Comments on the $W_{1+\infty}$ and Super W_∞ Algebras with $SU(N)$ Symmetry *

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Abstract

We extend the $W_{1+\infty}$ algebra by adding the affine $\widehat{su}(N)$ algebra with level k. The central charge c of this algebra, denoted by $\widehat{su}(N)_k$-$W_{1+\infty}$, turns out to be $c = Nk$. We also obtain the supersymmetric extension whose bosonic sector is $W_\infty \oplus \widehat{su}(N)_k$-$W_{1+\infty}$. In this report we comment on the algebraic structures, the geometric interpretations, the anomaly-free conditions, and the representation theories of these algebras.

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1 Introduction

The Virasoro algebra plays a central role in two-dimensional conformal field theories. For some years several extensions of the Virasoro algebra have been attempted. The typical one is the W_N algebra, which contains fields of conformal spin $2, \cdots, N$. But, for $N \geq 3$, W_N doesn't have the structure of Lie algebra. Recently new Lie algebraic extensions of the Virasoro algebra have been considered, which can be regarded as large N limit of W_N algebra. Pope, Romans, Shen etc. have constructed and studied the such algebras, what are called W_∞, $W_{1+\infty}$, and super W_∞[1,2,3].

In ref.[4], the new algebra $\widehat{su}(N)_k$-$W_{1+\infty}$ is constructed, which contains the $W_{1+\infty}$ algebra and the affine $\widehat{su}(N)_k$ algebra as subalgebras. The supersymmetric extension of the $\widehat{su}(N)_k$-$W_{1+\infty}$ is also constructed, and some properties (realization and spectral flow) of these new algebras are studied. In this report, we present some additional results of the study of these new algebras. Due to space limitations, we refer the readers to ref.[4] for the definitions of these algebras and follow the notations of [4].

2 Algebraic Structure

In ref.[1], Pope, Romans and Shen defined and studied *the lone-star product*, which is defined as follows:

$$V_m^i \star V_n^j \equiv \frac{1}{2} \sum_{r=-1}^{\infty} q^r g_r^{ij}(m,n,\mu)\; V_{m+n}^{i+j-r}, \tag{1}$$

where the coefficients $g_r^{ij}(m,n,\mu)$ are defined in ref.[1] and μ is an arbitrary real constant. This product turns out to be associative. The following relation was also pointed out:

$$\{V_m^i; |m| \leq i+1,\ \star\} \simeq \mathcal{U}(SL(2,\mathbf{R}))/I(Q-\mu), \tag{2}$$

where $\mathcal{U}(SL(2,\mathbf{R}))$ is the universal enveloping algebra of $SL(2,\mathbf{R})$, $Q = (L_0)^2 - \frac{1}{2}(L_+L_- + L_-L_+)$ is the quadratic Casimir operator of $SL(2,\mathbf{R})$, and $I(Q-\mu)$ is the ideal generated by $Q-\mu$. One can extend the associative algebra $\{V_m^i; |m| \leq i+1,\ \star\}$ out of the region $|m| \leq i+1$, and the central extension of the extended algebra is generally non-trivial. It is the algebra $W_\infty(\mu)$ defined in ref.[1]. In general, the $W_\infty(\mu)$ algebra contains the generators of negative spins, which are unphysical. The condition that the fields of negative spins do not appear imposes strong restrictions on the allowed value of μ. There are only two values of μ under which all negative-spin generators vanish; $\mu = 0$, or $-\frac{1}{4}$. $W_\infty(0)$ and $W_\infty(-\frac{1}{4})$ are W_∞ and $W_{1+\infty}$, respectively[1].

The higher-spin generators of $W_\infty(\mu)$ can be constructed from the low-spin generators by the lone-star product. In the cases of W_∞ and $W_{1+\infty}$, the higher-spin generators can be constructed in the following way[1]:

$$W_\infty : \quad \tilde{V}^0_m = \tilde{L}_m, \qquad \tilde{V}^1_{m+n} = 4q(\tilde{L}_m \star \tilde{L}_n - \tfrac{1}{2}(m-n)\tilde{L}_{m+n}), \tag{3}$$

$$\tilde{V}^{i+1}_m = 4q\tilde{L}_0 \star \tilde{V}^i_m + 2qm\tilde{V}^i_m + \frac{i(i+2)((i+1)^2 - m^2)}{4(4(i+1)^2-1)}(4q)^2\tilde{V}^{i-1}_m, \; i \geq 1, \tag{4}$$

$$W_{1+\infty} : j_{m+n} = j_m \star j_n, \quad V^{-1}_m = \tfrac{1}{4q}j_m, \quad V^0_m = L_m, \tag{5}$$

$$V^{i+1}_m = 4qL_0 \star V^i_m + 2qmV^i_m + \frac{(i+1)^2((i+1)^2 - m^2)}{4(4(i+1)^2-1)}(4q)^2V^{i-1}_m, \; i \geq 0. \tag{6}$$

In consequence, one can construct the $W_{1+\infty}$ algebra without central extension from the $U(1)$ Kac-Moody algebra and its derivation d as follows[1]:

$$[j_m, j_n] = 0, \qquad [d, j_m] = -mj_m, \tag{7}$$

$$V^{-1}_m = \tfrac{1}{4q}j_m, \tag{8}$$

$$V^0_m \equiv L_m = (d + \tfrac{1}{2}m)j_m, \tag{9}$$

$$\begin{aligned} V^i_m &= 4q(d + \frac{1}{2}m)V^{i-1}_m + \frac{i^2(i^2 - m^2)}{4(4i^2-1)}(4q)^2V^{i-2}_m, \quad i \geq 1 \\ &= P_i(d,m)j_m, \end{aligned} \tag{10}$$

where $P_i(d,m)$ are polynomials in d of degree $i+1$. The W_∞ algebra can be constructed from the Virasoro generators by the similar way[1]:

