Chubu Summer School
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BPS index of E-strings

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String Theory predicts the existence of nontrivial QFTs in 6D.

- The worldvolume theory of multiple M5 branes

--- (2,0) SUSY (16 supercharges)

What about theories with (1,0) SUSY?

- Heterotic string theories on K3 (16 ⟶ 8 supercharges)

What happens when instantons in K3 shrink to zero size?

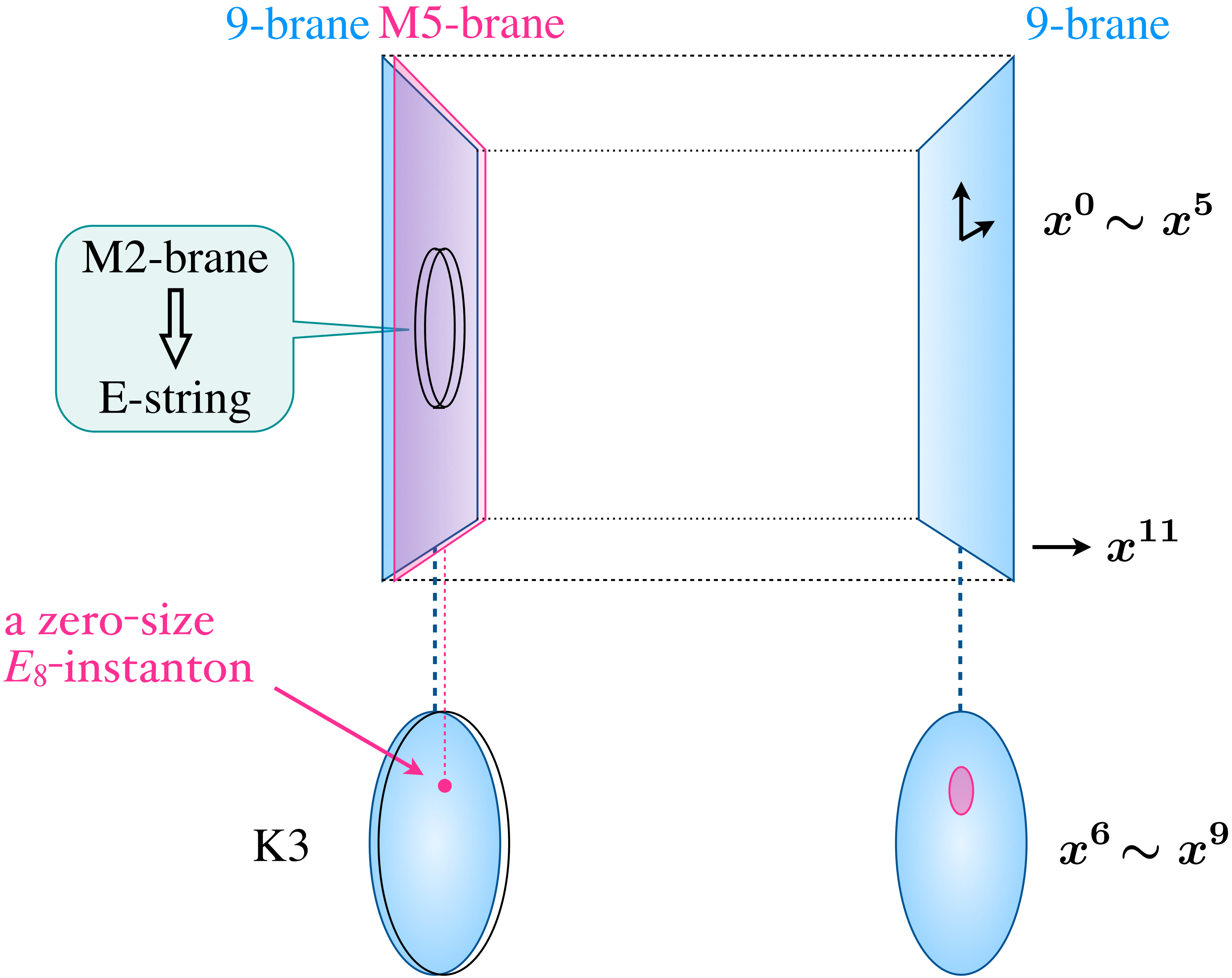
- Small SO(32) instantons

$\Rightarrow$ extra Sp(n) gauge symmetry (Witten '95)

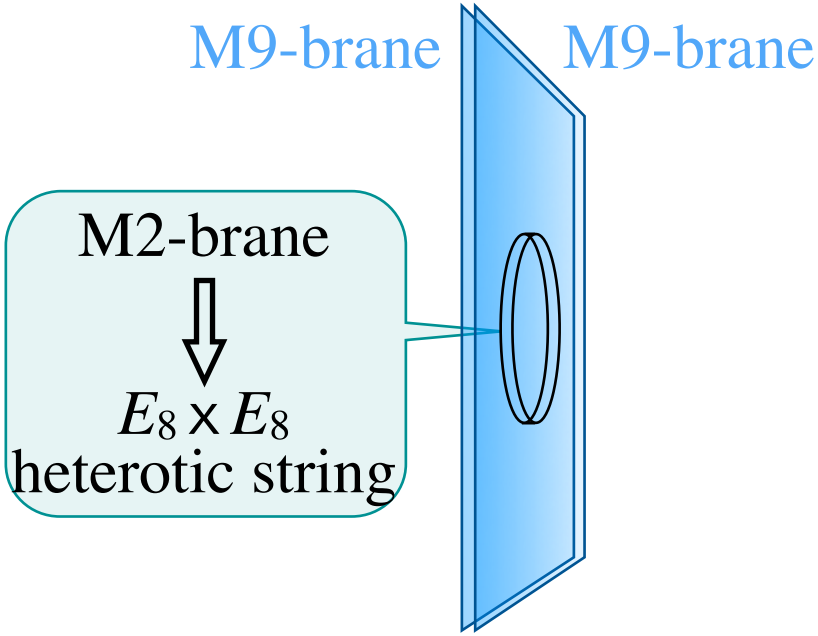
$\Leftrightarrow$ worldvolume theory of n type I D5-branes

- What about small E_8 instantons?

