Doctral Dissertation

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Search for extra dimensional model as physics beyond the standard model

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Abstract

In 2012, the Higgs boson, which is a last piece of the standard model (SM), was discovered by the experiments at Large Hadron Collider (LHC) in CERN. Hence, correctness of the SM has been proved. But, because the SM includes some problems, it is not a perfect theory. For example, the SM dose not include gravitational interaction and can not explain the origin of three families for quarks and leptons, the mass of neutrino, and the identity of dark matter and dark energy is unknown, and so on. In order to explain the history of universe, physics beyond the SM is needed.

Physics beyond the SM such as supersymmetry (SUSY), higher dimensional theories and technicolor (composite) theories has been proposed, and those theories predict new particles. However, no particles other than the SM ones have been found until now. This means that new physics should exist at a very high energy scale, and the SM should be effective up to such a scale.

In this thesis, we focus on a *family unification* and the origin and identity of unknown new particles. And, in order to solve those problem, we use the higher dimensional theory including extra dimensional spaces called *orbifold*.

In the SM, matter particles are composed of six types of quarks and leptons. However, in the early universe, those particles could not be distinguished in the framework of grand unified theories (GUTs). Therefore, we construct a unification model that all matter particles are unified under a large gauge group.

By considering $SU(N)$ gauge theory on six-dimensional (6D) space-time $M^4 \times T^2/\mathbb{Z}_M (M = 2, 3, 4, 6)$, we search the models to unify families and obtained enormous number of models with three families of $SU(5)$ matter multiplets and these with three families of the SM multiplets, from a single massless Dirac fermion with a higher-dimensional representation of $SU(N)$. We also study the relationship between the family number of chiral fermions and the Wilson line phases, based on the orbifold family unification. We show that flavor numbers are independent of Wilson line phases relating extra-dimensional component of higher-dimensional gauge field and this feature originates from a quantum-mechanical SUSY.

Next, we study phenomenological aspects of orbifold family unification models with $SU(9)$ gauge group on a 6D space-time including the orbifold $T^2/\mathbb{Z}_2$. Especially, we focus on a mass acquirement of the SM matter particles. And, we also predict relations among sfermion masses in the SUSY extension of models.

We explain the reason why new particles have not been discovered using gauge theory on 5D based on 1D orbifold $S^1/\mathbb{Z}_2$. We propose an idea that new particles can be separated according to gauge quantum numbers from the SM ones by the difference of boundary conditions (BCs) on extra dimensions, e.g. zero modes due to orbifold breaking by *inner automorphisms* correspond to the SM particles, and zero modes due to orbifold breaking by *outer automorphisms* correspond to new particles. We apply this idea on a *gauge-Higgs inflation scenario*. This model contains inflaton which causes the inflation and dark matter, but they hardly interact with the SM particles.

Contents

List of Figures

List of Tables

1 Introduction

The standard model (SM) contains by electromagnetic, weak and strong interaction and is constructed by gauge principle concerning the gauge group $G_{\rm SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ symmetry describes the strong interaction. The $SU(2)_L \times U(1)_Y$ symmetry is spontaneously broken down to $U(1)_{\rm EW}$ by the Higgs mechanism, and the unbroken symmetry describes the electromagnetic interaction and the broken one describes the weak interaction. Under this gauge group, the SM includes 12 matter particles (Table 1.1), 3 types of gauge bosons corresponding to $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ and the Higgs particles. Gauge quantum numbers of the SM particles are indicated in Table 1.2, 1.3 and 1.4. Gauge group, gauge couplings and gauge particles of the SM are summarized in Table 1.5.

		1$^{\rm st}$ generation	2$^{\rm nd}$ generation	3$^{\rm rd}$ generation
quarks	q_L^i	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$
	u_R^i	u_R	c_R	t_R
	d_R^i	d_R	s_R	b_R
leptons	l_L^i	$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$
	ν_R^i	ν_{eR}	$\nu_{\mu R}$	$\nu_{\tau R}$
	e_R^i	e_R	μ_R	τ_R

Table 1.1: The SM matter particles.

Matter particles	$SU(3)_C$	$SU(2)_L$	T_L^3	Y	$Q(=T_L^3+Y)$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	**3**	**2**	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$
u_R	**3**	**1**	0	$\frac{2}{3}$	$\frac{2}{3}$
d_R	**3**	**1**	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	**1**	**2**	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$
e_R	**1**	**1**	0	-1	-1
ν_{eR}	**1**	**1**	0	0	0

Table 1.2: Gauge quantum number of the SM matter particles.

The SM is verified with high accuracy by experiments. In 2012, *the Higgs particle* was discovered at the Large Hadron Collider (LHC) in the CERN. As a result, the SM has been completed. However, there are some unsolved problems in the SM frame. For example,

· Quantization of gravity

Gauge particles	$SU(3)_C$	$SU(2)_L$	T_L^3	Y	$Q(=T_L^3+Y)$
G_μ^α	$\mathbf{8}$	$\mathbf{1}$	0	0	0
$W_\mu^a \Rightarrow \begin{pmatrix} W_\mu^+ \\ W_\mu^0 \\ W_\mu^- \end{pmatrix}$	$\mathbf{1}$	$\mathbf{3}$	$\begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}$	0	$\begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}$
B_μ	$\mathbf{1}$	$\mathbf{1}$	0	0	0

Table 1.3: Gauge quantum number of the SM gauge particles.

Higgs particle	$SU(3)_C$	$SU(2)_L$	T_L^3	Y	$Q(=T_L^3+Y)$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Table 1.4: Gauge quantum number of the SM Higgs particles.

Gauge group	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
Gauge coupling	g_s	g	g'
Gauge particle	G_μ^α	W_μ^a	B_μ
Generator	$T_C^\alpha = \lambda^\alpha/2$	$T_L^\alpha = \sigma^a/2$	Y

Table 1.5: Gauge group, gauge couplings and gauge particles of the SM.

· Hierarchy problem (fune-tuning problem)
· Strong CP ploblem
· The number of family
· Neutrino mass
· Dark matter and energy
· Baryon asymmetry
· Grand unification

Those problems must be solved by considering new physics beyond the SM. Actually, physics beyond the SM such as *grand unified theories* (GUT), supersymmetry (SUSY), higher dimensional theories and technicolor theories have been proposed. In order to explain that why the SM gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$, GUT is proposed.

For example, in the case of $SU(5)$ GUT, when $SU(5)$ is broken down to subgroup G_{SM}, one generation matter particles of the SM is unified into $\mathbf{10}$, $\bar{\mathbf{5}}$ and $\mathbf{1}$ representation of $SU(5)$ such as

$$\mathbf{10} = \left(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{6}\sqrt{\frac{3}{5}}\right) : (u_R)^c \oplus \left(\mathbf{3}, \mathbf{2}, -\frac{2}{3}\sqrt{\frac{3}{5}}\right) : q_L \oplus \left(\bar{\mathbf{1}}, \mathbf{1}, \sqrt{\frac{3}{5}}\right) : (e_R)^c \ , \quad (1.1)$$

$$\bar{\mathbf{5}} = \left(\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3}\sqrt{\frac{3}{5}}\right) : (d_R)^c \oplus \left(\bar{\mathbf{1}}, \mathbf{2}, -\frac{1}{2}\sqrt{\frac{3}{5}}\right) : l_L \ , \quad (1.2)$$

$$\mathbf{1} = (\mathbf{1}, \mathbf{1}, 0) : (\nu_{eR})^c. \tag{1.3}$$

And, $SU(5)$ gauge field with **24** representation is decomposed to

$$\mathbf{24} = (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus \left(\mathbf{3}, \mathbf{2}, -\frac{5}{6}\sqrt{\frac{3}{5}}\right) \oplus \left(\bar{\mathbf{3}}, \mathbf{2}, \frac{5}{6}\sqrt{\frac{3}{5}}\right), \tag{1.4}$$

where $(\mathbf{8}, \mathbf{1}, 0)$, $(\mathbf{1}, \mathbf{3}, 0)$ and $(\mathbf{1}, \mathbf{1}, 0)_0$ are representation of $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ gauge boson, respectively. Therefore, the SM gauge bosons are unified, and three gauge couplings are unified as $g_s = g = \sqrt{5/3}g' = g_{\rm GUT}$ at GUT scale. The $\left(\mathbf{3}, \mathbf{2}, -\frac{5}{6}\sqrt{\frac{3}{5}}\right)$ and $\left(\bar{\mathbf{3}}, \mathbf{2}, \frac{5}{6}\sqrt{\frac{3}{5}}\right)$ are extra gauge bosons which can cause proton decay.

When $SO(10)$ gauge group are broken down to subgroup $SU(5) \times U(1)$ in $SO(10)$ GUT, **16** representation of $SO(10)$ is decomposed to

$$\mathbf{16} = \left(\bar{\mathbf{5}}, \frac{3}{2}\sqrt{\frac{1}{10}}\right) \oplus \left(\mathbf{10}, -\frac{1}{2}\sqrt{\frac{1}{10}}\right) \oplus \left(\mathbf{1}, -\frac{5}{2}\sqrt{\frac{1}{10}}\right). \tag{1.5}$$

Hence, one generation of matter particles are unified to a single multiplet with **16** representation of $SO(10)$. In GUT based on E_6 gauge group, **16** representation of $SO(10)$ gauge group is a part of **27** of E_6.

Furthermore, when exceptional group E_8 is broken down to subgroup $E_6 \times SU(3)$, **248** representation of E_8 is decomposed to

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \mathbf{3}). \tag{1.6}$$

Here, $(\mathbf{27}, \mathbf{3})$ includes all matter particles of the SM. However, there are a lot of extra particles which do not appear in the SM.

We have studied this problems by using higher-dimensional theories. The advantage of higher-dimensional theories is that substances including mirror particles can be reduces using the symmetry breaking concerning extra dimensions, as originally discusses in superstring theory [1–3]. Hence, a candidate realizing the family unification is GUTs on a higher-dimensional space-time including an orbifold as an extra space.[1]

Many physics beyond the SM have been proposed, but their evidences have not been discovered. In order to explain the history of universe, we should disclose the identity of unknown particles such as *dark matter* and *inflaton*. Because it is hard to detect such hidden particles directly, they are supposed to interact with the SM particles weakly. We also have studied this problems by using higher-dimensional theories.

The contexts of this thesis are as follows. In Sec. 2, we explain the properties of orbifold and orbifold breaking which is a kind of symmetry breaking. In Sec. 3, we review a family unification on the basis of $SU(N)$ gauge theory on 5D space-time,

[1] Five-dimensional supersymmetric GUTs on $M^4 \times S^1/\mathbb{Z}_2$ possess the attractive feature that the triplet-doublet splitting of Higgs multiplets is elegantly realized [4, 5].

$M^4 \times S^1/\mathbb{Z}_2$ [6]. In Sec. 4, we investigate the family unification on the basis of $SU(N)$ gauge theory on 6D space-time, $M^4 \times T^2/\mathbb{Z}_M$ $(M = 2, 3, 4, 6)$ [7]. In Sec. 5, we investigate the relationship between the family number of chiral fermion and the Wilson line phases, based on the orbifold family unification [8]. In Sec. 6, we predict orbifold family unification models with $SU(9)$ gauge group on a 6D space-time including the orbifold $T^2/\mathbb{Z}_2$, and obtain relations among sfermion masses in the SUSY extension of models [9]. In Sec. 7, we propose an idea that hidden particles can be separated according to gauge quantum numbers from the visible ones by the difference of boundary conditions (BCs) on extra dimensions [10]. Section 8 is devoted to conclusion and discussion.

2 Orbifold

Orbifold is the quotient space $M/\mathbb{H}$ which is obtained from a manifold M with some discrete transformation group $\mathbb{H}$, and the space has fixed point (or space). First, I consider how to generally construct orbifold.

For orbifold $M/\mathbb{H}$, the discrete group H is the direct production of the space group and discrete rotation group if space is a flat one $M = \mathbb{R}^n$. If the element of $\mathbb{H}$, $g = (\theta, v)$, acts on an arbitrary point $y^i (i = 1, 2, \cdots, m)$ of $\mathbb{R}^n$, it transforms as

$$g : y^i \to \theta^i_j y^j + v^j, \tag{2.1}$$

where v is a translation for space group and θ is a discrete rotation. Speaking in the language of the topological transformation group, a set of g is called the "orbit" of $\mathbb{H}$ for y^i, and because a "manifold" is divided by some discrete group, it is called *orbifold*. However, orbifold is not manifold because it has fixed points. In fixed points, the curvature diverges.

In the quotient space $\mathbb{R}^n/\mathbb{H}$, the coordinates has the equivalence relation as the following;

$$y^i \sim \theta^i_j y^j + v^j. \tag{2.2}$$

Because $\mathbb{R}^n$ is the flat space and $\mathbb{H}$ is the discrete group, $\mathbb{R}^n/\mathbb{H}$ is also flat and compact from the properties of space group. The quotient space, where the compact flat space is divided by the discrete rotation group, is orbifold.

The fixed points f for some (θ^k, v_0) is defined by points that satisfy the relation

$$f = \theta^k f + v_0 \tag{2.3}$$

in the fundamental region. The number of fixed points is defined as

$$\chi = \det(1 - \theta) = \prod_i 4\sin^2(\pi\phi_i) \tag{2.4}$$

by Lefschetz fixed point theorem. Here, θ is the integer representation matrix and $2\pi\phi_i$ are all angles that are integer multiples of $2\pi/M$ rotation obtained from $\mathbb{Z}_M$ symmetry up to π. If $\chi = 0$, $\phi_i = 0$, and the space is non-compact orbifold or fixed surface (torus). Therefore, the number of fixed points is automatically fixed when $\mathbb{Z}_M$ is determined.

2.1 $S^1/\mathbb{Z}_2$ orbifold

2.1.1 Property

The $S^1/\mathbb{Z}_2$ orbifold is obtained by dividing a circle S^1 whose radius is R with the identification,

$$S^1 : y \sim y + 2\pi R\ , \tag{2.5}$$

under the $\mathbb{Z}_2$ symmetry,

$$\mathbb{Z}_2 : y \sim -y\ , \tag{2.6}$$

which is shown in Fig. 2.1.

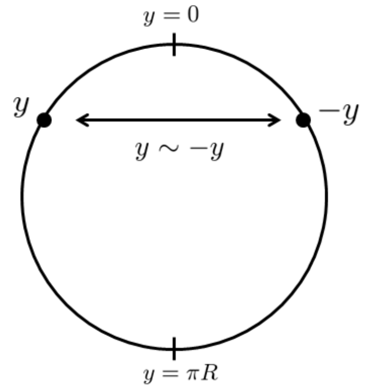


Figure 2.1: $S^1/\mathbb{Z}_2$ orbifold

It follows that when the point y is identified with the point $-y$ on $S^1/\mathbb{Z}_2$, the space is regarded as a line segment whose length is πR. The both end points $y = 0$ and πR are fixed points under the $\mathbb{Z}_2$ transformation. The transformations around those fixed points can be defined as

$$s_0 : y \to -y \ , \quad s_1 : y \to 2\pi R - y \ , \quad t : y \to y + 2\pi R \ . \tag{2.7}$$

They satisfy the relation,

$$s_0^2 = s_1^2 = I \ , \quad t = s_0 s_1 \ . \tag{2.8}$$

2.1.2 Orbifold breaking by inner automorphisms boundary condition

Let us discuss $SU(N)$ gauge theory to consider boundary conditions (BCs) of gauge, scalar and spinor field under the transformation, using inner auotomophisms. 5D Lagrangian density is given by

$$\mathcal{L}_{5D} = -\frac{1}{4} F^a_{MN} F^{aMN} + \bar{\psi} i \Gamma^M D_M \psi + |D_M \phi|^2 \ , \tag{2.9}$$

where $D_M = \partial_M - i g_5 A^a_M T^a$ and $F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b_M A^c_N$, and g_5 is 5D gauge coupling.

First, the BCs of gauge field $A_M \equiv A^a_M T^a$ are determined as

$$s_0 : \ A_\mu(x, -y) = P_0 A_\mu(x, y) P_0^\dagger, \ A_5(x, -y) = -P_0 A_5(x, y) P_0^\dagger \ , \tag{2.10}$$

$$s_1 : \ A_\mu(x, 2\pi R - y) = P_1 A_\mu(x, y) P_1^\dagger, \ A_5(x, 2\pi R - y) = -P_1 A_5(x, y) P_1^\dagger, \tag{2.11}$$

$$t_1 : \ A_M(x, y + 2\pi R) = U A_M(x, y) U^\dagger, \tag{2.12}$$

where P_0, P_1 and U stand for the representation matrices for the $\mathbb{Z}_2$, $\mathbb{Z}_2'$ and T transformation, respectively. Those matrices satisfy the relations,

$$P_0^2 = P_1^2 = I \ , \quad U U^\dagger = I \ , \quad U = P_0 P_1 \ . \tag{2.13}$$

where, P_0 and P_1 are hermitian matrices because of $P_0 = P_0^\dagger$ and $P_1 = P_1^\dagger$.

Next, the BCs of scalar field ϕ are determined as

$$s_0 : \phi(x,-y) = T_\Phi[P_0]\phi(x,y) \ , \tag{2.14}$$
$$s_1 : \phi(x,2\pi R-y) = T_\Phi[P_1]\phi(x,y) \ , \tag{2.15}$$
$$t_1 : \phi(x,y+2\pi R) = T_\Phi[U]\phi(x,y) \ , \tag{2.16}$$

where $T_\Phi[P_0]$, $T_\Phi[P_1]$ and $T_\Phi[U]$ represent appropriate representation matrices including arbitrary sign factors, with the matrices P_0, P_1 and U. The representation matrices satisfy

$$T_\Phi[P_0]^2 = T_\Phi[P_1]^2 = I \ , \quad T_\Phi[U] = T_\Phi[P_0]T_\Phi[P_1] \ . \tag{2.17}$$

For example, if ϕ is the fundamental representation of $SU(N)$ gauge symmetry,

$$\begin{aligned} &T_\Phi[P_0]\phi(x,y) = \eta_{\phi 0}P_0\phi(x,y) \ , \quad T_\Phi[P_1]\phi(x,y) = \eta_{\phi 1}P_1\phi(x,y) \ , \\ &T_\Phi[U]\phi(x,y) = \eta_{\phi 2}U\phi(x,y) \ , \end{aligned} \tag{2.18}$$

where η_0, η_1 and η_2 are intrinsic $\mathbb{Z}_2$ parity and they take 1 or -1.

The BCs of spinor field ψ are determined as

$$s_0 : \psi(x,-y) = i\Gamma^5 T_\Psi[P_0]\psi(x,y) \ , \tag{2.19}$$
$$s_1 : \psi(x,2\pi R-y) = i\Gamma^5 T_\Psi[P_1]\psi(x,y) \ , \tag{2.20}$$
$$t_1 : \psi(x,y+2\pi R) = T_\Psi[U]\psi(x,y) \ , \tag{2.21}$$

where $T_\Psi[P_0]$, $T_\Psi[P_1]$ and $T_\Psi[U]$ represent appropriate representation matrices including arbitrary sign factors, with the matrices P_0, P_1 and U. The representation matrices satisfy

$$T_\Psi[P_0]^2 = T_\Psi[P_1]^2 = I \ , \quad T_\Psi[U] = T_\Psi[P_0]T_\Psi[P_1] \ . \tag{2.22}$$

For example, if ψ is the fundamental representation of $SU(N)$ gauge symmetry,

$$\begin{aligned} &s_0 : \psi_L(x,-y) = -\eta_{\psi 0}P_0\psi_L(x,y) \ , \quad \psi_R(x,-y) = \eta_{\psi 0}P_0\psi_R(x,y) \\ &s_1 : \psi_L(x,2\pi R-y) = -\eta_{\psi 1}P_1\psi_L(x,y) \ , \quad \psi_R(x,2\pi R-y) = \eta_{\psi 1}P_1\psi_R(x,y) \\ &t_1 : \psi_L(x,y+2\pi R) = \eta_{\psi 2}U\psi_L(x,y) \ , \quad \psi_R(x,y+2\pi R) = \eta_{\psi 2}U\psi_R(x,y) \ . \end{aligned} \tag{2.23}$$

note that $\mathbb{Z}_2$ parity of ψ_L is different from that of ψ_R. This property is important to consider chiral theory on 4D.

Let $\varphi^{(\mathscr{P}_0,\mathscr{P}_1)}(x,y)$ be a component in a multiplet and have a definite $\mathbb{Z}_2$ parity $(\mathscr{P}_0,\mathscr{P}_1)$. Here, φ is a generic field and it is applied to scalar field ϕ, fermion field ψ or gauge field A_M. The Fourier expansion of $\varphi^{(\mathscr{P}_0,\mathscr{P}_1)}(x,y)$ is given by

$$\varphi^{(+1,+1)}(x,y) = \frac{1}{\sqrt{\pi R}}\varphi^{(0)}(x) + \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\varphi^{(n)}(x)\cos\frac{n}{R}y \ , \tag{2.24}$$

$$\varphi^{(+1,-1)}(x,y) = \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\varphi^{(n)}(x)\cos\frac{(n-\frac{1}{2})}{R}y, \tag{2.25}$$

$$\varphi^{(-1,+1)}(x,y) = \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \varphi^{(n)}(x) \sin \frac{(n-\frac{1}{2})}{R} y \ , \tag{2.26}$$

$$\varphi^{(-1,-1)}(x,y) = \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \varphi^{(n)}(x) \sin \frac{n}{R} y \ . \tag{2.27}$$

Upon compactification, massless mode $\varphi^{(0)}(x)$ appears on 4D when $\mathbb{Z}_2$ parities are $(\mathscr{P}_0, \mathscr{P}_1) = (+1,+1)$. The massive Kaluza-Kein (KK) modes $\varphi^{(n)}(x)$ do not appear in our low energy world because they have heavy masses of $\mathcal{O}(1/R)$, with the same magnitude as the unification scale. Unless all components of non-singlet field have a common $\mathbb{Z}_2$ parity, a symmetry reduction occurs upon compactification because zero modes are absent in fields with an odd parity. This type of symmetry breaking mechanism is called *orbifold breaking mechanism*.[2]

For example, if the representation matrices P_0 and P_1 are

$$\begin{aligned} P_0 &= \mathrm{diag}\big(\overbrace{+1,\cdots,+1,+1,\cdots,+1,-1,\cdots,-1,-1,\cdots,-1}^{N}\big) \ , \\ P_1 &= \mathrm{diag}\big(\underbrace{+1,\cdots,+1}_{p},\underbrace{-1,\cdots,-1}_{q},\underbrace{+1,\cdots,+1}_{r},\underbrace{-1,\cdots,-1}_{s}\big) \ , \end{aligned} \tag{2.28}$$

where $s = N - p - q - r$, $SU(N)$ gauge symmetry is broken down as

$$SU(N) \to SU(p) \times SU(q) \times SU(r) \times SU(s) \times U(1)^{3-\kappa} \tag{2.29}$$

where κ is the number of $SU(0)$ and $SU(1)$. The $SU(1)$ stands for $U(1)$ and $SU(0)$ means nothing. In this case, the gauge field $A_M^{\alpha(\mathscr{P}_0,\mathscr{P}_1)}$ are divided as

$$\begin{aligned} &A_\mu^{\alpha(+1,+1)} \ , \quad A_\mu^{\beta(+1,-1)} \ , \quad A_\mu^{\beta(-1,+1)} \ , \quad A_\mu^{\beta(-1,-1)} \ , \\ &A_5^{\alpha(-1,-1)} \ , \quad A_5^{\beta(-1,+1)} \ , \quad A_5^{\beta(+1,-1)} \ , \quad A_5^{\beta(+1,+1)} \ , \end{aligned} \tag{2.30}$$

where the index α indicates the gauge generators of unbroken gauge symmetry and the index β indicates the gauge generators of broken gauge symmetry. This shows that the gauge symmetry is unbroken when gauge field contains zero modes.

2.1.3 Orbifold breaking by outer automorphisms boundary condition

Let us discuss $SU(N)$ gauge theory to consider BCs of gauge, scalar and spinor field under the transformation, using outer automorphisms. The BCs of gauge field $A_M^a T^a$ are generated by a conjugation transformation,

$$\begin{aligned} s_0: \ & A_\mu^a(x,-y)T^a = -A_\mu(x,y)(T^a)^* \ , \\ & A_5^a(x,-y)T^a = A_5(x,y)(T^a)^* \ , \end{aligned} \tag{2.31}$$

$$t_1: \ A_M^a(x,y+2\pi R)T^a = A_M^a(x,y)T^a \ . \tag{2.32}$$

[2] The $\mathbb{Z}_2$ orbifolding was used in superstring theory [11] and heterotic M-theory [12, 13]. In field theoretical models, it was applied to the reduction of global SUSY [14,15], which is an orbifold version of Scherk-Schwarz mechanism [16,17], and then to the reduction of gauge symmetry [18].

This is an outer automorphism transformation. Such BCs relate particles with a representation $\mathbf{R}$ to that with the conjugated one $\overline{\mathbf{R}}$ as *conjugate BCs* [19]. In this case of BCs, $SU(N)$ gauge symmetry is broken down as

$$\begin{aligned} &U(1) \to \text{nothing} \ , \\ &SU(N) \to SO(N) \ , \end{aligned} \tag{2.33}$$

and the rank is reduced (for $n > 2$) [20]. In the case of other gauge symmetry, symmetries are broken down as

$$\begin{aligned} &SO(p+q) \to SO(p) \times SO(q) \ , \\ &SU(2n) \to Sp(n) \ , \\ &E_6 \to Sp(4) \ , \quad E_6 \to F_4 \ . \end{aligned}$$

Let us consider a $U(1)$ gauge theory as an example. In the case of $U(1)$, the BCs (2.31) and (2.32) are represented such as

$$s_0 : \ A_\mu(x,-y) = -A_\mu(x,y) \ , \quad A_5(x,-y) = A_5(x,y) \ , \tag{2.34}$$

$$t_1 : \ A_M(x, y+2\pi R) = A_M(x,y) \ . \tag{2.35}$$

The 5D $U(1)$ gauge fields A_M are given by the Fourier expansions:

$$A_\mu(x,y) = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{(n)}(x) \sin \frac{ny}{R} \ , \tag{2.36}$$

$$A_5(x,y) = \frac{1}{\sqrt{2\pi R}} A_5^{(0)}(x) + \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_5^{(n)}(x) \cos \frac{ny}{R} \ . \tag{2.37}$$

The BCs of scalar field ϕ and spinor field ψ are determined as

$$s_0 : \phi(x,-y) = \phi^*(x,y) \ , \tag{2.38}$$

$$t_1 : \phi(x, y+2\pi R) = e^{i\beta_\phi} \phi(x,y) \ , \tag{2.39}$$

$$s_0 : \psi(x,-y) = i\psi^c(x,y) \ , \tag{2.40}$$

$$t_1 : \psi(x, y+2\pi R) = e^{i\beta_\psi} \psi(x,y) \ , \tag{2.41}$$

where β_ϕ and β_ψ are arbitrary real constants and $\psi^c = e^{i\gamma_c}\Gamma^2\psi^*$. The ψ^c corresponds to a charge conjugation of ψ on 4D space-time, and γ_c is an arbitrary real number. From the BCs of (2.38) - (2.41), ϕ and ψ are given by the Fourier expansion:

$$\phi(x,y) = \frac{1}{2\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} \phi^{(n)}(x) e^{i\frac{2\pi n + \beta_\phi}{2\pi R} y} \ , \tag{2.42}$$

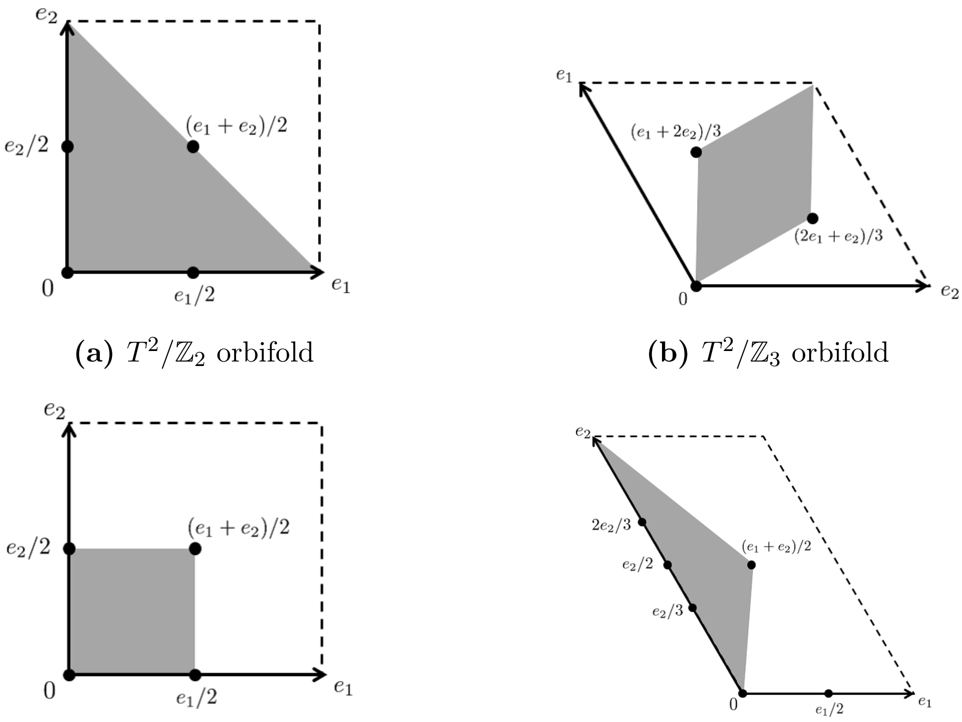
$$\psi(x,y) = \frac{1}{2\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} \begin{pmatrix} \xi_\alpha^{(n)}(x) \\ i\bar{\xi}^{(n)\dot{\alpha}}(x) \end{pmatrix} e^{i\frac{2\pi n + \beta_\psi}{2\pi R} y} \ , \tag{2.43}$$

where $\phi^{(n)}(x)$ are 4D real scalar fields ($\phi^{(n)*}(x) = \phi^{(n)}(x)$), $\xi_\alpha^{(n)}(x)$ are 4D 2-component spinor fields, and α and $\dot{\alpha}$ are spinor indices.