$$\tilde{V}^0_m = \tilde{L}_m, \tag{11}$$

$$\begin{aligned} \tilde{V}^i_m &= 4q(\tilde{L}_0 + \frac{1}{2}m)\tilde{V}^{i-1}_m + \frac{(i^2-1)(i^2 - m^2)}{4(4i^2-1)}(4q)^2\tilde{V}^{i-2}_m, \quad i \geq 1 \\ &= Q_i(\tilde{L}_0, m)\tilde{L}_m, \end{aligned} \tag{12}$$

where $Q_i(\tilde{L}_0, m)$ are polynomials in $\tilde{L}_0$ of degree i. Furthermore one get the following relations:

$$W_{1+\infty}(\text{without central extension}) \simeq \mathcal{U}(j_m, d)/I(j_mj_n - j_{m+n}), \tag{13}$$

$$W_\infty(\text{without central extension}) \simeq \mathcal{U}(\tilde{L}_m)/I(\tilde{L}_m\tilde{L}_n - (\tilde{L}_0 + m)\tilde{L}_{m+n}). \tag{14}$$

Now we can represent the $\widehat{su}(N)_0$-$W_{1+\infty}$ algebra by the above notations as:

$$V_m^i = P_i(d,m) j_m, \quad W_m^{i,a} = P_i(d,m) j_m t^a, \quad i \geq -1 \tag{15}$$

where t^a denote the basis of $su(N)$. The W_∞^N algebra, obtained by Bakas and Kiritsis[3], is also represented in a similar way:

$$\tilde{V}_m^i = Q_i(\tilde{L}_0, m)\tilde{L}_m, \quad \tilde{W}_m^{i,a} = Q_i(\tilde{L}_0, m)\tilde{L}_m t^a, \quad i \geq 0. \tag{16}$$

The $\widehat{su}(N)_k$-$W_{1+\infty}$ algebra contains the W_∞^N algebra as its subalgebra nontrivially, and then the central charge c of the former relates to the central charge $\tilde{c}$ of the latter by $\tilde{c} = -2c$. This relation is the same as that between W_∞ and $W_{1+\infty}$[1]. $W_\infty^N(\tilde{c}=0)$ and $\widehat{su}(N)_0$-$W_{1+\infty}$ are also realized as enveloping algebras:

$$\begin{aligned} W_\infty^N(\tilde{c}=0) &\simeq \mathcal{U}(\tilde{L}_m, t^a)/I(\tilde{L}_m\tilde{L}_n - (\tilde{L}_0 + m)\tilde{L}_{m+n}, \\ &\qquad\qquad t^a t^b - \tfrac{1}{2} i f^{abc} t^c - d^{abc} t^c - \tfrac{1}{N}\delta^{ab}), \qquad (17) \\ \widehat{su}(N)_0\text{-}W_{1+\infty} &\simeq \mathcal{U}(j_m, d, t^a)/I(j_m j_n - j_{m+n},\ t^a t^b - \tfrac{1}{2} i f^{abc} t^c - d^{abc} t^c - \tfrac{1}{N}\delta^{ab}). (18) \end{aligned}$$

Now we introduce the algebra $W_\infty^N(\mu)$, which is obtained by replacing the coefficients $g_r^{ij}(m,n)$ of $\widehat{su}(N)_k$-$W_{1+\infty}$ by $g_r^{ij}(m,n,\mu)$. Immediately we obtain the following results:

$$W_\infty^N = W_\infty^N(0), \quad \widehat{su}(N)_k\text{-}W_{1+\infty} = W_\infty^N(-\tfrac{1}{4}), \tag{19}$$

$$W_\infty = W_\infty^1(0), \qquad W_{1+\infty} = W_\infty^1(-\tfrac{1}{4}). \tag{20}$$

$W_\infty^N(\mu)$ without central extension contains so-called wedge subalgebra $W_\infty^N(\mu)_\wedge$ as subalgebra, which consists of generators $V_m^i, W_m^{i,a}; |m| \leq i+1$. We have the following relation:

$$W_\infty^N(\mu)_\wedge \simeq \mathcal{U}(SL(2,\mathbf{R}) \otimes u(N))/I(Q-\mu,\ t^a t^b - \tfrac{1}{2} i f^{abc} t^c - d^{abc} t^c - \tfrac{1}{N}\delta^{ab}). \tag{21}$$

Setting $j_m = e^{im\theta}$, $d = i\frac{d}{d\theta}$, and $\tilde{L}_m = ie^{im\theta}\frac{d}{d\theta}$, then W_∞^N and $\widehat{su}(N)_0$-$W_{1+\infty}$ can be regarded as the algebras of operators which act on the N-dimensional vector-valued functions on S^1.

3 Geometric Interpretation

The W_∞ algebras with non-zero q values are all isomorphic to each other. Any other algebra appeared in this report has this property. We can consider the $q \to 0$ limit of these algebras and the resultant algebras are not isomorphic to the original ones. For example, one obtains the w_∞ algebra[1] as the $q \to 0$ limit of the W_∞ algebra. The w_∞ algebra is geometrically the algebra of area-preserving diffeomorphisms of 2-surface. What are the geometric meanings of $\widehat{su}(N)_k$-$W_{1+\infty}$ and super $\widehat{su}(N)_k$-W_∞ ? After some redefinitions of generators, we can take the $q \to 0$ limit of the $\widehat{su}(N)_k$-$W_{1+\infty}$ algebra:

$$
\begin{aligned}
[v_m^i, v_n^j] &= (m(j+1) - n(i+1))v_{m+n}^{i+j} + \tfrac{c}{12}m(m^2-1)\delta^{ij}\delta^{i0}\delta_{m+n,0}, \quad i,j \geq 0, && (22)\\
[v_m^i, w_n^{j,a}] &= (m(j+1) - n(i+1))w_{m+n}^{i+j,a}, \quad i \geq 0, \quad j \geq -1, && (23)\\
[w_m^{i,a}, w_n^{j,b}] &= if^{abc}w_{m+n}^{i+j+1,c} + km\delta_{m+n,0}\delta^{ab}\delta^{ij}\delta^{i,-1}, \quad i,j \geq -1, && (24)\\
[v_m^{-1}, w_n^{j,a}] &= 0, \quad j \geq -1, && (25)\\
[v_m^{-1}, v_n^j] &= m\delta^{j,0}v_{m+n}^{-1}, \quad j \geq 0, && (26)\\
[v_m^{-1}, v_n^{-1}] &= c'm\delta_{m+n,0}, && (27)
\end{aligned}
$$

where $c = Nk = c'$ but the above algebra is closed for arbitrary c, k and c'. These commutators without center can be realized in the following way:

$$
\begin{aligned}
v_m^i &= (i+m+1)x^{i+m}y^{i+1}\frac{\partial}{\partial y} - (i+1)x^{i+m+1}y^i\frac{\partial}{\partial x}, && (28)\\
w_m^{i,a} &= t^a x^{i+m+1}y^{i+1}, && (29)\\
v_m^{-1} &= mx^{m-1}\delta(y). && (30)
\end{aligned}
$$

The vector fields v_m^i are the Hamiltonian vector fields which generate the canonical transformations (in other words, area-preserving diffeomorphisms) on 2-surface(x,y). We can regard $w_m^{i,a}$ as the generators of the local gauge transformations of $SU(N)$ bundle on 2-surface. The geometric meaning of v_m^{-1} is not clear.

The result of $q \to 0$ limit of the super $\widehat{su}(N)_k$-W_∞ algebra ($N > 1$) is more or less complex. After some redefinitions of generators we can take $q \to 0$ limit and obtain

$$
[v_m^i, v_n^j] = cm\delta_{m+n,0}\delta^{ij}\delta^{i,-1}, \qquad (31)
$$

$$[v^i_m, g^{j,\alpha}_n] = -g^{i+j+1,\alpha}_{m+n}, \quad (32)$$
$$[v^i_m, \bar{g}^{j,\alpha}_n] = \bar{g}^{i+j+1,\alpha}_{m+n}, \quad (33)$$
$$[w^{i,a}_m, g^{j,\alpha}_n] = -t^a_{\alpha\beta} g^{i+j+1,\beta}_{m+n}, \quad (34)$$
$$[w^{i,a}_m, \bar{g}^{j,\alpha}_n] = t^a_{\beta\alpha} \bar{g}^{i+j+1,\beta}_{m+n}, \quad (35)$$
$$[w^{i,a}_m, w^{j,b}_n] = if^{abc} w^{i+j+1,c}_{m+n} + km\delta^{ij}\delta^{i,-1}\delta^{ab}\delta_{m+n,0}, \quad (36)$$
$$\{g^{i,\alpha}_m, \bar{g}^{j,\beta}_n\} = 2t^a_{\alpha\beta} w^{i+j,a}_{m+n} + 2(\tfrac{1}{N}-1)\delta^{\alpha\beta} v^{i+j}_{m+n}, \quad (37)$$
$$[\tilde{v}^i_m, \tilde{v}^j_n] = (m(j+1)-n(i+1))\tilde{v}^{i+j}_{m+n} + \frac{c+\tilde{c}}{12} m(m^2-1)\delta^{ij}\delta^{i0}\delta_{m+n,0}, \quad (38)$$
$$[\tilde{v}^i_m, v^j_n] = (m(j+1)-n(i+1)) v^{i+j}_{m+n}, \quad (39)$$
$$[\tilde{v}^i_m, w^{j,a}_n] = (m(j+1)-n(i+1)) w^{i+j,a}_{m+n}, \quad (40)$$
$$[\tilde{v}^i_m, g^{j,\alpha}_n] = (m(j+\tfrac{1}{2})-n(i+1)) g^{i+j,\alpha}_{m+n}, \quad (41)$$
$$[\tilde{v}^i_m, \bar{g}^{j,\alpha}_n] = (m(j+\tfrac{1}{2})-n(i+1)) \bar{g}^{i+j,\alpha}_{m+n}, \quad (42)$$
$$\{g^{i,\alpha}_m, g^{j,\beta}_n\} = \{\bar{g}^{i,\alpha}_m, \bar{g}^{j,\beta}_n\} = [v^i_m, w^{j,a}_n] = 0. \quad (43)$$

This algebra without center can be realized as follows:

$$\tilde{v}^i_m = (i+m+1)x^{i+m}y^{i+1}\frac{\partial}{\partial y} - (i+1)x^{i+m+1}y^i\frac{\partial}{\partial x}, \quad (44)$$
$$v^i_m = x^{i+m+1}y^{i+1}\Big(\frac{DE'+D'E}{2(\frac{1}{N}-1)} - \theta^\alpha\frac{\partial}{\partial\theta^\alpha} + \bar{\theta}^\alpha\frac{\partial}{\partial\bar{\theta}^\alpha}\Big), \quad (45)$$
$$w^{i,a}_m = x^{i+m+1}y^{i+1}\Big(\theta^\alpha(-t^a_{\beta\alpha})\frac{\partial}{\partial\theta^\beta} + \bar{\theta}^\alpha t^a_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}^\beta}\Big), \quad (46)$$
$$g^{i,\alpha}_m = x^{i+m+\frac{1}{2}}y^{i+\frac{1}{2}}\Big(D\theta^\alpha + E\frac{\partial}{\partial\bar{\theta}^\alpha} + \frac{2}{E'}\theta^\alpha\theta^\beta\frac{\partial}{\partial\theta^\beta}\Big), \quad (47)$$
$$\bar{g}^{i,\alpha}_m = x^{i+m+\frac{1}{2}}y^{i+\frac{1}{2}}\Big(D'\bar{\theta}^\alpha + E'\frac{\partial}{\partial\theta^\alpha} - \frac{2}{E}\bar{\theta}^\alpha\bar{\theta}^\beta\frac{\partial}{\partial\bar{\theta}^\beta}\Big), \quad (48)$$

where D, D', E and E' are arbitrary real parameters. By rescaling the fermionic coordinates θ^α and $\bar{\theta}^\alpha$, two parameters can be absorbed. These generators generate the volume-preserving diffeomorphisms of the superspace $(x, y, \theta^\alpha, \bar{\theta}^\alpha; \alpha = 1, \ldots, N)$.