M-theory description of the $\boldsymbol{E_8} \times \boldsymbol{E_8}$ heterotic string theory on K3



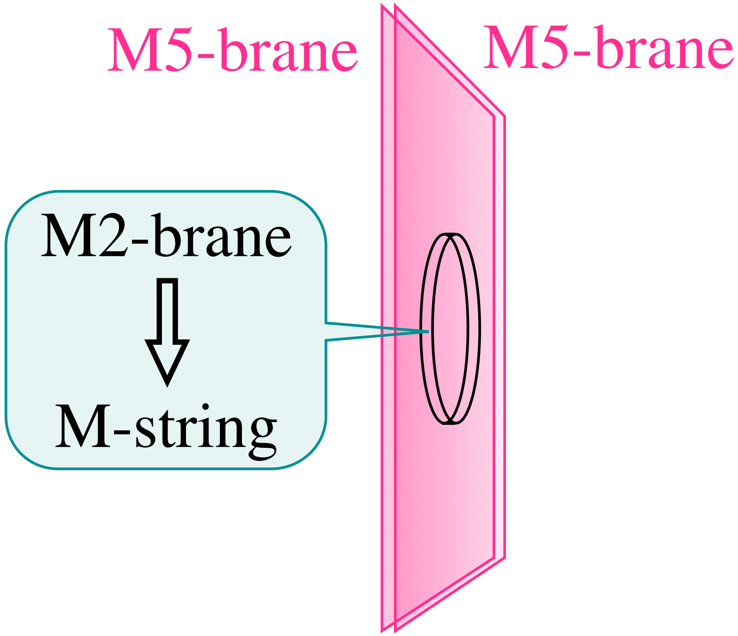
What is E-string?


M9-brane M9-brane

M2-brane

$\Downarrow$

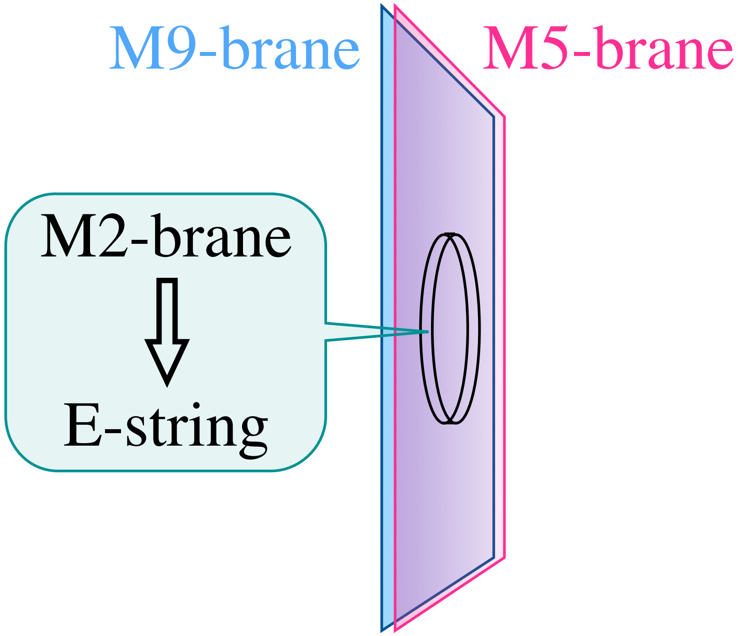
$E_8 \times E_8$
heterotic string


M5-brane M5-brane

M2-brane

$\Downarrow$

M-string


M9-brane M5-brane

M2-brane

$\Downarrow$

E-string

6次元理論の分類

構成要素（零質量の場）

- ハイパー多重項（物質場）
- ベクトル多重項（ゲージ場）
- テンソル多重項

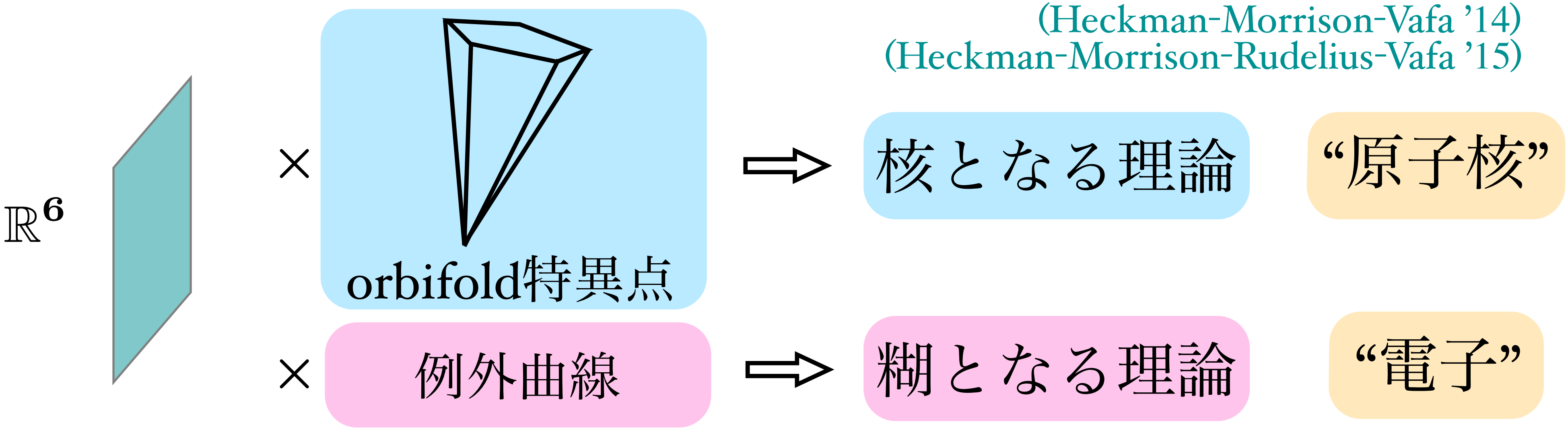
アノマリー相殺条件：場の構成に制限を与える

例）SU(2) ゲージ理論

許される物質場の数はN_f= 4,10,16あるいはN_a=1の４種類

F理論コンパクト化による6次元SCFTの構成と分類（提案）

(Heckman-Morrison-Vafa ’14)
(Heckman-Morrison-Rudelius-Vafa ’15)



$\mathbb{R}^6$ × orbifold特異点 ⇨ 核となる理論　“原子核”

× 例外曲線 ⇨ 糊となる理論　“電子”

E-string theory

(Ganor-Hanany ’96) (Seiberg-Witten ’96) (Klemm-Mayr-Vafa ’96)
(Ganor-Morrison-Seiberg ’96) (Minahan-Nemeschansky-Vafa-Warner ’98)

- 6d (1,0)-supersymmetric field theory
- decoupled from gravity
- no vector multiplets
- Coulomb branch --- a tensor multiplet
- Higgs branch $\cong$ the moduli space of an E_8 instanton
- fundamental excitations --- strings
- global E_8 symmetry
- no Lagrangian description

Why is the E-string theory interesting?

- An elementary supersymmetric field theory in 6d.

