平成17年度（1次募集)解答

川口賢二
答案確認者氏名 高畠信弘、斎藤高

平成17 年6 月30 日

平成17年度(1次募集) 力学

(1)

$$\text{バネの伸びは}\, x - \sin\omega t - L \tag{1}$$

(2)

(2)

$$\text{運動方程式は}\, m\frac{d^2x}{dt^2} = -k(x - \sin\omega t - L) \tag{3}$$

$$A = 0\,\text{として運動方程式を変形すると、}\rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}(x - L) \tag{4}$$

$$k\,\text{これは単振動の式で、}x = \alpha\cos\sqrt{\frac{k}{m}}t + \beta\sin\sqrt{\frac{k}{m}}t + L \tag{5}$$

$(\alpha, \beta$は定数)

(3)

$$\text{運動方程式から特解を予想して、}x = T\sin\omega t\,\text{とおく。}(T\,\text{は定数}) \tag{6}$$

運動方程式に代入すると、

$$m\cdot(-\omega^2 T\sin\omega t) = -k(T - A)\sin\omega t \rightarrow (k - m\omega^2)T = kA \tag{7}$$

$$\therefore T = \frac{kA}{k - m\omega^2} \tag{8}$$

$$\therefore x = \frac{kA}{k - m\omega^2}\sin\omega t + L \tag{9}$$

よって(2),(3)の和をとると、一般解は

$$x = \alpha\cos\sqrt{\frac{k}{m}}t + \beta\sin\sqrt{\frac{k}{m}}t + \frac{kA}{k - m\omega^2}\sin\omega t + 2L \tag{10}$$

[1]

(1) 入射波、反射波、透過波の電場と磁場の強さをそれぞれ

$E_0, E_1, E_2, H_0, H_1, H_2$とすると、

$$H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_0 \quad , \quad H_1 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_1 \quad , \quad H_2 = \sqrt{\frac{\epsilon_2}{\mu_2}} E_2 \qquad (11)$$

境界面の両側で$\vec{E}$と$\vec{H}$の境界面の方向の成分は連続であるから、

$$E_0 + E_1 = E_2 \qquad (12)$$

$$H_0 - H_1 = H_2 \qquad (13)$$

これらを解くと、

$$E_1 = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}} - \sqrt{\frac{\epsilon_2}{\mu_2}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}} E_0 \quad , \quad E_2 = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}} E_0 \qquad (14)$$

$$H_0 = \sqrt{\frac{\epsilon_1}{\mu_1}} E_0 \quad , \quad H_1 = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}} - \sqrt{\frac{\epsilon_2}{\mu_2}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}} \sqrt{\frac{\epsilon_1}{\mu_1}} E_0 \quad , \quad H_2 = \frac{2\sqrt{\frac{\epsilon_1}{\mu_1}}\sqrt{\frac{\epsilon_2}{\mu_2}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}} E_0 \qquad (15)$$

(2)

$$\text{エネルギー流の大きさは、} |\vec{E} \times \vec{H}| = |EH| \qquad (16)$$

$$(\because \vec{E} \perp \vec{H})$$

$$\text{反射率}\, r = \left| \frac{E_1 H_1}{E_0 H_0} \right| = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}} E_1^2}{\sqrt{\frac{\epsilon_1}{\mu_1}} E_0^2} = \left(\frac{\sqrt{\frac{\epsilon_1}{\mu_1}} - \sqrt{\frac{\epsilon_2}{\mu_2}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}} \right)^2 \qquad (17)$$

$$\text{透過率}\, t = \left| \frac{E_2 H_2}{E_0 H_0} \right| = \frac{\sqrt{\frac{\epsilon_2}{\mu_2}} E_2^2}{\sqrt{\frac{\epsilon_1}{\mu_1}} E_0^2} = \frac{4\sqrt{\frac{\epsilon_1 \epsilon_2}{\mu_1 \mu_2}}}{\left(\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}} \right)^2} \qquad (18)$$

[1]

(1)

$$e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = \widehat{B} + [\widehat{A}, \widehat{B}] + \frac{1}{2!}[\widehat{A}, [\widehat{A}, \widehat{B}]] + ... \tag{19}$$

上の公式で、

$$\widehat{A} = -\frac{i}{\hbar}a\widehat{p} \quad , \quad \widehat{B} = x \tag{20}$$