2.2 $T^2/\mathbb{Z}_M$ orbifold

In this subsection, let us explain $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_M$. Because the properties of $T^2/\mathbb{Z}_M$ orbifold is similar to previous subsection, we easily summarize it. The details of the properties and orbifold breaking mechanism of $T^2/\mathbb{Z}_M$ orbifold are described in appendix.

Let z be the complex coordinate of $T^2/\mathbb{Z}_M$. Here, T^2 is constructed from a two-dimensional $SO(4)$, $SU(3)$, $SO(5)$ and G_2 lattice on $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$, respectively (Fig. 2.2).



(a) $T^2/\mathbb{Z}_2$ orbifold **(b)** $T^2/\mathbb{Z}_3$ orbifold

(c) $T^2/\mathbb{Z}_4$ orbifold **(d)** $T^2/\mathbb{Z}_6$ orbifold

Figure 2.2: $T^2/\mathbb{Z}_M$ orbifold

On T^2, the point z is equivalent to the points $z + e_1$ and $z + e_2$ where e_1 and e_2 are the basis vectors. The orbifold $T^2/\mathbb{Z}_M$ is obtained by dividing T^2 by the $\mathbb{Z}_M$ transformation: $z \to \theta z (\theta^M = 1)$. As the point z is identified with the point θz on $T^2/\mathbb{Z}_M$, the space is regarded as a dark area in Fig. 2.2, respectively. The fixed point z_f for the $\mathbb{Z}_M$ transformation satisfies

$$z_f = \theta^k z_f + n e_1 + m e_2 \ , \tag{2.44}$$

where k, n and m are integers. In Fig. 2.2, the fixed points is shown by filled circle. Basis vector, transformation properties and their representation matrices of $T^2/\mathbb{Z}_M$ are summarized in Table 2.1. [21, 22]

M	Basis vectors (e_1, e_2)	Transformation properties	Representation matrices
2	$1, i$	$z \to -z, z \to e_1 - z, z \to e_2 - z$	P_0, P_1, P_2
3	$1, e^{2\pi i/3}$	$z \to e^{2\pi i/3} z, z \to e^{2\pi i/3} z + e_1$	Θ_0, Θ_1
4	$1, i$	$z \to iz, z \to e_1 - z$	Q_0, P_1
6	$1, (-3 + i\sqrt{3})/2$	$z \to e^{\pi i/3} z, z \to e_1 - z$	Ξ_0, P_1

Table 2.1: The characters of $T^2/\mathbb{Z}_M$

3 Review of Orbifold Family Unification on $M^4 \times S^1/\mathbb{Z}_2$

In this section, we review family unification on the basis of $SU(N)$ gauge theory on 5D space-time, $M^4 \times S^1/\mathbb{Z}_2$ [6].

With suitable diagonal representation matrices P_0, P_1 such as (2.28), the $SU(N)$ gauge group is broken down into its subgroup such that

$$SU(N) \to SU(p) \times SU(q) \times SU(r) \times SU(s) \times U(1)^{3-\kappa} , \tag{3.1}$$

where $s = N - p - q - r$ and κ is the number of $SU(0)$ and $SU(1)$, and SU(1) stand for $U(1)$ and $SU(0)$ means nothing.

A fermion with spin 1/2 in 5D is regarded as a Dirac fermion or a pair of Weyl fermions with opposite chiralities in 4D. After the breakdown of $SU(N)$, Weyl fermion with the rank k totally antisymmetric tensor representation $[N, k]_{L(R)}$, whose dimension is ${}_N C_k$, is decomposed as

$$[N, k]_L = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left({}_p C_{l_1}, {}_q C_{l_2}, {}_r C_{l_3}, {}_s C_{l_4}\right)_L , \tag{3.2}$$

$$[N, k]_R = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left({}_p C_{l_1}, {}_q C_{l_2}, {}_r C_{l_3}, {}_s C_{l_4}\right)_R , \tag{3.3}$$

where $l_4 = k - l_1 - l_2 - l_3$, and our notation is that ${}_n C_l$ for $l > n$ and $n > 0$. The $\mathbb{Z}_2$ parity of the representation $\left({}_p C_{l_1}, {}_q C_{l_2}, {}_r C_{l_3}, {}_s C_{l_4}\right)_{L(R)}$ are given by

$$\mathscr{P}_{0L(R)} = (-1)^{k+l_1+l_2} \eta_{[N,k]_{L(R)}} , \quad \mathscr{P}_{1L(R)} = (-1)^{k+l_1+l_3} \eta'_{[N,k]_{L(R)}} , \tag{3.4}$$

where $\eta_{[N,k]_{L(R)}}$ and $\eta'_{[N,k]_{L(R)}}$ are the intrinsic $\mathbb{Z}_2$ and $\mathbb{Z}'_2$, respectively. In order to the kinetic term is invariant under the $\mathbb{Z}_2$ parity transformation, $\left({}_p C_{l_1}, {}_q C_{l_2}, {}_r C_{l_3}, {}_s C_{l_4}\right)_L$ and $\left({}_p C_{l_1}, {}_q C_{l_2}, {}_r C_{l_3}, {}_s C_{l_4}\right)_R$ should have opposite $\mathbb{Z}_2$ parity to each other:

$$\eta_{[N,k]_L} = -\eta_{[N,k]_R} , \quad \eta'_{[N,k]_L} = -\eta'_{[N,k]_R} . \tag{3.5}$$

Therefore, $\mathscr{P}_{0L} = -\mathscr{P}_{0R}$ and $\mathscr{P}_{1L} = -\mathscr{P}_{1R}$ 4D Weyl fermions having even $\mathbb{Z}_2$ parities $\mathscr{P}_{0L(R)} = \mathscr{P}_{1L(R)} = +1$ compose chiral fermions in the SM.

In order to remove zero mode of unwelcome particles such as mirror particles from low-energy spectrum, the *survival hypothesis* [23, 24], which is proposed by Georgi, is adopted. Here, the survival hypothesis is the assumption that if a symmetry is broken down into a smaller symmetry at a scale $M_{\rm SB}$, then any fermion masses terms invariant under the smaller group induce fermion masses of $\mathcal{O}(M_{\rm SB})$.

Let consider two gauge symmetry breaking pattern:

$$SU(N) \to SU(5) \times SU(q) \times SU(r) \times SU(s) \times U(1)^{3-\kappa}$$
$$SU(N) \to SU(3) \times SU(2) \times SU(r) \times SU(s) \times U(1)^{3-\kappa}$$

In the case of the gauge symmetry breaking pattern $SU(N) \to SU(5) \times SU(q) \times SU(r) \times SU(s)$, using the survival hypothesis and the equivalence of $(\mathbf{5}_R)^c$ and $(\bar{\mathbf{10}}_R)^c$ with $\bar{\mathbf{5}}_L$ and $\mathbf{10}_L$, respectively, the number of $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations for left-handed Weyl fermions are

$$\begin{aligned} n_{\bar{5}} &\equiv \sharp\bar{\mathbf{5}}_L - \sharp\mathbf{5}_L + \sharp\mathbf{5}_R - \sharp\bar{\mathbf{5}}_R \\ &= \sum_{l_1=1,4} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1} (P_L - P_R)\, {}_qC_{l_2} \cdot {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.6}$$

$$\begin{aligned} n_{10} &\equiv \sharp\mathbf{10}_L - \sharp\overline{\mathbf{10}}_L + \sharp\overline{\mathbf{10}}_R - \sharp\mathbf{10}_R \\ &= \sum_{l_1=2,4} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1} (P_L - P_R)\, {}_qC_{l_2} \cdot {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.7}$$

where $\sharp$ represents the number of each multiplet and

$$P_{L(R)} = \frac{1 - \mathscr{P}_{0L(R)}}{2} \frac{1 - \mathscr{P}_{1L(R)}}{2} \ . \tag{3.8}$$

In [6], many solutions which give rise to three families $n_{\bar{5}} = n_{10} = 3$ have been found.

Next, in the case of gauge symmetry breaking pattern $SU(N) \to SU(3) \times SU(2) \times SU(r) \times SU(s) \times U(1)^{3-\kappa}$, using the survival hypothesis and the equivalence on charge conjugation, the flavor number of each chiral fermion are

$$\begin{aligned} n_{\bar{d}} &= \sharp\,(\bar{\mathbf{3}},\mathbf{1})_L - \sharp\,(\mathbf{3},\mathbf{1})_L + \sharp\,(\mathbf{3},\mathbf{1})_R - \sharp\,(\bar{\mathbf{3}},\mathbf{1})_R \\ &= \sum_{(l_1,l_2)=(2,2),(1,0)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.9}$$

$$\begin{aligned} n_{l} &= \sharp\,(\mathbf{1},\mathbf{2}) - \sharp\,(\mathbf{1},\mathbf{2})_L + \sharp\,(\mathbf{1},\mathbf{2})_R - \sharp\,(\mathbf{1},\mathbf{2})_R \\ &= \sum_{(l_1,l_2)=(3,1),(0,1)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.10}$$

$$\begin{aligned} n_{\bar{u}} &= \sharp\,(\bar{\mathbf{3}},\mathbf{1}) - \sharp\,(\mathbf{3},\mathbf{1})_L + \sharp\,(\mathbf{3},\mathbf{1})_R - \sharp\,(\bar{\mathbf{3}},\mathbf{1})_R \\ &= \sum_{(l_1,l_2)=(2,0),(1,2)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.11}$$

$$\begin{aligned} n_{\bar{e}} &= \sharp\,(\mathbf{1},\mathbf{1}) - \sharp\,(\mathbf{1},\mathbf{1})_L + \sharp\,(\mathbf{1},\mathbf{1})_R - \sharp\,(\mathbf{1},\mathbf{1})_R \\ &= \sum_{(l_1,l_2)=(0,2),(3,0)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ , \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
n_q &= \sharp\,(\mathbf{3},\mathbf{2}) - \sharp\,(\bar{\mathbf{3}},\mathbf{2})_L + \sharp\,(\bar{\mathbf{3}},\mathbf{2})_R - \sharp\,(\mathbf{3},\mathbf{2})_R \\
&= \sum_{(l_1,l_2)=(1,1),(2,1)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ .
\end{aligned} \tag{3.13}
$$

The total number of heavy neutrino singlets $(\nu_e)^c$ is

$$
\begin{aligned}
n_{\bar{\nu}} &= \sharp\,(\mathbf{1},\mathbf{1}) + \sharp\,(\mathbf{1},\mathbf{1})_L + \sharp\,(\mathbf{1},\mathbf{1})_R + \sharp\,(\mathbf{1},\mathbf{1})_R \\
&= \sum_{(l_1,l_2)=(0,0),(3,2)} \sum_{l_3=0}^{k-l_1-l_2} (-1)^{l_1+l_2} (P_L - P_R)\, {}_rC_{l_3} \cdot {}_sC_{l_4} \ .
\end{aligned} \tag{3.14}
$$

In [6], it is found that there is no solution satisfying $n_{\bar{d}} = n_l = n_{\bar{u}} = n_{\bar{e}} = n_q = n_{\bar{\nu}} = 3$.

Therefore, we think that it is important to expand space dimension. In next section, we study family unification on 6D $M^4 \times T^2/\mathbb{Z}_M$.

4 Orbifold family unification on $M^4 \times T^2/\mathbb{Z}_M$

In this section, we study the possibility of family unification on basis of $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_M$ ($M = 2, 3, 4, 6$), in the framework of 6D $SU(N)$ GUTs.

4.1 $\mathbb{Z}_M$ orbifold breaking and formulas for numbers of species

Fields possess discrete charges relating eigenvalues of representation matrices for $\mathbb{Z}_M$ transformation. The discrete charges are assigned as numbers n/M ($n = 0, 1, \cdots, M-1$) and $e^{2\pi i n/M}$ are elements of $\mathbb{Z}_M$ transformation. We refer to them as $\mathbb{Z}_M$ elements.

A fermion with spin 1/2 in 6D is regarded as a Dirac fermion or a pair of Weyl fermions with opposite chiralities in 4D. There are two choices in a 6D Weyl fermion, i.e.,

$$\Psi_+ = \frac{1+\Gamma_7}{2}\Psi = \begin{pmatrix} \frac{1-\gamma_5}{2} & 0 \\ 0 & \frac{1+\gamma_5}{2} \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \begin{pmatrix} \Psi^1_L \\ \Psi^2_R \end{pmatrix} , \tag{4.1}$$

$$\Psi_- = \frac{1-\Gamma_7}{2}\Psi = \begin{pmatrix} \frac{1+\gamma_5}{2} & 0 \\ 0 & \frac{1-\gamma_5}{2} \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix} = \begin{pmatrix} \Psi^1_R \\ \Psi^2_L \end{pmatrix} , \tag{4.2}$$

where Ψ_+ and Ψ_- are fermions with positive and negative chirality, respectively, and Γ_7 and γ_5 are the chirality operators for 6D fermions and 4D ones, respectively. [3] Here and hereafter, the subscript $\pm$ stands for the chiralities on 6D.

From the $\mathbb{Z}_M$ invariance of kinetic term and the transformation property of the covariant derivatives $\mathbb{Z}_M : D_z \to \overline{\rho} D_z$ and $D_{\overline{z}} \to \rho D_{\overline{z}}$ with $\overline{\rho} = e^{-2\pi i/M}$ and $\rho = e^{2\pi i/M}$, the following relations hold between the $\mathbb{Z}_M$ element of $\Psi^1_{L(R)}$ and $\Psi^2_{R(L)}$,

$$\mathscr{P}_{\Psi^2_R} = \rho \mathscr{P}_{\Psi^1_L} , \quad \mathscr{P}_{\Psi^1_R} = \overline{\rho} \mathscr{P}_{\Psi^2_L} , \tag{4.3}$$

where $z \equiv x^5 + ix^6$ and $\overline{z} \equiv x^5 - ix^6$.

Chiral gauge theories including Weyl fermions on even dimensional space-time become, in general, anomalous in the presence of gauge anomalies, gravitational anomalies, mixed anomalies and/or global anomaly [26, 27]. In $SU(N)$ GUTs on 6D space-time, the global anomaly is absent because of $\Pi_6(SU(N)) = 0$ for $N \geq 4$. Here, $\Pi_6(SU(N))$ is the 6-th homotopy group of $SU(N)$. In our analysis, we consider a massless Dirac fermion (Ψ_+, Ψ_-) under the $SU(N)$ gauge group ($N \geq 8$) on 6D space-time. In this case, anomalies are canceled out by the contributions from fermions with different chiralities

4.2 Formulae for numbers of species

With suitable diagonal representation matrices R_a ($a = 0, 1, 2$ for $T^2/\mathbb{Z}_2$ and $a = 0, 1$ for $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$), the $SU(N)$ gauge group is broken down into its subgroup such that

$$SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-m-1} , \tag{4.4}$$

[3] For more detailed explanations for 6D fermions, see Ref. [25].

where $N = \sum_{i=1}^{n} p_i$. Here and hereafter, $SU(1)$ unconventionally stands for $U(1)$, $SU(0)$ means nothing and m is a sum of the number of $SU(0)$ and $SU(1)$. The concrete form of R_a will be given in the next section.

After the breakdown of $SU(N)$, the rank k totally antisymmetric tensor representation $[N, k]$, whose dimension is ${}_N C_k$, is decomposed into a sum of multiplets of the subgroup $SU(p_1) \times \cdots \times SU(p_n)$ as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n}) , \tag{4.5}$$

where $l_n = k - l_1 - \cdots - l_{n-1}$ and our notation is that ${}_n C_l = 0$ for $l > n$ and $l < 0$. Here and hereafter, we use ${}_n C_l$ instead of $[n, l]$ in many cases. We sometimes use the ordinary notation for representations too, e.g., $\mathbf{5}$ and $\overline{\mathbf{5}}$ in place of ${}_5C_1$ and ${}_5C_4$.

The $[N, k]$ is constructed by the antisymmetrization of k-ple product of the fundamental representation $\boldsymbol{N} = [N, 1]$:

$$[N, k] = (\boldsymbol{N} \times \cdots \times \boldsymbol{N})_{\rm A} . \tag{4.6}$$

We define the intrinsic $\mathbb{Z}_M$ elements η_k^a such that

$$(\boldsymbol{N} \times \cdots \times \boldsymbol{N})_{\rm A} \to \eta_k^a (R_a \boldsymbol{N} \times \cdots \times R_a \boldsymbol{N})_{\rm A} . \tag{4.7}$$

By definition, η_k^a take a value of $\mathbb{Z}_M$ elements, i.e., $e^{2\pi i n/M}$ $(n = 0, 1, \cdots, M-1)$. Note that η_k^a for Ψ_+ are not necessarily same as those of Ψ_-, and the chiral symmetry is still respected.

Let us investigate the family unification in two cases. Each breaking pattern is given by

$$SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-m-1} , \tag{4.8}$$
$$SU(N) \to SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_n) \times U(1)^{n-m-1} , \tag{4.9}$$

where $SU(3)$ and $SU(2)$ are identified with $SU(3)_C$ and $SU(2)_L$ in the SM gauge group.

4.2.1 Formulae for $SU(5)$ multiplets

We study the breaking pattern (4.8). After the breakdown of $SU(N)$, $[N, k]$ is decomposed as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_5C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n}) . \tag{4.10}$$

As mentioned before, ${}_5C_0$, ${}_5C_1$, ${}_5C_2$, ${}_5C_3$, ${}_5C_4$ and ${}_5C_5$ stand for representations $\mathbf{1}$, $\mathbf{5}$, $\mathbf{10}$, $\overline{\mathbf{10}}$, $\overline{\mathbf{5}}$ and $\overline{\mathbf{1}}$. [4]

[4] We denote the $SU(5)$ singlet relating to ${}_5C_5$ as $\overline{\mathbf{1}}$, for convenience sake, to avoid the confusion over singlets.

Utilizing the survival hypothesis and the equivalence of $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ with $\overline{\mathbf{5}}_L$ and $\mathbf{10}_L$, respectively, [5] we write the numbers of $\overline{\mathbf{5}}$ and $\mathbf{10}$ representations for left-handed Weyl fermions as

$$n_{\bar{5}} \equiv \sharp\overline{\mathbf{5}}_L - \sharp\mathbf{5}_L + \sharp\mathbf{5}_R - \sharp\overline{\mathbf{5}}_R \ , \tag{4.11}$$

$$n_{10} \equiv \sharp\mathbf{10}_L - \sharp\overline{\mathbf{10}}_L + \sharp\overline{\mathbf{10}}_R - \sharp\mathbf{10}_R \ , \tag{4.12}$$

where $\sharp$ represents the number of each multiplet.

The $SU(5)$ singlets are regarded as the right-handed neutrinos, which can obtain heavy Majorana masses among themselves as well as the Dirac masses with left-handed neutrinos. Some of them can be involved in see-saw mechanism [28–30]. The total number of $SU(5)$ singlets (with heavy masses) is given by

$$n_1 \equiv \sharp\mathbf{1}_L + \sharp\overline{\mathbf{1}}_L + \sharp\overline{\mathbf{1}}_R + \sharp\mathbf{1}_R \ . \tag{4.13}$$

Formulae for $n_{\bar{5}}$, n_{10} and n_1 from a Dirac fermion (Ψ_+, Ψ_-) whose intrinsic $\mathbb{Z}_M$ elements are $(\eta^a_{k+}, \eta^a_{k-})$are given by

$$n_{\bar{5}} = \sum_{\pm} \sum_{l_1=1,4} (-1)^{l_1} \left(\sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 L\pm}}} - \sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n} \ , \tag{4.14}$$

$$n_{10} = \sum_{\pm} \sum_{l_1=2,3} (-1)^{l_1} \left(\sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 L\pm}}} - \sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n} \ , \tag{4.15}$$

$$n_1 = \sum_{\pm} \sum_{l_1=0,5} \left(\sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 L\pm}}} + \sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 R\pm}}} \right) {}_{p_2}C_{l_2} \cdots {}_{p_n}C_{l_n} \ , \tag{4.16}$$

where $p_n = N - \sum_{i=1}^{n-1} p_i$ and $l_n = N - \sum_{i=1}^{n-1} l_i$. $\sum_{\pm}$ represents the summation of contributions from Ψ_+ and Ψ_-. Furthermore, $\sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 L\pm}}}$ means that the summations over $l_j = 0, \cdots, k - l_1 - \cdots - l_{j-1}$ $(j = 2, \cdots, n-1)$ are carried out under the condition that l_j should satisfy specific relations on $T^2/\mathbb{Z}_M$ given in Table 4.1. The relations will be confirmed in the next section. In the same way, $\sum_{\{l_2,\cdots,l_{n-1}\}_{n^a_{l_1 R\pm}}}$ means that the summations over $l_j = 0, \cdots, k - l_1 - \cdots - l_{j-1}$ $(j = 2, \cdots, n-1)$ are carried out under the condition that l_j should satisfy specific relations $n^a_{l_1 R\pm} = n^a_{l_1 L\pm} \mp 1 \pmod{M}$ for $\Psi_\pm$. The formulae (4.14) – (4.16) will be rewritten in more concrete form for each $T^2/\mathbb{Z}_M$ $(M = 2, 3, 4, 6)$, by the use of projection operators, in the next section.

[5] As usual, $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ represent the charge conjugate of $\mathbf{5}_R$ and $\overline{\mathbf{10}}_R$, respectively. Note that $(\mathbf{5}_R)^c$ and $(\overline{\mathbf{10}}_R)^c$ transform as the left-handed Weyl fermions under the 4-dimensional Lorentz transformations.

Orbifolds	$\overline{\rho}^k \eta^a_{k\pm}$	Specific relations
$T^2/\mathbb{Z}_2$	$(-1)^k \eta^0_{k\pm} = (-1)^{\alpha_\pm}$ $(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$ $(-1)^k \eta^2_{k\pm} = (-1)^{\gamma_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + l_3 + l_4 = 2 - l_1 - \alpha_\pm \pmod 2$ $n^1_{l_1 L\pm} \equiv l_2 + l_5 + l_6 = 2 - l_1 - \beta_\pm \pmod 2$ $n^2_{l_1 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \gamma_\pm \pmod 2$
$T^2/\mathbb{Z}_3$	$(e^{-2\pi i/3})^k \eta^0_{k\pm} = (e^{2\pi i/3})^{\alpha_\pm}$ $(e^{-2\pi i/3})^k \eta^1_{k\pm} = (e^{2\pi i/3})^{\beta_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + l_3 + 2(l_4 + l_5 + l_6)$ $= 3 - l_1 - \alpha_\pm \pmod 3$ $n^1_{l_1 L\pm} \equiv l_4 + l_7 + 2(l_2 + l_5 + l_8)$ $= 3 - l_1 - \beta_\pm \pmod 3$
$T^2/\mathbb{Z}_4$	$(-i)^k \eta^0_{k\pm} = i^{\alpha_\pm}$ $(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + 2(l_3 + l_4) + 3(l_5 + l_6)$ $= 4 - l_1 - \alpha_\pm \pmod 4$ $n^1_{l_1 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \beta_\pm \pmod 2$
$T^2/\mathbb{Z}_6$	$(e^{-\pi i/3})^k \eta^0_{k\pm} = (e^{\pi i/3})^{\alpha_\pm}$ $(-1)^k \eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^0_{l_1 L\pm} \equiv l_2 + 2(l_3 + l_4) + 3(l_5 + l_6)$ $+4(l_7 + l_8) + 5(l_9 + l_{10})$ $= 6 - l_1 - \alpha_\pm \pmod 6$ $n^1_{l_1 L\pm} \equiv l_3 + l_5 + l_7 + l_9 + l_{11}$ $= 2 - l_1 - \beta_\pm \pmod 2$

Table 4.1: The specific relations for l_j for $SU(5)$ multiplets.

4.2.2 Formulae for the SM multiplets

We study the breaking pattern (4.9). After the breakdown of $SU(N)$, $[N, k]$ is decomposed as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} ({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \cdots, {}_{p_n}C_{l_n}) \ . \tag{4.17}$$

The flavor numbers of down-type anti-quark singlets $(d_R)^c$, lepton doublets l_L, up-type anti-quark singlets $(u_R)^c$, positron-type lepton singlets $(e_R)^c$, and quark doublets q_L are denoted as $n_{\bar{d}}$, n_l, $n_{\bar{u}}$, $n_{\bar{e}}$ and n_q. Using the survival hypothesis and the equivalence on charge conjugation, we define the flavor number of each chiral fermion as

$$n_{\bar{d}} \equiv \sharp({}_3C_2, {}_2C_2)_L - \sharp({}_3C_1, {}_2C_0)_L + \sharp({}_3C_1, {}_2C_0)_R - \sharp({}_3C_2, {}_2C_2)_R \ , \tag{4.18}$$
$$n_l \equiv \sharp({}_3C_3, {}_2C_1)_L - \sharp({}_3C_0, {}_2C_1)_L + \sharp({}_3C_0, {}_2C_1)_R - \sharp({}_3C_3, {}_2C_1)_R \ , \tag{4.19}$$
$$n_{\bar{u}} \equiv \sharp({}_3C_2, {}_2C_0)_L - \sharp({}_3C_1, {}_2C_2)_L + \sharp({}_3C_1, {}_2C_2)_R - \sharp({}_3C_2, {}_2C_0)_R \ , \tag{4.20}$$
$$n_{\bar{e}} \equiv \sharp({}_3C_0, {}_2C_2)_L - \sharp({}_3C_3, {}_2C_0)_L + \sharp({}_3C_3, {}_2C_0)_R - \sharp({}_3C_0, {}_2C_2)_R \ , \tag{4.21}$$
$$n_q \equiv \sharp({}_3C_1, {}_2C_1)_L - \sharp({}_3C_2, {}_2C_1)_L + \sharp({}_3C_2, {}_2C_1)_R - \sharp({}_3C_1, {}_2C_1)_R \ , \tag{4.22}$$

where $\sharp$ again represents the number of each multiplet. The total number of (heavy) neutrino singlets $(\nu_R)^c$ is denoted $n_{\bar{\nu}}$ and defined as

$$n_{\bar{\nu}} \equiv \sharp({}_3C_0, {}_2C_0)_L + \sharp({}_3C_3, {}_2C_2)_L + \sharp({}_3C_3, {}_2C_2)_R + \sharp({}_3C_0, {}_2C_0)_R. \tag{4.23}$$

Formulae for the SM species including neutrino singlets are given by

$$n_{\bar{d}} = \sum_{\pm} \sum_{(l_1,l_2)=(2,2),(1,0)} (-1)^{l_1+l_2} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.24}$$

$$n_{l} = \sum_{\pm} \sum_{(l_1,l_2)=(3,1),(0,1)} (-1)^{l_1+l_2} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.25}$$

$$n_{\bar{u}} = \sum_{\pm} \sum_{(l_1,l_2)=(2,0),(1,2)} (-1)^{l_1+l_2} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.26}$$

$$n_{\bar{e}} = \sum_{\pm} \sum_{(l_1,l_2)=(0,2),(3,0)} (-1)^{l_1+l_2} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.27}$$

$$n_{q} = \sum_{\pm} \sum_{(l_1,l_2)=(1,1),(2,1)} (-1)^{l_1+l_2} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} - \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.28}$$

$$n_{\bar{\nu}} = \sum_{\pm} \sum_{(l_1,l_2)=(0,0),(3,2)} \left(\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}} + \sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}} \right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.29}$$

where $\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 L\pm}}}$ means that the summations over $l_j = 0, \cdots, k-l_1-\cdots-l_{j-1}$ ($j = 3, \cdots, n-1$) are carried out under the condition that l_j should satisfy specific relations on $T^2/\mathbb{Z}_M$ given in Table 4.2. The relations will be confirmed in the next section. In the same way, $\sum_{\{l_3,\cdots,l_{n-1}\}_{n^a_{l_1 l_2 R\pm}}}$ means that the summations over $l_j = 0, \cdots, k - l_1 - \cdots - l_{j-1}$ ($j = 3, \cdots, n-1$) are carried out under the condition that l_j should satisfy specific relations $n^a_{l_1 l_2 R\pm} = n^a_{l_1 l_2 L\pm} \mp 1 \pmod M$ for $\Psi_\pm$. The formulae (4.24) – (4.29) will be also rewritten in more concrete form for each $T^2/\mathbb{Z}_M$, by the use of projection operators, in the next section.