4 Anomaly-free Condition

In refs.[5,6] the anomaly-free conditions for W_∞, $W_{1+\infty}$, and super W_∞ are considered by the BRS formalism and it is shown that their critical central charges are -2, 0 and -3, respectively. We do not ask here whether there exist the physical string models which possess the symmetries according to those algebras. However, it is interesting that we can extract the significant results in spite of the appearance of infinite number of ghost fields, and it is expected that there may be the theories in higher dimensions which consist of finite number of fields[5,6]. In this section we will compute the anomaly-free conditions for $\widehat{su}(N)$-$W_{1+\infty}$ and super $\widehat{su}(N)$-W_∞.

Let us fix our notation. A field $A(z)$ with conformal weight h_A is expanded as $A(z) = \sum A_n z^{-n-h_A}$, where sum is taken over $n \in \mathbf{Z} - h_A$. We consider only the case $2h_A \in \mathbf{Z}$ and denote the Grassmann parity of $A(z)$ by $(-1)^A$. (Anti-)commutation relation is defined by $[A_m, B_n\}$=$A_m B_n - (-1)^{AB} B_n A_m$. Normal ordering $(AB)(z)$ of the two fields $A(z)$ and $B(z)$ is defined by $(AB)(z) = \oint_z \frac{dx}{2\pi i}\frac{1}{x-z}A(x)B(z)$ and its mode expansion is

$$(AB)_m = \sum_{p\le -h_A} A_p B_{m-p} + (-1)^{AB} \sum_{p>-h_A} B_{m-p} A_p. \tag{49}$$

The step function $\theta(P)$ takes 1 if the proposition P is true, and 0 if P is false.

At first we review the construction of BRS charge. Let us consider the following Lie algebra generated by the field $J^A(z)$ with conformal weight h_A ($\in \frac{1}{2}\mathbf{N}$):

$$[J^A_m, J^B_n\} \;=\; i f^{AB}{}_C(m,n,l) J^C_l + \delta^{AB}\delta_{m+n,0} k_A(m), \tag{50}$$

$$k_A(m) \;=\; k_A \prod_{j=0}^{2h_A-2} (m + j - h_A + 1), \tag{51}$$

where the structure constants $f^{AB}{}_C(m,n,l)$= $f^{AB}{}_C(m,n)\delta_{m+n,l}$ and the central terms $k_A(m)$ satisfy the Jacobi identity. For each $J^A(z)$, we introduce the ghost field $c_A(z)$ with conformal weight $1-h_A$ and antighost field $b^A(z)$ with conformal weight h_A. $c_A(z)$ and $b^A(z)$ are both Grassmann $(-1)^{A+1}$ and their (anti-)commutation relations are

$$[c_{A,m}\ , b^B_n\} = \delta_A{}^B \delta_{m+n,0}\ , \quad \left(\ c_A(z) b^B(w) \sim \frac{\delta_A{}^B}{z-w}\ \right). \tag{52}$$

Setting $h_{ABC} = h_A + h_B - h_C$, we can rewrite the structure constants $f^{BA}{}_C(n,m)$ as

$$i f^{BA}{}_C(n,m) = \sum_{k=0}^{h_{ABC}-1} i\tilde{f}^{BA}{}_{C,k} [n + h_B - 1]_{h_{ABC}-1-k} [-n - m - h_C]_k, \tag{53}$$

and using these coefficients we define the ghost currents $J^A_{gh}(z)$ as follows:

$$J^A_{gh}(z) = \sum_{k=0}^{h_{ABC}-1} i\tilde{f}^{BA}{}_{C,k}(-1)^B(\partial^{h_{ABC}-1-k}c_B\partial^k b^C)(z), \quad (54)$$

$$\Big(\; J^A_{gh,m} = if^{BA}{}_C(n,m,l)(-1)^B c_{B,-n}b^C_l \; \Big). \quad (55)$$

The above equations need a little explanation. Of course the second equation needs some regularization, and we define the regularization of $J^A_{gh,m}$ by the first equation [1].

The algebra generated by $J^A_{gh}(z)$ is the same one generated by $J^A(z)$ except for the center. The center of ghost currents are

$$k^{gh}_A = -\sum_{k=0}^{h_{ABC}-1}\sum_{k'=0}^{h_{ACB}-1} i\tilde{f}^{BA}{}_{C,k}\, i\tilde{f}^{CA}{}_{B,k'}(-1)^{B+h_{ABC}} \times \frac{(h_{ABC}-1-k+k')!\,(h_{ACB}-1-k'+k)!}{(2h_A-1)!}. \quad (56)$$

The BRS current $J^{BRS}(z)$ and the BRS charge Q_{BRS} are given as:

$$J^{BRS}(z) = (c_A J^A)(z) + \tfrac{1}{2}(c_A J^A_{gh})(z) + \theta(h_A \in \mathbf{N})\kappa_A \partial^{h_A} c_A(z), \quad (57)$$

$$Q_{BRS} = \oint_0 \frac{dz}{2\pi i} J^{BRS}(z) \quad (58)$$

$$\Big(\; = c_{A,-m}J^A_m + \tfrac{1}{2} if^{BA}{}_C(n,m,l)(-1)^B c_{A,-m}c_{B,-n}b^C_l \; \Big). \quad (59)$$

Here we define again the regularization of the third line by the second line. The coefficients κ_A, which do not contribute to Q_{BRS}, are determined by the requirement that $J^{BRS}(z)$ should be a primary field of conformal weight 1 with respect to the total Virasoro generator. The nilpotency of Q_{BRS} ($Q^2_{BRS}=0$) is equivalent to the condition that the central charge $k^{total}_A{=}k_A + k^{gh}_A$ of the total current $J^A_{total}(z){=}J^A(z)+J^A_{gh}(z)$ is equal to 0. So, to obtain the anomaly-free condition, we have only to compute the center of the ghost current. The total current can be expressed as $[Q_{BRS}, b^A(z)\}{=}J^A_{total}(z)$.