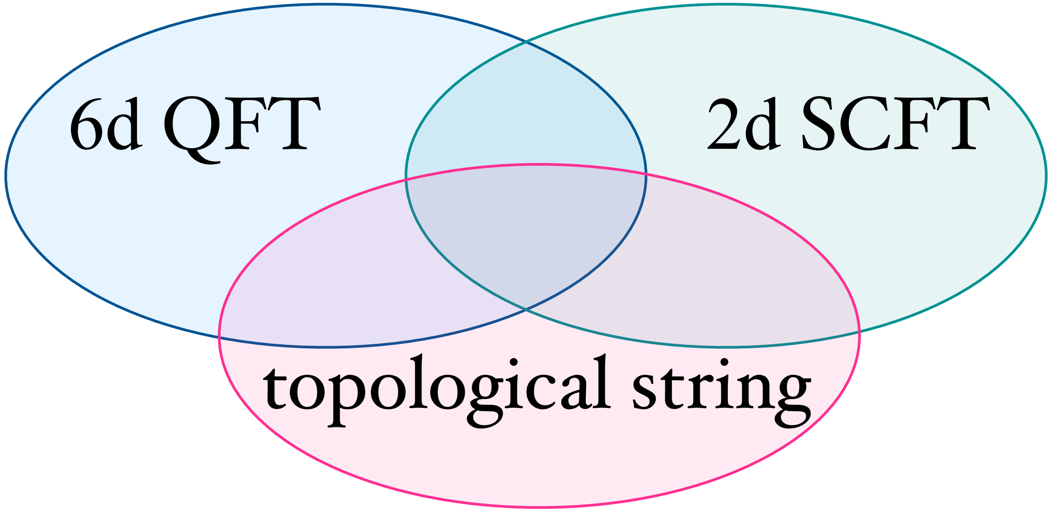
  It consists of just one tensor multiplet.

  The “glue” which holds together all of the other (1,0) theories.

  (Heckman-Morrison-Vafa ’14)
  (Heckman-Morrison-Rudelius-Vafa ’15)

- The mother theory of almost all gauge theories with rank-one gauge groups in 5d and 4d.

- The place where three entirely different physics meet.



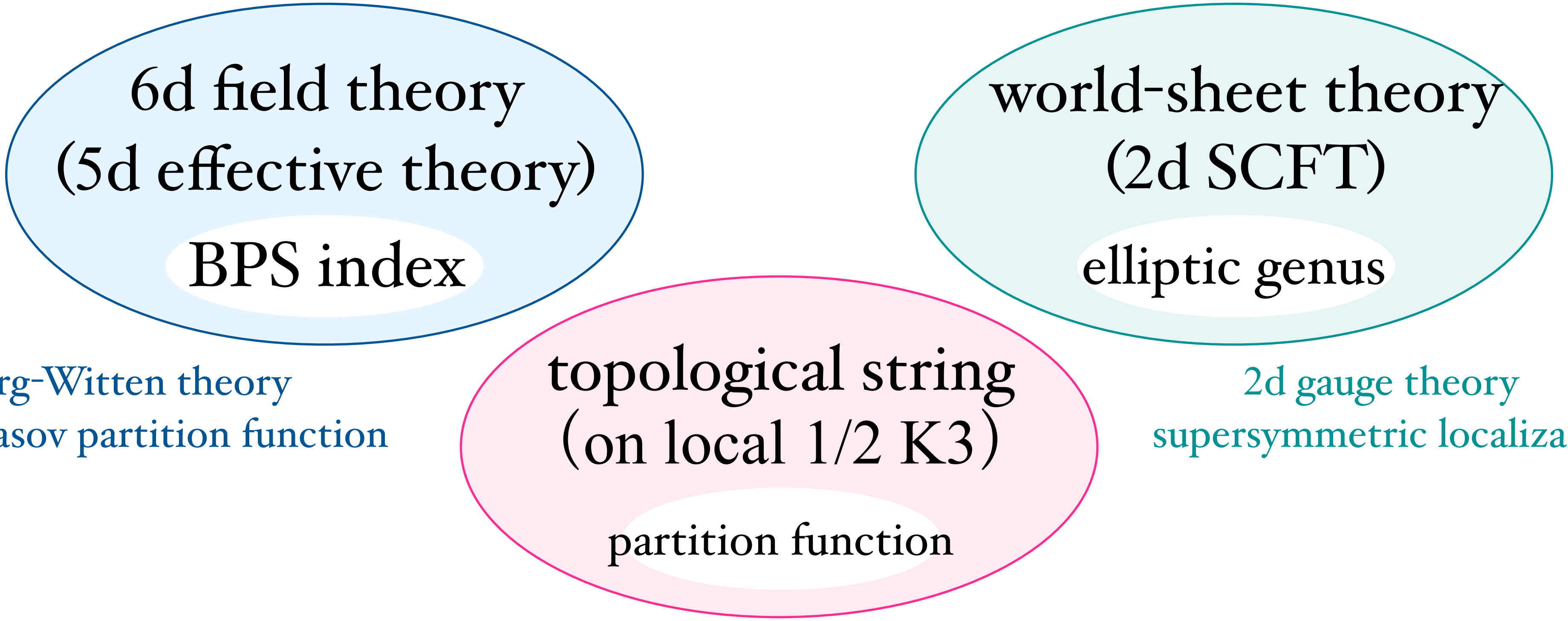
- Supersymmetric (BPS) index

$$I(\mu) = \mathrm{Tr}(-1)^F e^{i\sum_a \mu_a J_a}$$

F : fermion number
J_a : global symmetry charges
μ_a : chemical potentials

$$\left(\begin{array}{c}\text{partition function} \\ Z = \mathrm{Tr} e^{-\beta H}\end{array}\right)$$

BPS(i.e. supersymmetry protected) quantities of E-strings



6d field theory
(5d effective theory)
BPS index

world-sheet theory
(2d SCFT)
elliptic genus

topological string
(on local 1/2 K3)
partition function

Seiberg-Witten theory
Nekrasov partition function

2d gauge theory
supersymmetric localization

Topological vertex
Holomorphic anomaly equation

BPS index

E-string theory on $\mathbb{R}^5 \times S^1$

= effective 5d theory with infinitely many of particles

$$Z(\phi, \tau, \vec{m}, \epsilon_1, \epsilon_2) := \mathrm{Tr}\,(-1)^{2J_{\mathrm{L}}+2J_{\mathrm{R}}} y_{\mathrm{L}}^{J_{\mathrm{L}}} y_{\mathrm{R}}^{J_{\mathrm{R}}+J_{\mathrm{I}}} p^n q^k e^{i\vec{\Lambda}\cdot\vec{m}}$$

$$y_{\mathrm{L}} := e^{i(\epsilon_1-\epsilon_2)}, \quad y_{\mathrm{R}} := e^{i(\epsilon_1+\epsilon_2)}, \quad p := e^{-\phi}, \quad q := e^{2\pi i\tau}$$

$J_{\mathrm{L}}, J_{\mathrm{R}}, J_{\mathrm{I}}$: spins of Lorentz and R-symmetries