4.3 Total numbers of models with three families

We investigate the family unification in $SU(N)$ GUTs for each $T^2/\mathbb{Z}_M$ ($M = 2, 3, 4, 6$). Let us present total numbers of models with the three families, for reference. Total numbers of models with the three families of $SU(5)$ multiplets and the SM multiplets, which originate from a Dirac fermion whose representation is $[N, k]$ ($k \leq N/2$) of $SU(N)$, are summarized up to $SU(12)$ in Table 4.5 and up to $SU(13)$

Orbifolds	$\overline{\rho}^k\eta^a_{k\pm}$	Specific relations
$T^2/\mathbb{Z}_2$	$(-1)^k\eta^0_{k\pm} = (-1)^{\alpha_\pm}$ $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$ $(-1)^k\eta^2_{k\pm} = (-1)^{\gamma_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv l_3 + l_4 = 2 - l_1 - l_2 - \alpha_\pm \pmod 2$ $n^1_{l_1 l_2 L\pm} \equiv l_5 + l_6 = 2 - l_1 - l_2 - \beta_\pm \pmod 2$ $n^2_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \gamma_\pm \pmod 2$
$T^2/\mathbb{Z}_3$	$(e^{-2\pi i/3})^k\eta^0_{k\pm} = (e^{2\pi i/3})^{\alpha_\pm}$ $(e^{-2\pi i/3})^k\eta^1_{k\pm} = (e^{2\pi i/3})^{\beta_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv l_3 + 2(l_4 + l_5 + l_6)$ $= 3 - l_1 - l_2 - \alpha_\pm \pmod 3$ $n^1_{l_1 l_2 L\pm} \equiv l_4 + l_7 + 2(l_5 + l_8)$ $= 3 - l_1 - 2l_2 - \beta_\pm \pmod 3$
$T^2/\mathbb{Z}_4$	$(-i)^k\eta^0_{k\pm} = i^{\alpha_\pm}$ $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv 2(l_3 + l_4) + 3(l_5 + l_6)$ $= 4 - l_1 - l_2 - \alpha_\pm \pmod 4$ $n^1_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 = 2 - l_1 - \beta_\pm \pmod 2$
$T^2/\mathbb{Z}_6$	$(e^{-\pi i/3})^k\eta^0_{k\pm} = (e^{\pi i/3})^{\alpha_\pm}$ $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$	$n^0_{l_1 l_2 L\pm} \equiv 2(l_3 + l_4) + 3(l_5 + l_6)$ $+4(l_7 + l_8) + 5(l_9 + l_{10})$ $= 6 - l_1 - l_2 - \alpha_\pm \pmod 6$ $n^1_{l_1 l_2 L\pm} \equiv l_3 + l_5 + l_7 + l_9 + l_{11}$ $= 2 - l_1 - \beta_\pm \pmod 2$

Table 4.2: The specific relations for l_j for the SM multiplets.

in Table 4.10, respectively. In the Tables, the hyphen (-) means no models. We omit the total numbers of models from $[N, N-k]$, because each flavor number from $[N, k]$ with intrinsic $\mathbb{Z}_M$ elements $\eta^a_{k\pm}$ is equal to that from $[N, N-k]$ with appropriate ones $\eta^a_{N-k\pm}$.

4.3.1 Numbers of $SU(5)$ multiplets on $T^2/\mathbb{Z}_M$

After the breakdown $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_n) \times U(1)^{n-m+1}$, $[N, k]_\pm$ is decomposed as

$$[N, k]_\pm = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-1}} ({}_5C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n})_\pm \ , \tag{4.30}$$

where $l_n = k - l_1 - l_2 - \cdots - l_{n-1}$.

The $\mathbb{Z}_M$ elements of the representation $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n})_\pm$ are given by Table 4.3. Using the assignment of $\mathbb{Z}_M$ elements, we find that zero modes appear if the specific relations of Table 4.1 are satisfied.

Utilizing the survival hypothesis and equivalence of charge conjugation, we obtain the formulate the formulae (4.14) – (4.16). The $\mathbb{Z}_M$ projection operator that picks up zero modes of left- and right-handed ones represents $P_{M\pm}$. For each $T^2/\mathbb{Z}_M$, the $\mathbb{Z}_M$ projection operators are defined as

$$P^{(\theta,\theta,\theta)}_{2\pm} \equiv \frac{1}{8}(1 + \bar{\theta}\mathscr{P}_{0\pm})(1 + \bar{\theta}\mathscr{P}_{1\pm})(1 + \bar{\theta}\mathscr{P}_{2\pm}) \text{ for } T^2/\mathbb{Z}_2 \ , \tag{4.31}$$

Orbifolds	n	the $\mathbb{Z}_M$ elements
$T^2/\mathbb{Z}_2$	8	$\mathscr{P}_{0\pm} = (-1)^{l_1+l_2+l_3+l_4+\alpha_\pm}$ $\mathscr{P}_{1\pm} = (-1)^{l_1+l_2+l_5+l_6+\beta_\pm}$ $\mathscr{P}_{2\pm} = (-1)^{l_1+l_3+l_5+l_7+\gamma_\pm}$
$T^2/\mathbb{Z}_3$	9	$\mathscr{P}_{0\pm} = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)+\alpha_\pm}$ $\mathscr{P}_{1\pm} = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)+\beta_\pm}$
$T^2/\mathbb{Z}_4$	8	$\mathscr{P}_{0\pm} = i^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)+\alpha_\pm}$ $\mathscr{P}_{1\pm} = (-1)^{l_1+l_3+l_5+l_7+\beta_\pm}$
$T^2/\mathbb{Z}_6$	12	$\mathscr{P}_{0\pm} = \rho^{l_1+l_2+2(l_3+l_4)+3(l_5+l_6)+4(l_7+l_8)+5(l_9+l_{10})+\alpha_\pm}$ $\mathscr{P}_{1\pm} = (-1)^{l_1+l_3+l_5+l_7+l_9+l_{11}+\beta_\pm}$

Table 4.3: The $\mathbb{Z}_M$ elements for each $T^2/\mathbb{Z}_M$

$$P_{3\pm}^{(\theta,\theta)} \equiv \frac{1}{9}(1+\bar{\theta}\mathscr{P}_{0\pm}+\bar{\theta}^2\mathscr{P}_{0\pm}^2)(1+\bar{\theta}\mathscr{P}_{1\pm}+\bar{\theta}^2\mathscr{P}_{1\pm}^2) \text{ for } T^2/\mathbb{Z}_3 , \tag{4.32}$$

$$P_{4\pm}^{(\theta,\theta')} \equiv \frac{1}{8}(1+\bar{\theta}\mathscr{P}_{0\pm}+\bar{\theta}^2\mathscr{P}_{0\pm}^2+\bar{\theta}^3\mathscr{P}_{0\pm}^3)(1+\bar{\theta}'\mathscr{P}_{1\pm}) \text{ for } T^2/\mathbb{Z}_4 , \tag{4.33}$$

$$P_{6\pm}^{(\theta,\theta')} \equiv \frac{1}{12}(1+\bar{\theta}\mathscr{P}_{0\pm}+\bar{\theta}^2\mathscr{P}_{0\pm}^2+\bar{\theta}^3\mathscr{P}_{0\pm}^3+\bar{\theta}^4\mathscr{P}_{0\pm}^4+\bar{\theta}^5\mathscr{P}_{0\pm}^5) \times (1+\bar{\theta}'\mathscr{P}_{1\pm}) \text{ for } T^2/\mathbb{Z}_6 . \tag{4.34}$$

Using the $\mathbb{Z}_M$ projection operators, the formulae (4.14) – (4.16) are rewritten as

$$n_{\bar{5}} = \sum_{l_1=1,4}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}}(-1)^{l_1} \times \left(P_{M+}-P'_{M+}+P_{M-}-P'_{M-}\right){}_{p_2}C_{l_2}\cdots{}_{p_n}C_{l_n} , \tag{4.35}$$

$$n_{10} = \sum_{l_1=2,3}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}}(-1)^{l_1} \times \left(P_{M+}-P'_{M+}+P_{M-}-P'_{M-}\right){}_{p_2}C_{l_2}\cdots{}_{p_n}C_{l_n} , \tag{4.36}$$

$$n_1 = \sum_{l_1=0,5}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}}\left(P_{M+}+P'_{M+}+P_{M-}+P'_{M-}\right){}_{p_2}C_{l_2}\cdots{}_{p_n}C_{l_n} . \tag{4.37}$$

Here, we give a list of $\mathbb{Z}_M$ projection operator in Table 4.4.

Total numbers of models with the three families of $SU(5)$ multiplets, which originate from a Dirac fermion whose representation is $[N,k]$ ($k \leq N/2$) of $SU(N)$, are summarized up to $SU(12)$ in Table 4.5.

Here, we give some examples for representations and BCs to derive $n_{\bar{5}} = n_{10} = 3$, for each $T^2/\mathbb{Z}_M$ orbifold, in Table 4.6 – 4.9.

Orbifolds	P_{M+}	P'_{M+}	P_{M-}	P'_{M-}
$T^2/\mathbb{Z}_2$	$P_{2+}^{(1,1,1)}$	$P_{2+}^{(-1,-1,-1)}$	$P_{2-}^{(1,1,1)}$	$P_{2-}^{(-1,-1,-1)}$
$T^2/\mathbb{Z}_3$	$P_{3+}^{(1,1)}$	$P_{3+}^{(\omega,\omega)}$	$P_{3-}^{(1,1)}$	$P_{3-}^{(\bar{\omega},\bar{\omega})}$
$T^2/\mathbb{Z}_4$	$P_{4+}^{(1,1)}$	$P_{4+}^{(i,-1)}$	$P_{4-}^{(1,1)}$	$P_{4-}^{(-i,-1)}$
$T^2/\mathbb{Z}_6$	$P_{6+}^{(1,1)}$	$P_{6+}^{(\rho,-1)}$	$P_{6-}^{(1,1)}$	$P_{6-}^{(\bar{\rho},-1)}$

Table 4.4: The $\mathbb{Z}_M$ projection operator for picking up zero modes.

	$T^2/\mathbb{Z}_2$	$T^2/\mathbb{Z}_3$	$T^2/\mathbb{Z}_4$	$T^2/\mathbb{Z}_6$
$SU(8)$	-	[8,3]:24 [8,4]:12	[8,3]:14 [8,4]:16	[8,3]:28 [8,4]:20
$SU(9)$	[9,3]:192	[9,3]:182 [9,4]:348	[9,3]:142 [9,4]:32	[9,3]:512 [9,4]:800
$SU(10)$	-	[10,3]:852 [10,4]:1308 [10,5]:48	[10,3]:160 [10,4]:92	[10,3]:2484 [10,4]:2654 [10,5]:1532
$SU(11)$	[11,3]:768 [11,4]:768	[11,3]:1608 [11,4]:1716 [11,5]:1794	[11,3]:456 [11,4]:436 [11,5]:186	[11,3]:6530 [11,4]:6768 [11,5]:5540
$SU(12)$	[12,3]:1104	[12,3]:2214 [12,4]:1020	[12,3]:748 [12,4]:676 [12,5]:534 [12,6]:632	[12,3]:17084 [12,4]:13692 [12,5]:10498 [12,6]:13188

Table 4.5: Total numbers of models with the three families of $SU(5)$ multiplets.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	$(\alpha_+,\beta_+,\gamma_+)$	$(\alpha_-,\beta_-,\gamma_-)$
[9,3]	(5,0,0,0,3,0,0,1)	(0,1,1)	(0,0,1)
[11,3]	(5,0,1,0,4,0,1,0)	(0,0,1)	(1,1,0)
[11,4]	(5,0,3,1,0,1,1,0)	(0,0,0)	(0,0,1)
[12,3]	(5,2,0,0,2,0,1,2)	(1,0,1)	(0,0,0)

Table 4.6: Examples for the three families of $SU(5)$ from $T^2/\mathbb{Z}_2$.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9)$	(α_+,β_+)	(α_-,β_-)
[8,3]	(5,0,0,0,3,0,0,0,0)	(2,0)	(2,2)
[8,4]	(5,1,1,0,1,0,0,0,0)	(0,0)	(2,2)
[9,3]	(5,0,0,2,0,1,0,0,1)	(2,0)	(2,1)
[9,4]	(5,0,2,0,0,0,0,2,0)	(2,2)	(0,2)
[10,3]	(5,0,0,0,3,2,0,0,0)	(2,0)	(2,2)
[10,4]	(5,0,0,1,0,1,1,1,1)	(2,2)	(2,2)
[10,5]	(5,1,0,0,1,0,2,0,1)	(0,0)	(0,0)
[11,3]	(5,1,0,0,1,4,0,0,0)	(0,0)	(2,1)
[11,4]	(5,2,2,0,0,1,0,1,0)	(1,2)	(2,1)
[11,5]	(5,1,1,1,1,0,0,0,2)	(0,1)	(1,1)
[12,3]	(5,0,0,3,3,0,0,0,1)	(2,0)	(0,2)
[12,4]	(5,0,3,1,0,1,0,2,0)	(1,2)	(0,1)

Table 4.7: Examples for the three families of $SU(5)$ from $T^2/\mathbb{Z}_3$.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	(α_+,β_+)	(α_-,β_-)
[8,3]	(5,0,0,0,0,0,3,0)	(2,1)	(0,0)
[8,4]	(5,0,0,3,0,0,0,0)	(0,0)	(2,0)
[9,3]	(5,3,0,0,0,0,0,1)	(1,0)	(0,1)
[9,4]	(5,0,2,0,0,0,1,1)	(2,0)	(2,0)
[10,3]	(5,0,0,0,3,0,0,2)	(1,0)	(2,0)
[10,4]	(5,0,0,0,0,4,0,1)	(0,0)	(2,1)
[11,3]	(5,0,0,1,2,2,0,1)	(3,1)	(2,0)
[11,4]	(5,0,3,1,2,0,0,0)	(2.0)	(1,1)
[11,5]	(5,0,0,2,0,0,1,3)	(0,1)	(3,0)
[12,3]	(5,4,0,1,0,0,0,2)	(3,1)	(1,0)
[12,4]	(5,0,4,0,1,2,0,0)	(2,0)	(3,0)
[12,5]	(5,1,2,0,2,2,0,0)	(3,1)	(1,1)
[12,6]	(5,0,3,0,1,0,3,0)	(2,0)	(2,1)

Table 4.8: Examples for the three families of $SU(5)$ from $T^2/\mathbb{Z}_4$.

$[N,k]$	$(p_1,p_2,p_3,\cdots,p_{11},p_{12})$	(α_+,β_+)	(α_-,β_-)
[8,3]	(5,0,0,3,0,0,0,0,0,0,0,0)	(0,1)	(2,0)
[8,4]	(5,0,0,1,0,0,0,2,0,0,0,0)	(0,0)	(2,0)
[9,3]	(5,0,0,0,0,0,3,0,0,0,0,1)	(0,1)	(5,0)
[9,4]	(5,2,0,1,0,0,1,0,0,0,0,0)	(2,0)	(2,0)
[10,3]	(5,0,0,1,1,0,0,0,0,0,3,0)	(0,1)	(4,1)
[10,4]	(5,0,1,0,1,1,0,0,0,1,1,0)	(5,0)	(2,0)
[10,5]	(5,0,0,0,0,0,1,2,0,2,0,0)	(4,1)	(1,0)
[11,3]	(5,0,0,1,0,0,0,0,0,1,4,0)	(3,1)	(4,1)
[11,4]	(5,0,0,0,0,2,0,0,2,1,0,1)	(5,0)	(2,0)
[11,5]	(5,3,0,0,0,0,0,0,0,0,3,0)	(1,1)	(1,1)
[12,3]	(5,3,0,1,0,0,0,0,0,0,0,3)	(0,1)	(3,0)
[12,4]	(5,0,0,0,0,0,0,1,0,4,1,1)	(5,0)	(2,0)
[12,5]	(5,0,0,0,0,0,2,1,2,1,1,0)	(1,1)	(1,1)
[12,6]	(5,0,0,0,0,3,1,1,2,0,0,0)	(3,0)	(0,0)

Table 4.9: Examples for the three families of $SU(5)$ from $T^2/\mathbb{Z}_6$.

4.3.2 Numbers of the SM multiplets on $T^2/\mathbb{Z}_M$

After the breakdown $SU(N) \to SU(3) \times SU(3) \times SU(p_3) \times \cdots \times SU(p_n) \times U(1)^{n-m+1}$, $[N,k]_\pm$ is decomposed as

$$[N,k]_\pm = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-1}} \left({}_3C_{l_1}, {}_2C_{l_2}, {}_{p_3}C_{l_3}, \cdots, {}_{p_n}C_{l_n}\right)_\pm \ , \tag{4.38}$$

where $l_n = k - l_1 - l_2 - \cdots - l_{n-1}$.

Using the $\mathbb{Z}_M$ projection operators (4.31) – (4.34), the formulae (4.24) – (4.29) are rewritten as

$$n_{\bar{d}} = \sum_{(l_1,l_2)=(2,2),(1,0)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} (-1)^{l_1+l_2} \times \left(P_{M+} - P'_{M+} + P_{M-} - P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.39}$$

$$n_{l} = \sum_{(l_1,l_2)=(3,1),(0,1)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} (-1)^{l_1+l_2} \times \left(P_{M+} - P'_{M+} + P_{M-} - P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.40}$$

$$n_{\bar{u}} = \sum_{(l_1,l_2)=(2,0),(1,2)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} (-1)^{l_1+l_2}$$

$$\times \left(P_{M+} - P'_{M+} + P_{M-} - P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.41}$$

$$n_{\bar{e}} = \sum_{(l_1,l_2)=(0,2),(3,0)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} (-1)^{l_1+l_2}$$
$$\times \left(P_{M+} - P'_{M+} + P_{M-} - P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.42}$$

$$n_q = \sum_{(l_1,l_2)=(1,1),(2,1)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} (-1)^{l_1+l_2}$$
$$\times \left(P_{M+} - P'_{M+} + P_{M-} - P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.43}$$

$$n_{\bar{\nu}} = \sum_{(l_1,l_2)=(0,0),(3,2)} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6}$$
$$\times \left(P_{M+} + P'_{M+} + P_{M-} + P'_{M-}\right) {}_{p_3}C_{l_3} \cdots {}_{p_n}C_{l_n} \ , \tag{4.44}$$

where each $\mathbb{Z}_M$ projection operator are listed in Table 4.4. Total numbers of models with the three families of the SM multiplets, which originate from a Dirac fermion whose representation is $[N, k]$ ($k \leq N/2$) of $SU(N)$, are summarized up to $SU(13)$ in Table 4.10.

	$T^2/\mathbb{Z}_2$	$T^2/\mathbb{Z}_3$	$T^2/\mathbb{Z}_4$	$T^2/\mathbb{Z}_6$
$SU(8)$	-	-	-	-
$SU(9)$	[9,3]:32	-	[9,3]:8	[9,3]:8 [9,4]:32
$SU(10)$	-	-	-	[10,3]:80 [10,4]:108
$SU(11)$	[11,3]:80 [11,4]:80	[11,4]:80	[11,3]:20 [11,4]:20	[11,3]:84 [11,4]:144 [11,5]:156
$SU(12)$	[12,3]:120	[12,3]:80	[12,4]:88 [12,6]:240	[12,3]:392 [12,4]:120 [12,5]:72 [12,6]:552
$SU(13)$	[13,3]:144	-	[13,4]:40	[13,3]:712 [13,4]:88 [13,5]:140 [13,6]:200

Table 4.10: Total numbers of models with the three families of SM multiplets.

Here, we give a list of all BCs to derive three families of SM fermions from $[9, 3]$ from $T^2/\mathbb{Z}_2$, in Table 4.11, and some examples for representations and BCs to derive three families of SM fermions from $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_3$ and $T^2/\mathbb{Z}_6$, in Table 4.12 – 4.14.

4.4 Generic features of flavor numbers

We list generic features of flavor numbers.

(i) *Each flavor number from $[N, k]$ with intrinsic $\mathbb{Z}_M$ elements $\eta^a_{k\pm}$ is equal to that from $[N, N-k]$ with appropriate ones $\eta^a_{N-k\pm}$.*

Let us explain this feature using the $SU(5)$ multiplets. From (4.10) and the decomposition of $[N, N-k]$ such that

$$[N, N-k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-\cdots-l_{n-2}} \left({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \cdots, {}_{p_n}C_{p_n-l_n}\right) , \tag{4.45}$$

there is a one-to-one correspondence between $({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \cdots, {}_{p_n}C_{p_n-l_n})$ in $[N, N-k]$ and $({}_5C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n})$ in $[N, k]$. The right-handed Weyl fermion whose representation is $({}_5C_{5-l_1}, {}_{p_2}C_{p_2-l_2}, \cdots, {}_{p_n}C_{p_n-l_n})$ is regarded as the left-handed one whose representation is the conjugate representation $({}_5C_{l_1}, {}_{p_2}C_{l_2}, \cdots, {}_{p_n}C_{l_n})$, and hence we obtain the same numbers for (4.14) – (4.16) with a suitable assignment of intrinsic $\mathbb{Z}_M$ elements for $[N, N-k]$.

Here, we give an example for $T^2/\mathbb{Z}_2$. Each flavor number obtained from $[N, k]$ with $(-1)^k\eta^0_{k\pm} = (-1)^{\alpha_\pm}$, $(-1)^k\eta^1_{k\pm} = (-1)^{\beta_\pm}$ and $(-1)^k\eta^2_{k\pm} = (-1)^{\gamma_\pm}$ agrees with that from $[N, N-k]$ with $(-1)^{N-k}\eta^0_{N-k\pm} = (-1)^{\alpha'_\pm}$, $(-1)^{N-k}\eta^1_{N-k\pm} = (-1)^{\beta'_\pm}$ and $(-1)^{N-k}\eta^2_{N-k\pm} = (-1)^{\gamma'_\pm}$, where $\alpha'_\pm$, $\beta'_\pm$ and $\gamma'_\pm$ satisfy the relations $\alpha'_\pm = \alpha_\pm + p_2 + p_3 + p_4 (\mathrm{mod} 2)$, $\beta'_\pm = \beta_\pm + p_2 + p_5 + p_6 (\mathrm{mod} 2)$ and $\gamma'_\pm = \gamma_\pm + p_3 + p_5 + p_7 (\mathrm{mod} 2)$, respectively.

(ii) *Each flavor number from $[N, k]$ with intrinsic $\mathbb{Z}_2$ elements $(-1)^k\eta^a_{k\pm} = (-1)^{\delta^a_\pm}$ is equal to that from $[N, k]$ with the exchanged ones ($\delta^a_+ \leftrightarrow \delta^a_-$), i.e., $(-1)^k\eta^a_{k\pm} = (-1)^{\delta^a_\mp}$.*

This feature is understood from the fact that specific relations on l_j for Ψ_+ change into those of Ψ_- and vice versa, under the exchange of $\mathbb{Z}_2$ parity of Ψ_+ and that of Ψ_-.

Here, we give an example for $T^2/\mathbb{Z}_2$. Under the exchange of α_+ and α_-, $n^0_{l_1 L+}$ and $n^0_{l_1 R+}$ change into $n^0_{l_1 L-}$ and $n^0_{l_1 R-}$ (mod2), respectively. Each flavor number remains the same, because the summation is taken for Ψ_+ and Ψ_-.

(iii) *Each flavor number from $[N, k]$ is invariant under several types of exchange among p_j and intrinsic $\mathbb{Z}_M$ elements.*

From specific relations in Table 4.1, we find that the same number for each $SU(5)$ multiplet is obtained under the exchange,

$$(p_3, p_4, \alpha_\pm) \Longleftrightarrow (p_5, p_6, \beta_\pm) \ ,$$
$$(p_2, p_6, \beta_\pm) \Longleftrightarrow (p_3, p_7, \gamma_\pm) \ ,$$

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	$(\alpha_+,\beta_+,\gamma_+)$	$(\alpha_-,\beta_-,\gamma_-)$
	(3,2,0,0,0,3,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,0,3,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,0,3,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,0,3,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,0,3,0,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,3,0,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,3,0,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,3,0,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,3,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,0,3,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,0,3,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,0,3,0,0,1,0)	(1,0,0)	(1,0,1)
	(3,2,3,0,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,3,0,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,3,0,0,0,1,0)	(1,0,1)	(1,0,0)
[9,3]	(3,2,3,0,0,0,1,0)	(1,0,0)	(1,0,1)
	(3,2,0,0,1,2,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,1,2,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,1,2,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,1,2,1,0)	(0,1,0)	(0,1,1)
	(3,2,0,0,2,1,0,1)	(0,1,1)	(0,1,0)
	(3,2,0,0,2,1,0,1)	(0,1,0)	(0,1,1)
	(3,2,0,0,2,1,1,0)	(0,1,1)	(0,1,0)
	(3,2,0,0,2,1,1,0)	(0,1,0)	(0,1,1)
	(3,2,1,2,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,1,2,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,1,2,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,1,2,0,0,1,0)	(1,0,0)	(1,0,1)
	(3,2,2,1,0,0,0,1)	(1,0,1)	(1,0,0)
	(3,2,2,1,0,0,0,1)	(1,0,0)	(1,0,1)
	(3,2,2,1,0,0,1,0)	(1,0,1)	(1,0,0)
	(3,2,2,1,0,0,1,0)	(1,0,0)	(1,0,1)

Table 4.11: The three families of SM multiplets from $[9,3]$ on $T^2/\mathbb{Z}_2$.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9)$	(α_+,β_+)	(α_-,β_-)
[11,4]	(3,2,0,0,1,2,3,0,0)	(0,1)	(0,1)
[12,3]	(3,2,0,1,1,0,1,2,2)	(1,0)	(0,1)

Table 4.12: Examples for the three families of SM multiplets from $T^2/\mathbb{Z}_3$.

$[N,k]$	$(p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8)$	(α_+,β_+)	(α_-,β_-)
[9,3]	(3,2,1,0,0,0,2,1)	(0,1)	(0,0)
[11,3]	(3,2,1,1,0,4,0,0)	(1,0)	(1,1)
[11,4]	(3,2,0,0,3,1,1,1)	(0,1)	(0,0)
[12,4]	(3,2,1,0,2,1,3,0)	(0,1)	(0,0)
[12,6]	(3,2,1,2,0,0,0,4)	(0,1)	(1,1)
[13,4]	(3,2,1,2,2,2,0,1)	(0,1)	(0,0)

Table 4.13: Examples for the three families of SM multiplets from $T^2/\mathbb{Z}_4$.

$[N,k]$	$(p_1,p_2,p_3,\cdots,p_{11},p_{12})$	(α_+,β_+)	(α_-,β_-)
[9,3]	(3,2,0,1,0,0,0,0,0,0,1,2)	(0,0)	(0,1)
[9,4]	(3,2,0,0,0,1,0,0,1,2,0,0)	(1,1)	(1,0)
[10,3]	(3,2,0,0,3,0,0,0,0,0,1,1)	(1,0)	(1,1)
[10,4]	(3,2,0,1,1,2,0,0,0,0,1,0)	(0,1)	(0,0)
[11,3]	(3,2,1,1,1,0,0,0,0,1,1,1)	(0,1)	(0,0)
[11,4]	(3,2,0,1,0,2,0,0,0,3,0,0)	(0,1)	(1,0)
[11,5]	(3,2,0,0,1,0,4,0,1,0,0,0)	(0,1)	(0,0)
[12,3]	(3,2,0,1,3,1,0,1,0,0,0,1)	(1,0)	(1,1)
[12,4]	(3,2,0,0,0,1,1,2,0,2,1,0)	(1,1)	(1,0)
[12,5]	(3,2,1,1,0,3,1,1,0,0,0,0)	(1,0)	(1,1)
[12,6]	(3,2,0,0,0,1,0,0,3,0,0,3)	(1,1)	(1,1)
[13,3]	(3,2,1,0,0,0,0,3,2,0,0,2)	(0,0)	(0,1)
[13,4]	(3,2,2,0,1,1,1,1,0,0,1,1)	(1,0)	(1,1)
[13,5]	(3,2,1,0,0,4,0,0,0,3,0,0)	(1,1)	(1,0)
[13,6]	(3,2,1,0,0,0,0,2,4,0,0,1)	(0,0)	(0,1)

Table 4.14: Examples for the three families of SM multiplets from $T^2/\mathbb{Z}_6$.

$$(p_2, p_4, \alpha_\pm) \Longleftrightarrow (p_5, p_7, \gamma_\pm) \qquad \text{for } T^2/\mathbb{Z}_2 , \tag{4.46}$$

$$(p_2, p_3, p_6, \alpha_\pm) \Longleftrightarrow (p_4, p_7, p_8, \beta_\pm) \qquad \text{for } T^2/\mathbb{Z}_3 , \tag{4.47}$$

where the exchange is done independently.

In the same way, from specific relations in Table 4.2, we find that the same number for each SM multiplet is obtained under the exchange,

$$(p_3, p_4, \alpha_\pm) \Longleftrightarrow (p_5, p_6, \beta_\pm) , \qquad \text{for } T^2/\mathbb{Z}_2 . \tag{4.48}$$

Under the above exchanges, although the unbroken gauge symmetry remains, the numbers of zero modes for extra-dimensional components of gauge bosons are, in general, different and hence a model is transformed into a different one.

(iv) *Each flavor number obtained from $[N, k]$ is invariant in the introduction of Wilson line phases.*

Let us give some examples.