To avoid the repetition, we present the expressions for the ghost currents corresponding to the super $\widehat{su}(N)$-W_∞ algebra. The coefficients $N^{x,y}_r(m,n)$ (eq.(9) in ref.[4]) can be rewritten as follows [2]:

$$N^{x,y}_r(m,n) = \sum_{k=0}^{r+1}(-1)^k\binom{r+1}{k}(2x+2-r)_k[2y+2-k]_{r+1-k}$$

[1] This regularization may be different from the usual normal ordering : : only by constant in zero mode, which gives an intercept.

[2] eq. (60) is needed when we express the (anti-)commutation relations in OPE form[1].

$$\times[x+1+m]_{r+1-k}[y+1+n]_k \quad (60)$$

$$= \sum_{k=0}^{r+1} \tilde{N}^{x,y}_{r,k}[n+y+1]_{r+1-k}[-n-m-(x+y-r+2)]_k \quad (61)$$

$$= \sum_{k=0}^{r+1} \tilde{N}'^{x,y}_{r,k}[m+x+1]_{r+1-k}[-n-m-(x+y-r+2)]_k, \quad (62)$$

$$\tilde{N}^{x,y}_{r,k} = (-1)^{r+1}\tilde{N}'^{y,x}_{r,k}$$
$$= (-1)^{r+1}\binom{r+1}{k}(2y+2-r)_k[2x+2y+4-r]_{r+1-k}. \quad (63)$$

We replace the coefficients $N^{x,y}_r(m,n)$ in the structure constants $d^{ij}_r(m,n)$ ($d = g,\tilde{g},a,\tilde{a},b,\tilde{b}$) by $\tilde{N}^{x,y}_{r,k}$ and $\tilde{N}'^{x,y}_{r,k}$, and denote the resultant coefficients by $d^{ij}_{r,k}$ and $d'^{ij}_{r,k}$ respectively. Let us introduce the ghost fields ($c^{i,(\alpha\beta)}$, $b^{i,(\alpha\beta)}$), ($\tilde{c}^i$, $\tilde{b}^i$), ($\gamma^{i,\alpha}$, $\beta^{i,\alpha}$), and ($\bar{\gamma}^{i,\alpha}$, $\bar{\beta}^{i,\alpha}$) corresponding to $W^{i,(\alpha\beta)}$, $\tilde{V}^i$, $G^{i,\alpha}$, and $\bar{G}^{i,\alpha}$ respectively. These fields satisfy the following OPE's:

$$c^{i,(\alpha\beta)}(z)b^{j,(\gamma\delta)}(w) \sim \frac{\delta^{ij}\delta^{\alpha\delta}\delta^{\beta\gamma}}{z-w} \quad , \quad \tilde{c}^i(z)\tilde{b}^j(w) \sim \frac{\delta^{ij}}{z-w}, \quad (64)$$

$$\gamma^{i,\alpha}(z)\bar{\beta}^{j,\beta}(w) \sim \frac{\delta^{ij}\delta^{\alpha\beta}}{z-w} \quad , \quad \bar{\gamma}^{i,\alpha}(z)\beta^{j,\beta}(w) \sim \frac{\delta^{ij}\delta^{\alpha\beta}}{z-w}. \quad (65)$$

The ghost currents for the super $\widehat{su}(N)$-W_∞ algebra are constructed by the method mentioned above. The results are

$$W^{i,(\alpha\beta)}_{gh}(z) = -\frac{1}{2}\sum_{j\geq -1}\sum_{r=-1}^{i+j+1}\sum_{k=0}^{r+1} q^r g^{ij}_{r,k}((\partial^{r+1-k}c^{j,(\gamma\beta)}\partial^k b^{i+j-r,(\alpha\gamma)})(z)$$
$$+(-1)^r(\partial^{r+1-k}c^{j,(\alpha\gamma)}\partial^k b^{i+j-r,(\gamma\beta)})(z))$$
$$+\sum_{j\geq 0}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r a^{ij}_{r,k}((\partial^{r+1-k}\bar{\gamma}^{j,\alpha}\partial^k\beta^{i+j-r,\alpha})(z)$$
$$+(-1)^r(\partial^{r+1-k}\gamma^{j,\alpha}\partial^k\bar{\beta}^{i+j-r,\alpha})(z)), \quad (66)$$

$$\tilde{V}^i_{gh}(z) = -\sum_{j\geq 0}\sum_{r\geq 0,even}^{i+j}\sum_{k=0}^{r+1} q^r \tilde{g}^{ij}_{r,k}(\partial^{r+1-k}\tilde{c}^j\partial^k\tilde{b}^{i+j-r})(z)$$
$$+\sum_{j\geq 0}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r \tilde{a}^{ij}_{r,k}((\partial^{r+1-k}\bar{\gamma}^{j,\alpha}\partial^k\beta^{i+j-r,\alpha})(z)$$
$$+(-1)^r(\partial^{r+1-k}\gamma^{j,\alpha}\partial^k\bar{\beta}^{i+j-r,\alpha})(z)), \quad (67)$$

$$G^{i,\alpha}_{gh}(z) = \sum_{j\geq -1}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r a'^{ji}_{r,k}(\partial^{r+1-k}c^{j,(\beta\alpha)}\partial^k\beta^{i+j-r,\beta})(z)$$