$\vec{\Lambda}$: weight of E_8, n: winding number, k: momentum

Expansions

$$\ln Z = \sum_{n=0}^{\infty}\sum_{g=0}^{\infty}(\epsilon_1+\epsilon_2)^{2n}(-\epsilon_1\epsilon_2)^{g-1}\boxed{F^{(n,g)}}$$

topological string amplitudes

$F^{(0,0)}$: Seiberg-Witten prepoptential

$$Z = 1 + \sum_{n=1}^{\infty} p^n \boxed{Z_n}$$

Elliptic genus of n E-strings

Elliptic genera

The worldsheet theory of the E-string theory
= 2d (0,4) superconformal field theory (SCFT)

Elliptic genus

(Shellekens-Warner '86)
(Witten '87)

$$\mathrm{Tr}_{\mathrm{R}}(-1)^F q^{H_L} \bar{q}^{H_R} \prod_a x_a^{K_a}$$

F :	fermion number
H_L, H_R :	left/right Hamiltonians
K_a :	Cartan generators of the global symmetry
x_a :	fugacities

$q = e^{2\pi i\tau}$, τ : complex structure of a torus

Seiberg-Witten theory

(Seiberg-Witten '94)

- Exact solution to the low energy theory of 4d $\mathcal{N}=2$ SYM

Low energy effective Lagrangian

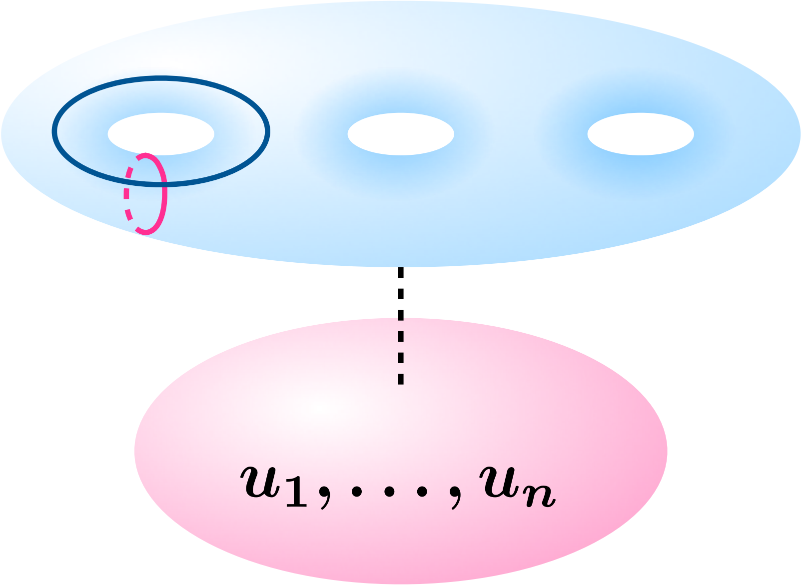
$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\text{Im}\left[\int d^4\theta \frac{\partial F_0(A)}{\partial A^i}\bar{A}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 F_0(A)}{\partial A^i \partial A^j} W^i_\alpha W^{\alpha j}\right]$$

prepotential $F_0(a_1, \ldots, a_n)$: holomorphic function

Seiberg-Witten curve

λ_{SW} : Seiberg-Witten differential

$$a_i = \oint_{\alpha_i} \lambda_{\text{SW}}, \qquad \frac{\partial F}{\partial a_i} = \oint_{\beta_i} \lambda_{\text{SW}}$$



4d pure SU(2) super Yang-Mills theory

Seiberg-Witten curve

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x$$

$$a(u,\Lambda) = \int du \oint_{\tilde{\alpha}} \frac{dx}{y}, \qquad \frac{\partial F_0}{\partial a}(u,\Lambda) = \int du \oint_{\tilde{\beta}} \frac{dx}{y}$$

⇨

$$F_0(a) = \frac{1}{2}\tau_0 a^2 + \frac{i}{\pi} a^2 \ln \frac{a^2}{\Lambda^2} + \frac{1}{2\pi i} a^2 \sum_{k=1}^{\infty} c_k \left(\frac{\Lambda}{a}\right)^4$$

$$c_1 = \frac{1}{2^5}, \quad c_2 = \frac{5}{2^{14}}, \quad c_3 = \frac{3}{2^{18}}, \quad c_4 = \frac{1469}{2^{31}}, \quad \cdots$$

E-string theory (with unbroken E_8 symmetry)

Seiberg-Witten curve

$$y^2 = 4x^3 - \frac{1}{12}E_4 u^4 x - \frac{1}{216}E_6 u^6 + 4u^5$$

$$\varphi(u,\tau) = \int du \oint_{\tilde{\alpha}} \frac{dx}{y}, \qquad \frac{\partial F_0}{\partial \varphi}(u,\tau) = \int du \oint_{\tilde{\beta}} \frac{dx}{y}$$

⇨

$$F_0(\varphi,\tau) = \sum_{n=1}^{\infty} e^{2\pi i n \varphi} q^{\frac{n}{2}} Z_n(\tau)$$

$$Z_1 = \frac{E_4}{\eta^{12}}, \qquad Z_2 = \frac{E_2 E_4^2 + 2E_4 E_6}{24\eta^{24}}, \qquad \cdots$$

$$Z_n = \frac{P_{6n-2}(E_2, E_4, E_6)}{\eta^{12n}}$$

(Minahan-Nemeschansky-Warner '97)

The most general SW curve has been determined. (Eguchi-K.S. '02)

Modular forms

- Eisenstein series E_{2n} $(n \geq 2)$ satisfy

$$E_{2n}(\tau+1) = E_{2n}(\tau), \qquad E_{2n}\left(-\frac{1}{\tau}\right) = \tau^{2n} E_{2n}(\tau)$$

- E_{2n} $(n \geq 4)$ are generated by E_4, E_6

- E_2 is anomalous:

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 \left(E_2(\tau) + \frac{6}{\pi i \tau}\right)$$

- $\hat{E}_2$ transforms as a modular form of weight 2

$$\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) + \frac{6}{\pi i(\tau - \bar{\tau})}$$

modular anomaly ⟺ holomorphic anomaly

Nekrasov partition function

(Nekrasov '02)

(for pure U(N) theory; instanton part; $\epsilon_1 = -\epsilon_2 = \hbar$)