On $T^2/\mathbb{Z}_2$, the numbers $n_{\bar{5}}$ and n_{10} obtained from the breaking pattern $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are same as those from $SU(N) \to SU(5) \times SU(p_2') \times \cdots \times SU(p_8') \times U(1)^{7-m}$, if the following relations are satisfied,

$$\begin{aligned} &p_2' - p_2 = p_7' - p_7 = p_3 - p_3' = p_6 - p_6' , \\ &p_4' = p_4 , \quad p_5' = p_5 , \quad p_8' = p_8 , \end{aligned} \tag{4.49}$$

or

$$\begin{aligned} &p_2' - p_2 = p_7' - p_7 = p_4 - p_4' = p_5 - p_5' , \\ &p_3' = p_3 , \quad p_6' = p_6 , \quad p_8' = p_8 , \end{aligned} \tag{4.50}$$

or

$$\begin{aligned} &p_3' - p_3 = p_6' - p_6 = p_4 - p_4' = p_5 - p_5' , \\ &p_2' = p_2 , \quad p_7' = p_7 , \quad p_8' = p_8 . \end{aligned} \tag{4.51}$$

The above BCs are connected by a singular gauge transformation, and they are regarded as equivalent in the presence of Wilson line phases. This equivalence originates from the Hosotani mechanism [31–34], and is shown by the following relations among the diagonal representatives for 2×2 submatrices of (P_0, P_1, P_2) [22],

$$(\tau_3, \tau_3, \tau_3) \sim (\tau_3, \tau_3, -\tau_3) \sim (\tau_3, -\tau_3, \tau_3) \sim (\tau_3, -\tau_3, -\tau_3) , \tag{4.52}$$

where τ_3 is the third component of Pauli matrices.

In our present case, we assume that the BC is chosen as a physical one, i.e., the system with the physical vacuum is realized with the vanishing Wilson line phases after a suitable gauge transformation is performed. Hence, it is understood that each net flavor number obtained from $[N, k]$ does not change even though the vacuum changes different ones in the presence of Wilson line phases.

In the same way, the numbers $n_{\bar{d}}$, n_l, $n_{\bar{u}}$, $n_{\bar{e}}$ and n_q obtained from the breaking pattern $SU(N) \to SU(3) \times SU(2) \times SU(p_3) \times \cdots \times SU(p_8) \times U(1)^{7-m}$ are same as those from $SU(N) \to SU(3) \times SU(2) \times SU(p'_3) \times \cdots \times SU(p'_8) \times U(1)^{7-m}$, if the following relations are satisfied,

$$p'_3 - p_3 = p'_6 - p_6 = p_4 - p'_4 = p_5 - p'_5 \ , \quad p'_7 = p_7 \ , \quad p'_8 = p_8 \ . \tag{4.53}$$

On $T^2/\mathbb{Z}_3$, the numbers $n_{\bar{5}}$ and n_{10} obtained from the breaking pattern $SU(N) \to SU(5) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-m}$ are same as those from $SU(N) \to SU(5) \times SU(p'_2) \times \cdots \times SU(p'_9) \times U(1)^{8-m}$, if the following relations are satisfied,

$$\begin{aligned} &p'_2 - p_2 = p'_6 - p_6 = p'_7 - p_7 = p_3 - p'_3 = p_4 - p'_4 = p_8 - p'_8 \ , \\ &p'_5 = p_5 \ , \quad p'_9 = p_9 \ . \end{aligned} \tag{4.54}$$

The above BCs are also connected by a singular gauge transformation, and they are regarded as equivalent in the presence of Wilson line phases. The equivalence is shown using the following relations among the diagonal representatives for 3×3 submatrices of (Θ_0, Θ_1) on $T^2/\mathbb{Z}_3$ [22],

$$(X, X) \sim (X, \overline{\omega} X) \sim (X, \omega X) \ , \tag{4.55}$$

where $\omega = e^{2\pi i/3}$, $\overline{\omega} = e^{4\pi i/3}$, and $X = \mathrm{diag}(1, \omega, \overline{\omega})$.

For these cases, it is also understood that each net flavor number does not change even though the vacuum changes different ones in the presence of Wilson line phases.

Although this feature holds for models on $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$, there are no examples in our setting, because of the absence of Wilson line phases changing BCs but keeping $SU(5)$ or the SM gauge group for $T^2/\mathbb{Z}_4$ and because of the absence of equivalence relations between diagonal representatives for $T^2/\mathbb{Z}_6$ [22].

5 Relationship between the family number of chiral fermions and the Wilson line phase

In this section, we study the relationship between the family number of chiral fermions and Wilson line phases, based on the orbifold family unification of previous section.

5.1 Family number in orbifold family unification

In section 4, we assume that the BCs are chosen as physical ones, i.e., the system with the physical vacuum is realized with the vanishing Wilson line phases after a suitable gauge transformation is performed. Then, the feature is expressed by

$$N_{\boldsymbol{r}}|_{(\{p_i\},a_k=0)} = N_{\boldsymbol{r}}|_{(\{p_i'\},a_k=0)} \ , \tag{5.1}$$

where $N_{\boldsymbol{r}}$ is a net chiral fermion number (flavor number) for 4D fermions with the representation $\boldsymbol{r}$ of the gauge group, unbroken even in the presence of the Wilson line phases ($2\pi a_k$), and it is defined by

$$N_{\boldsymbol{r}} \equiv n^0_{\mathrm{L}\boldsymbol{r}} - n^0_{\mathrm{R}\boldsymbol{r}} - n^0_{\mathrm{L}\overline{\boldsymbol{r}}} + n^0_{\mathrm{R}\overline{\boldsymbol{r}}} \ . \tag{5.2}$$

Here, $n^0_{\mathrm{L}\boldsymbol{r}}$, $n^0_{\mathrm{R}\boldsymbol{r}}$, $n^0_{\mathrm{L}\overline{\boldsymbol{r}}}$ and $n^0_{\mathrm{R}\overline{\boldsymbol{r}}}$ are the numbers of 4D left-handed massless fermions with $\boldsymbol{r}$, 4D right-handed one with $\boldsymbol{r}$, 4D left-handed one with the complex conjugate representation $\overline{\boldsymbol{r}}$ and 4D right-handed one with $\overline{\boldsymbol{r}}$, respectively. Note that 4D right-handed fermion with $\overline{\boldsymbol{r}}$ and 4D left-handed one with $\boldsymbol{r}$ are transformed into each other under the charge conjugation.

On the other hand, the equivalence due to the dynamical rearrangement is expressed by

$$N_{\boldsymbol{r}}|_{(\{p_i\},a_k\neq 0)} = N_{\boldsymbol{r}}|_{(\{p_i'\},a_k=0)} \ . \tag{5.3}$$

From (5.1) and (5.3), we obtain the relation,

$$N_{\boldsymbol{r}}|_{(\{p_i\},a_k=0)} = N_{\boldsymbol{r}}|_{(\{p_i\},a_k\neq 0)} \ , \tag{5.4}$$

and find that each flavor number obtained from $[N,k]$ does not change even though the vacuum changes different ones in the presence of the Wilson line phases.

In this way, we arrive at the conjecture that each flavor number in the SM is independent of the Wilson line phases that respect the SM gauge group. If there were a Wilson line phase with a non-vanishing SM gauge quantum number, (a part of) the SM gauge symmetry can be broken down. Hence, we assume that such a Wilson line phase is vanishing or switched off.

5.2 Fermion numbers and hidden supersymmetry

On a higher-dimensional space-time $M^4 \times \mathbb{K}^{D-4}$, a massless fermion $\Psi = \Psi(x,y)$ satisfies the equation,

$$i\Gamma^M D_M \Psi = 0 \ , \tag{5.5}$$

where $\mathbb{K}^{D-4}$ is an $(D-4)$-dimensional extra space, Γ^M $(M = 0,1,2,3,5,\cdots,D)$ are matrices that satisfy the Clifford algebra $\Gamma^M\Gamma^N + \Gamma^N\Gamma^M = 2\eta^{MN}$, $D_M \equiv \partial_M + igA_M$ and Ψ is a fermion with $2^{[D/2]}$-components. Here, g is a gauge coupling constant, $A_M (= A_M^\alpha T^\alpha)$ are gauge bosons, and $[*]$ is the Gauss symbol. The coordinates x^μ $(\mu = 0,1,2,3)$ on M^4 and x^m $(m = 5,\cdots,D)$ on $\mathbb{K}^{D-4}$ are denoted by x and y, respectively.

After the breakdown of gauge symmetry, Ψ is decomposed as

$$\Psi(x,y) = \sum_{\boldsymbol{r}_H}\sum_{\{n_i\}}\left[\psi_{\mathrm{L}\boldsymbol{r}_H}^{\{n_i\}}(x)\phi_{\mathrm{L}\boldsymbol{r}_H}^{\{n_i\}}(y) + \psi_{\mathrm{R}\boldsymbol{r}_H}^{\{n_i\}}(x)\phi_{\mathrm{R}\boldsymbol{r}_H}^{\{n_i\}}(y)\right] , \tag{5.6}$$

where $\psi_{\mathrm{L}\boldsymbol{r}_H}^{\{n_i\}}(x)$ and $\psi_{\mathrm{R}\boldsymbol{r}_H}^{\{n_i\}}(x)$ are 4D left-handed spinors and right-handed ones, respectively. The subscript $\boldsymbol{r}_H$ stands for some representation of the unbroken gauge group H, and the superscript $\{n_i\}$ represents a set of numbers relating massive modes and those concerning components of multiplet $\boldsymbol{r}_H$. The functions $\phi_{\mathrm{L}\boldsymbol{r}_H}^{\{n_i\}}(y)$ and $\phi_{\mathrm{R}\boldsymbol{r}_H}^{\{n_i\}}(y)$ form complete sets on $\mathbb{K}^{D-4}$.

We define the chiral fermion number relating $\boldsymbol{r}$ as

$$n_{\boldsymbol{r}} \equiv n_{\mathrm{L}\boldsymbol{r}}^0 - n_{\mathrm{R}\boldsymbol{r}}^0 , \tag{5.7}$$

where $\boldsymbol{r}$ is a representation of the subgroup unbroken in the presence of the Wilson line phases. The net chiral fermion number $N_{\boldsymbol{r}}$ is given by $N_{\boldsymbol{r}} = n_{\boldsymbol{r}} - n_{\overline{\boldsymbol{r}}}$.

In case that $n_{\boldsymbol{r}}$ is independent of the Wilson line phases $(2\pi a_k)$, $n_{\mathrm{L}\boldsymbol{r}}^0$ and $n_{\mathrm{R}\boldsymbol{r}}^0$ must be expressed as

$$n_{\mathrm{L}\boldsymbol{r}}^0 = n_{\mathrm{L}\boldsymbol{r}}'^0 + f_{\boldsymbol{r}}(a_k) \quad \text{and} \quad n_{\mathrm{R}\boldsymbol{r}}^0 = n_{\mathrm{R}\boldsymbol{r}}'^0 + f_{\boldsymbol{r}}(a_k) , \tag{5.8}$$

respectively. Here, $n_{\mathrm{L}\boldsymbol{r}}'^0$ and $n_{\mathrm{R}\boldsymbol{r}}'^0$ are some constants irrelevant to a_k and $f_{\boldsymbol{r}}(a_k)$ is a function of a_k.

5.2.1 An example

Let us calculate $n_{\mathrm{L}\boldsymbol{r}}^0$ and $n_{\mathrm{R}\boldsymbol{r}}^0$, and verify the relations (5.8), using an $SU(3)$ gauge theory on $M^4 \times S^1/\mathbb{Z}_2$.

On 5D space-time, Ψ is expressed as

$$\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} , \tag{5.9}$$

where ψ_{L} and ψ_{R} are components containing 4D left-handed fermions and 4D right-handed ones, respectively.

The equation (5.5) is divided into two parts,

$$i\overline{\sigma}^\mu D_\mu \psi_{\mathrm{L}} - D_y\psi_{\mathrm{R}} = 0 , \quad i\sigma^\mu D_\mu\psi_{\mathrm{R}} + D_y\psi_{\mathrm{L}} = 0 , \tag{5.10}$$

where $D_y \equiv \partial_y + igA_y$. For ψ_{L} and ψ_{R}, the BCs are given by

$$\psi_{\mathrm{L}}(x,-y) = \eta^0 P_0 \psi_{\mathrm{L}}(x,y) , \quad \psi_{\mathrm{L}}(x, 2\pi R - y) = \eta^1 P_1 \psi_{\mathrm{L}}(x,y) , \tag{5.11}$$

$$\psi_{\mathrm{R}}(x,-y) = -\eta^0 P_0 \psi_{\mathrm{R}}(x,y) , \quad \psi_{\mathrm{R}}(x, 2\pi R - y) = -\eta^1 P_1 \psi_{\mathrm{R}}(x,y) , \tag{5.12}$$

where P_0 and P_1 are the representation matrices for the $\mathbb{Z}_2$ transformation $y \to -y$ and the $\mathbb{Z}_2$ transformation $y \to 2\pi R - y$, respectively. η^0 and η^1 are the intrinsic $\mathbb{Z}_2$ parities for the left-handed component. Note that $\mathbb{Z}_2$ parities for the right-handed one are opposite to those of the left-handed one. For the gauge bosons, the BCs are given by

$$A_\mu(x,-y) = P_0 A_\mu(x,y) P_0^\dagger \ , \quad A_\mu(x, 2\pi R - y) = P_1 A_\mu(x,y) P_1^\dagger \ , \tag{5.13}$$

$$A_y(x,-y) = -P_0 A_y(x,y) P_0^\dagger \ , \quad A_y(x, 2\pi R - y) = -P_1 A_y(x,y) P_1^\dagger \ . \tag{5.14}$$

We take the representation matrices,

$$P_0 = \mathrm{diag}(1,1,-1) \ , \quad P_1 = \mathrm{diag}(1,1,-1) \ . \tag{5.15}$$

Then $SU(3)$ is broken down to $SU(2) \times U(1)$. We consider the fermion with the representation **3** of $SU(3)$ and $(\eta^0, \eta^1) = (1,1)$. Then, ψ_{L} and ψ_{R} are expanded as

$$\psi_{\mathrm{L}} = \begin{pmatrix} \sum_{n=0}^{\infty} \psi_{\mathrm{L}n}^1(x) \cos \frac{n}{R} y \\ \sum_{n=0}^{\infty} \psi_{\mathrm{L}n}^2(x) \cos \frac{n}{R} y \\ \sum_{n=1}^{\infty} \psi_{\mathrm{L}n}^3(x) \sin \frac{n}{R} y \end{pmatrix} \ , \quad \psi_{\mathrm{R}} = \begin{pmatrix} \sum_{n=1}^{\infty} \psi_{\mathrm{R}n}^1(x) \sin \frac{n}{R} y \\ \sum_{n=1}^{\infty} \psi_{\mathrm{R}n}^2(x) \sin \frac{n}{R} y \\ \sum_{n=0}^{\infty} \psi_{\mathrm{R}n}^3(x) \cos \frac{n}{R} y \end{pmatrix} . \tag{5.16}$$

After a suitable $SU(2)$ gauge transformation, the vacuum expectation value (VEV) of A_y is parameterized as

$$\langle A_y \rangle = \frac{-i}{gR} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} \ , \tag{5.17}$$

where $2\pi a$ is the Wilson line phase. From the periodicity, we limit the domain of definition for a as $0 \le a < 1$. In case with $a \ne 0$, $SU(2)$ is broken down to $U(1)$, and then every 4D fermion becomes a singlet.

Inserting (5.16) and (5.17) into (5.10), we obtain a set of 4D equations,

$$i\bar{\sigma}^\mu D_\mu \psi_{\mathrm{L}0}^1 - \frac{a}{R}\psi_{\mathrm{R}0}^3 = 0 \ , \quad i\sigma^\mu D_\mu \psi_{\mathrm{R}0}^3 - \frac{a}{R}\psi_{\mathrm{L}0}^1 = 0 \ , \tag{5.18}$$

$$i\bar{\sigma}^\mu D_\mu \psi_{\mathrm{L}0}^2 = 0 \ , \tag{5.19}$$

$$i\bar{\sigma}^\mu D_\mu \psi_{\mathrm{L}n}^1 - \frac{n}{R}\psi_{\mathrm{R}n}^1 - \frac{a}{R}\psi_{\mathrm{R}n}^3 = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.20}$$

$$i\bar{\sigma}^\mu D_\mu \psi_{\mathrm{L}n}^2 - \frac{n}{R}\psi_{\mathrm{R}n}^2 = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.21}$$

$$i\bar{\sigma}^\mu D_\mu \psi_{\mathrm{L}n}^3 + \frac{n}{R}\psi_{\mathrm{R}n}^3 + \frac{a}{R}\psi_{\mathrm{R}n}^1 = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.22}$$

$$i\sigma^\mu D_\mu \psi_{\mathrm{R}n}^1 - \frac{n}{R}\psi_{\mathrm{L}n}^1 + \frac{a}{R}\psi_{\mathrm{L}n}^3 = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.23}$$

$$i\sigma^\mu D_\mu \psi_{\mathrm{R}n}^2 - \frac{n}{R}\psi_{\mathrm{L}n}^2 = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.24}$$

$$i\sigma^\mu D_\mu \psi_{\mathrm{R}n}^3 + \frac{n}{R}\psi_{\mathrm{L}n}^3 - \frac{a}{R}\psi_{\mathrm{L}n}^1 = 0 \quad (n = 1, 2, \cdots) \ . \tag{5.25}$$

Using the equations (5.20), (5.22), (5.23) and (5.25), we derive a set of 4D equations,

$$i\bar{\sigma}^\mu D_\mu(\psi^1_{\mathrm{L}n} + \psi^3_{\mathrm{L}n}) - \frac{n-a}{R}(\psi^1_{\mathrm{R}n} - \psi^3_{\mathrm{R}n}) = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.26}$$

$$i\bar{\sigma}^\mu D_\mu(\psi^1_{\mathrm{L}n} - \psi^3_{\mathrm{L}n}) - \frac{n+a}{R}(\psi^1_{\mathrm{R}n} + \psi^3_{\mathrm{R}n}) = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.27}$$

$$i\sigma^\mu D_\mu(\psi^1_{\mathrm{R}n} + \psi^3_{\mathrm{R}n}) - \frac{n+a}{R}(\psi^1_{\mathrm{L}n} - \psi^3_{\mathrm{L}n}) = 0 \quad (n = 1, 2, \cdots) \ , \tag{5.28}$$

$$i\sigma^\mu D_\mu(\psi^1_{\mathrm{R}n} - \psi^3_{\mathrm{R}n}) - \frac{n-a}{R}(\psi^1_{\mathrm{L}n} + \psi^3_{\mathrm{L}n}) = 0 \quad (n = 1, 2, \cdots) \ . \tag{5.29}$$

From (5.18), $\psi^1_{\mathrm{L}0}$ and $\psi^3_{\mathrm{R}0}$ form a 4D Dirac fermion. In the same way, we find that $(\psi^2_{\mathrm{L}n}, \psi^2_{\mathrm{R}n})$, $(\psi^1_{\mathrm{L}n} + \psi^3_{\mathrm{L}n}, \psi^1_{\mathrm{R}n} - \psi^3_{\mathrm{R}n})$ and $(\psi^1_{\mathrm{L}n} - \psi^3_{\mathrm{L}n}, \psi^1_{\mathrm{R}n} + \psi^3_{\mathrm{R}n})$ form 4D Dirac fermions for $n = 1, 2, \cdots$ from (5.21) and (5.24), (5.26) and (5.29), and (5.27) and (5.28), respectively.

The numbers of 4D massless fermions are evaluated as

$$n^0_{\mathrm{L}} = 1 + \delta_{0a} \ , \quad n^0_{\mathrm{R}} = \delta_{0a} \ , \tag{5.30}$$

where δ_{0a} represents the Kronecker delta. From (5.30), we confirm that the fermion number $n(\equiv n^0_{\mathrm{L}} - n^0_{\mathrm{R}} = 1)$ does not depend on the Wilson line phase. The mass spectrum for 4D fermions in this model is depicted as Figure 5.1.

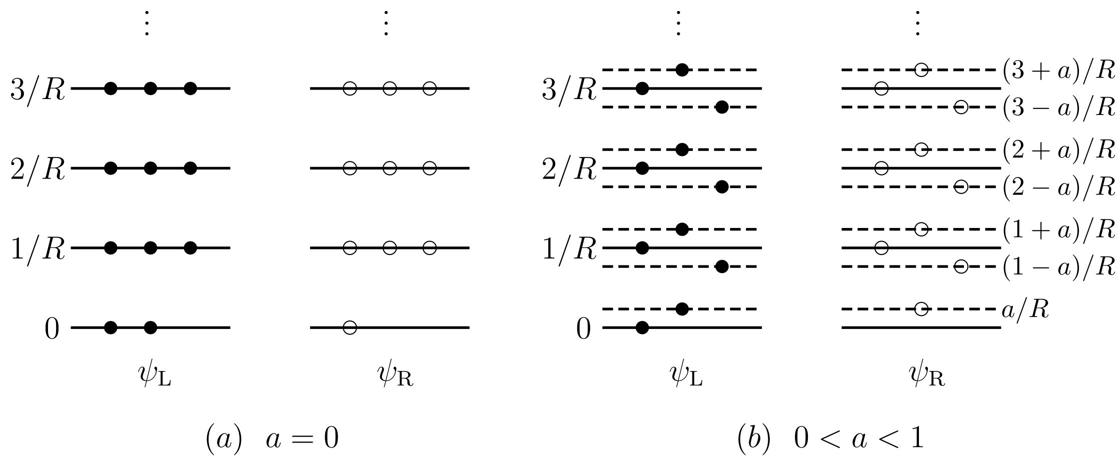


Figure 5.1: Mass spectrum of 4D fermions. The filled circles and the open ones represent left-handed fermions and right-handed ones, respectively.

5.2.2 Hidden quantum-mechanical supersymmetry

We explore a physics behind the feature that the fermion numbers are independent of the Wilson line phases.

From Figure 5.1, we anticipate that the feature originates from a hidden quantum-mechanical SUSY. Here, the quantum-mechanical SUSY means the symmetry generated by the supercharge Q that satisfies the algebraic relations [35, 36],

$$H = Q^2 \ , \quad \{Q, (-1)^F\} = 0 \ , \quad \left((-1)^F\right)^2 = I \ , \tag{5.31}$$

where H, F and I are the Hamiltonian, the "fermion" number operator and the identity operator, respectively. The eigenvalue of $(-1)^F$ is given by $+1$ for "bosonic" states and -1 for "fermionic" states, and Tr $(-1)^F$ is a topological invariant, called the Witten index [37].

It is known that the system with 4D fermions has the hidden SUSY where the 4D Dirac operator plays the role of Q [38,39]. The correspondences are given by

$$Q \leftrightarrow i\gamma^\mu D_\mu = \begin{pmatrix} 0 & i\sigma^\mu D_\mu \\ i\overline{\sigma}^\mu D_\mu & 0 \end{pmatrix} \ , \quad (-1)^F \leftrightarrow \gamma_5 \ , \tag{5.32}$$

where γ_5 is the chirality operator defined by $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. The trace of γ_5 is the index of the 4D Dirac operator, and the following relations hold,

$$\begin{aligned} \mathrm{Tr}\ \gamma_5|_{\boldsymbol{r}} &= n^0_{\mathrm{R}\boldsymbol{r}}[A_\mu] - n^0_{\mathrm{L}\boldsymbol{r}}[A_\mu] = \dim\ker \sigma^\mu D_\mu|_{\boldsymbol{r}} - \dim\ker \overline{\sigma}^\mu D_\mu|_{\boldsymbol{r}} \\ &= \frac{1}{32\pi^2}\int \mathrm{tr}_{\boldsymbol{r}}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}d^4x \ , \end{aligned} \tag{5.33}$$

from the Atiyah-Singer index theorem. Here, $n^0_{\mathrm{R}\boldsymbol{r}}[A_\mu]$ and $n^0_{\mathrm{L}\boldsymbol{r}}[A_\mu]$ are the numbers of normalizable solutions (massless fermions) satisfying $i\sigma^\mu D_\mu \psi_{\mathrm{R}\boldsymbol{r}} = 0$ and $i\overline{\sigma}^\mu D_\mu \psi_{\mathrm{L}\boldsymbol{r}} = 0$, respectively. Note that massive fermions exist in pairs ($\psi_{\mathrm{R}\boldsymbol{r}}$ and $\psi_{\mathrm{L}\boldsymbol{r}}$) and do not contribute to the index. The integral quantity in (5.33) is called the Pontryagin number, and it is deeply connected to the configuration of gauge bosons A_μ on 4D space-time.

It is pointed out that higher-dimensional theories with extra dimensions also possess the hidden SUSY [40,41]. In the system with a 5D fermion, the Dirac operator relating the fifth-coordinate plays the role of Q and there are the correspondences,

$$Q \leftrightarrow \tilde{D}_y = \begin{pmatrix} 0 & D_y \\ -D_y & 0 \end{pmatrix} \ , \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{5.34}$$

Note that $\tilde{\Gamma} = -\gamma_5$. The counterpart of the Witten index is given by

$$\mathrm{Tr}\ \tilde{\Gamma}\Big|_{\boldsymbol{r}} = \tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a) - \tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a) \ , \tag{5.35}$$

where $\tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a)$ and $\tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a)$ are the numbers of eigenfunctions, that satisfy the equations,

$$\tilde{D}_y \begin{pmatrix} 0 \\ \psi_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} D_y\psi_{\mathrm{R}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.36}$$

and

$$\tilde{D}_y \begin{pmatrix} \psi_{\mathrm{L}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -D_y\psi_{\mathrm{L}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \tag{5.37}$$

respectively. Note that the eigenvalue equations are given by $D_y\psi_{\mathrm{R}} = \lambda\psi_{\mathrm{R}}$ and $D_y\psi_{\mathrm{L}} = \lambda'\psi_{\mathrm{L}}$, eigenfunctions with non-zero eigenvalues exist in pairs, which correspond to 4D massive fermions as seen from (5.10), and they do not contribute to the index. From the equations (5.10), there is a one-to-one correspondence such that

$$D_y\psi_{\mathrm{R}} = 0 \leftrightarrow i\overline{\sigma}^\mu D_\mu\psi_{\mathrm{L}} = 0 \ , \quad D_y\psi_{\mathrm{L}} = 0 \leftrightarrow i\sigma^\mu D_\mu\psi_{\mathrm{R}} = 0 \ . \tag{5.38}$$

Let us generalize to a system with a fermion on a higher-dimensional space-time. For the case that $D = 2n$ $(n = 3, 4, \cdots)$, the correspondences are given by

$$Q \leftrightarrow \tilde{D} \equiv \sum_{m=5}^{D} i\Gamma^m D_m \ , \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv -\Gamma_{D+1} \ , \tag{5.39}$$

where Γ_{D+1} is the chirality operator defined by $\Gamma_{D+1} = (-i)^{n+1}\Gamma^0\Gamma^1 \cdots \Gamma^D$.

For the case that $D = 2n + 1$ $(n = 2, 3, \cdots)$, the correspondences are given by

$$Q \leftrightarrow \tilde{D} \equiv U^\dagger \sum_{m=5}^{D} i\Gamma^m D_m U \ , \quad (-1)^F \leftrightarrow \tilde{\Gamma} \equiv i\Gamma^D \ , \tag{5.40}$$

where U is the unitary matrix that satisfies the relation $i\Gamma^D = U^\dagger\Gamma^1 U$, and $i\Gamma^D$ is a diagonal matrix with the same form as the chirality operator on $D(= 2n)$-dimensions up to a sign factor.

The equation (5.5) is written by

$$i\Gamma^\mu D_\mu \Psi + \sum_{m=5}^{D} i\Gamma^m D_m \Psi = 0 \ . \tag{5.41}$$

For the case that $D = 2n + 1$, after the unitary transformation $\Gamma'^M = U^\dagger \Gamma^M U$ and $\Psi' = U^\dagger \Psi$ is performed, Γ'^M and Ψ' are again denoted as Γ^M and Ψ in (5.41). The counterpart of the Witten index is given by

$$\mathrm{Tr}\ \tilde{\Gamma}\Big|_{\boldsymbol{r}} = \tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a_k) - \tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a_k) \ , \tag{5.42}$$

where $\tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a_k)$ and $\tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a_k)$ are the numbers of eigenfunctions, that satisfy $\tilde{D}\psi_{\mathrm{R}} = 0$ and $\tilde{D}\psi_{\mathrm{L}} = 0$, respectively. From (5.41), there is a one-to-one correspondence such that

$$\tilde{D}\psi_{\mathrm{R}} = 0 \leftrightarrow i\Gamma^\mu D_\mu \psi_{\mathrm{L}} = 0 \ , \quad \tilde{D}\psi_{\mathrm{L}} = 0 \leftrightarrow i\Gamma^\mu D_\mu \psi_{\mathrm{R}} = 0 \ . \tag{5.43}$$

Here ψ_{R} and ψ_{L} are a 4D right-handed spinor component and a 4D left-handed one in Ψ, that are eigenspinors of the 4D chirality operator $\Gamma_5 \equiv i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$ whose eigenvalues are 1 and -1, respectively. Note that components with a different 4D chirality involve each other through the equation (5.41), because Γ_5 is anti-commutable to $i\Gamma^\mu D_\mu$ but it is commutable to $\tilde{D}$.