$$
\begin{aligned}
&+\sum_{j\geq 0}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r \tilde{a}'^{ji}_{r,k}(\partial^{r+1-k}\tilde{c}^j\partial^k\beta^{i+j-r,\alpha})(z) \\
&-\sum_{j\geq 0}\sum_{r=0}^{i+j+1}\sum_{k=0}^{r} q^r b^{ij}_{r,k}(\partial^{r-k}\gamma^{j,\beta}\partial^k b^{i+j-r,(\beta\alpha)})(z) \\
&-\sum_{j\geq 0}\sum_{r=0}^{i+j}\sum_{k=0}^{r} q^r \tilde{b}^{ij}_{r,k}(\partial^{r-k}\gamma^{j,\alpha}\partial^k\tilde{b}^{i+j-r})(z),
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
\bar{G}^{i,\alpha}_{gh}(z) \;=\;& \sum_{j\geq -1}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r(-1)^r a'^{ji}_{r,k}(\partial^{r+1-k}c^{j,(\alpha\beta)}\partial^k\bar{\beta}^{i+j-r,\beta})(z) \\
&+\sum_{j\geq 0}\sum_{r=-1}^{i+j}\sum_{k=0}^{r+1} q^r(-1)^r \tilde{a}'^{ji}_{r,k}(\partial^{r+1-k}\tilde{c}^j\partial^k\bar{\beta}^{i+j-r,\alpha})(z) \\
&-\sum_{j\geq 0}\sum_{r=0}^{i+j+1}\sum_{k=0}^{r} q^r b'^{ji}_{r,k}(\partial^{r-k}\bar{\gamma}^{j,\beta}\partial^k b^{i+j-r,(\alpha\beta)})(z) \\
&-\sum_{j\geq 0}\sum_{r=0}^{i+j}\sum_{k=0}^{r} q^r \tilde{b}'^{ji}_{r,k}(\partial^{r-k}\bar{\gamma}^{j,\alpha}\partial^k\tilde{b}^{i+j-r})(z).
\end{aligned}
\tag{69}
$$

When we calculate the center, we must regularize the summation with respect to the field index, since there exist the infinite number of fields. The method of the regularization is as follows. We take W_∞, i.e. the first term of eq. (67), for illustration. The center of ghost takes the form $\sum_{j\geq 0}$(rational function in j), where the origin of j is $\tilde{V}^j$. We divide this rational function into the polynomial and fractional parts. We expand the polynomial part in $j+\frac{3}{2}$ and decompose the fractional part into the partial fractions. If the origin of j is the fermionic generator, we expand the polynomial part in $j+1$ [3]. After the explicit calculation, the expressions to be summed with respect to j are

$$
\sum_{r=0}^{i+1} p_{ir}(j+\frac{3}{2})^{2r} + \sum_{r=0}^{i-1} p'_{ir}(\frac{1}{2j+2r+5} - \frac{1}{2j-2r+1}). \tag{70}
$$

We regularize these as follows:

$$
\tilde{c}^{gh}_i = \sum_{r=0}^{i+1} p_{ir}\zeta(-2r,\frac{3}{2}) + \sum_{r=0}^{i-1} p'_{ir}\sum_{j=0}^{2r+1}\frac{-1}{2j-2r+1}, \tag{71}
$$

[3]Actual calculation shows that the polynomial parts have only even power terms with respect to $j+\frac{3}{2}$ (or $j+1$). $j+\frac{3}{2}$ (or $j+1$) are chosen in order to respect the symmetry of (b,c) system: $h \leftrightarrow 1-h$ [5].

where $\zeta(s,a)$ is the Hurwitz's zeta function, and the sum with respect to j in the second term is equal to $-\frac{1}{2r+3}-\frac{1}{2r+1}$. The Hurwitz's zeta function is defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad (\mathrm{Re}s>1, a\notin \mathbf{Z}_{\leq 0}) \tag{72}$$

and analytically continued to the whole complex plane in s (except $s=1$). When s takes a nonpositive integer value, ζ is represented by the Bernoulli polynomial:

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1} \quad (n\geq 0), \qquad \frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{73}$$

The special values which we will use are

$$\zeta(-2n,\frac{1}{2})=0, \quad \zeta(-2n,\frac{3}{2})=-\frac{1}{2^{2n}}, \quad (n\geq 0), \quad \zeta(0,1)=-\frac{1}{2}. \tag{74}$$

In the case of W_∞, we have $p_{01}=-1$, $p_{00}=\frac{1}{12}$ for $i=0$, $p_{12}=-\frac{4}{3}$, $p_{11}=0$, $p_{10}=-\frac{3}{20}$, $p'_{10}=\frac{3}{40}$ for $i=1$, $p_{23}=-\frac{32}{45}$, $p_{22}=-\frac{64}{45}$, $p_{21}=\frac{2}{75}$, $p_{20}=-\frac{43}{350}$, $p'_{21}=\frac{15}{448}$, $p'_{20}=\frac{321}{5600}$ for $i=2$, etc. We have checked $\tilde{c}^{gh}=2$ for $i\leq 5$ (we have also explicitly checked the results given in the rest of this section for $i\leq 5$.). In refs. [5,6], Q^2_{BRS} is directly evaluated and $\tilde{c}^{gh}=2$ is checked for $i\leq 16$[6].

