], there are (super-)wedge subalgebras which consist of $\{V^i_m, W^{i,a}_m : |m| \leq i+1\}$ in the case of $\widehat{su}(N)$-$W_{1+\infty}$, and $\{V^i_m, \tilde{V}^i_m, W^{i,a}_m, G^{j,\alpha}_n, \bar{G}^{j,\alpha}_n : |m| \leq i+1, |n| \leq j+\frac{1}{2}\}$ in the case of super $\widehat{su}(N)$-W_∞ (NS sector), respectively. We can take the limit $q \to 0$ after some redefinitions of the generators (for example, we replace $W^{i,a}_m$ with $\frac{1}{4q}W^{i,a}_m$). In the case of $\widehat{su}(N)_k$-$W_{1+\infty}$, we obtain the algebra which has non-trivial centers only in the Virasoro and Kac-Moody sectors.

We are able to construct $\widehat{su}(N)_k$-$W_{1+\infty}$ with no centers by applying the method provided in ref.[5]. Here we take $\widehat{su}(N)_k$ instead of $U(1)$ Kac-Moody algebra. Then the remainder of the construction is performed in the same way as [5].

So far we have developed only the case of $SU(N)$, but similar extensions for other Lie algebras, such as $SO(N)$, could be straightforwardly constructed. Some problems are left as future subjects. For example, we have not discussed the representation theory, the field theory corresponding to W_∞-gravity[11, 12] and the existence of the $\widehat{su}(N)$-$W_{1+\infty}$ type algebra for W_N.

Acknowledgments

The authors would like to thank K. Igi for careful reading the manuscript. The authors also would like to acknowledge useful discussions with T. Eguchi, H. Kawai and M. Ono. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan No.01790191.

Note Added

After this work was complete, we became aware of reference[13], where Bakas and Kiritsis constructed the $U(N)$ matrix generalization of W_∞ algebra with central charge $2N$, and gave its realization using N complex free bosons.

References

[1] A.B.Zamolodchikov, Theor.Math.Phys. **63** (1986) 1205;
V.A.Fateev and S.L.Lykyanov, Int.J.Mod.Phys. **A3** (1988) 507.

[2] I. Bakas, Phys. Lett. **B242** (1989) 57.

[3] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B236** (1990) 173.

[4] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B242** (1990) 401.

[5] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **B245** (1990) 72.

[6] C.N. Pope, X. Shen and L.J. Romans, Nucl. Phys. **B339** (1990) 191.

[7] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen,
Phys. Lett. **B245** (1990) 447.

[8] I. Bakas and E. Kiritsis, "Bosonic Realizations of a Universal W-Algebra and Z_∞ parafermions", LBL preprint, LBL-28714.

[9] M. Bershadsky, Phys. Lett. **B174** (1986) 285,
V.G. Knizhnik, Theor. Math. Phys. **66** (1986) 68.

[10] A. Schwimmer and N. Seiberg, Phys. Lett. **B184** (1987) 191.

[11] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen and K.S. Stelle,
Phys. Lett. **B243** (1990) 350;
E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen,
"W_∞ gravity and super-W_∞ gravity", CERN preprint, CERN-TH.5743/90.

[12] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Stony Brook preprint,
"Covariant w_∞ gravity and its reduction to W_N gravity", ITP-SB-90-39;
"Covariant formulation of classical W-gravity", ITP-SB-90-26.

[13] I. Bakas and E. Kiritsis, "Grassmannian Coset Model and Unitary Representations of W_∞", LBL preprint, LBL-28969.