として、

$$\widehat{U}(a)\widehat{U}^{\dagger}(a) = \exp\left(-\frac{i}{\hbar}a\widehat{p}\right) x \exp\left(\frac{i}{\hbar}a\widehat{p}\right) \tag{21}$$

$$= x + \left[-\frac{i}{\hbar}a\widehat{p}, x\right] + \frac{1}{2!}\left[-\frac{i}{\hbar}a\widehat{p}, \left[-\frac{i}{\hbar}a\widehat{p}, x\right]\right] + ... \tag{22}$$

$$= x - \frac{i}{\hbar}a[\widehat{p}, x] \tag{23}$$

$$= x - \frac{i}{\hbar}a \cdot \frac{\hbar}{i} \tag{24}$$

$$= x - a \tag{25}$$

(2)

$$\widehat{U}(a) = \exp(-\frac{i}{\hbar}a\widehat{p}) \tag{26}$$

$$= \exp(-a\frac{\partial}{\partial x}) \tag{27}$$

$$= 1 + (-a)\frac{\partial}{\partial x} + \frac{1}{2!}(-a)^2\frac{\partial^2}{\partial^2 x} + ... \tag{28}$$

∴

$$\widehat{U}(a)u_n(x) = (1 + (-a)\frac{\partial}{\partial x} + \frac{1}{2!}(-a)^2\frac{\partial^2}{\partial^2 x} + ...)u_n(x) \tag{29}$$

$$= u_n(x) + (-a)\frac{\partial u_n(x)}{\partial x} + \frac{1}{2!}(-a)^2\frac{\partial^2 u_n(x)}{\partial^2 x} + ... \tag{30}$$

$$= u_n(x - a) \tag{31}$$

(3)

$$
\begin{aligned}
(2)\text{ の左辺に}\widehat{U}(a)\text{ を作用} \quad &\rightarrow \quad \widehat{U}(a)\widehat{H}(x)u_n(x) && (32)\\
&= \quad \widehat{U}(a)\widehat{H}(x)\widehat{U}^\dagger(a)\widehat{U}(a)u_n(x) && (33)\\
\widehat{U}(a)\widehat{H}(x)\widehat{U}^\dagger(a) &= \quad \widehat{U}(a)\left(\frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2x^2\right)\widehat{U}^\dagger(a) && (34)\\
&= \quad \frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2x^2 && (35)\\
&\quad\quad +\left[-\frac{i}{\hbar}a\widehat{p},\frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2x^2\right] && (36)\\
&\quad\quad +\frac{1}{2!}\left[-\frac{i}{\hbar}a\widehat{p},\left[-\frac{i}{\hbar}a\widehat{p},\frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2x^2\right]\right]\cdots && (37)\\
&= \quad \frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2x^2-2a\cdot\frac{1}{2}m\omega^2x && (38)\\
&\quad\quad +a^2\cdot\frac{1}{2}m\omega^2 && (39)\\
&= \quad \frac{\widehat{p}^2}{2m}+\frac{1}{2}m\omega^2(x-a)^2 && (40)\\
&= \quad \widehat{H}(x-a) && (41)\\
\widehat{U}(a)u_n(x) &= \quad u_n(x-a) && (42)\\
\therefore\widehat{H}(x-a)u_n(x-a) &= \quad E_nu_n(x-a) && (43)
\end{aligned}
$$

[2]

(1)

(9) より

$$
\widehat{p}=\frac{\sqrt{2m\hbar\omega}}{-2i}(\widehat{a}^\dagger-\widehat{a}) \tag{44}
$$

$$
\begin{aligned}
\therefore\widehat{U}(a)=\exp-\frac{i}{\hbar}a\widehat{p} \quad &= \quad \exp-\frac{i}{\hbar}a\cdot\frac{\sqrt{2m\hbar\omega}}{-2i}(\widehat{a}^\dagger-\widehat{a}) && (45)\\
&= \quad \exp\left[\frac{a}{\sqrt{2}x_0}(\widehat{a}^\dagger-\widehat{a})\right] && (46)
\end{aligned}
$$

(2)

$$\hat{U}(a)\hat{a}\hat{U}(a)^\dagger = \hat{a} + \left[\frac{a}{\sqrt{2}x_0}(\hat{a}^\dagger - \hat{a}), \hat{a}\right] + \frac{1}{2!}\left[\frac{a}{\sqrt{2}x_0}(\hat{a}^\dagger - \hat{a}), \left[\frac{a}{\sqrt{2}x_0}(\hat{a}^\dagger - \hat{a}), \hat{a}\right]\right] + \cdots \quad (47)$$

$$= \hat{a} - \frac{a}{\sqrt{2}x_0} \quad (48)$$

(3)

$$\text{(12) の左辺に}\hat{U}(a)\text{ を作用} \rightarrow \hat{U}(a)\widehat{H}(a)|n\rangle \quad (49)$$