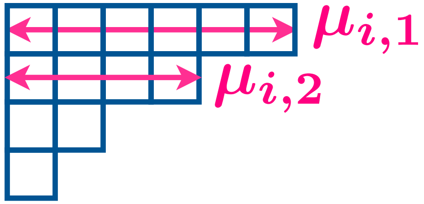
for 5D theory (in $\mathbb{R}^4 \times S^1$)

$$Z = \sum_{R} \Lambda^{|R|} \prod_{i,j=1}^{N} \prod_{k,l=1}^{\infty} \frac{\sinh \beta\left(a_{ij} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)\right)}{\sinh \beta\left(a_{ij} + \hbar(l - k)\right)}$$

$$R = (R_1, \ldots, R_N)$$

R_i : partition

$(a_{ij} := a_i - a_j)$



$$Z = \exp \sum_{g=0}^{\infty} F_g \hbar^{2g-2}$$

⇨ F_0 : prepotential

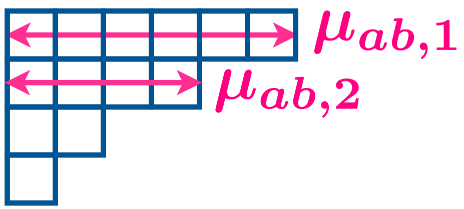
Nekrasov-type expression for the prepotential

(K.S. '12)

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

$$\mathcal{Z} = \sum_{\vec{R}} Q^{|\vec{R}|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$\vec{R} = (R_{11}, R_{10}, R_{00}, R_{01})$ $\quad$ R_{ab} : partition

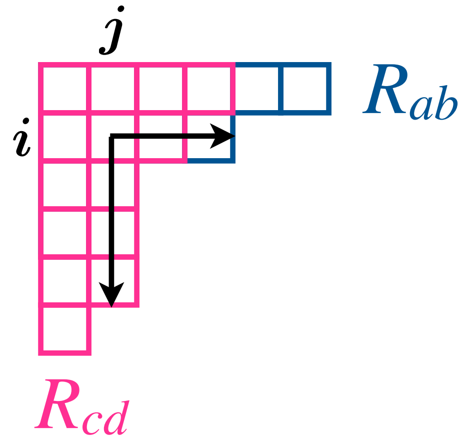


$a, b, c, d = 0, 1$

$\vartheta_{ab}(z, \tau)$: Jacobi theta functions

$h_{ab,cd}(i,j) := \mu_{ab,i} + \mu^{\vee}_{cd,j} - i - j + 1$

(relative hook-length)



$Q = q^{1/2}p$

Higher genus amplitudes

- F_g is determined by he modular anomaly eq.

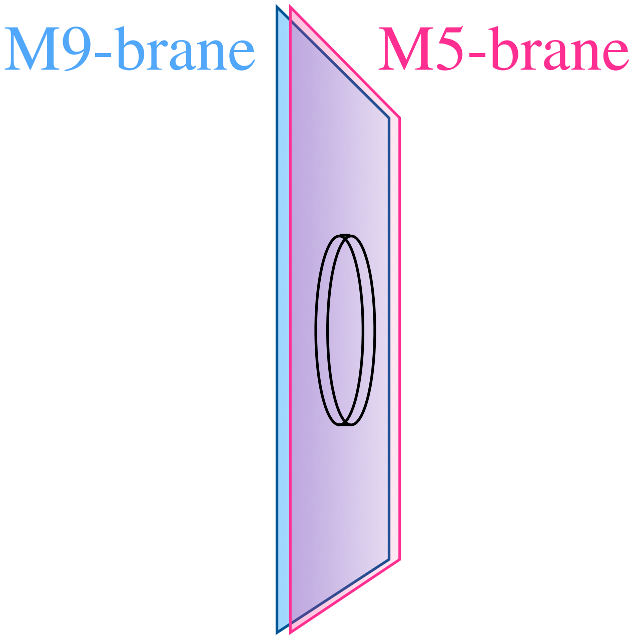
$$\partial_{E_2} Z = -\frac{1}{24}\left[\epsilon_1\epsilon_2\partial_\phi\left(\partial_\phi - 1\right) + (\epsilon_1+\epsilon_2)^2\partial_\phi\right] Z$$

(Hosono-Saito-Takahashi '99) (Huang-Klemm-Poretschkin'13)

and the data of suitable "boundary conditions"

- An improved ansatz based on the domain-wall construction

(Haghighat-Lockhart-Vafa '14)



A reduced BPS index

The general refined BPS index is a function in twelve variables:

(Klemm-Mayr-Vafa '96) (Minahan-Nemeschansky-Vafa-Warner '98)

$$Z(\phi,\tau,\vec{m},\epsilon_1,\epsilon_2) := \mathrm{Tr}\,(-1)^{2J_\mathrm{L}+2J_\mathrm{R}} y_\mathrm{L}^{J_\mathrm{L}} y_\mathrm{R}^{J_\mathrm{R}+J_\mathrm{I}} p^n q^k e^{i\vec{\Lambda}\cdot\vec{m}}$$

$\epsilon_1,\epsilon_2,m_1,\ldots,m_8$: chemical potentials for global symmetry SO(4) × E_8

ϕ,τ : tention of E-strings, inverse of the radius of S^1

- a very complicated function

$\Longrightarrow$ consider special cases with less parameters

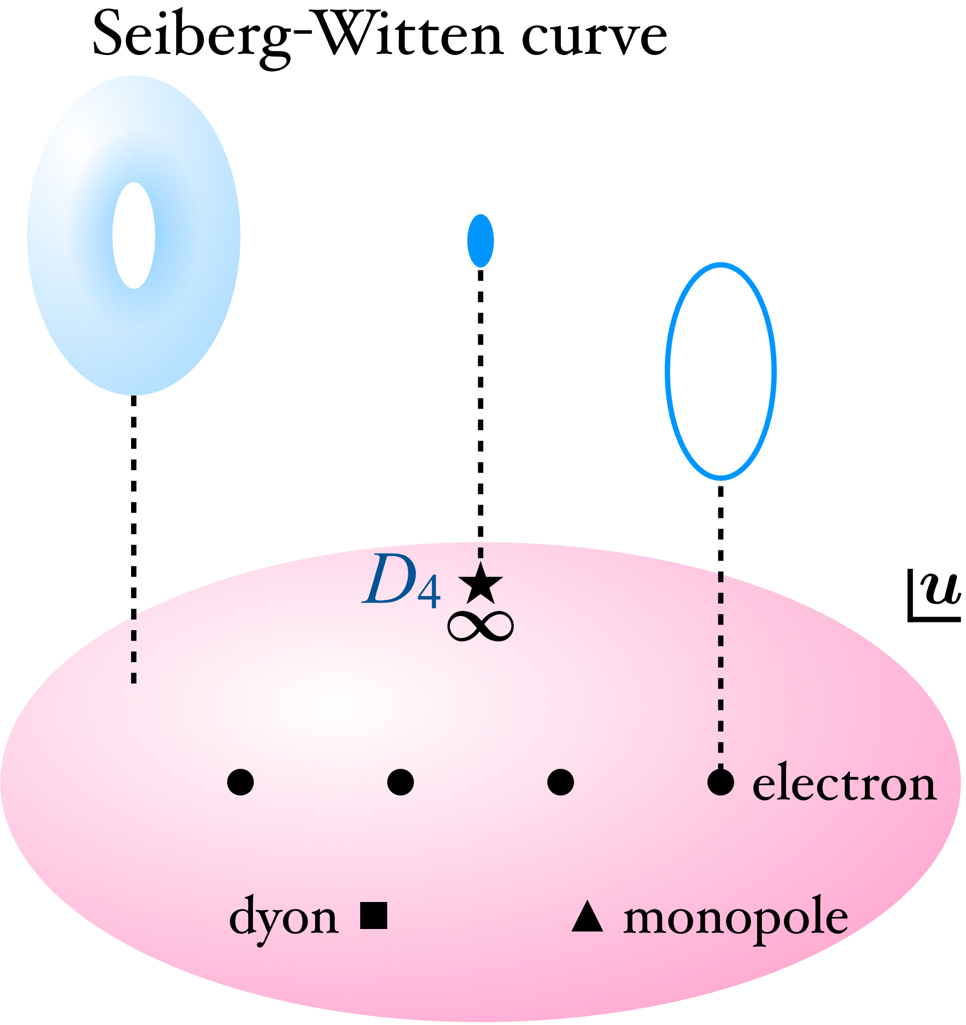
until now: unbroken E_8 $\boxed{m_1=\cdots=m_8=0}$