The contribution from the first term of eq. (66), which is the ghost current of $\widehat{su}(N)$-$W_{1+\infty}$, has the same form as eq. (70) [4]. But the sum is taken over the region $j\geq -1$, because the origin of j is $W^{j,(\alpha\beta)}$. In the expression corresponding to eq. (71), $\zeta(-2r,\frac{1}{2})$ and $\sum_{j=-1}^{2r}\frac{-1}{2j-2r+1}$ appear. Since both of these are 0, we obtain

$$c^{gh}=Nk^{gh}=0. \tag{75}$$

In the case of the super $\widehat{su}(N)$-W_∞, the ghost central charge c^{gh} of $W_{1+\infty}$ contains the contribution from the bosonic ghost (the second term of eq. (66)). This contribution is written in the form of the sum of

$$2N\times\Big(p_i(j+1)^0+\sum_{r=0}^{i}p''_{ir}(\frac{1}{2j+2r+3}-\frac{1}{2j-2r+1})\Big) \tag{76}$$

over $j\geq 0$ (the origin of this j is the fermionic generator $G^{j,\alpha}$.). We regularize this as follows:

$$c^{gh}_{i,fermionic}=2N\times\Big(p_i\zeta(0,1)+\sum_{r=0}^{i}p''_{ir}\sum_{j=0}^{2r}\frac{-1}{2j-2r+1}\Big), \tag{77}$$

[4]Of course coefficients p,p' are different and upper bound of r may be different.

where $\sum_{j=0}^{2r} \frac{-1}{2j-2r+1} = -\frac{1}{2r+1}$. We have, for example, $p_{-1} = -\frac{1}{16}$ for $i = -1$, $p_0 = -\frac{1}{8}$, $p''_{00} = \frac{1}{48}$ for $i = 0$, $p_1 = -\frac{11}{72}$, $p''_{11} = \frac{9}{640}$, $p''_{10} = \frac{157}{5760}$ for $i = 1$, etc. and we obtain $c^{gh}_{fermionic} = Nk^{gh}_{fermionic} = N$. Taking into account of the contribution from the fermionic ghost $c^{gh}_{bosonic} = Nk^{gh}_{bosonic} = 0$, we obtain

$$c^{gh} = Nk^{gh} = N. \tag{78}$$

From the Jacobi identity, we have $\tilde{c}^{gh} = 2$ and $\check{c}^{gh} = 3$. These could be also evaluated by the explicit calculation: in fact we have $\tilde{c}^{gh}_{fermionic} = 0$, $\check{c}^{gh} = N \times 0 + \frac{3}{2} + N \times 0 + \frac{3}{2}$ [5]. The total central charge of the ghost currents for the super $\widehat{su}(N)$-W_∞ algebra is

$$c^{gh} + \tilde{c}^{gh} = N + 2. \tag{79}$$

In conclusion, the anomaly-free conditions turn out to be $c = 0$ for $\widehat{su}(N)$-$W_{1+\infty}$, and $c + \tilde{c} = -N - 2$ for super $\widehat{su}(N)$-W_∞.

5 Representation Theory

In this section we consider the irreducible unitary representations of the $\widehat{su}(N)_k$-$W_{1+\infty}$ algebra which can be realized by free fields. We especially analyze the case of $k = 1$, because then the Virasoro generator $V^0(z)$ is equal to the sum of the Sugawara forms of the $\widehat{u}(1)$ and $\widehat{su}(N)_1$ currents[7], so the analysis becomes very easy.

The highest weight state (HWS) of $\widehat{su}(N)$-$W_{1+\infty}$ is the state which vanishes by the actions of $\{V^i_m, W^{i,a}_m : m > 0\}$ and $W^{i,a}_0$ of positive roots, and is the eigenstate of $W^{i,a}_0$ contained in the Cartan subalgebra and V^i_0. The HWS is also the eigenstate of $\widehat{su}(N)_1$. The HWS's of $\widehat{su}(N)_1$ contained in the fermion Fock space of the free field realization (eqs.(38,39) in ref.[4]) are

$$|l\rangle = \begin{cases} \psi^1_{-\frac{1}{2}}\psi^2_{-\frac{1}{2}}\cdots\psi^l_{-\frac{1}{2}}|0\rangle & l = 1, 2, \cdots, N-1 \\ |0\rangle & l = 0 \\ \bar{\psi}^1_{-\frac{1}{2}}\bar{\psi}^2_{-\frac{1}{2}}\cdots\bar{\psi}^{-l}_{-\frac{1}{2}}|0\rangle & l = -1, -2, \cdots, -(N-1). \end{cases} \tag{80}$$

[5]The first (second, third, fourth) term is the contribution from the first (second, third, fourth) term of eq. (68) and the third (fourth, first, second) term of eq. (69) respectively.

The state $|l\rangle$ is the HWS of the l'-th rank antisymmetric representation, where $l' = l + \theta(l < 0)N$. The state $|l\rangle$ is also the HWS of $\widehat{su}(N)_1$-$W_{1+\infty}$. The eigenvalues of $W_0^{i,(\alpha\alpha)}$ (no sum over α) is obtained as follows:

$$W_0^{i,(\alpha\alpha)}|l\rangle = \frac{2^{i-1}((i+1)!)^2}{(2i+1)!!}(-1)^i q^i |l\rangle \times \begin{cases} 1 & l > 0,\ \alpha = 1, 2, \cdots, l \\ -1 & l < 0,\ \alpha = 1, 2, \cdots, -l \\ 0 & \text{otherwise.} \end{cases} \quad (81)$$

The conformal weight of $|l\rangle$ is $h_l = \frac{1}{2}|l|$. Neglecting the dependence on the eigenvalues of higher-spin generators, we obtain the character formula:

$$\chi_l(\tau; \theta_0, \vec{\theta}) = \mathrm{tr}_{|l\rangle} q^{V_0^0 - \frac{N}{24}} e^{i\theta_0 J_0 + i\vec{\theta}\cdot\vec{J}_0^{Cartan}} = \frac{e^{il\theta_0}}{\eta(\tau)} \chi_{l'}^{\widehat{su}(N)_1}(\vec{\theta}, \tau), \quad (82)$$

where $q = e^{2\pi i\tau}$, $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n)$, and $\chi_{l'}^{\widehat{su}(N)_1}(\vec{\theta}, \tau)$ is the character formula of the l'-th rank antisymmetric representation of $\widehat{su}(N)_1$. In derivation we have used the fact that the generators of $\widehat{su}(N)$-$W_{1+\infty}$ do not change the $U(1)$ charge.