$$= \hat{U}(a)\widehat{H}(a)\hat{U}^\dagger(a)\hat{U}(a)|n\rangle \quad (50)$$

$$\hat{U}(a)\widehat{H}(a)\hat{U}^\dagger(a) = \hat{U}(a)\cdot\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{U}^\dagger(a) \quad (51)$$

$$= \hbar\omega\left(\hat{U}(a)\hat{a}^\dagger\hat{a}\hat{U}^\dagger(a) + \frac{1}{2}\right) \quad (52)$$

$$= \hbar\omega\left(\hat{U}(a)\hat{a}^\dagger\hat{U}^\dagger(a)\hat{U}(a)\hat{a}\hat{U}^\dagger(a) + \frac{1}{2}\right) \quad (53)$$

$$= \hbar\omega\left(\hat{\alpha}^\dagger\hat{\alpha} + \frac{1}{2}\right) \quad (54)$$

$$= \widehat{H}(\hat{\alpha}) \quad (55)$$

$$\hat{U}(a)|n\rangle = \hat{U}(a)\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad (56)$$

$$= \hat{U}(a)\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\hat{U}^\dagger(a)\hat{U}(a)|0\rangle \quad (57)$$

$$= \frac{1}{\sqrt{n!}}(\hat{\alpha}^\dagger)^n|0(a)\rangle \quad (58)$$

$$= |n(a)\rangle \quad (59)$$

$$\text{右辺} \rightarrow \hat{U}(a)E_n|n\rangle \quad (60)$$

$$= E_n\hat{U}(a)|n\rangle \quad (61)$$

$$= E_n|n(a)\rangle \quad (62)$$

$$\therefore \widehat{H}(\hat{\alpha})|n(a)\rangle = E_n|n(a)\rangle \quad (63)$$

(4)

$$(23)\ \text{より、}\hat{\alpha}|0(a)\rangle = 0\ \text{で、}\hat{\alpha} = \hat{a} - \frac{a}{\sqrt{2}x_0}\ \text{より} \tag{64}$$

$$\left(\hat{a} - \frac{a}{\sqrt{2}x_0}\right)|0(a)\rangle = 0 \to \hat{a}|0(a)\rangle = \frac{a}{\sqrt{2}x_0}|0(a)\rangle \tag{65}$$

(問 1)
(1)

$$H = \frac{p^2}{2m} + \frac{1}{2}m(2\pi\nu)^2 q^2 \tag{66}$$

$$\begin{aligned} E &= \frac{p^2}{2m} + \frac{1}{2}m(2\pi\nu)^2 q^2 & (67) \\ \rightarrow &\ \frac{p^2}{2mE} + \frac{m(2\pi\nu)^2}{2E} q^2 = 1 & (68) \\ \rightarrow &\ \left(\frac{p}{\sqrt{2mE}}\right)^2 + \left(2\pi\nu\sqrt{\frac{m}{2E}}q\right)^2 = 1 & (69) \end{aligned}$$

よって

$$a = \sqrt{2mE}, b = \frac{1}{2\pi\nu}\sqrt{\frac{2E}{m}} \tag{70}$$

とすると、これの面積は

$$\pi ab = \pi \times \sqrt{2mE} \times \frac{1}{2\pi\nu}\sqrt{\frac{2E}{m}} = \frac{E}{\nu} \tag{71}$$

(2)

$$E = nh\nu, J = \frac{E}{\nu} \tag{72}$$

$$\therefore J = \frac{E}{\nu} = \frac{nh\nu}{\nu} = nh \tag{73}$$

(問 2)
(1)

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d}{dt}(\vec{r} \times \vec{p}) & (74) \\ &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 & (75) \end{aligned}$$

(2)

$$x = r\cos\theta, y = r\sin\theta \tag{76}$$

$$\begin{aligned} |\vec{L}| &= |\vec{r}\times\vec{p}| & (77)\\ &= \left|\begin{pmatrix} x \\ y \end{pmatrix} \times \begin{pmatrix} m\dot{x} \\ m\dot{y} \end{pmatrix}\right| & (78)\\ &= mx\dot{y} - my\dot{x} & (79)\\ &= m(x\dot{y} - y\dot{x}) & (80)\\ &= m\left(r\cos\theta r\cos\theta\frac{d\theta}{dt} + r\sin\theta r\sin\theta\frac{d\theta}{dt}\right) & (81)\\ &= mr^2\frac{d\theta}{dt} & (82) \end{aligned}$$

(3)

$$\begin{aligned} J &= \int_0^{2\pi} p_\theta d\theta & (83)\\ &= \int_0^{2\pi} L d\theta & (84)\\ &= 2\pi L & (85)\\ &= nh & (86) \end{aligned}$$

$$\therefore L = \frac{nh}{2\pi} \tag{87}$$