From (5.43), the following relations hold,

$$\tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a_k) = n^0_{\mathrm{L}\boldsymbol{r}} \ , \quad \tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a_k) = n^0_{\mathrm{R}\boldsymbol{r}} \ , \tag{5.44}$$

and, using (5.44), we derive the relation,

$$\mathrm{Tr}\ \tilde{\Gamma}\Big|_{\boldsymbol{r}} = \tilde{n}^0_{\mathrm{R}\boldsymbol{r}}(a_k) - \tilde{n}^0_{\mathrm{L}\boldsymbol{r}}(a_k) = n^0_{\mathrm{L}\boldsymbol{r}} - n^0_{\mathrm{R}\boldsymbol{r}} \ . \tag{5.45}$$

Because $\mathrm{Tr}\ \tilde{\Gamma}\Big|_{\boldsymbol{r}}$ is a topological invariant and the Wilson line phases determine the vacuum with $\langle F_{mn} \rangle = 0$ globally in our orbifold family unification models,

$n_{\boldsymbol{r}}(= n^0_{\mathrm{L}\boldsymbol{r}} - n^0_{\mathrm{R}\boldsymbol{r}})$ is independent of the Wilson line phases. Hence, $N_{\boldsymbol{r}}(= n_{\boldsymbol{r}} - n_{\bar{\boldsymbol{r}}})$ is also independent of the Wilson line phases.

Finally, we give a comment on $\mathrm{Tr}\, \tilde{\Gamma}\Big|_{\boldsymbol{r}}$. As seen from the Atiyah-Singer index theorem relating the Dirac operator for extra-dimensions, fermion numbers are deeply connected to the topological structure on $\mathbb{K}^{D-4}$ including the configurations of A_m on $\mathbb{K}^{D-4}$. From this point of view, the family number has been studied in the Kaluza-Klein theory [42] and superstring theory [20].

6 Prediction of $SU(9)$ orbifold family unification

In this section, we study predictions of orbifold family unification models with $SU(9)$ gauge group on a 6D space-time including the orbifold $T^2/\mathbb{Z}_2$. For the predictions, we search specific relations among sfermion masses on the SUSY extention of models.

6.1 $SU(9)$ orbifold family unification

We have found 32 possibilities that just three families of the SM fermions survive as zero modes from a pair of Weyl fermions with the $\mathbf{84}(= {}_9C_3)$ representation of $SU(9)$. For the list of $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$ to derive them, see Table 4.11. They are classified into two cases based on the pattern of gauge symmetry breaking such that $SU(9) \to SU(3)_C \times SU(2)_L \times SU(3)_F \times U(1)^3$ and $SU(9) \to SU(3)_C \times SU(2)_L \times SU(2)_F \times U(1)^4$. We study how well the three families of fermions in the SM are embedded into Ψ_+ and Ψ_-, in the following.

6.1.1 $SU(9) \to SU(3)_C \times SU(2)_L \times SU(3)_F \times U(1)^3$

For the case that $p_1 = 3$, $p_2 = 2$, either of p_3, p_4, p_5 or p_6 is 3 and either of p_7 or p_8 is 1, $SU(9)$ is broken down as

$$SU(9) \to SU(3)_C \times SU(2)_L \times SU(3)_F \times U(1)_1] \times U(1)_2 \times U(1)_3, \tag{6.1}$$

where $SU(3)_F$ is the gauge group concerning the family of fermions, $U(1)_1$ belongs to a subgroup of $SU(5)$ and is identified with $U(1)_Y$ in the SM, and others are originated from $SU(9)$ and $SU(4)$ as

$$SU(9) \supset SU(5) \times SU(4) \times U(1)_2, \tag{6.2}$$

$$SU(4) \supset SU(3) \times U(1)_3. \tag{6.3}$$

Let us illustrate the survival of three families in the SM, using two typical BCs.

$(\mathbf{BC1})$: $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3, 2, 3, 0, 0, 0, 0, 1)$
In this case, $\mathbf{84}$ is decomposed into particles with the SM gauge quantum numbers and its opposite ones, and their $U(1)$ charges and $\mathbb{Z}_2$ parities are listed in Table 6.1. In the first and second columns, particles are denoted by using the symbols in the SM, and those with primes are regarded as mirror particles. Here, mirror particles are particles with opposite quantum numbers under the SM gauge group $G_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$. The $U(1)$ charges are given up to the normalization. The $\mathbb{Z}_2$ parities of $\psi_L^{1(2)}$ are given by omitting the subscript $k(=3)$ in the last column. The $\mathbb{Z}_2$ parities of $\psi_R^{2(1)}$ are opposite to those of $\psi_L^{1(2)}$.

When we assign the intrinsic $\mathbb{Z}_2$ parities of ψ_L^1 and ψ_L^2 as

$$(\eta_+^0, \eta_+^1, \eta_+^2) = (+1, -1, +1), \quad (\eta_-^0, \eta_-^1, \eta_-^2) = (+1, -1, -1), \tag{6.4}$$

all mirror particles have an odd $\mathbb{Z}_2$ parity and disappear in the low-energy world. Then, just three sets of SM fermions $(q_L^i, (u_R^i)^c, (d_R^i)^c, l_L^i, (e_R^i)^c)$ survive as zero modes

$\psi_L^{1(2)}$	$\psi_R^{1(2)}$	$SU(3)_C \times SU(2)_L \times SU(3)_F$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$(\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2)$
$(e'_R)^c$	e_R	$({}_3C_3, {}_2C_0, {}_3C_0) = (\mathbf{1}, \mathbf{1}, \mathbf{1})$	-6	12	0	$(+\eta^0, +\eta^1, +\eta^2)$
q'_L	$(q_L)^c$	$({}_3C_2, {}_2C_1, {}_3C_0) = (\overline{\mathbf{3}}, \mathbf{2}, \mathbf{1})$	-1	12	0	$(+\eta^0, +\eta^1, -\eta^2)$
$(u'_R)^c$	u_R	$({}_3C_1, {}_2C_2, {}_3C_0) = (\mathbf{3}, \mathbf{1}, \mathbf{1})$	4	12	0	$(+\eta^0, +\eta^1, +\eta^2)$
$(u_R)^c$	u'_R	$({}_3C_2, {}_2C_0, {}_3C_1) = (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3})$	-4	3	1	$(+\eta^0, -\eta^1, +\eta^2)$
$(u_R)^c$	u'_R	$({}_3C_2, {}_2C_0, {}_3C_0) = (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1})$	-4	3	-3	$(-\eta^0, -\eta^1, -\eta^2)$
q_L	$(q'_L)^c$	$({}_3C_1, {}_2C_1, {}_3C_1) = (\mathbf{3}, \mathbf{2}, \mathbf{3})$	1	3	1	$(+\eta^0, -\eta^1, -\eta^2)$
q_L	$(q'_L)^c$	$({}_3C_1, {}_2C_1, {}_3C_0) = (\mathbf{3}, \mathbf{2}, \mathbf{1})$	1	3	-3	$(-\eta^0, -\eta^1, +\eta^2)$
$(e_R)^c$	e'_R	$({}_3C_0, {}_2C_2, {}_3C_1) = (\mathbf{1}, \mathbf{1}, \mathbf{3})$	6	3	1	$(+\eta^0, -\eta^1, +\eta^2)$
$(e_R)^c$	e'_R	$({}_3C_0, {}_2C_2, {}_3C_0) = (\mathbf{1}, \mathbf{1}, \mathbf{1})$	6	3	-3	$(-\eta^0, -\eta^1, -\eta^2)$
$(d'_R)^c$	d_R	$({}_3C_1, {}_2C_0, {}_3C_2) = (\mathbf{3}, \mathbf{1}, \overline{\mathbf{3}})$	-2	-6	2	$(+\eta_0, +\eta_1, +\eta_2)$
$(d'_R)^c$	d_R	$({}_3C_1, {}_2C_0, {}_3C_1) = (\mathbf{3}, \mathbf{1}, \mathbf{3})$	-2	-6	-2	$(-\eta^0, +\eta^1, -\eta^2)$
l'_L	$(l_L)^c$	$({}_3C_0, {}_2C_1, {}_3C_2) = (\mathbf{1}, \mathbf{2}, \overline{\mathbf{3}})$	3	-6	2	$(+\eta^0, +\eta^1, -\eta^2)$
l'_L	$(l_L)^c$	$({}_3C_0, {}_2C_1, {}_3C_1) = (\mathbf{1}, \mathbf{2}, \mathbf{3})$	3	-6	-2	$(-\eta^0, +\eta^1, +\eta^2)$
$(\nu_R)^c$	$\hat{\nu}_R$	$({}_3C_0, {}_2C_0, {}_3C_3) = (\mathbf{1}, \mathbf{1}, \mathbf{1})$	0	-15	3	$(+\eta^0, -\eta^1, +\eta^2)$
$(\nu_R)^c$	$\hat{\nu}_R$	$({}_3C_0, {}_2C_0, {}_3C_2) = (\mathbf{1}, \mathbf{1}, \overline{\mathbf{3}})$	0	-15	-1	$(-\eta^0, -\eta^1, -\eta^2)$

Table 6.1: Decomposition of **84** for $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3, 2, 3, 0, 0, 0, 0, 1)$.

and they belong to the following chiral fermions,

$$\psi_L^1 \supset (u_R^i)^c, (e_R^i)^c, (\nu_R)^c, \quad \psi_R^2 \supset d_R^i, \quad \psi_R^1 \supset (l_L^i)^c, \ \psi_L^2 \supset q_L^i, \tag{6.5}$$

where $i(= 1, 2, 3)$ stands for the family index. By exchanging η_+^a for η_-^a, ψ_L^1 and ψ_R^2 are exchanged for ψ_L^2 and ψ_R^1, respectively. Note that a right-handed neutrino $(\nu_R)^c$ appears alone. We obtain the same result (6.5) by assigning the intrinsic $\mathbb{Z}_2$ parities suitably, in case with p_4, p_5 or $p_6 = 3$ in place of $p_3 = 3$.

(**BC2**) : $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) = (3, 2, 3, 0, 0, 0, 1, 0)$

In this case, **84** is decomposed into particles with the same gauge quantum numbers but sightly different $\mathbb{Z}_2$ parities from those of (BC1). Concretely, the third $\mathbb{Z}_2$ parity $\mathscr{P}_2$ of fields with $l_7 = 1$ is opposite to that with $l_8 = 1$, i.e., $\mathscr{P}_2$ of $({}_3C_2, {}_2C_0, {}_3C_0)$, $({}_3C_1, {}_2C_1, {}_3C_0)$, $({}_3C_0, {}_2C_2, {}_3C_0)$, $({}_3C_1, {}_2C_0, {}_3C_1)$, $({}_3C_0, {}_2C_1, {}_3C_1)$ and $({}_3C_0, {}_2C_0, {}_3C_2)$ is given by $+\eta^2$, $-\eta^2$, $+\eta^2$, $+\eta^2$, $-\eta^2$ and $+\eta^2$, respectively.

Under the same assignment of the intrinsic $\mathbb{Z}_2$ parities as (6.4), all mirror particles have an odd $\mathbb{Z}_2$ parity and disappear in the low-energy world. Then, just three sets of SM fermions survive as zero modes such that

$$\psi_L^1 \supset (u_R^i)^c, (e_R^i)^c, (\nu_R)^c, \quad \psi_R^2 \supset (l_L^i)^c, \quad \psi_R^1 \supset d_R^i, \quad \psi_L^2 \supset q_L^i. \tag{6.6}$$

Note that $(l_L^i)^c$ and d_R^i are embedded into ψ_R^2 and ψ_R^1, respectively, different from the case of (BC1). We obtain the same result (6.6) by assigning the intrinsic $\mathbb{Z}_2$ parities suitably, in case with p_4, p_5 or $p_6 = 3$ in place of $p_3 = 3$.

We summarize fermions with zero modes and those gauge quantum numbers in Table 6.2. Here, $G_{323} = SU(3)_C \times SU(2)_L \times SU(3)_F$, l_a is a number appearing in a representation ${}_{p_a}C_{l_a}$ of $SU(3)_F$ for $a = 3, 4, 5$ or 6, and, in the 7-th and 8-th columns, the way of embeddings for the SM species are shown for $p_8 = 1$ and $p_7 = 1$, respectively.

species	G_{323}	(l_1, l_2, l_a)	$U(1)_1$	$U(1)_2$	$U(1)_3$	$p_8 = 1$	$p_7 = 1$
q_L^i	$(\mathbf{3}, \mathbf{2}, \mathbf{3})$	$(1, 1, 1)$	1	3	1	$\psi_L^{2(1)}$	$\psi_L^{2(1)}$
$(u_R^i)^c$	$(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3})$	(2,0,1)	-4	3	1	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$
d_R^i	$(\mathbf{3}, \mathbf{1}, \mathbf{3})$	$(1, 0, 1)$	-2	-6	-2	$\psi_R^{2(1)}$	$\psi_R^{1(2)}$
$(l_L^i)^c$	$(\mathbf{1}, \mathbf{2}, \mathbf{3})$	$(0, 1, 1)$	3	-6	-2	$\psi_R^{1(2)}$	$\psi_R^{2(1)}$
$(e_R^i)^c$	$(\mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, 2, 1)$	6	3	1	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$
$(\nu_R)^c$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, 0, 3)$	0	-15	3	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$

Table 6.2: Gauge quantum numbers of fermions with even $\mathbb{Z}_2$ parities for $SU(9) \to G_{323} \times U(1)_1 \times U(1)_2 \times U(1)_3$.

6.1.2 $SU(9) \to SU(3)_C \times SU(2)_L \times SU(2)_F \times U(1)^4$

For the case that $p_1 = 3$, $p_2 = 2$, either of (p_3, p_4) or (p_5, p_6) is $(2, 1)$ or $(1, 2)$ and either of p_7 or p_8 is 1, $SU(9)$ is broken down as

$$SU(9) \to SU(3)_C \times SU(2)_L \times SU(2)_F \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_4, \quad (6.7)$$

where $U(1)_1$ belongs to a subgroup of $SU(5)$ and is identified with $U(1)_Y$ in the SM, and others are originated from $SU(9)$, $SU(4)$ and $SU(3)$ as

$$SU(9) \supset SU(5) \times SU(4) \times U(1)_2, \quad (6.8)$$
$$SU(4) \supset SU(3) \times U(1)_3, \quad (6.9)$$
$$SU(3) \supset SU(2) \times U(1)_4. \quad (6.10)$$

The embedding of species are classified into two types, according to $p_8 = 1$ or $p_7 = 1$.

(**BC3**)

For the case with $p_8 = 1$, just three sets of SM fermions survive as zero modes such that

$$\psi_L^{1(2)} \supset (u_R^i)^c, (e_R^i)^c, q_L, \quad \psi_R^{2(1)} \supset d_R^i, (l_L)^c,$$
$$\psi_R^{1(2)} \supset d_R, (l_L^i)^c, \quad \psi_L^{2(1)} \supset (u_R)^c, (e_R)^c, q_L^i, (\nu_R)^c, \quad (6.11)$$

where $i = 1, 2$.

(**BC4**)

For the case with $p_7 = 1$, just three sets of SM fermions survive as zero modes such that

$$\psi_L^{1(2)} \supset (u_R^i)^c, (e_R^i)^c, q_L, \quad \psi_R^{2(1)} \supset d_R, (l_L^i)^c,$$
$$\psi_R^{1(2)} \supset d_R^i, (l_L)^c, \quad \psi_L^{2(1)} \supset (u_R)^c, (e_R)^c, q_L^i, (\nu_R)^c, \tag{6.12}$$

where $i = 1, 2$.

We summarize fermions with zero modes and those gauge quantum numbers in Table 6.3. Here, $G_{322} = SU(3)_C \times SU(2)_L \times SU(2)_F$.

species	G_{322}	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$	$p_8 = 1$	$p_7 = 1$
$(u_R^1)^c, (u_R^2)^c$	$(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{2})$	-4	3	1	1	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$
$(u_R)^c$	$(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{1})$	-4	3	1	-2	$\psi_L^{2(1)}$	$\psi_L^{2(1)}$
q_L^1, q_L^2	$(\mathbf{3}, \mathbf{2}, \mathbf{2})$	1	3	1	1	$\psi_L^{2(1)}$	$\psi_L^{2(1)}$
q_L	$(\mathbf{3}, \mathbf{2}, \mathbf{1})$	1	3	1	-2	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$
$(e_R^1)^c, (e_R^2)^c$	$(\mathbf{1}, \mathbf{1}, \mathbf{2})$	6	3	1	1	$\psi_L^{1(2)}$	$\psi_L^{1(2)}$
$(e_R)^c$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$	6	3	1	-2	$\psi_L^{2(1)}$	$\psi_L^{2(1)}$
d_R^1, d_R^2	$(\mathbf{3}, \mathbf{1}, \mathbf{2})$	-2	-6	-2	1	$\psi_R^{2(1)}$	$\psi_R^{1(2)}$
d_R	$(\mathbf{3}, \mathbf{1}, \mathbf{1})$	-2	-6	-2	-2	$\psi_R^{1(2)}$	$\psi_R^{2(1)}$
$(l_L^1)^c, (l_L^2)^c$	$(\mathbf{1}, \mathbf{2}, \mathbf{2})$	3	-6	-2	1	$\psi_R^{1(2)}$	$\psi_R^{2(1)}$
$(l_L)^c$	$(\mathbf{1}, \mathbf{2}, \mathbf{1})$	3	-6	-2	1	$\psi_R^{2(1)}$	$\psi_R^{1(2)}$
$(\nu_L)^c$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$	0	-15	3	0	$\psi_L^{2(1)}$	$\psi_L^{2(1)}$

Table 6.3: Gauge quantum numbers of fermions with even $\mathbb{Z}_2$ parities for $SU(9) \to G_{322} \times U(1)_1 \times U(1)_2 \times U(1)_3 \times U(1)_4$.

6.2 Predictions

6.2.1 Yukawa interactions

We examine whether four types of $SU(9)$ orbifold family unification models, where the embedding of the SM fermions are realized as (6.5), (6.6), (6.11) and (6.12), are realistic or not,by adopting the appearance of Yukawa interactions from interactions in the 6D bulk as a selection rule. This rule is not almighty to select models, because Yukawa interactions can also be constructed on the fixed points of $T^2/\mathbb{Z}_2$. Here, we carry out the analysis under the assumption that such brane interactions are small compared with the bulk ones in the absence of SUSY.

We assume that the Yukawa interactions in the SM come from interaction terms containing fermions in the bilinear form and products of scalar fields in the 6D bulk. [6] From the Lorentz, gauge and $\mathbb{Z}_2$ invariance, the Lagrangian density containing

[6] We assume that fermion condensations and Lorentz tensor fields are not involved with the generation of Yukawa interactions.

interactions among a pair of Weyl fermions (Ψ_+, Ψ_-) and scalar fields Φ^I on 6D space-time is, in general, written as

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \sum_{a,\cdots,f} \overline{\Psi}_{+abc}\Psi_-^{def} F^{abc}{}_{def}(\Phi^I) + \sum_{a,\cdots,f} \Psi_+^{Tabc} E \Psi_-^{def} G_{abcdef}(\Phi^I) + \text{h.c.} \\ &= \sum \left(\overline{\psi}_L^1 \psi_R^1 + \overline{\psi}_R^2 \psi_L^2\right) F(\Phi^I) + \sum \left((\psi_L^1)^{c\dagger}\psi_L^2 + (\psi_R^1)^{c\dagger}\psi_R^2\right) G(\Phi^I) + \text{h.c.},\end{aligned} \tag{6.13}$$

where $\overline{\Psi}_+ \equiv \Psi_+^\dagger \Gamma^0$, $\overline{\psi}_{L(R)}^{1(2)} = \psi_{L(R)}^{1(2)\dagger}\gamma^0$, and $(\psi_{L(R)}^{1(2)})^c = i\gamma^0\gamma^2\psi_{L(R)}^{1(2)*}$. In the final expression of (6.13), we omit indices of $SU(9)$ such as a, b, $\cdots$, f designating the components to avoid complications. The $F(\Phi^I)$ and $G(\Phi^I)$ are some polynomials of Φ^I, e.g., $F(\Phi^I)$ is expressed by

$$F(\Phi^I) = \sum_{I_1} f_{I_1}\Phi^{I_1} + \sum_{I_1,I_2} f_{I_1 I_2}\Phi^{I_1}\Phi^{I_2} + \cdots = \sum_n \sum_{I_1,\cdots,I_n} f_{I_1\cdots I_n}\Phi^{I_1}\cdots\Phi^{I_n}, \tag{6.14}$$

where $f_{I_1\cdots I_n}$ are coupling constants. Note that mass terms of $\Psi_\pm$ such as $m_{\text{D}}\overline{\Psi}_+\Psi_-$ and $m_{\text{M}}\Psi_+^T E\Psi_-$ are forbidden at the tree level, in case that Ψ_+ and Ψ_- have different intrinsic $\mathbb{Z}_2$ parities. Using the representation given by 6D gamma matrices, E is written as

$$E \equiv \Gamma^1\Gamma^3\Gamma^6 = \begin{pmatrix} 0 & 0 & i\sigma^2 & 0 \\ 0 & 0 & 0 & i\sigma^2 \\ -i\sigma^2 & 0 & 0 & 0 \\ 0 & -i\sigma^2 & 0 & 0 \end{pmatrix}, \tag{6.15}$$

where σ^2 is the second element of Pauli matrices. It is shown that $\mathcal{L}_{\text{int}}$ is invariant under the 6D Lorentz transformation, $\Psi_\pm \to \exp\left[-\frac{i}{4}\omega_{MN}\Sigma^{MN}\right]\Psi_\pm$, where $\Sigma^{MN} = \frac{i}{2}[\Gamma^M, \Gamma^N]$ and ω_{MN} are parameters relating 6D Lorentz boosts and rotations.

After the dimensional reduction occurs and some components acquire the vacuum expectation values (VEVs) generating the breakdown of extra gauge symmetries, the linear terms of the Higgs doublet ϕ_h and its charge conjugated one $\tilde{\phi}_h$ can appear in $F(\Phi^I)$ and $G(\Phi^I)$ and then the Yukawa interactions are derived. For instance, the linear term $\tilde{f}\phi_h$ appears from $F(\Phi^I) = f\Phi_1\Phi_3\Phi_5$ where Φ_m are scalar fields whose representations are $\binom{9}{m}$, after some SM singlets in Φ_3 and Φ_5 acquire the VEVs.

From the above observations, we impose the selection rule that *Yukawa interactions $f_{ij}^u\overline{q}_L^i u_R^j\tilde{\phi}_h$, $f_{ij}^d\overline{q}_L^i d_R^j\phi_h$ and $f_{ij}^e\overline{l}_L^i e_R^j\phi_h$ in the SM can be derived from* $\mathcal{L}_{\text{int}}$ on orbifold family unification models.

For (BC1), the following Lagrangian density is derived at the compactification scale M_{C},

$$\mathcal{L}_{(\text{BC1})} = \sum_{i,j=1}^{3}\overline{d}_R^i q_L^j \tilde{F}_{1ij}^{(1)}(\phi) + \sum_{i,j=1}^{3}\overline{l}_L^i e_R^j \tilde{F}_{2ij}^{(1)}(\phi) + \sum_{i,j=1}^{3}\overline{u}_R^i q_L^j \tilde{G}_{ij}^{(1)}(\phi) + \text{h.c.}, \tag{6.16}$$

using (6.5), and Yukawa interactions in the SM can be obtained, after some SM singlet scalar fields in the polynomials $\tilde{F}_1^{(1)}(\phi)$, $\tilde{F}_2^{(1)}(\phi)$ and $\tilde{G}^{(1)}(\phi)$ acquire the

VEVs. Because all gauge quantum numbers of the operator $\overline{q}^i_L d^j_R$ are same as those of $\overline{l}^i_L e^j_R$, there is a possibility that $\tilde{F}^{(1)}_1(\phi)$ is identical with $\tilde{F}^{(1)}_2(\phi)$ as a simple case. In this case, we have the relations $f^d_{ij} = f^e_{ji}$ at the extra gauge symmetry breaking scale.

For (BC2), the following Lagrangian density is derived,

$$\mathcal{L}_{(\mathrm{BC2})} = \sum_{i,j=1}^{3} \overline{u}^i_R q^j_L \tilde{G}^{(2)}_{ij}(\phi) + \mathrm{h.c.}, \tag{6.17}$$

using (6.6). In this case, down-type quark and charged leptons masses cannot be obtained from $\mathcal{L}_{\mathrm{int}}$ at the tree level at M_{C}.

For (BC3), the following Lagrangian density is derived,

$$\mathcal{L}_{(\mathrm{BC3})} = \sum_{i,j=1}^{2} \overline{d}^i_R q^j_L \tilde{F}^{(3)}_{1ij}(\phi) + \overline{q}_L d_R \tilde{F}^{(3)}_2(\phi) + \sum_{i,j=1}^{2} \overline{l}^i_L e^j_R \tilde{F}^{(3)}_{3ij}(\phi) + \overline{e}_R l_L \tilde{F}^{(3)}_4(\phi) + \mathrm{h.c.}$$
$$+ \sum_{i,j=1}^{2} \overline{u}^i_R q^j_L \tilde{G}^{(3)}_{1ij}(\phi) + \overline{q}_L u_R \tilde{G}^{(3)}_2(\phi) + \mathrm{h.c.}, \tag{6.18}$$

using (6.11). For (BC4), the following Lagrangian density is derived,

$$\mathcal{L}_{(\mathrm{BC4})} = \sum_{i=1}^{2} \left(\overline{d}_R q^i_L \tilde{F}^{(4)}_{1i}(\phi) + \overline{q}_L d^i_R \tilde{F}^{(4)}_{2i}(\phi) + \overline{l}_L e^i_R \tilde{F}^{(4)}_{3i}(\phi) + \overline{e}_R l^i_L \tilde{F}^{(4)}_{4i}(\phi) \right) + \mathrm{h.c.}$$
$$+ \sum_{i,j=1}^{2} \overline{u}^i_R q^j_L \tilde{G}^{(4)}_{1ij}(\phi) + \overline{q}_L u_R \tilde{G}^{(4)}_2(\phi) + \mathrm{h.c.}, \tag{6.19}$$

using (6.12). In both cases, the full flavor mixing cannot be realized at the tree level at M_{C}.

In this way, we find that the model based on the embedding (6.5) is a possible candidate to realize the fermion mass hierarchy and flavor mixing, in case that radiative corrections are too small to generate mixing terms with suitable size for (BC2), (BC3) and (BC4). In any case, we have no powerful principle to determine the polynomials of scalar fields, and hence we obtain no useful predictions from the fermion sector.

6.2.2 Sfermion masses

The SUSY grand unified theories on an orbifold have a desirable feature that the triplet-doublet splitting of Higgs multiplets is elegantly realized [4, 5]. Hence, it would be interesting to construct a SUSY extension of orbifold family unification models.

In the presence of SUSY, the model with (BC1) does not obtain advantages of fermion sector over that with (BC2), (BC3) or (BC4), because any interactions other than gauge interactions are not allowed in the bulk and Yukawa interactions must appear from brane interactions. In SUSY models, complex scalar fields (Φ_+, Φ_-)

are introduced as superpartners of (Ψ_+, Ψ_-), and they consist of two sets of complex scalar fields $\Phi_+ = (\phi_+^1, \phi_+^2)$ and $\Phi_- = (\phi_-^1, \phi_-^2)$, where ϕ_+^1, ϕ_+^2, ϕ_-^1 and ϕ_-^2 are superpartners of ψ_L^1, ψ_R^2, ψ_R^1 and ψ_L^2, respectively. Here, we pay attention to superpartners of the SM fermions called sfermions and study predictions of models.

Based on the assignment (6.5) for (BC1), sfermions are embedded into scalar fields as follows,

$$\phi_+^1 \supset \tilde{u}_R^{i*}, \quad \tilde{e}_R^{i*}, \quad \tilde{\nu}_R^*, \quad \phi_+^2 \supset \tilde{d}_R^i, \quad \phi_-^1 \supset \tilde{l}_L^{i*}, \quad \phi_-^2 \supset \tilde{q}_L^i. \tag{6.20}$$

Gauge quantum numbers for sfermions are given in Table 6.4. Here, the charge conjugation is performed for scalar fields $\tilde{d}_R^i$ and $\tilde{l}_L^{i*}$ corresponding to the right-handed fermions, and $G_{323} = SU(3)_C \times SU(2)_L \times SU(3)_F$. Note that (l_1, l_2, l_a) is untouched by change as a mark of the place of origin in **84**.

species	G_{323}	(l_1, l_2, l_a)	$U(1)_1$	$U(1)_2$	$U(1)_3$
$\tilde{q}_L^i$	$(\mathbf{3}, \mathbf{2}, \mathbf{3})$	$(1, 1, 1)$	1	3	1
$\tilde{u}_R^{i*}$	$(\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3})$	$(2, 0, 1)$	-4	3	1
$\tilde{d}_R^{i*}$	$(\overline{\mathbf{3}}, \mathbf{1}, \overline{\mathbf{3}})$	$(1, 0, 1)$	2	6	2
$\tilde{l}_L^i$	$(\mathbf{1}, \mathbf{2}, \overline{\mathbf{3}})$	$(0, 1, 1)$	-3	6	2
$\tilde{e}_R^{i*}$	$(\mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, 2, 1)$	6	3	1
$\tilde{\nu}_R^*$	$(\mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, 0, 3)$	0	-15	3

Table 6.4: Gauge quantum numbers of sfermions with even $\mathbb{Z}_2$ parities for $SU(9) \to G_{323} \times U(1)_1 \times U(1)_2 \times U(1)_3$.

We study the sfermion masses based on the following two assumptions.

1) The SUSY is broken down by some mechanism and sfermions acquire the soft SUSY breaking masses respecting $SU(9)$ gauge symmetry. Then, $\tilde{u}_R^{i*}$, $\tilde{e}_R^{i*}$, $\tilde{\nu}_R^*$ and $\tilde{d}_R^{i*}$ get a common mass m_+, and $\tilde{q}_L^i$ and $\tilde{l}_L^i$ get a common mass m_- at some scale M_S.