this time: D_4+D_4 twist

$$\begin{aligned} &m_1=m_2=0, && m_3=m_4=\pi,\\ &m_5=m_6=-\pi-\pi\tau, && m_7=m_8=\pi\tau. \end{aligned}$$

- simpler than the unbroken E_8 case!
- modular properties are kept intact

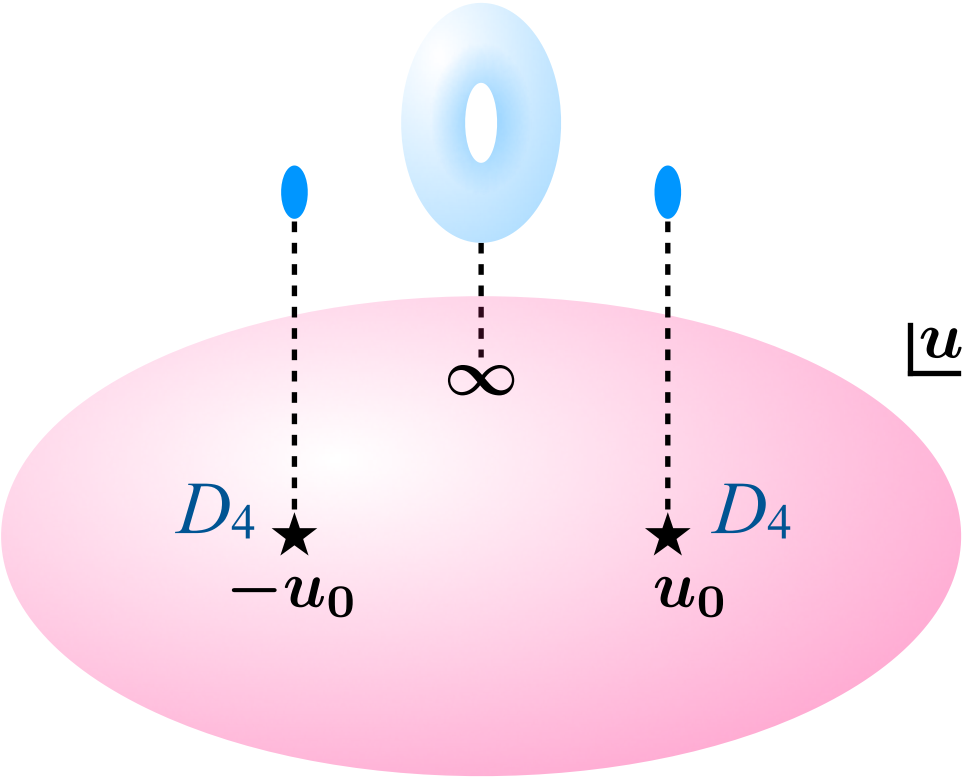
E-strings under the D_4+D_4 twist vs. 4d SU(2) N_f=4 theory



Moduli space of
SU(2) N_f=4 theory

Monodromy around D_4 singularity

$$\begin{pmatrix} a_\mathrm{D}(u) \\ a(u) \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_\mathrm{D}(u) \\ a(u) \end{pmatrix}$$



Moduli space of the E-string
theory under the D_4+D_4 twist

Main results

- We propose a reduced index of E-strings
    - corresponding to the D_4+D_4 twist
    - very much simplified, easy to deal with
    - The index satisfies the same modular anomaly equation as that of the original BPS index

- We study the index by performing three kinds of expansions that correspond to three entirely different physical pictures

- We find that the reduced index gives a novel trigonometric generalization of the Nekrasov partition function for 4d $\mathcal{N} = 2$ SU(2) super Yang-Mills with massless $N_\mathrm{f} = 4$ flavors.

Definition of the reduced index of E-strings

$$Z(\phi,\tau,\epsilon_1,\epsilon_2) :=$$

D_4+D_4 twist

$$\mathrm{Tr}\,(-1)^{2J_\mathrm{L}+2J_\mathrm{R}\boxed{+\Lambda_3+\Lambda_4-\Lambda_5-\Lambda_6}}\,y_\mathrm{L}^{J_\mathrm{L}}\,y_\mathrm{R}^{J_\mathrm{R}+J_\mathrm{I}}\,\boxed{p^n q^k}^{\boxed{+(-\Lambda_5-\Lambda_6+\Lambda_7+\Lambda_8)/2}}$$

$$y_\mathrm{L} := e^{i(\epsilon_1-\epsilon_2)}, \quad y_\mathrm{R} := e^{i(\epsilon_1+\epsilon_2)}, \quad \boxed{p := e^{-\phi}, \quad q := e^{2\pi i\tau}}$$

$J_\mathrm{L}, J_\mathrm{R}, J_\mathrm{I}$: spins of Lorentz and R symmetries

$\Lambda_1, \ldots, \Lambda_8$: E_8 weight

n, k : winding number and momentum along S^1

Expansions	expansion coefficients
(1) expansion in ϵ_1, ϵ_2 (genus expansion)	topological string amplitudes
(2) expansion in q (instanton expansion)	instanton corrections (spacetime picture)
(3) expansion in p (winding # expansion)	elliptic genera (worldsheet picture)

(1) genus expansion and topological string amplitudes

$$\ln Z = F = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\phi, \tau)$$

F_g : topological string amplitudes for local 1/2 K3

(evaluated under D_4+D_4 twist)

(We set $\epsilon_1 = -\epsilon_2 = \hbar$)

- F_g is determined by the modular anomaly eq.