In the rest of this section we give some comments on the irreducible unitary representations of $W_{1+\infty}(c = 1)$ and $W_\infty(\tilde{c} = 2)$ realized by the free fields. The free field realization of $W_{1+\infty}$ with central charge $c = 1$ is constructed by one fermion(eq.(40) in ref.[4])[1]. The HWS of $U(1)$ current $J(z)$ contained in the fermion Fock space are

$$|l\rangle = \begin{cases} \psi_{-\frac{1}{2}}\psi_{-\frac{3}{2}} \cdots \psi_{-\frac{2l-1}{2}}|0\rangle & l \geq 1 \\ |0\rangle & l = 0 \\ \bar{\psi}_{-\frac{1}{2}}\bar{\psi}_{-\frac{3}{2}} \cdots \bar{\psi}_{-\frac{-2l-1}{2}}|0\rangle & l \leq -1, \end{cases} \quad (83)$$

where the $U(1)$ charge of $|l\rangle$ is l. These states are well known in Sato theory[8]. The states $|l\rangle$ are also the HWS of $W_{1+\infty}$, and the eigenvalue of V_0^i are computed as follows:

$$V_0^i|l\rangle = \frac{2^{i-1}(i+1)!}{(2i+1)!!} q^i \sum_{r=0}^{\min(i+1,l-1)} \sum_{s=0}^{l-1-r} \frac{(i+1+s)!}{s!} \binom{i+1}{r}^2 |l\rangle \times \begin{cases} (-1)^i & l > 0 \\ 0 & l = 0 \\ 1 & l < 0. \end{cases} \quad (84)$$

The conformal weight of $|l\rangle$ is $h_l = \frac{1}{2}l^2$. Taking into account of the fact that V_m^i does not change the $U(1)$ charge and neglecting the dependence on the eigenvalues of higher-spin generators, we obtain the following character formula:

$$\chi_l(\tau; \theta) = \mathrm{tr}_{|l\rangle} q^{V_0^0 - \frac{1}{24}} e^{i\theta J_0} = \frac{e^{il\theta}}{\eta(\tau)}. \quad (85)$$

The free field realization of W_∞ with central charge $\tilde{c} = 2$ is constructed by one free boson(eq.(42) in ref.[4])[3]. The Fock space of boson is obtained by the actions of negative-mode oscillators on the momentum eigenstate $|\alpha, \bar{\alpha}\rangle$ ($p|\alpha, \bar{\alpha}\rangle = \alpha|\alpha, \bar{\alpha}\rangle$, $\bar{p}|\alpha, \bar{\alpha}\rangle = \bar{\alpha}|\alpha, \bar{\alpha}\rangle$). The state $|\alpha, \bar{\alpha}\rangle$ is created from the vacuum state $|0, 0\rangle$ by vertex operator $: e^{i\bar{\alpha}\varphi + i\alpha\bar{\varphi}} : (z)$. The unitarity requires $\bar{\alpha} = \alpha^*$. The HWS of the Virasoro generator $\tilde{V}^0(z)$ in the boson Fock space are $|\alpha, \bar{\alpha}\rangle$, $(\alpha_{-1})^l|0, 0\rangle$ and $(\bar{\alpha}_{-1})^l|0, 0\rangle$, $(l \geq 1)$, and these states are also the HWS of W_∞. The eigenvalues of $\tilde{V}_0^i$ are

$$\tilde{V}_0^i|\alpha, \bar{\alpha}\rangle = \frac{2^i i!(i+1)!}{(2i+1)!!} q^i \alpha\bar{\alpha} \frac{1 + (-1)^i}{2} |\alpha, \bar{\alpha}\rangle, \tag{86}$$

$$\tilde{V}_0^i \left\{ \begin{array}{l} (\alpha_{-1})^l|0, 0\rangle \\ (\bar{\alpha}_{-1})^l|0, 0\rangle \end{array} \right. = \frac{2^{i-1}(i+1)!(i+2)!}{(2i+1)!!} q^i l \times \left\{ \begin{array}{l} (-1)^i(\alpha_{-1})^l|0, 0\rangle \\ (\bar{\alpha}_{-1})^l|0, 0\rangle. \end{array} \right. \tag{87}$$

For the states $|\alpha, \bar{\alpha}\rangle$, Bakas and Kiritsis have obtained, by using the Z_∞ parafermion, the following character formula[3]:

$$\chi_\alpha(\tau) = \mathrm{tr}_{|\alpha,\bar{\alpha}\rangle} q^{\tilde{V}_0^0 - \frac{2}{24}} = \frac{q^{\alpha\bar{\alpha}}}{\eta(\tau)^2}. \tag{88}$$

For $|0, 0\rangle$, $(\alpha_{-1})^l|0, 0\rangle$ and $(\bar{\alpha}_{-1})^l|0, 0\rangle$, we must subtract the zero norm states. For example, the character of $|0, 0\rangle$ is

$$\begin{aligned} \chi_{vac}(\tau) &= q^{-\frac{2}{24}}(1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 11q^6 + \cdots) \\ &< q^{-\frac{2}{24}} \prod_{n\geq 1}(1 - q^n)^{1-n}, \end{aligned} \tag{89}$$

where $A < B$ means that $B - A$ is q-series with positive coefficients.

Representation theories given in this section are insufficient. We need further investigations (higher level, higher center, super, etc.).

Recently, in the context of the two dimensional quantum gravity, the relation between the $W_{1+\infty}$ algebra and KP hierarchy has been studied by Fukama, Nakayama and Kawai (Tokyo preprint UT-572). We expect that our $\widehat{su}(N)_k$-$W_{1+\infty}$ algebra relate to multi-component KP hierarchy.

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