2) Extra gauge symmetries $SU(3)_F \times U(1)_2 \times U(1)_3$ are broken down by the VEVs of some scalar fields at M_S. Then, the D-term contributions to the scalar masses can appear as a dominant source of mass splitting.

The D-term contributions, in general, originate from D-terms related to broken gauge symmetries when the soft SUSY breaking parameters possess non-universal structure and the rank of gauge group decreases after the breakdown of gauge symmetry [43–46]. The contributions for scalar fields specifying by (l_1, l_2, l_a) are given by

$$m^2_{D(l_1,l_2,l_a)} = (-1)^{l_1+l_2}[Q_1 D_{F1} + Q_2 D_{F2} + \{9(l_1 + l_2) - 15\} D_2 + \{4l_a - 3(3 - l_1 - l_2)\} D_3], \tag{6.21}$$

where Q_1 and Q_2 are the diagonal charges (up to normalization) of $SU(3)_F$ for the triplet, i.e., $(Q_1, Q_2) = (1, 1)$, $(-1, 1)$ and $(0, -2)$. D_{F1}, D_{F2}, D_2 and D_3 are parameters including D-term condensations for broken symmetries.

Using m_+, m_- and $m^2_{D(l_1,l_2,l_a)}$, we derive the following formulae of mass square for each species at M_S: [7]

$$m^2_{\tilde{u}_R^{1*}} = m_+^2 + D_{F1} + D_{F2} + 3D_2 + D_3, \tag{6.22}$$
$$m^2_{\tilde{u}_R^{2*}} = m_+^2 - D_{F1} + D_{F2} + 3D_2 + D_3, \tag{6.23}$$
$$m^2_{\tilde{u}_R^{3*}} = m_+^2 - 2D_{F2} + 3D_2 + D_3, \tag{6.24}$$
$$m^2_{\tilde{e}_R^{1*}} = m_+^2 + D_{F1} + D_{F2} + 3D_2 + D_3, \tag{6.25}$$
$$m^2_{\tilde{e}_R^{2*}} = m_+^2 - D_{F1} + D_{F2} + 3D_2 + D_3, \tag{6.26}$$
$$m^2_{\tilde{e}_R^{3*}} = m_+^2 - 2D_{F2} + 3D_2 + D_3, \tag{6.27}$$
$$m^2_{\tilde{d}_R^{1*}} = m_+^2 - D_{F1} - D_{F2} + 6D_2 + 2D_3, \tag{6.28}$$
$$m^2_{\tilde{d}_R^{2*}} = m_+^2 + D_{F1} - D_{F2} + 6D_2 + 2D_3, \tag{6.29}$$
$$m^2_{\tilde{d}_R^{3*}} = m_+^2 + 2D_{F2} + 6D_2 + 2D_3, \tag{6.30}$$
$$m^2_{\tilde{q}_L^{1}} = m_-^2 + D_{F1} + D_{F2} + 3D_2 + D_3, \tag{6.31}$$
$$m^2_{\tilde{q}_L^{2}} = m_-^2 + D_{F1} - D_{F2} + 3D_2 + D_3, \tag{6.32}$$
$$m^2_{\tilde{q}_L^{3}} = m_-^2 - 2D_{F2} + 3D_2 + D_3, \tag{6.33}$$
$$m^2_{\tilde{l}_L^{1}} = m_-^2 - D_{F1} - D_{F2} + 6D_2 + 2D_3, \tag{6.34}$$
$$m^2_{\tilde{l}_L^{2}} = m_-^2 - D_{F1} + D_{F2} + 6D_2 + 2D_3, \tag{6.35}$$
$$m^2_{\tilde{l}_L^{3}} = m_-^2 + 2D_{F2} + 6D_2 + 2D_3. \tag{6.36}$$

By eliminating unknown parameters such as m_+^2, m_-^2, D_{F1}, D_{F2}, D_2 and D_3, we obtain 15 kinds of relations [8]

$$m^2_{\tilde{u}_R^{1*}} = m^2_{\tilde{e}_R^{1*}}, \quad m^2_{\tilde{u}_R^{2*}} = m^2_{\tilde{e}_R^{2*}}, \quad m^2_{\tilde{u}_R^{3*}} = m^2_{\tilde{e}_R^{3*}}, \tag{6.37}$$
$$\begin{aligned} m^2_{\tilde{d}_R^{1*}} - m^2_{\tilde{l}_L^{1}} &= m^2_{\tilde{d}_R^{2*}} - m^2_{\tilde{l}_L^{2}} = m^2_{\tilde{d}_R^{3*}} - m^2_{\tilde{l}_L^{3}} \\ &= m^2_{\tilde{u}_R^{1*}} - m^2_{\tilde{q}_L^{1}} = m^2_{\tilde{u}_R^{2*}} - m^2_{\tilde{q}_L^{2}} = m^2_{\tilde{u}_R^{3*}} - m^2_{\tilde{q}_L^{3}}, \end{aligned} \tag{6.38}$$
$$m^2_{\tilde{q}_L^{1}} + m^2_{\tilde{l}_L^{1}} = m^2_{\tilde{q}_L^{2}} + m^2_{\tilde{l}_L^{2}} = m^2_{\tilde{q}_L^{3}} + m^2_{\tilde{l}_L^{3}}, \tag{6.39}$$
$$\begin{aligned} m^2_{\tilde{q}_L^{1}} + m^2_{\tilde{d}_R^{1*}} &= m^2_{\tilde{q}_L^{2}} + m^2_{\tilde{d}_R^{2*}} = m^2_{\tilde{q}_L^{3}} + m^2_{\tilde{d}_R^{3*}} \\ &= m^2_{\tilde{l}_L^{1}} + m^2_{\tilde{u}_R^{1*}} = m^2_{\tilde{l}_L^{2}} + m^2_{\tilde{u}_R^{2*}} = m^2_{\tilde{l}_L^{3}} + m^2_{\tilde{u}_R^{3*}}. \end{aligned} \tag{6.40}$$

They are compactly rewritten as

$$m^2_{\tilde{u}_R^{i*}} = m^2_{\tilde{e}_R^{i*}}, \quad m^2_{\tilde{d}_R^{i*}} - m^2_{\tilde{u}_R^{i*}} = m^2_{\tilde{l}_L^{i}} - m^2_{\tilde{q}_L^{i}}, \tag{6.41}$$
$$m^2_{\tilde{u}_R^{i*}} - m^2_{\tilde{u}_R^{j*}} = -m^2_{\tilde{d}_R^{i*}} + m^2_{\tilde{d}_R^{j*}} = m^2_{\tilde{q}_L^{i}} - m^2_{\tilde{q}_L^{j}} = -m^2_{\tilde{l}_L^{i}} + m^2_{\tilde{l}_L^{j}}, \tag{6.42}$$

[7] In case that the extra gauge symmetry breaking scale (M_F) is lower than M_S, $m^2_\pm$ receive radiative corrections between M_S and M_F, and the mass formulae should be modified. Here, we consider the simplest case to avoid complications.

[8] Sum rules among sfermion masses have also been derived using the orbifold family unification models on five-dimensional (5D) space-time [47–49].

where $i, j = 1, 2, 3$.

In the same way, based on (6.6) for (BC2), we obtain the relations,

$$m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{e}^{i*}_R}, \quad m^2_{\tilde{l}^i_L} - m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{d}^{i*}_R} - m^2_{\tilde{q}^i_L}, \tag{6.43}$$

$$m^2_{\tilde{u}^{i*}_R} - m^2_{\tilde{u}^{j*}_R} = -m^2_{\tilde{d}^{i*}_R} + m^2_{\tilde{d}^{j*}_R} = m^2_{\tilde{q}^i_L} - m^2_{\tilde{q}^j_L} = -m^2_{\tilde{l}^i_L} + m^2_{\tilde{l}^j_L}, \tag{6.44}$$

where $i, j = 1, 2, 3$. Note that these relations are obtained by exchanging $m^2_{\tilde{d}^{i*}_R}$ for $m^2_{\tilde{l}^i_L}$ in those for (BC1).

Furthermore, we obtain the specific relations,

$$m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{e}^{i*}_R}, \quad m^2_{\tilde{d}^{i*}_R} - m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{l}^i_L} - m^2_{\tilde{q}^i_L}, \tag{6.45}$$

$$m^2_{\tilde{u}^{i*}_R} - m^2_{\tilde{u}^{j*}_R} = -m^2_{\tilde{l}^i_L} + m^2_{\tilde{l}^j_L}, \quad m^2_{\tilde{q}^i_L} - m^2_{\tilde{q}^j_L} = -m^2_{\tilde{d}^{i*}_R} + m^2_{\tilde{d}^{j*}_R}, \tag{6.46}$$

$$m^2_{\tilde{u}^{1*}_R} - m^2_{\tilde{u}^{2*}_R} = m^2_{\tilde{q}^1_L} - m^2_{\tilde{q}^2_L}, \tag{6.47}$$

$$m^2_{\tilde{u}^{1*}_R} + m^2_{\tilde{u}^{3*}_R} = m^2_{\tilde{q}^1_L} + m^2_{\tilde{q}^3_L}, \quad m^2_{\tilde{d}^{1*}_R} + m^2_{\tilde{d}^{3*}_R} = m^2_{\tilde{l}^1_L} + m^2_{\tilde{l}^3_L} \tag{6.48}$$

for (BC3) and

$$m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{e}^{i*}_R}, \quad m^2_{\tilde{l}^i_L} - m^2_{\tilde{u}^{i*}_R} = m^2_{\tilde{d}^{i*}_R} - m^2_{\tilde{q}^i_L}, \tag{6.49}$$

$$m^2_{\tilde{u}^{i*}_R} - m^2_{\tilde{u}^{j*}_R} = -m^2_{\tilde{d}^{i*}_R} + m^2_{\tilde{d}^{j*}_R}, \quad m^2_{\tilde{q}^i_L} - m^2_{\tilde{q}^j_L} = -m^2_{\tilde{l}^i_L} + m^2_{\tilde{l}^j_L}, \tag{6.50}$$

$$m^2_{\tilde{u}^{1*}_R} - m^2_{\tilde{u}^{2*}_R} = m^2_{\tilde{q}^1_L} - m^2_{\tilde{q}^2_L}, \tag{6.51}$$

$$m^2_{\tilde{u}^{1*}_R} + m^2_{\tilde{u}^{3*}_R} = m^2_{\tilde{q}^1_L} + m^2_{\tilde{q}^3_L}, \quad m^2_{\tilde{d}^{1*}_R} + m^2_{\tilde{d}^{3*}_R} = m^2_{\tilde{l}^1_L} + m^2_{\tilde{l}^3_L} \tag{6.52}$$

for (BC4). Here, $i, j = 1, 2, 3$ and we denote $\tilde{u}^*_R$, $\tilde{e}^*_R$, $\tilde{d}^*_R$, $\tilde{l}_L$ and $\tilde{q}_L$ as $\tilde{u}^{3*}_R$, $\tilde{e}^{3*}_R$, $\tilde{d}^{3*}_R$, $\tilde{l}^3_L$ and $\tilde{q}^3_L$. The relations for (BC4) are obtained by exchanging $m^2_{\tilde{d}^{i*}_R}$ for $m^2_{\tilde{l}^i_L}$ in those for (BC3).

The above relations become predictions to probe models because they are specific to models, in case that the extra gauge symmetry breaking scale is near $M_{\rm S}$.

7 Separation of the SM and hidden particles on 5D

In this section, we propose that hidden particles can be separated according to gauge quantum numbers from the visible ones by the different BCs. Especially, we show that the separation of visible and hidden particles can be realized in gauge interactions using a 5D extension of the SM with an extra $U(1)$ gauge symmetry coexisting different types of BCs. Furthermore, we also study models that hidden particles relating to conjugate BCs are identified with dark matter or inflaton.

7.1 Why hidden

In order to obtain some hints to explore the origin of dark matter and the identity of inflaton and to address the reason for their existence, we search for an factor that it is hard to detect hidden particles based on the following assumptions.

- There is an extra gauge group G_{hidden} other than the SM one G_{SM} (or some extension such as a grand unified group G_{GUT}), and G_{hidden} leaves little trace behind around the terascale.
- Hidden particles such as dark matter and inflaton possess gauge quantum numbers of G_{hidden} or are some components of gauge bosons in a hidden sector, and they are gauge singlets of G_{SM} (or G_{GUT}).
- The SM particles are gauge singlets of G_{hidden}.

Gauge quantum numbers are suitably assigned to construct a realistic model, but in most cases, it would be done without any foundation except for symmetry principle. We expect a reason or a mechanism that a subtle separation of gauge quantum numbers in the above assumptions is realized naturally, and propose a hypothesis that *hidden particles can be separated according to gauge quantum numbers from the visible ones by the difference of BCs on extra dimensions.* [9]

To embody our hypothesis, we consider a 5D theory with $G_{\text{SM}} \times U(1)_C$ gauge group as an extension of the SM with an extra $U(1)$ gauge boson $C_M = C_M(x,y)$ and an extra matter $\tilde{\varphi} = \tilde{\varphi}(x,y)$. For simplicity, we pay attention to scalar fields and $U(1)$ gauge bosons and treat the Lagrangian density,

$$\begin{aligned}\mathcal{L}_{5\text{D}} = (D_M H)^*(D^M H) - m_H^2|H|^2 - \frac{1}{4}B_{MN}B^{MN} \\ + (D_M\tilde{\varphi})^*(D^M\tilde{\varphi}) - m_{\tilde{\varphi}}^2|\tilde{\varphi}|^2 - \frac{1}{4}C_{MN}C^{MN} \\ - \lambda\left(|H|^2\right)^2 - \lambda_{\tilde{\varphi}}\left(|\tilde{\varphi}|^2\right)^2 - \lambda_{\text{mix}}|H|^2|\tilde{\varphi}|^2 + \cdots ,\end{aligned} \tag{7.1}$$

where $H = H(x,y)$ is 5D complex scalar field containing the SM Higgs doublet as its zero mode ($H^{(0)}$), and λ, $\lambda_{\tilde{\varphi}}$ and λ_{mix} are quartic couplings of scalar fields.

[9] According to a similar idea that a dark matter possesses different features from the SM particles on extra dimensions, a truncated-inert-doublet model has been constructed that the SM ones belong to $\mathbb{Z}_2$ even zero modes and the dark matter is one of $\mathbb{Z}_2$ odd zero modes on a warped extra dimension [50].

If B_M which is the 5D extension of the $U(1)_Y$ gauge boson in the SM satisfies the BCs such as (2.10) – (2.12) and C_M satisfies the BCs such as (2.34) and (2.35), H and $\tilde{\varphi}$ cannot own both non-zero $U(1)$ charges. In other words, H is separated from $\tilde{\varphi}$ in gauge interactions through the difference of BCs.

After the dimensional reduction, we obtain the following 4D Lagrangian density for zero modes $H^{(0)}$, $\tilde{\varphi}^{(0)}$, $B_\mu^{(0)}$ and $C_5^{(0)}$, at the tree level,

$$\begin{aligned}\mathcal{L}_{4\mathrm{D}}^{(0)} &= (D_\mu^{(0)} H^{(0)})^*(D^{(0)\mu} H^{(0)}) - m_H^2 |H^{(0)}|^2 - \frac{1}{4} B_{\mu\nu}^{(0)} B^{(0)\mu\nu} \\ &\quad + \frac{1}{2}\partial_\mu \tilde{\varphi}^{(0)} \partial^\mu \tilde{\varphi}^{(0)} - \frac{1}{2}\left\{ m_{\tilde{\varphi}}^2 + \left(\frac{\beta_{\tilde{\varphi}} - \tilde{q}_{\tilde{\varphi}}\theta}{2\pi R}\right)^2 \right\} (\tilde{\varphi}^{(0)})^2 + \frac{1}{2}\partial_\mu C_5^{(0)} \partial^\mu C_5^{(0)} \\ &\quad - \lambda \left(|H^{(0)}|^2\right)^2 - \frac{1}{4}\lambda_{\tilde{\varphi}} (\tilde{\varphi}^{(0)})^4 - \frac{1}{2}\lambda_{\mathrm{mix}} |H^{(0)}|^2 (\tilde{\varphi}^{(0)})^2 + \cdots ,\end{aligned} \tag{7.2}$$

where where θ is the Wilson line phase defined by

$$\theta = \tilde{g}_5 \int_{-\pi R}^{\pi R} \frac{1}{\sqrt{2\pi R}} C_5^{(0)} dy = \sqrt{2\pi R}\, \tilde{g}_5 C_5^{(0)}, \tag{7.3}$$

and the ellipse in (7.2) stands for parts containing Kaluza-Klein modes of gauge bosons and the kinetic term of $C_5^{(0)}$. Note that the $U(1)$ gauge symmetry is broken by orbifolding, and θ is a remnant of the $U(1)$. And, we use the Fourier expansion (2.24) for H and (2.42) for $\tilde{\varphi}$.

As seen from (7.2), $C_5^{(0)}$ is massless at the tree level. After receiving radiative corrections, the effective potential relating to $C_5^{(0)}$ is induced and $C_5^{(0)}$ acquires a mass through the Hosotani mechanism [31, 32]. Concretely, the one-loop effective potential for the Wilson line phase $\theta(= \sqrt{2\pi R}\tilde{g}_5 C_5^{(0)})$ is derived as

$$\begin{aligned}V_{\mathrm{eff}}[\theta] &= \frac{1}{2}\int \frac{d^4 p_{\mathrm{E}}}{(2\pi)^4} \sum_{n=-\infty}^{\infty} \ln \left\{ p_{\mathrm{E}}^2 + m_{\tilde{\varphi}}^2 + \left(\frac{2\pi n + \beta_{\tilde{\varphi}} - \tilde{q}_{\tilde{\varphi}}\theta}{2\pi R}\right)^2 \right\} \\ &= E_0 - \frac{3}{64\pi^6 R^4} \sum_{n=1}^{\infty} \left(\frac{1}{n^5} + \frac{r_{\tilde{\varphi}}}{n^4} + \frac{r_{\tilde{\varphi}}^2}{3n^3}\right) e^{-n r_{\tilde{\varphi}}} \cos\left\{ n\left(\beta_{\tilde{\varphi}} + \tilde{q}_{\tilde{\varphi}}\theta\right)\right\} ,\end{aligned} \tag{7.4}$$

where p_{E} is a 4D Euclidean momentum, E_0 is a θ-independent constant and $r_{\tilde{\varphi}} = 2\pi R m_{\tilde{\varphi}}$. The physical vacuum is realized at $\beta_{\tilde{\varphi}} - \tilde{q}_{\tilde{\varphi}}\theta = 0$ and $C_5^{(0)}$ decouples in the low-energy theory, if R is small enough, by acquiring the mass of $O(1/R)$.

The scalar field $\tilde{\varphi}^{(0)}(x)$ survives in a post-SM at the terascale for $\beta_{\tilde{\varphi}} - \tilde{q}_{\tilde{\varphi}}\theta = 0$ and $m_{\tilde{\varphi}} < O(1)$TeV, and we find that our Lagrangian density agrees with that containing a dark matter in a specific model called the *New Minimal Standard Model* (NMSM) [51, 52]. Then, $\tilde{\varphi}^{(0)}(x)$ becomes a possible candidate of dark matter.

The $\tilde{\varphi}^{(0)}(x)$ couples to the SM Higgs doublet through the quartic interaction $-(1/2)\lambda_{\mathrm{mix}}|H^{(0)}|^2(\tilde{\varphi}^{(0)})^2$. In the presence of this term as the Higgs portal, the running of λ based on the renormalization group equation changes compared with that in the SM, and the vacuum stability of Higgs potential can be improved [52, 53].

Here, as a complementary comment on our hypothesis, we state a feature that *matters are not necessarily classified into the visible ones and the hidden ones, even* *if a system has two $U(1)$ gauge bosons B_M and C_M with different types of BCs, because there can exist particles that possess both $U(1)$ charges.* Let us show it using a model described by the Lagrangian density,

$$\mathcal{L}_{\tilde{\varphi}_a} = \sum_{a=1,2} \{(D_M\tilde{\varphi}_a)^*(D^M\tilde{\varphi}_a) - m^2_{\tilde{\varphi}_a}|\tilde{\varphi}_a|^2\} - \frac{1}{4}B_{MN}B^{MN} - \frac{1}{4}C_{MN}C^{MN}, \quad (7.5)$$

where $D_M = \partial_M - ig_5 q_{\tilde{\varphi}_a} B_M - i\tilde{g}_5\tilde{q}_{\tilde{\varphi}_a} C_M$ for a pair of complex scalar fields $\tilde{\varphi}_a = \tilde{\varphi}_a(x,y)$ $(a = 1,2)$. In case that $q_{\tilde{\varphi}_1} = q_{\tilde{\varphi}_2}$, $\tilde{q}_{\tilde{\varphi}_1} = -\tilde{q}_{\tilde{\varphi}_2}$ and $m_{\tilde{\varphi}_1} = m_{\tilde{\varphi}_2}$, $\mathcal{L}_{\tilde{\varphi}_a}$ is a single-valued function under the BCs (2.10) – (2.12), (2.34), (2.35) and

$$\tilde{\varphi}_a(x, y + 2\pi R) = e^{i\beta_{\tilde{\varphi}}}\tilde{\varphi}_a(x,y), \quad \tilde{\varphi}_1(x,-y) = \eta_{\tilde{\varphi}}\tilde{\varphi}_2(x,y), \quad (7.6)$$

where $\beta_{\tilde{\varphi}}$ takes 0 or π and $\eta_{\tilde{\varphi}}$ takes 1 or -1. We refer to the $U(1)$ gauge symmetry concerning the BCs (2.34), (2.35) and (7.6) as an *exotic $U(1)$ symmetry* [54, 55]. [10] Then, we find that $\tilde{\varphi}_a$ own both $U(1)$ gauge quantum numbers. A similar feature holds on a theory containing non-abelian gauge symmetries: matters can possess both gauge quantum numbers whose gauge bosons satisfy different types of BCs if the theory is vector-like.

7.2 Gauge-higgs inflation

7.2.1 Inflation

Inflation has been proposed to solve some problems in Big Bang cosmology such as horizon problem, flatness problem and magnetic-monopole problem by K. Sato and A. Guth in the early 1980s [57, 58]. Inflation is an exponential expansion of space in the early universe. It is realized by a vacuum energy of inflaton potential. Here, inflaton is any scalar field.

Especially, slow-roll inflation models which have been proposed by A. Linde is one of the most important model [59]. Inflation can be estimated by inflation parameters, which are observable, only using inflaton potential. From observation and theoretical analysis, inflation parameters are restricted as follow:

· Minimum value of inflaton potential $V(\phi)$ is almost zero:

$$V(\langle\phi\rangle) \simeq 0 \,. \quad (7.7)$$

· *The slow-roll conditions*:

$$\epsilon \equiv \frac{M_G^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1 \,, \quad \eta \equiv M_G^2\left|\frac{V''(\phi)}{V(\phi)}\right| \ll 1 \,, \quad (7.8)$$

$$M_G = 2.4 \times 10^{18}\text{GeV} : \text{the reduced Planck scale} \,,$$

$$V'(\phi) = \frac{\partial V(\phi)}{\partial \phi} \,, \quad V''(\phi) = \frac{\partial^2 V(\phi)}{\partial \phi^2} \,.$$

[10] The orbifolding due to these BCs is regarded as a variant of the diagonal embedding proposed in [56].

· *The e-folding number*:

$$N \equiv \int_{t_*}^{t_e} H dt = \frac{1}{M_G^2} \left| \int_{\phi_*}^{\phi_e} \frac{V(\phi)}{V'(\phi)} d\phi \right| \simeq 50 \sim 60 \ , \tag{7.9}$$

H : the Hubble constant

ϕ_* : the value of inflaton field in the start point of inflation

ϕ_e : the value of inflaton field in the end point of inflation

· *The scalar power spectrum*:

$$\mathcal{P}_\zeta \equiv \frac{1}{12\pi^2 M_G^6} \frac{(V(\phi))^3}{(V'(\phi))^2} \bigg|_{\phi=\phi_*} = (2.196 \pm 0.079) \times 10^{-9} \tag{7.10}$$

· *The spectral index*:

$$n_s = 1 - 6\epsilon_* + 2\eta_* = 0.9655 \pm 0.0062 \ , \tag{7.11}$$

$\epsilon_*, \ \eta_*$: the quantities at the horizon exit

· *The tensor-to-scalar ratio*:

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 16\epsilon_* < 0.12 \ , \tag{7.12}$$

$$\mathcal{P}_h = \frac{2V(\phi)}{3\pi^2 M_G^4}$$

The first conditions are assumed because the current cosmological constant is very small value. The slow-roll conditions are demanded from the flatness of potential. The e-folding number represents that how much exponential expansion continued. In order to realize our universe, the e-folding number should be taken $N = 50 \sim 60$. The constraint of the scalar power spectrum, the spectral index and the tensor-to-scalar ratio are given by Planck observation in 2015 [60].

Many slow-roll inflation models have been proposed, but in most of models, inflaton potential have been given by hand. This causes problems such as the origin of inflaton and fine-tuning problem of parameters. Higher-dimensional theories may solve those problems. On 5D gauge theory, *gauge-Higgs field* which is 5-th component of 5D gauge field dose not have its potential in the classical level, but, in 1-loop level, gauge-Higgs potential is generated by radiative corrections. Fine-tuning problem is solved because this potential is finite due to 5D gauge symmetry. N. Arkani-Hamed have proposed inflation model that gauge-Higgs field are identified with inflaton [61]. This model can solve the origin of inflaton and the fine-tuning problem, under the condition that the value of relevant gauge coupling constant is tiny enough.

Recently, the models with 5D gauge theory added to 5D gravitational theory has been constructed, and investigated fine-tuning problem and the origin of inflaton [62–64]. These models may solve problems of fine-tuning and the origin of inflaton with a same magnitude of gauge coupling constant as the SM ones. On 5D gravitational theory, a scalar field called *radion*, which is an extra-dimensional component of 5D gravity field, is included, and it may be also inflaton.

7.2.2 Gauge-higgs inflation

We apply a model with conjugate BCs on a *gauge-Higgs inflation scenario*. Let us consider a gravity theory coupled to a $U(1)_C$ gauge theory defined on a 5D space-time whose classical background is $M^4 \times S^1/\mathbb{Z}_2$. The starting action is given by

$$S_{\rm 5D}^{\rm gr} = \int d^5x \sqrt{-\hat{g}_5}\Bigg[\frac{1}{16\pi G_5}\hat{R}_5 - \frac{1}{4}\hat{g}^{MP}\hat{g}^{NL}C_{MN}C_{PL} + \sum_{a=1}^{c_1}\bar{\tilde{\psi}}_a^{\rm n}(-i\hat{g}^{MN}\hat{\Gamma}_M\nabla_N - \mu_a)\tilde{\psi}_a^{\rm n} + \sum_{b=1}^{c_2}\bar{\tilde{\psi}}_b^{\rm ch}(-i\hat{g}^{MN}\hat{\Gamma}_M D_N - m_b)\tilde{\psi}_b^{\rm ch}\Bigg] , \tag{7.13}$$

where $\hat{g}_5 = \det \hat{g}_{MN}$, $\hat{g}^{MN}$ is the inverse of 5D metric $\hat{g}_{MN}$, G_5 is the 5D Newton constant, $\hat{R}_5$ is the 5D Ricci scalar, $C_{MN} = \partial_M C_N - \partial_N C_M$, $\hat{\Gamma}_M = E_M^k\Gamma_k$ ($E_M^k = E_M^k(x,y)$ is the fünf bein, Γ_k are 5D gamma matrices, and k is the space-time index in the local Lorentz frame), $\nabla_N = \partial_N - (i/4)\hat{\omega}_N^{kl}\Sigma_{kl}$ ($\hat{\omega}_N^{kl}$ is the spin connection and $\Sigma_{kl} = i[\Gamma_k,\Gamma_l]/2$), $D_N = \partial_N - (i/4)\hat{\omega}_N^{kl}\Sigma_{kl} - i\tilde{g}_5\tilde{q}_b C_N$ for $\tilde{\psi}_b^{\rm ch}$, C_N is a 5D $U(1)_C$ gauge boson in the hidden sector and we assume that it satisfies the conjugate BCs (2.31) and (2.32), $\tilde{\psi}_a^{\rm n}$ are neutral fermions, $\tilde{\psi}_b^{\rm ch}$ are $U(1)_C$ charged fermions whose $U(1)_C$ charge is $\tilde{q}_b$, and c_1 and c_2 stand for numbers of neutral and charged fermions, respectively. The $\tilde{g}_5$ is a 5D gauge coupling constant.

If the SM gauge bosons satisfy the ordinary BCs such as (2.10) – (2.12) and both $\tilde{\psi}_a^{\rm n}$ and $\tilde{\psi}_b^{\rm ch}$ satisfy the BCs (2.40) and (2.41) with β_a and β_b as a twisted phase ($\beta_{\tilde{\psi}}$), $\tilde{\psi}_a^{\rm n}$ and $\tilde{\psi}_b^{\rm ch}$ should be singlets of the SM gauge group, as a consequence in the previous section.