$$\partial_{E_2} Z = -\frac{1}{24}\left[\epsilon_1\epsilon_2\partial_\phi\left(\partial_\phi - 1\right) + (\epsilon_1+\epsilon_2)^2\partial_\phi\right] Z$$

and the data of q-expansion (next slide)

$$F_0 = 0,$$

$$F_1 = -\frac{1}{2}\ln(1-e^{-2\phi}),$$

$$F_2 = \boxed{\frac{1}{32}\frac{E_2}{\sinh^2\phi}},$$

$$F_3 = \frac{1}{768}\frac{3E_2^2+E_4}{\sinh^2\phi} + \boxed{\frac{1}{384}\frac{2E_2^2+E_4}{\sinh^4\phi}},$$

$$F_4 = \frac{1}{46080}\frac{15E_2^3+15E_2E_4+4E_6}{\sinh^2\phi} + \frac{1}{55296}\frac{107E_2^3+144E_2E_4+55E_6}{\sinh^4\phi} + \boxed{\frac{1}{6144}\frac{11E_2^3+16E_2E_4+7E_6}{\sinh^6\phi}},$$

$$\vdots$$

- infinite power series in $p^2 = e^{-2\phi}$ are expressed in closed forms

- $\phi \to \beta a,\ \hbar \to -i\beta\hbar,\ \beta \to 0$ limit = 4d SU(2) N_f= 4 Nekrasov

(2) q-expansion: perturbative term and instanton corrections

$$F = \boxed{F^{\text{pert}}(\phi, \epsilon_1, \epsilon_2)} + \sum_{k=1}^{\infty} q^k \boxed{F_k^{q\text{-inst}}(\phi, \epsilon_1, \epsilon_2)}$$

- perturbative term (conjecture; verified for $\phi \to \pm\infty,\ \phi \to 0$)

$$F^{\text{pert}} = -\sum_{n=1}^{\infty} \frac{\sin^2(n\epsilon_1/2) + \sin^2(n\epsilon_2/2) + \sin^2(n(\epsilon_1+\epsilon_2)/2)}{n \sin(n\epsilon_1)\sin(n\epsilon_2)} e^{-2n\phi}$$

$\Leftrightarrow$

$$Z^{\text{pert}} = \exp\left(F^{\text{pert}}\right) = \frac{\left[\prod_{k,l=1}^{\infty}\left(1 - q_1^{2k-1} t_2^{2l-1} p^2\right)\right]^8}{\left[\prod_{k,l=1}^{\infty}\left(1 - q_1^{k-1} t_2^{l} p^2\right)\right]\left[\prod_{k,l=1}^{\infty}\left(1 - q_1^{k} t_2^{l-1} p^2\right)\right]}$$

$(q_1 := e^{i\epsilon_1},\ t_2 := e^{-i\epsilon_2})$

  - similar to 5d SU(2) $N_{\rm f} = 4$ $Z^{\text{pert}}_{\text{Nekrasov}}$

- instanton corrections (here we present numerators only; $\epsilon_1 = -\epsilon_2 = \hbar$)

$$\left[F_1^{q\text{-inst}}\right]_{\text{denom}} = \cos^2\frac{\hbar}{2}\left(\sinh^2\phi + \sin^2\frac{\hbar}{2}\right)^2$$

$$\left[F_2^{q\text{-inst}}\right]_{\text{denom}} = 2\cos^2\frac{\hbar}{2}\cos^2\frac{2\hbar}{2}\left(\sinh^2\phi + \sin^2\frac{\hbar}{2}\right)^4\left(\sinh^2\phi + \sin^2\frac{3\hbar}{2}\right)^2$$

  - this gives us a hint about how to generalize 4d Z_{Nekrasov} to 6d

(3) winding number expansion and elliptic genera

$$Z = 1 + \sum_{n=1}^{\infty} p^{2n} Z_{2n}(\tau, \epsilon_1, \epsilon_2)$$

Z_{2n} : Elliptic genus of $2n$ E-strings

(evaluated under D_4+D_4 twist)

- Explicit form of Z_2 :

$$\boxed{Z_2 = 2\frac{\vartheta_1(\epsilon_1)\vartheta_1(\epsilon_2)\vartheta_1(\epsilon_3)}{\eta^3\vartheta_1(2\epsilon_1)\vartheta_1(2\epsilon_2)}\left(\zeta(\epsilon_1)+\zeta(\epsilon_2)+\zeta(\epsilon_3)\right)}$$

$(\epsilon_3 := -\epsilon_1 - \epsilon_2)$

  - It follows from the general form of Z_2 (Haghighat-Lockhart-Vafa '14)

- Conjecture about Z_{2n} (based on modularity and singularity structure)

$$\boxed{Z_{2n} = \frac{1}{\eta^{8(n^3-n)}}\frac{\vartheta_1(\hbar)^{(8n^3-2n)/3}}{\prod_{k=1}^{n}\vartheta_1(2k\hbar)^2}\left[\text{polynomial in } \wp(\hbar), E_4, E_6 \text{ of weight } \frac{8(n^3-n)}{3}\right]}$$

(We set $\epsilon_1 = -\epsilon_2 = \hbar$)

- Z_4 is determined as

$$\boxed{Z_4 = \frac{1}{2\eta^{48}}\frac{\vartheta_1(\hbar)^{20}}{\vartheta_1(4\hbar)^2\vartheta_1(2\hbar)^2}\left(72\wp'^4\wp^2 - 18\wp''^2\wp'^2\wp + 2\wp''\wp'^4 + \wp''^4\right)}$$

$$\wp = \wp(\hbar), \wp' = \wp''(\hbar), \wp'' = \wp''(\hbar)$$

World-sheet approach

(Kim-Kim-Lee-Park-Vafa '14)

11d M

	0	1	2	3	4	5	6	7	8	9	10
M5	●	●		●	●	●	●				
M9	●	●		●	●	●	●	●	●	●	●
M2	●	●	●								

↓

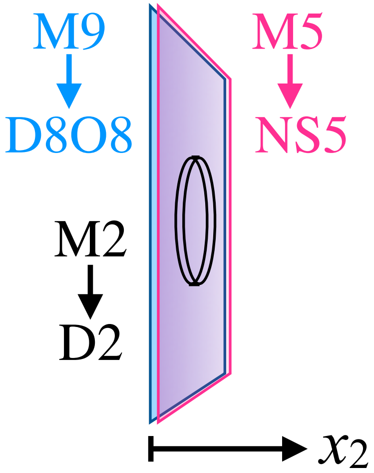
10d IIA

	0	1	2	3	4	5	6	7	8	9
NS5	●	●		●	●	●	●			
8 D8 + O8	●	●		●	●	●	●	●	●	●
n D2	●	●	●							

2d (0,4) supersymmetric gauge theory

O(*n*) antisymmetric vector
O(*n*) symmetric hyper
O(*n*) x SO(16) bifundamental Fermi

IR limit = world-sheet theory of E-strings!



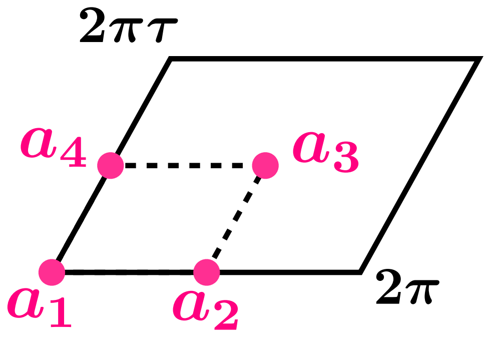
Elliptic genera of general 2d $\mathcal{N} = 2$ gauge theories can be computed by using the supersymmetric localization technique.

(Benini-Eager-Hori-Tachikawa '13)

There appear compact zero modes from the path integral, coming from the flat connections on T^2 (Wilson lines).

Classification of flat connections (O(n) group elements U_1, U_2)
= Classification of possible (fractional) D0 brane configurations

$$n \text{ D2 and an O2 on } T^2 \xrightarrow[\text{T-dual}]{} n \text{ D0 on } T^2/Z_2$$



There are disconnected sectors $\Longrightarrow$ E_8 symmetry enhancement

$$Z_n = \sum_a \frac{1}{|W_a|} \cdot \frac{1}{(2\pi i)^r} \oint Z^{(a)}_{\text{1-loop}}, \qquad Z^{(a)}_{\text{1-loop}} = Z^{(a)}_{\text{vector}} Z^{(a)}_{\text{hyper}} Z^{(a)}_{\text{Fermi}}$$

$$Z^{(a)}_{\text{vector}} = \prod_{i=1}^{r} \left(\frac{2\pi\eta^2 du_i}{i} \cdot \frac{\vartheta_1(\epsilon_1+\epsilon_2)}{i\eta} \right) \cdot \prod_{\alpha\in\text{root}} \frac{\vartheta_1(\alpha(u))\vartheta_1(\epsilon_1+\epsilon_2+\alpha(u))}{i^2\eta^2},$$

$$Z^{(a)}_{\text{hyper}} = \prod_{\rho\in\text{sym}} \frac{i\eta}{\vartheta_1(\epsilon_1+\rho(u))} \cdot \frac{i\eta}{\vartheta_1(\epsilon_2+\rho(u))}, \qquad Z^{(a)}_{\text{Fermi}} = \prod_{\rho\in\text{fund}} \prod_{l=1}^{8} \frac{\vartheta_1(m_l+\rho(u))}{i\eta}$$

The contour integral over the continuous parameters is defined (and can be evaluated) by using the Jeffrey-Kirwan residues.

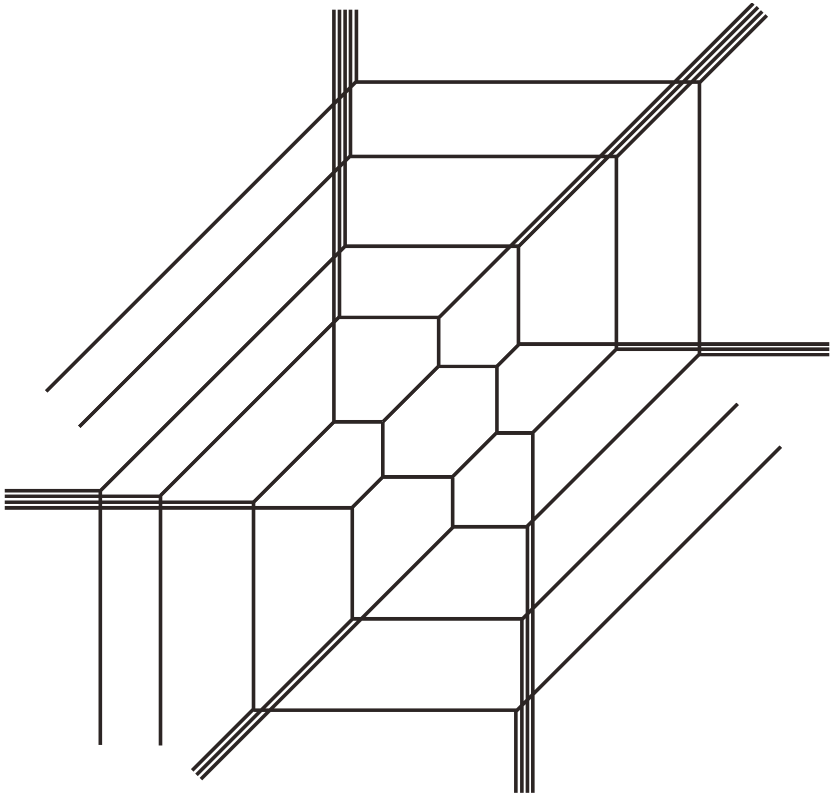
- Kim-Kim-Lee-Park-Vafa computed Z_n with n=1,2,3,4.
  Z_4 in the D4+D4 limit is in perfect agreement with our result.

- Z_n can in principle be computed for arbitrary n.
  The computation quickly becomes complicated for higher n.

'Toric' Web diagram for E-string theory

(Kim-Taki-Yagi '15)

- Tao (p,q) 5-brane web
  - One can make full use of the topological vertex formalism!



  - Sum over infinitely many of partitions
  - E_8 symmetry is not manifest

Outlook

- Generalization to the massive $N_\mathrm{f} = 4$ case
  (nonzero D_4 chemical potentials)
- Nekrasov-type formula for the BPS index
  (with general ϵ_1, ϵ_2)
- AGT correspondence
  (2d CFT interpretation of the BPS index of E-strings)

Flavor symmetries of possible effective U(1) theories

$$\begin{tikzcd}
\text{6D} & \hat{E}_8 \arrow[dd] & & & & & & & & \\
 & & & & & & & & E_1 \arrow[r] & E_0 \\
\text{5D} & E_8 \arrow[r] \arrow[dd] & E_7 \arrow[r] \arrow[dd] & E_6 \arrow[r] \arrow[dd] & E_5 \arrow[r] \arrow[dd] & E_4 \arrow[r] \arrow[dd] & E_3 \arrow[r] \arrow[dd] & E_2 \arrow[ur] \arrow[dr] \arrow[dd] & & \\
 & & & & & & & & \tilde{E}_1 \arrow[d] & \\
\text{4D} & E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & D_1 \arrow[r] \arrow[d] & D_0 & \\
 & & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & &
\end{tikzcd}$$

(Minahan-Nemeschansky ’96)

(Argyres-Plesser-Seiberg-Witten ’95)

$$E_9 = \hat{E}_8$$