The BCs of $\hat{g}_{MN}$ are given by

$$\hat{g}_{MN}(x, y + 2\pi R) = \hat{g}_{MN}(x,y) , \tag{7.14}$$

$$\hat{g}_{\mu\nu}(x,-y) = \hat{g}_{\mu\nu}(x,y) , \quad \hat{g}_{\mu 5}(x,-y) = -\hat{g}_{\mu 5}(x,y) ,$$
$$\hat{g}_{55}(x,-y) = \hat{g}_{55}(x,y) , \tag{7.15}$$

and then the Fourier expansions of $\hat{g}_{MN}$ are presented as

$$\hat{g}_{\mu\nu}(x,y) = \hat{g}_{\mu\nu}^{(0)}(x) + \sum_{n=1}^{\infty}\hat{g}_{\mu\nu}^{(n)}(x)\cos\frac{ny}{R} , \tag{7.16}$$

$$\hat{g}_{\mu 5}(x,y) = \sum_{n=1}^{\infty}\hat{g}_{\mu 5}^{(n)}(x)\sin\frac{ny}{R}y , \tag{7.17}$$

$$\hat{g}_{55}(x,y) = \hat{g}_{55}^{(0)}(x) + \sum_{n=1}^{\infty}\hat{g}_{55}^{(n)}(x)\cos\frac{ny}{R} . \tag{7.18}$$

The spin connection $\hat{\omega}_M^{kl}$ satisfy the ordinary BCs such that

$$\hat{\omega}_M^{kl}(x, y + 2\pi R) = \hat{\omega}_M^{kl}(x,y) , \tag{7.19}$$

$$\hat{\omega}_\mu^{kl}(x,-y) = \hat{\omega}_\mu^{kl}(x,y)\ ,\ \ \hat{\omega}_5^{kl}(x,-y) = -\hat{\omega}_5^{kl}(x,y), \tag{7.20}$$

and then the full Lagrangian density containing both visible and hidden sectors becomes a single-valued function on $S^1/\mathbb{Z}_2$.

On the Minkowski background, $\hat{g}_{\mu\nu}^{(0)}$ takes the classical value such as $\langle g_{\mu\nu}^{(0)}\rangle = \eta_{\mu\nu}$, and other zero modes are assumed to have the following classical values:

$$\langle \hat{g}_{55}^{(0)}\rangle = \phi^{2/3}\ ,\ \ \langle C_5^{(0)}\rangle = \frac{\theta}{\sqrt{2\pi R}\tilde{g}_5}, \tag{7.21}$$

where ϕ is the radion and θ is the Wilson line phase. The Kaluza-Klein modes are assumed to have zero classical values.

According to a usual procedure, the following effective potential is obtained at the one-loop level,

$$\begin{aligned} V(\rho,\theta) = \frac{3L^2m^6}{2\pi^2\rho^2}\Bigg[&-2\zeta(5) + c_1\sum_{n=1}^{\infty}\left(\frac{1}{n^5} + r_m\frac{\rho^{1/3}}{n^4} + r_m^2\frac{\rho^{2/3}}{3n^3}\right)e^{-nr_m\rho^{1/3}} \\ &+ c_2\sum_{n=1}^{\infty}\left(\frac{1}{n^5} + \frac{\rho^{1/3}}{n^4} + \frac{\rho^{2/3}}{3n^3}\right)e^{-n\rho^{1/3}}\cos\left\{n\left(\beta-\tilde{q}\theta\right)\right\}\Bigg] \\ &+ \frac{L^2m}{\rho^{1/3}}\tilde{a} + \cdots\ , \end{aligned} \tag{7.22}$$

where we take common masses $\mu = \mu_a$ and $m = m_b$, a common twisted phase $\beta = \beta_b$ and a common charge $\tilde{q} = \tilde{q}_b$ for simplicity, $L = 2\pi R$, $\rho = L^3m^3\phi$, $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, $r_m = \mu/m$ and $\tilde{a}$ is some constant.

The above potential has the same form as that obtained in [63] except overall factor and β, and hence both radion and Wilson line phase are stabilized in case with $c_1 > 2+c_2$, and θ is, in particular, fixed as $\beta - \tilde{q}\theta = \pi$. Furthermore, the gauge-Higgs field θ can give rise to inflation in accord with the astrophysical data [64].

We need some modification of our model to explain the origin of the Big Bang after inflaton decays into the SM particles. The direct coupling between inflaton and some SM particles is necessary to produce radiations at a very early universe, but it is difficult due to the mismatch of BCs, as explained in the previous section. As a way out, if some SM particles or its extension form a pair of vector-like multiplet for $U(1)_C$ and satisfy the BCs such as (7.6) or counterparts of fermions, they can directly couple to $C_5^{(0)}$. For instance, if there exist two Higgs doublets H_a as a vector-like pair of $U(1)_C$, there can appear the coupling such as $\tilde{g}_5^2\tilde{q}_H^2|H_a^{(0)}|^2(C_5^{(0)})^2$. In this case, although the contributions from H_a are added to the potential (7.22), θ might remain inflaton because they are not dominated.

8 Conclusion and Discussion

First, we have explained feature of the orbifold $S^1/\mathbb{Z}_2$, $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$. And, we have reviewed orbifold family unification on the basis of $SU(N)$ gauge theories on five-dimensional space-time, $M^4 \times S^1/\mathbb{Z}_2$. Orbifold family unification model on the basis of $SU(N)$ gauge theories which is broken down to $SU(5)$ gauge group by orbifold breaking have been found, but orbifold family unification model on the basis of $SU(N)$ gauge theories which is directly broken down to the SM gauge group by orbifold breaking have not been found.

Second, we have studied the possibility of family unification on the basis of $SU(N)$ gauge theory on 6 dimensional space-time, $M^4 \times T^2/\mathbb{Z}_M$. We have obtained enormous numbers of models with three families of $SU(5)$ matter multiplets and those with three families of the SM multiplets from a single massless Dirac fermion with a higher-dimensional representation of $SU(N)$, after the orbifold breaking. The total numbers of models with the three families of $SU(5)$ multiplets and the SM multiplets are summarized in Table 4.5 and 4.10, respectively.

Third, we have also studied the relationship between the family number of chiral fermions and the Wilson line phases, based on the orbifold family unification. We have found that flavor numbers are independent of the Wilson line phases relating extra-dimensional components of gauge boson, as far as the SM gauge symmetry is respected. This feature originates from a hidden quantum-mechanical SUSY. The relationship of left-handed fermions and right-handed ones corresponds to that of bosons and fermions in quantum-mechanical SUSY.

Fourth, we have taken orbifold family unification models base on $SU(9)$ gauge symmetry on $M^4 \times T^2/\mathbb{Z}_2$ and have examined the reality of models by checking the appearance of Yukawa interactions from the interactions in the 6D bulk as a selection rule. We have picked out a candidate of model compatible with the observed fermion masses and flavor mixing. The model has a feature that just three families of fermions in the SM exist as zero modes and any mirror particles of fermions do not appear in the low energy world after the breakdown of gauge symmetry $SU(9) \to SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(3)_F \times U(1)^3$ or $SU(9) \to SU(3)_C \times SU(2)_L \times U(1)_Y \times SU(2)_F \times U(1)^4$ by orbifold breaking. Depending on the assignment of intrinsic Z_2 parities, u_R^i, e_R^i, d_R^i, l_L^i and q_L^i belong to $\Psi_\pm$ and $\Psi_\mp$ with **84** of $SU(9)$, respectively. We have found out specific relations among sfermion masses as model-dependent predictions in the SUSY extension of models.

The massless degrees of freedom relating to a family symmetry must be made massive by further breaking. For example, extra scalar fields can play the role of Higgs fields for the breakdown of extra gauge symmetries including non-Abelian gauge symmetries. As a result, extra massless fields including the family gauge bosons can be massive.

Fifth, we have formulated 5D $U(1)$ gauge theories yielding different types of BCs on $S^1/\mathbb{Z}_2$. On the conjugate BCs, the 4D components of $U(1)_C$ gauge boson have odd Z_2 parities and their zero modes are projected out through the dimensional reduction. Then, the $U(1)_C$ gauge symmetry is broken down by orbifolding. In contrast, the 5-th component of $U(1)_C$ gauge boson has even Z_2 parities, and its zero mode $C_5^{(0)}$ survives and becomes a dynamical field. It is massless at the tree level, but the effective potential relating to $C_5^{(0)}$ is induced after receiving radiative corrections. Then, $C_5^{(0)}$ acquires a mass of $\mathcal{O}(1/R)$ and decouples to the low energy theory if R is small enough. Matter fields transform into the charge conjugated ones under the Z_2 transformation. Then, only real fields such as real scalar and Majorana fermions appear after compactification.

We have also shown that the separation of visible and hidden particles can be realized in the gauge interactions using a 5D extension of the SM with an extra $U(1)$ gauge symmetry and an extra scalar field coexisting different types of BCs. We also have derived the Lagrangian density containing a dark matter in the NMSM. The zero mode of extra scalar field yielding the conjugate BCs becomes a possible candidate of dark matter.

Furthermore, we have applied a 5D gravity theory coupled to a $U(1)$ gauge theory with conjugate BCs on a gauge-Higgs inflation scenario. We have found that the effective potential containing the radion ϕ and Wilson line phase θ plays a role of an inflaton potential and θ become inflaton.

We give a comment on the right-handed neutrinos. Because, the right-handed neutrinos are singlets of the SM gauge group and they have Majorana masses, we guess that there might be hidden matters obeying conjugate BCs. But, it is difficult to realize it, because we cannot construct a Z_2 invariant term in 5D Lagrangian density to derive the 4D Yukawa interaction relating to neutrino, due to the mismatch of BCs between the SM non-singlets and singlets. Nevertheless, it would also be interesting to examine the origin of the right-handed neutrinos from the viewpoint of BCs.

In this thesis, we have studied the possibility of extra dimensional theories as the physics beyond the SM choosing orbifolds as an extra dimensional space-time. Especially, we have focused on the mystery of family number and the origin of undiscovered particles. Our models can be attractive from the phenomenological point of view. However, we should investigate other phenomenological and cosmological verifications from the view point of the mass of the SM particles and observables.

It would be interesting to construct GUT models with a large gauge group because gauge theories on higher-dimensional space-time satisfying conjugate BCs lower the rank of gauge symmetries after orbifold breaking. Extra dimensional models satisfying conjugate BCs have not been studied very much. It would be interesting to combine orbifold family unification models with orbifold with conjugate BCs. In this case, there can be family unification models without family symmetry after orbifold breaking.

Extra dimensional theories relate sting theory, which is the candidate of ultimate theory. If our models are considered as effective theories of string theory, it is interested to reconsider our models in the framework of string theory.

A Notation

We use the natural unit system. The speed of light c and the reduced Planck constant $\hbar$ are

$$c = \hbar = 1. \tag{A.1}$$

· Pauli matrix

$$\begin{aligned}\sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .\end{aligned} \tag{A.2}$$

· 4D gamma matrix: γ^μ $(\mu = 0, 1, 2, 3)$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu , \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.3}$$

$$\sigma^\mu = \begin{pmatrix} \sigma^0 & \sigma^i \end{pmatrix} , \quad \bar{\sigma}^\mu = \begin{pmatrix} \sigma^0 & -\sigma^i \end{pmatrix} ,$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1}_{2\times2} & 0 \\ 0 & \mathbf{1}_{2\times2} \end{pmatrix} , \tag{A.4}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \quad \{\gamma^\mu, \gamma^5\} = 0 . \tag{A.5}$$

· 5D gamma matrix: Γ^M $(M = 0, 1, 2, 3, 5)$

$$\Gamma^\mu = \gamma^\mu , \quad \Gamma^5 = i\gamma^5 , \tag{A.6}$$

$$\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN} . \tag{A.7}$$

· 6D gamma matrix: Γ^M $(M = 0, 1, 2, 3, 5, 6)$

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^3 = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{pmatrix} , \tag{A.8}$$

$$\Gamma^5 = \mathbf{1}_{4\times4} \otimes i\sigma^1 = \begin{pmatrix} 0 & i\mathbf{1}_{4\times4} \\ i\mathbf{1}_{4\times4} & 0 \end{pmatrix} , \tag{A.9}$$

$$\Gamma^6 = \mathbf{1}_{4\times4} \otimes i\sigma^2 = \begin{pmatrix} 0 & \mathbf{1}_{4\times4} \\ -\mathbf{1}_{4\times4} & 0 \end{pmatrix} , \tag{A.10}$$

$$\Gamma^7 \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 = -\gamma^5 \otimes \sigma^3 = \begin{pmatrix} -\gamma^5 & 0 \\ 0 & \gamma^5 \end{pmatrix} , \tag{A.11}$$

$$\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN} , \quad \left\{\Gamma^5, \Gamma^M\right\} = 0 , \tag{A.12}$$

$$\Gamma^z \equiv \Gamma^5 + i\Gamma^6 , \quad \Gamma^{\bar{z}} \equiv \Gamma^5 - i\Gamma^6 . \tag{A.13}$$

B The Properties of $T^2/\mathbb{Z}_M$ orbifold

In this section, let us discuss $SU(N)$ gauge thoery on $M^4 \times \mathbb{Z}_M$ in detail. Especially, we explain the properties of orbifold $M^4 \times \mathbb{Z}_M$ and orbifold breaking mechanism by inner automophisms boundary conditions.

B.1 $T^2/\mathbb{Z}_2$ orbifold

B.1.1 Property

Let us discuss $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_2$. On $T^2/\mathbb{Z}_2$, the T^2 is constructed by $SU(2)\times SU(2)$ lattice, and its basis vector takes $e_1 = 1$, $e_2 = i$. The point z is equivalent to the points $z+e_1$ and $z+e_2$, and the point $-z$ on $T^2/\mathbb{Z}_2$. In this case, the fixed points are

$$0\ ,\quad \frac{e_1}{2}\ ,\quad \frac{e_2}{2}\ ,\quad \frac{e_1+e_2}{2}\ . \tag{B.1}$$

The transformation around those fixed points can be defined as

$$\begin{aligned}
&s_{20} : z \to -z\ ,\quad s_{21} : z \to -z+e_1\ ,\quad s_{22} : z \to -z+e_2\ ,\\
&s_{23} : z \to -z+e_1+e_2\ ,\quad t_1 : z \to z+e_1\ ,\quad t_2 : z \to z+e_2\ .
\end{aligned} \tag{B.2}$$

They satisfy the relations,

$$\begin{aligned}
&s_{20}^2 = s_{21}^2 = s_{22}^2 = s_{23}^2 = I\ ,\quad s_{21} = t_1 s_{20}\ ,\quad s_{22} = t_2 s_{20}\ ,\\
&s_{23} = t_1 t_2 s_{20} = s_{21} s_{20} s_{22} = s_{22} s_{20} s_{21}\ ,\quad t_1 t_2 = t_2 t_1\ .
\end{aligned} \tag{B.3}$$

At this time, the BCs of bulk fields are characterized by matrices $(P_0, P_1, P_2, P_3, U_1, U_2)$. Those matrices satisfy the relations,

$$\begin{aligned}
&P_0^2 = P_1^2 = P_2^2 = P_3^2 = I\ ,\quad P_1 = U_1 P_0\ ,\quad P_2 = U_2 P_0\ ,\\
&P_3 = U_1 U_2 P_0 = P_1 P_0 P_2 = P_2 P_0 P_1\ ,\quad U_1 U_2 = U_2 U_1\ .
\end{aligned} \tag{B.4}$$

Since three of those matrices is independent, we choose three matrices P_0, P_1, P_2 which are unitary and hermitian matricies.

B.1.2 Orbifold breaking by inner automophisms boundary conditions

The BCs of gauge field are determined as

$$\begin{aligned}
s_{20} :\ & A_\mu(x,-z,-\bar{z}) = P_0 A_\mu(x,z,\bar{z}) P_0^\dagger,\\
& A_z(x,-z,-\bar{z}) = -P_0 A_z(x,z,\bar{z}) P_0^\dagger,\\
& A_{\bar{z}}(x,-z,-\bar{z}) = -P_0 A_{\bar{z}}(x,z,\bar{z}) P_0^\dagger,
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
s_{21} :\ & A_\mu(x,e_1-z,\bar{e}_1-\bar{z}) = P_1 A_\mu(x,z,\bar{z}) P_1^\dagger,\\
& A_z(x,e_1-z,\bar{e}_1-\bar{z}) = -P_1 A_z(x,z,\bar{z}) P_1^\dagger,\\
& A_{\bar{z}}(x,e_1-z,\bar{e}_1-\bar{z}) = -P_1 A_{\bar{z}}(x,z,\bar{z}) P_1^\dagger,
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
s_{22} :\ & A_\mu(x,e_2-z,\bar{e}_2-\bar{z}) = P_2 A_\mu(x,z,\bar{z}) P_2^\dagger,\\
& A_z(x,e_2-z,\bar{e}_2-\bar{z}) = -P_2 A_z(x,z,\bar{z}) P_2^\dagger,\\
& A_{\bar{z}}(x,e_2-z,\bar{e}_2-\bar{z}) = -P_2 A_{\bar{z}}(x,z,\bar{z}) P_2^\dagger,
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
s_{23} :\ & A_\mu(x,e_1+e_2-z,\bar{e}_1+\bar{e}_2-\bar{z}) = P_3 A_\mu(x,z,\bar{z}) P_3^\dagger,\\
& A_z(x,e_1+e_2-z,\bar{e}_1+\bar{e}_2-\bar{z}) = -P_3 A_z(x,z,\bar{z}) P_3^\dagger,
\end{aligned}$$

$$A_{\bar{z}}(x, e_1 + e_2 - z, \bar{e}_1 + \bar{e}_2 - \bar{z}) = -P_3 A_{\bar{z}}(x, z, \bar{z}) P_3^\dagger, \tag{B.8}$$

$$t_1 : \; A_M(x, z + e_1, \bar{z} + \bar{e}_1) = U_1 A_M(x, z, \bar{z}) U_1^\dagger, \tag{B.9}$$

$$t_1 : \; A_M(x, z + e_2, \bar{z} + \bar{e}_2) = U_2 A_M(x, z, \bar{z}) U_2^\dagger, \tag{B.10}$$

where $z = x^5 + ix^6$, $\bar{z} = x^5 - ix^6$, $A_z = A_5 + iA_6$ and $A_{\bar{z}} = A_5 - iA_6$. The BCs of scalar field ϕ and spinor field ψ are determined as

$$s_{20} : \; \phi(x, -z, -\bar{z}) = T_\Phi[P_0]\phi(x, z, \bar{z}) \; , \tag{B.11}$$

$$s_{21} : \; \phi(x, e_1 - z, \bar{e}_1 - \bar{z}) = T_\Phi[P_1]\psi(x, z, \bar{z}) \; , \tag{B.12}$$

$$s_{22} : \; \phi(x, e_2 - z, \bar{e}_2 - \bar{z}) = T_\Phi[P_2]\psi(x, z, \bar{z}) \; , \tag{B.13}$$

$$s_{23} : \; \phi(x, e_1 - e_2 - z, \bar{e}_1 - \bar{e}_2 - \bar{z}) = T_\Phi[P_3]\phi(x, z, \bar{z}) \; , \tag{B.14}$$

$$t_1 : \; \phi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Phi[U_1]\phi(x, z, \bar{z}) \; , \tag{B.15}$$

$$t_2 : \; \phi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Phi[U_2]\phi(x, z, \bar{z}) \; , \tag{B.16}$$

$$s_{20} : \; \psi(x, -z, -\bar{z}) = T_\Psi[P_0]\psi(x, z, \bar{z}) \; , \tag{B.17}$$

$$s_{21} : \; \psi(x, e_1 - z, \bar{e_1} - \bar{z}) = T_\Psi[P_1]\psi(x, z, \bar{z}) \; , \tag{B.18}$$

$$s_{22} : \; \psi(x, e_2 - z, \bar{e_2} - \bar{z}) = T_\Psi[P_2]\psi(x, z, \bar{z}) \; . \tag{B.19}$$

$$s_{23} : \; \psi(x, e_1 - e_2 - z, \bar{e}_1 - \bar{e}_2 - \bar{z}) = T_\Psi[P_3]\phi(x, z, \bar{z}) \; , \tag{B.20}$$

$$t_1 : \; \psi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Psi[U_1]\psi(x, z, \bar{z}) \; , \tag{B.21}$$

$$t_2 : \; \psi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Psi[U_2]\psi(x, z, \bar{z}) \; , \tag{B.22}$$

where $T_{\Phi(\Psi)}[P_i]$ and $T_{\Phi(\Psi)}[U_i]$ represent appropriate representation matrices including arbitrary sign factors, with the matices P_i and U_i. The eigenvalues of $T_\Phi[P_0]$, $T_\Phi[P_1]$ and $T_\Phi[P_2]$ are interpreted as the $\mathbb{Z}_2$ parities for the extra space. The representation matrices $T_\Sigma[P](\Sigma = \Phi, \Psi, P = P_0, P_1, P_2, P_3, U_1, U_2)$ satisfy

$$\begin{aligned}
&T_\Sigma[P_0]^2 = T_\Sigma[P_1]^2 = T_\Sigma[P_2]^2 = I \; , \quad T_\Sigma[U_1]T_\Sigma[U_2] = T_\Sigma[U_2]T_\Sigma[U_1] \; , \\
&T_\Sigma[P_1] = T_\Sigma[U_1]T_\Sigma[P_0] \; , \quad T_\Sigma[P_2] = T_\Sigma[U_2]T_\Sigma[P_0] \; , \\
&T_\Sigma[P_3] = T_\Sigma[U_1]T_\Sigma[U_2]T_\Sigma[P_0] = T_\Sigma[P_1]T_\Sigma[P_0]T_\Sigma[P_2] = T_\Sigma[P_2]T_\Sigma[P_0]T_\Sigma[P_1] \; .
\end{aligned} \tag{B.23}$$

Let $\varphi^{(\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2)}(x, z, \bar{z})$ be a component in a multiplet and have a definite $\mathbb{Z}_2$ parity $(\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2)$. Here, φ is a generic field and it is applied to scalar field ϕ, fermion field ψ or gauge field A_M. The Fourier expansion of $\varphi^{(\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2)}(x, z, \bar{z})$ is given by

$$\begin{aligned}
\varphi^{(+1,+1,+1)}(x, z, \bar{z}) = {} & \frac{1}{\pi\sqrt{R_1 R_2}} \varphi^{(0,0)}(x) \\
& + \frac{2}{\pi\sqrt{R_1 R_2}} \sum_{\substack{n,m=0 \\ (n+m \neq 0)}}^{\infty} \varphi^{(n,m)}(x) [\cos]_{n,m}(z, \bar{z}) \; ,
\end{aligned} \tag{B.24}$$

$$\varphi^{(+1,+1,-1)}(x, z, \bar{z}) = \frac{2}{\pi\sqrt{R_1 R_2}} \sum_{\substack{n,m=0 \\ (n+m \neq 0)}}^{\infty} \varphi^{(n,m)}(x) [\cos]_{n,m+1/2}(z, \bar{z}) \; , \tag{B.25}$$

$$\varphi^{(+1,-1,+1)}(x, z, \bar{z}) = \frac{2}{\pi\sqrt{R_1 R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) [\cos]_{n+1/2,m}(z, \bar{z}) \; , \tag{B.26}$$

$$\varphi^{(-1,+1,+1)}(x,z,\bar{z}) = \frac{2}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)[\sin]_{n+1/2,m+1/2}(z,\bar{z}) \ , \tag{B.27}$$

$$\varphi^{(+1,-1,-1)}(x,z,\bar{z}) = \frac{2}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)[\cos]_{n+1/2,m+1/2}(z,\bar{z}) \ , \tag{B.28}$$

$$\varphi^{(-1,+1,-1)}(x,z,\bar{z}) = \frac{2}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)[\sin]_{n+1/2,m}(z,\bar{z}) \ , \tag{B.29}$$

$$\varphi^{(-1,-1,+1)}(x,z,\bar{z}) = \frac{2}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)[\sin]_{n,m+1/2}(z,\bar{z}) \ , \tag{B.30}$$

$$\varphi^{(-1,-1,-1)}(x,z,\bar{z}) = \frac{2}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)[\sin]_{n,m}(z,\bar{z}) \ , \tag{B.31}$$

where

$$\begin{aligned}
[\sin]_{n+\alpha,m+\beta}(z,\bar{z}) &= \sin\Bigg[-\frac{1}{2}\left\{\left(\frac{n+\alpha}{R_1} - i\frac{m+\beta}{R_2}\right)\right\} z \\
&\qquad + \frac{1}{2}\left\{\left(\frac{n+\alpha}{R_1} + i\frac{m+\beta}{R_2}\right)\right\} \bar{z}\Bigg] \ , \\
[\cos]_{n+\alpha,m+\beta}(z,\bar{z}) &= \cos\Bigg[-\frac{1}{2}\left\{\left(\frac{n+\alpha}{R_1} - i\frac{m+\beta}{R_2}\right)\right\} z \\
&\qquad + \frac{1}{2}\left\{\left(\frac{n+\alpha}{R_1} + i\frac{m+\beta}{R_2}\right)\right\} \bar{z}\Bigg] \ .
\end{aligned} \tag{B.32}$$

Upon compactification, massless zero mode $\varphi^{(0,0)}(x)$ appears on 4D when $\mathbb{Z}_2$ parities are $(\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2) = (+1,+1,+1)$. And, the massive KK modes $\varphi^{(n,m)}(x)$ do not appear in our low energy world because they have heavy masses. Here, zero modes mean 4-dimensional massless fields surviving after compactification. KK modes do not appear in our low-energy world, because they have heavy masses of $O(1/R)$, with the same magnitude as the unification scale.

If the representation matrices P_0, P_1 and P_2 are given by

$$\begin{aligned}
P_0 &= \mathrm{diag}(\overbrace{[+1]_{p_1},[+1]_{p_2},[+1]_{p_3},[+1]_{p_4},[-1]_{p_5},[-1]_{p_6},[-1]_{p_7},[-1]_{p_8}}^{N}) \ , \\
P_1 &= \mathrm{diag}([+1]_{p_1},[+1]_{p_2},[-1]_{p_3},[-1]_{p_4},[+1]_{p_5},[+1]_{p_6},[-1]_{p_7},[-1]_{p_8}) \ , \\
P_2 &= \mathrm{diag}([+1]_{p_1},[-1]_{p_2},[+1]_{p_3},[-1]_{p_4},[+1]_{p_5},[-1]_{p_6},[+1]_{p_7},[-1]_{p_8}) \ ,
\end{aligned} \tag{B.33}$$

where $[\pm 1]_{p_i}$ represents ± 1 for all elements and $N = \sum_{i=1}^{8} p_i$, the $SU(N)$ gauge group is broken down into its subgroup such as

$$SU(N) \to SU(p_1)\times SU(p_2)\times\cdots\times SU(p_8)\times U(1)^{7-\kappa} \ , \tag{B.34}$$

by orbifold breaking mechanism. In this case, the gauge fields $A_M^{\alpha(\mathscr{P}_0,\mathscr{P}_1,\mathscr{P}_2)}$ are divided as

$$A_\mu^{\alpha(+1,+1,+1)} \ , \quad A_\mu^{\beta(+1,+1,-1)} \ , \quad A_\mu^{\beta(+1,-1,+1)} \ , \quad A_\mu^{\beta(-1,+1,+1)} \ ,$$

$$
\begin{aligned}
&A_\mu^{\beta(+1,-1,-1)}\ , \quad A_\mu^{\beta(-1,+1,-1)}\ , \quad A_\mu^{\beta(-1,-1,+1)}\ , \quad A_\mu^{\beta(-1,-1,-1)}\ , \\
&A_z^{\beta(+1,+1,+1)}\ , \quad A_z^{\beta(+1,+1,-1)}\ , \quad A_z^{\beta(+1,-1,+1)}\ , \quad A_z^{\beta(-1,+1,+1)}\ , \\
&A_z^{\beta(+1,-1,-1)}\ , \quad A_z^{\beta(-1,+1,-1)}\ , \quad A_z^{\beta(-1,-1,+1)}\ , \quad A_z^{\alpha(-1,-1,-1)}\ , \\
&A_{\bar{z}}^{\beta(+1,+1,+1)}\ , \quad A_{\bar{z}}^{\beta(+1,+1,-1)}\ , \quad A_{\bar{z}}^{\beta(+1,+1,-1)}\ , \quad A_{\bar{z}}^{\beta(+1,+1,-1)}\ , \\
&A_{\bar{z}}^{\beta(+1,-1,-1)}\ , \quad A_{\bar{z}}^{\beta(-1,+1,-1)}\ , \quad A_{\bar{z}}^{\beta(-1,-1,+1)}\ , \quad A_{\bar{z}}^{\alpha(-1,-1,-1)}\ ,
\end{aligned}
\tag{B.35}
$$

where the index α indicates the gauge generators of unbroken gauge symmetry and the index β indicates the gauge generators of broken gauge symmetry.

B.2 $T^2/\mathbb{Z}_3$ orbifold

B.2.1 Property

Let us discuss $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_3$. On $T^2/\mathbb{Z}_3$, T^2 is constructed by $SU(3)$ lattice, and its basic vectors takes $e_1 = 1$ and $e_2 = e^{2\pi i/3} \equiv \omega$. The point z is equivalent to the points $z + e_1$ and $z + e_2$, and the points ωz on $M^4 \times T^2/\mathbb{Z}_3$. The fixed points for the $\mathbb{Z}_3$ transformation $z \to \omega z$ are

$$
0\ , \quad \frac{2e_1 + e_2}{3}\ , \quad \frac{e_1 + 2e_2}{3}\ . \tag{B.36}
$$

The transformation around those fixed points can be defined as

$$
\begin{aligned}
&s_{30} : z \to \omega z\ , \quad s_{31} : z \to \omega z + e_1\ , \quad s_{32} : z \to \omega z + e_2\ , \\
&t_1 : z \to z + e_1\ , \quad t_2 : z \to z + e_2\ ,
\end{aligned}
\tag{B.37}
$$

where satisfy the relation,

$$
\begin{aligned}
&s_{30}^3 = s_{31}^3 = s_{32}^3 = s_{30}s_{31}s_{32} = s_{31}s_{32}s_{30} = s_{32}s_{30}s_{31} = I\ , \\
&s_{31} = t_1 s_{30}\ , \quad s_{32} = t_2 t_1 s_{30}\ , \quad t_1 t_2 = t_2 t_1\ .
\end{aligned}
\tag{B.38}
$$

At this time, the BCs of bulk fields are characterized by matrices $(\Theta_0, \Theta_1, \Theta_2, \Theta_3, U_1, U_2)$. Those matrices satisfy the relations,

$$
\begin{aligned}
&\Theta_0^3 = \Theta_1^3 = \Theta_2^3 = \Theta_0\Theta_1\Theta_2 = \Theta_1\Theta_2\Theta_0 = \Theta_2\Theta_0\Theta_1 = I\ , \\
&\Theta_1 = U_1\Theta_0\ , \quad \Theta_2 = U_2U_1\Theta_0\ , \quad U_1U_2 = U_2U_1\ .
\end{aligned}
\tag{B.39}
$$

Since two of those matrices is independent, we choose two matrices Θ_0, Θ_1 which are unitary matrices.

B.2.2 Orbifold breaking by inner automophisms boundary conditions

The BCs of gauge field are determined as

$$
\begin{aligned}
s_{30}:\ & A_\mu(x, \omega z, \bar{\omega}\bar{z}) = \Theta_0 A_\mu(x, z, \bar{z})\Theta_0^\dagger, \\
& A_z(x, \omega z, \bar{\omega}\bar{z}) = \bar{\omega}\Theta_0 A_z(x, z, \bar{z})\Theta_0^\dagger, \\
& A_{\bar{z}}(x, \omega z, \bar{\omega}\bar{z}) = \omega\Theta_0 A_{\bar{z}}(x, z, \bar{z})\Theta_0^\dagger,
\end{aligned}
\tag{B.40}
$$

$$s_{31}: \; A_\mu(x, \omega z + e_1, \bar{\omega}\bar{z} + \bar{e}_1) = \Theta_1 A_\mu(x, z, \bar{z})\Theta_1^\dagger,$$
$$A_z(x, \omega z + e_1, \bar{\omega}\bar{z} + \bar{e}_1) = \bar{\omega}\Theta_1 A_z(x, z, \bar{z})\Theta_1^\dagger,$$
$$A_{\bar{z}}(x, \omega z + e_1, \bar{\omega}\bar{z} + \bar{e}_1) = \omega\Theta_1 A_{\bar{z}}(x, z, \bar{z})\Theta_1^\dagger, \quad \text{(B.41)}$$
$$s_{32}: \; A_\mu(x, \omega z + e_1 + e_2, \bar{\omega}\bar{z} + \bar{e}_1 + \bar{e}_2) = \Theta_2 A_\mu(x, z, \bar{z})\Theta_2^\dagger,$$
$$A_z(x, \omega z + e_1 + e_2, \bar{\omega}\bar{z} + \bar{e}_1 + \bar{e}_2) = \bar{\omega}\Theta_2 A_z(x, z, \bar{z})\Theta_2^\dagger,$$
$$A_{\bar{z}}(x, \omega z + e_1 + e_2, \bar{\omega}\bar{z} + \bar{e}_1 + \bar{e}_2) = \omega\Theta_2 A_{\bar{z}}(x, z, \bar{z})\Theta_2^\dagger, \quad \text{(B.42)}$$
$$t_1: \; A_M(x, z + e_1, \bar{z} + \bar{e}_1) = U_1 A_M(x, z, \bar{z})U_1^\dagger, \quad \text{(B.43)}$$
$$t_2: \; A_M(x, z + e_2, \bar{z} + \bar{e}_2) = U_2 A_M(x, z, \bar{z})U_2^\dagger, \quad \text{(B.44)}$$

where $z = x^5 + ix^6$, $\bar{z} = x^5 - ix^6$, $A_z = A_5 + iA_6$ and $A_{\bar{z}} = A_5 - iA_6$, and $\omega \equiv e^{2\pi i/3}$ and $\bar{\omega} \equiv e^{4\pi i/3}$. The BCs of scalar field ϕ areand spinor field ψ are determined as

$$s_{30}: \; \phi(x, \omega z, \bar{\omega}\bar{z}) = T_\Phi[\Theta_0]\phi(x, z, \bar{z}) \;, \quad \text{(B.45)}$$
$$s_{31}: \; \phi(x, \omega z + e_1, \bar{\omega}\bar{z} + \bar{e}_1) = T_\Phi[\Theta_1]\psi(x, z, \bar{z}) \;, \quad \text{(B.46)}$$
$$s_{32}: \; \phi(x, \omega z + e_1 + e_2, \bar{\omega}\bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Phi[\Theta_2]\psi(x, z, \bar{z}) \;, \quad \text{(B.47)}$$
$$t_1: \; \phi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Phi[\Xi_1]\phi(x, z, \bar{z}) \;, \quad \text{(B.48)}$$
$$t_2: \; \phi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Phi[\Xi_2]\phi(x, z, \bar{z}) \;, \quad \text{(B.49)}$$

$$s_{30}: \; \psi(x, \omega z, \bar{\omega}\bar{z}) = T_\Psi[\Theta_0]\psi(x, z, \bar{z}) \;, \quad \text{(B.50)}$$
$$s_{31}: \; \psi(x, \omega z + e_1, \bar{\omega}\bar{z} + \bar{e}_1) = T_\Psi[\Theta_1]\psi(x, z, \bar{z}) \;, \quad \text{(B.51)}$$
$$s_{32}: \; \psi(x, \omega z + e_1 + e_2, \bar{\omega}\bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Psi[\Theta_2]\psi(x, z, \bar{z}) \;. \quad \text{(B.52)}$$
$$t_1: \; \psi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Psi[U_1]\psi(x, z, \bar{z}) \;, \quad \text{(B.53)}$$
$$t_2: \; \psi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Psi[U_2]\psi(x, z, \bar{z}) \;, \quad \text{(B.54)}$$

where $T_{\Phi(\Psi)}[\Theta_i]$ and $T_{\Phi(\Psi)}[U_i]$ represent appropriate representation matrices including arbitrary sign factors, with the matices Θ_i and U_i. The representation matrices $T_\Sigma[P](\Sigma = \Phi, \Psi, P = \Theta_0, \Theta_1, \Theta_2, U_1, U_2)$ satisfy

$$\begin{aligned}
&T_\Sigma[\Theta_0]^3 = T_\Sigma[\Theta_1]^3 = T_\Sigma[\Theta_2]^3 \\
&\quad = T_\Sigma[\Theta_0]T_\Sigma[\Theta_1]T_\Sigma[\Theta_2] = T_\Sigma[\Theta_1]T_\Sigma[\Theta_2]T_\Sigma[\Theta_0] = T_\Sigma[\Theta_2]T_\Sigma[\Theta_0]T_\Sigma[\Theta_1] = I \;, \\
&T_\Sigma[\Theta_1] = T_\Sigma[U_1]T_\Sigma[\Theta_0] \;, \quad T_\Sigma[\Theta_2] = T_\Sigma[U_2]T_\Sigma[U_1]T_\Sigma[\Theta_0] \;, \\
&T_\Sigma[U_1]T_\Sigma[U_2] = T_\Sigma[U_2]T_\Sigma[U_1] \;.
\end{aligned} \quad \text{(B.55)}$$

Let $\varphi^{(\mathscr{P}_0, \mathscr{P}_1)}(x, z, \bar{z})$ be a component in a multiplet and have a definite the $\mathbb{Z}_3$ elements $\mathscr{P}_0$ and $\mathscr{P}_1$ which relate the representation matrices Θ_0 and Θ_1, and take 1, ω or $\bar{\omega}$, respectively. Here, φ is a generic field and it is applied to scalar field ϕ, fermion field ψ or gauge field A_M. The Fourier expansion of $\varphi^{(\mathscr{P}_0, \mathscr{P}_1)}(x, z, \bar{z})$ is given by

$$\varphi^{(1,1)}(x, z, \bar{z}) = \frac{3^{1/4}}{\pi\sqrt{2R_1R_2}}\varphi^{(0,0)}(x) + \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{\substack{n,m=0 \\ (n+m\neq 0)}}^{\infty} \varphi^{(n,m)}(x)\mathscr{F}^{(0)}_{n,m}(z, \bar{z}) \;, \quad \text{(B.56)}$$

$$\varphi^{(1,\omega)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(0)}_{n+1/3,m+1/3}(z,\bar{z}) \ , \tag{B.57}$$

$$\varphi^{(1,\bar{\omega})}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(0)}_{n+2/3,m+2/3}(z,\bar{z}) \ , \tag{B.58}$$

$$\varphi^{(\omega,\omega)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{\substack{n,m=0 \\ (n+m\neq 0)}}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(1)}_{n,m}(z,\bar{z}) \ , \tag{B.59}$$

$$\varphi^{(\omega,\bar{\omega})}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(1)}_{n+1/3,m+1/3}(z,\bar{z}) \ , \tag{B.60}$$

$$\varphi^{(\omega,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(1)}_{n+2/3,m+2/3}(z,\bar{z}) \ , \tag{B.61}$$

$$\varphi^{(\bar{\omega},\bar{\omega})}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{\substack{n,m=0 \\ (n+m\neq 0)}}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(2)}_{n,m}(z,\bar{z}) \ , \tag{B.62}$$

$$\varphi^{(\bar{\omega},+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(2)}_{n+1/3,m+1/3}(z,\bar{z}) \ , \tag{B.63}$$

$$\varphi^{(\bar{\omega},\omega)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x) \mathscr{F}^{(2)}_{n+2/3,m+2/3}(z,\bar{z}) \ , \tag{B.64}$$

where

$$\begin{aligned}
&\mathscr{F}^{(0)}_{n+\alpha,m+\beta}(z,\bar{z}) = \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\bar{\omega} z,\omega\bar{z}) \ , \\
&\mathscr{F}^{(1)}_{n+\alpha,m+\beta}(z,\bar{z}) = \bar{\omega}\mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \omega\mathscr{F}_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\bar{\omega} z,\omega\bar{z}) \ , \\
&\mathscr{F}^{(2)}_{n+\alpha,m+\beta}(z,\bar{z}) = \omega\mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \bar{\omega}\mathscr{F}_{n+\alpha,m+\beta}(\omega z,\bar{\omega}\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\bar{\omega} z,\omega\bar{z}) \ , \\
&\mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) = \exp\Bigg[-\frac{i}{2}\Bigg\{\left(\frac{n+\alpha}{R_1} - i\frac{n+\alpha}{\sqrt{3}R_1} - i\frac{2(m+\beta)}{\sqrt{3}R_2}\right) z \\
&\qquad\qquad + \left(\frac{n+\alpha}{R_1} + i\frac{n+\alpha}{\sqrt{3}R_1} + i\frac{2(m+\beta)}{\sqrt{3}R_2}\right) \bar{z}\Bigg\}\Bigg] \ .
\end{aligned} \tag{B.65}$$

Upon compactification, massless zero mode $\varphi^{(0,0)}(x)$ appears on 4D when $\mathbb{Z}_3$ elements are $(\mathscr{P}_0, \mathscr{P}_1) = (1,1)$. The massive KK modes $\varphi^{(n,m)}(x)$ do not appear in our low energy world because they have heavy masses.

If the representation matrices Θ_0 and Θ_1 are given by

$$\begin{aligned}
\Theta_0 &= \mathrm{diag}(\overbrace{[1]_{p_1},[1]_{p_2},[1]_{p_3},[\omega]_{p_4},[\omega]_{p_5},[\omega]_{p_6},[\bar{\omega}]_{p_7},[\bar{\omega}]_{p_8},[\bar{\omega}]_{p_9}}^{N}) \ , \\
\Theta_1 &= \mathrm{diag}([1]_{p_1},[\omega]_{p_2},[\bar{\omega}]_{p_3},[1]_{p_4},[\omega]_{p_5},[\bar{\omega}]_{p_6},[1]_{p_7},[\omega]_{p_8},[\bar{\omega}]_{p_9}) \ ,
\end{aligned} \tag{B.66}$$

where $[1]_{p_i}$, $[\omega]_{p_i}$ and $[\bar{\omega}]_{p_i}$ represent $+1$, ω and $\bar{\omega}$ for all elements and $N = \sum_{i=1}^{9} p_i$, the $SU(N)$ gauge group is broken down into its subgroup such as

$$SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_9) \times U(1)^{8-\kappa} \ , \tag{B.67}$$

by orbifold breaking mechanism. In this case, the gauge fields $A_M^{\alpha(\mathscr{P}_0,\mathscr{P}_1)}$ are divided as

$$\begin{aligned}
&A_\mu^{\alpha(1,1)} , \quad A_\mu^{\beta(1,\omega)} , \quad A_\mu^{\beta(1,\bar\omega)} , \quad A_\mu^{\beta(\omega,\omega)} , \quad A_\mu^{\beta(\omega,\bar\omega)} , \\
&A_\mu^{\beta(\omega,1)} , \quad A_\mu^{\beta(\bar\omega,\bar\omega)} , \quad A_\mu^{\beta(\bar\omega,1)} , \quad A_\mu^{\beta(\bar\omega,\omega)} , \\
&A_z^{\beta(1,1)} , \quad A_z^{\beta(1,\omega)} , \quad A_z^{\beta(1,\bar\omega)} , \quad A_z^{\alpha(\omega,\omega)} , \quad A_z^{\beta(\omega,\bar\omega)} , \\
&A_z^{\beta(\omega,1)} , \quad A_z^{\beta(\bar\omega,\bar\omega)} , \quad A_z^{\beta(\bar\omega,1)} , \quad A_z^{\beta(\bar\omega,\omega)} , \\
&A_{\bar z}^{\beta(1,1)} , \quad A_{\bar z}^{\beta(1,\omega)} , \quad A_{\bar z}^{\beta(1,\bar\omega)} , \quad A_{\bar z}^{\beta(\omega,\omega)} , \quad A_{\bar z}^{\beta(\omega,\bar\omega)} , \\
&A_{\bar z}^{\beta(\omega,1)} , \quad A_{\bar z}^{\alpha(\bar\omega,\bar\omega)} , \quad A_{\bar z}^{\beta(\bar\omega,1)} , \quad A_{\bar z}^{\beta(\bar\omega,\omega)} ,
\end{aligned} \tag{B.68}$$

where the index α indicates the gauge generators of unbroken gauge symmetry and the index β indicates the gauge generators of broken gauge symmetry.

B.3 $T^2/\mathbb{Z}_4$ orbifold

B.3.1 Property

Let us discuss $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_4$. On $T^2/\mathbb{Z}_4$, T^2 is constructed by $SU(2) \times SU(2) (\simeq SO(4))$ lattice, and its basic vectors are $e_1 = 1$ and $e_2 = i$, The point z is equivalent to the points $z + e_1$ and $z + e_2$, and the point z is equivalent to the points $-z$ and iz. The fixed points for the $\mathbb{Z}_4$ transformation $z \to \theta z = iz$ are

$$0 , \quad \frac{e_1 + e_2}{2} , \tag{B.69}$$

and it for the $\mathbb{Z}_2$ transformation $z \to \theta z = -z$ are

$$0 , \quad \frac{e_1}{2} , \quad \frac{e_2}{2} , \quad \frac{e_1 + e_2}{2} . \tag{B.70}$$

The transformation around those fixed points can be defined as

$$\begin{aligned}
&s_{40} : z \to iz , \quad s_{41} : z \to iz + e_1 , \quad s_{20} : z \to -z , \\
&s_{21} : z \to -z + e_1 , \quad s_{22} : z \to -z + e_2 , \quad s_{23} : z \to -z + e_1 + e_2 , \\
&t_1 : z \to z + e_1 , \quad t_2 : z \to z + e_2 ,
\end{aligned} \tag{B.71}$$

They satisfy the relations,

$$\begin{aligned}
&s_{40}^4 = s_{41}^4 = s_{20}^2 = s_{21}^2 = s_{22}^2 = s_{23}^2 = I , \quad s_{41} = t_1 s_{40} , \quad s_{21} = t_1 s_{20} , \\
&s_{22} = t_2 s_{20} , \quad s_{20} = s_{40}^2 , \quad s_{21} = s_{41} s_{40} , \quad s_{22} = s_{40} s_{41} , \\
&s_{23} = t_1 t_2 s_{20} = s_{21} s_{20} s_{22} = s_{22} s_{20} s_{21} , \quad t_1 t_2 = t_2 t_1 .
\end{aligned} \tag{B.72}$$

At this time, the BCs of bulk fields are characterized by matrices $(Q_0, Q_1, P_0, P_1, P_2, P_3, U_1, U_2)$. Those matrices satisfy the relations,

$$\begin{aligned}
&Q_0^4 = Q_1^4 = P_0^2 = P_1^2 = P_2^2 = P_3^2 = I , \quad Q_1 = U_1 Q_0 , \quad P_1 = U_1 P_0 , \\
&P_2 = U_2 P_0 , \quad P_0 = Q_0^2 , \quad P_1 = Q_1 Q_0 , \quad P_2 = Q_0 Q_1 , \\
&P_3 = U_1 U_2 P_0 = P_1 P_0 P_2 = P_2 P_0 P_1 , \quad U_1 U_2 = U_2 U_1 ,
\end{aligned} \tag{B.73}$$

where Q_i are unitary matrices, and P_i are unitary and hermitian matrices. Since two of those matrices is independent, we choose two matrices Q_0, P_1.

B.3.2 Orbifold breaking by inner automophisms boundary conditions

The BCs of gauge field are determined as

$$\begin{aligned}
s_{40}: \;& A_\mu(x, iz, -i\bar{z}) = Q_0 A_\mu(x, z, \bar{z}) Q_0^\dagger, \\
& A_z(x, iz, -i\bar{z}) = -i Q_0 A_z(x, z, \bar{z}) Q_0^\dagger, \\
& A_{\bar{z}}(x, iz, -i\bar{z}) = i Q_0 A_{\bar{z}}(x, z, \bar{z}) Q_0^\dagger, \qquad \text{(B.74)} \\
s_{41}: \;& A_\mu(x, iz + e_1, -i\bar{z} + \bar{e}_1) = Q_1 A_\mu(x, z, \bar{z}) Q_1^\dagger, \\
& A_z(x, iz + e_1, -i\bar{z} + \bar{e}_1) = -i Q_1 A_z(x, z, \bar{z}) Q_1^\dagger, \\
& A_{\bar{z}}(x, iz + e_1, -i\bar{z} + \bar{e}_1) = i Q_1 A_{\bar{z}}(x, z, \bar{z}) Q_1^\dagger, \qquad \text{(B.75)} \\
s_{20}: \;& A_\mu(x, -z, -\bar{z}) = P_0 A_\mu(x, z, \bar{z}) P_0^\dagger, \\
& A_z(x, -z, -\bar{z}) = -P_0 A_z(x, z, \bar{z}) P_0^\dagger, \\
& A_{\bar{z}}(x, -z, -\bar{z}) = -P_0 A_{\bar{z}}(x, z, \bar{z}) P_0^\dagger, \qquad \text{(B.76)} \\
s_{21}: \;& A_\mu(x, -z + e_1, -\bar{z} + \bar{e}_1) = P_1 A_\mu(x, z, \bar{z}) P_1^\dagger, \\
& A_z(x, -z + e_1, -\bar{z} + \bar{e}_1) = -P_1 A_z(x, z, \bar{z}) P_1^\dagger, \\
& A_{\bar{z}}(x, -z + e_1, -\bar{z} + \bar{e}_1) = -P_1 A_{\bar{z}}(x, z, \bar{z}) P_1^\dagger, \qquad \text{(B.77)} \\
s_{22}: \;& A_\mu(x, -z + e_2, -\bar{z} + \bar{e}_2) = P_2 A_\mu(x, z, \bar{z}) P_2^\dagger, \\
& A_z(x, -z + e_2, -\bar{z} + \bar{e}_2) = -P_2 A_z(x, z, \bar{z}) P_2^\dagger, \\
& A_{\bar{z}}(x, -z + e_2, -\bar{z} + \bar{e}_2) = -P_2 A_{\bar{z}}(x, z, \bar{z}) P_2^\dagger, \qquad \text{(B.78)} \\
s_{23}: \;& A_\mu(x, -z + e_1 + e_2, -\bar{z} + \bar{e}_1 + \bar{e}_2) = P_3 A_\mu(x, z, \bar{z}) P_3^\dagger, \\
& A_z(x, -z + e_1 + e_2, -\bar{z} + \bar{e}_1 + \bar{e}_2) = -P_3 A_z(x, z, \bar{z}) P_3^\dagger, \\
& A_{\bar{z}}(x, -z + e_1 + e_2, -\bar{z} + \bar{e}_1 + \bar{e}_2) = -P_3 A_{\bar{z}}(x, z, \bar{z}) P_3^\dagger, \qquad \text{(B.79)} \\
t_1: \;& A_M(x, z + e_1, \bar{z} + \bar{e}_1) = U_1 A_M(x, z, \bar{z}) U_1^\dagger, \qquad \text{(B.80)} \\
t_1: \;& A_M(x, z + e_2, \bar{z} + \bar{e}_2) = U_2 A_M(x, z, \bar{z}) U_2^\dagger, \qquad \text{(B.81)}
\end{aligned}$$

where $z = x^5 + ix^6$, $\bar{z} = x^5 - ix^6$, $A_z = A_5 + iA_6$ and $A_{\bar{z}} = A_5 - iA_6$. The BCs of scalar field ϕ and spinor field ψ are determined as

$$\begin{aligned}
s_{40}: \;& \phi(x, iz, -i\bar{z}) = T_\Phi[Q_0]\phi(x, z, \bar{z}) \;, \qquad \text{(B.82)} \\
s_{41}: \;& \phi(x, iz + e_1, -i\bar{z} + \bar{e}_1) = T_\Phi[Q_1]\psi(x, z, \bar{z}) \;, \qquad \text{(B.83)} \\
s_{20}: \;& \phi(x, -z, -\bar{z}) = T_\Phi[P_0]\phi(x, z, \bar{z}) \;, \qquad \text{(B.84)} \\
s_{21}: \;& \phi(x, -z + e_1, -\bar{z} + \bar{e}_1) = T_\Phi[P_1]\phi(x, z, \bar{z}) \;, \qquad \text{(B.85)} \\
s_{22}: \;& \phi(x, -z + e_2, -\bar{z} + \bar{e}_2) = T_\Phi[P_2]\phi(x, z, \bar{z}) \;, \qquad \text{(B.86)} \\
s_{23}: \;& \phi(x, -z + e_1 + e_2, -\bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Phi[P_3]\psi(x, z, \bar{z}) \;, \qquad \text{(B.87)} \\
t_1: \;& \phi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Phi[U_1]\phi(x, z, \bar{z}) \;, \qquad \text{(B.88)} \\
t_2: \;& \phi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Phi[U_2]\phi(x, z, \bar{z}) \;, \qquad \text{(B.89)}
\end{aligned}$$

$$\begin{aligned}
s_{40}: \;& \psi(x, iz, -i\bar{z}) = T_\Psi[Q_0]\psi(x, z, \bar{z}) \;, \qquad \text{(B.90)} \\
s_{41}: \;& \psi(x, iz + e_1, -i\bar{z} + \bar{e}_1) = T_\Psi[Q_1]\psi(x, z, \bar{z}) \;, \qquad \text{(B.91)} \\
s_{20}: \;& \psi(x, -z, -\bar{z}) = T_\Psi[P_0]\psi(x, z, \bar{z}) \;, \qquad \text{(B.92)}
\end{aligned}$$

$$s_{21}:\ \psi(x,-z+e_1,-\bar{z}+\bar{e}_1)=T_\Psi[P_1]\psi(x,z,\bar{z})\ , \tag{B.93}$$
$$s_{22}:\ \psi(x,-z+e_2,-\bar{z}+\bar{e}_2)=T_\Psi[P_2]\psi(x,z,\bar{z})\ , \tag{B.94}$$
$$s_{23}:\ \psi(x,-z+e_1+e_2,-\bar{z}+\bar{e}_1+\bar{e}_2)=T_\Psi[P_3]\psi(x,z,\bar{z})\ . \tag{B.95}$$
$$t_1:\ \psi(x,z+e_1,\bar{z}+\bar{e}_1)=T_\Psi[U_1]\psi(x,z,\bar{z})\ , \tag{B.96}$$
$$t_2:\ \psi(x,z+e_2,\bar{z}+\bar{e}_2)=T_\Psi[U_2]\psi(x,z,\bar{z})\ , \tag{B.97}$$

where $T_{\Phi(\Psi)}[P_i]$, $T_{\Phi(\Psi)}[Q_i]$ and $T_{\Phi(\Psi)}[U_i]$ represent appropriate representation matrices including arbitrary sign factors, with the matices P_i, Q_i and U_i. The representation matrices $T_\Sigma[P](\Sigma=\Phi,\Psi,P=Q_0,Q_1,P_0,P_1,P_2,P_3,U_1,U_2)$ satisfy

$$\begin{aligned}
&T_\Sigma[Q_0]^4=T_\Sigma[Q_1]^4=T_\Sigma[P_0]^2=T_\Sigma[P_1]^2=T_\Sigma[P_2]^2=T_\Sigma[P_3]^2=I\\
&T_\Sigma[Q_1]=T_\Sigma[U_1]T_\Sigma[Q_0]\ ,\quad T_\Sigma[P_1]=T_\Sigma[U_1]T_\Sigma[P_0]\ ,\\
&T_\Sigma[P_2]=T_\Sigma[U_2]\ ,\quad T_\Sigma[P_1]=T_\Sigma[Q_1]T_\Sigma[Q_0]\ ,\quad T_\Sigma[P_2]=T_\Sigma[Q_0]T_\Sigma[Q_1]\ ,\\
&T_\Sigma[P_3]=T_\Sigma[U_1]T_\Sigma[U_2]T_\Sigma[P_0]=T_\Sigma[P_1]T_\Sigma[P_0]T_\Sigma[P_2]=T_\Sigma[P_2]T_\Sigma[P_0]T_\Sigma[P_1]\ ,\\
&T_\Sigma[U_1]T_\Sigma[U_2]=T_\Sigma[U_2]T_\Sigma[U_1]\ .
\end{aligned} \tag{B.98}$$

Let $\varphi^{(\mathscr{P}_0,\mathscr{P}_1)}(x,z,\bar{z})$ be a component in a multiplet and have a definite the $\mathbb{Z}_4$ elements $\mathscr{P}_0$ and $\mathscr{P}_1$ which relate the representation matrices Q_0 and P_1, respectively. The eigenvalue of Q_0 takes $+1$, -1, $+i$ or $-i$ under the $\mathbb{Z}_4$ symmetry, and of P_1 takes $+1$ or -1 under the $\mathbb{Z}_2$ symmetry. Here, φ is a generic field and it is appied to scalar field ϕ, fermion field ψ or gauge field A_M. The Fourier expansion of $\varphi^{(\mathscr{P}_0,\mathscr{P}_1)}(x,z,\bar{z})$ is given by

$$\begin{aligned}
\varphi^{(+1,+1)}(x,z,\bar{z})=&\frac{\sqrt{2}}{\pi\sqrt{R_1R_2}}\varphi^{(0,0)}(x)\\
&+\frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\{[\cos]_{n,m}(z,\bar{z})+[\cos]_{n,m}(iz,-i\bar{z})\}\ ,
\end{aligned} \tag{B.99}$$

$$\begin{aligned}
\varphi^{(+1,-1)}(x,z,\bar{z})=&\frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\{[\cos]_{n+1/2,m+1/2}(z,\bar{z})\\
&+[\cos]_{n+1/2,m+1/2}(iz,-i\bar{z})\}\ ,
\end{aligned} \tag{B.100}$$

$$\begin{aligned}
\varphi^{(+i,+1)}(x,z,\bar{z})=&\frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\{[\sin]_{n+1/2,m+1/2}(z,\bar{z})\\
&+i[\sin]_{n,m}(iz,-i\bar{z})\}\ ,
\end{aligned} \tag{B.101}$$

$$\begin{aligned}
\varphi^{(+i,-1)}(x,z,\bar{z})=&\frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\{[\sin]_{n,m}(z,\bar{z})\\
&+i[\sin]_{n+1/2,m+1/2}(iz,-i\bar{z})\}\ ,
\end{aligned} \tag{B.102}$$

$$\varphi^{(-1,+1)}(x,z,\bar{z})=\frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\{[\cos]_{n,m}(z,\bar{z})$$

$$- [\cos]_{n,m}(iz, -i\bar{z})\} \ , \tag{B.103}$$

$$\varphi^{(-1,-1)}(x,z,\bar{z}) = \frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x)\{[\cos]_{n+1/2,m+1/2}(z,\bar{z}) - [\cos]_{n+1/2,m+1/2}(iz, -i\bar{z})\} \ , \tag{B.104}$$

$$\varphi^{(-i,+1)}(x,z,\bar{z}) = \frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x)\{[\sin]_{n+1/2,m+1/2}(z,\bar{z}) - i[\sin]_{n,m}(iz, -i\bar{z})\} \ , \tag{B.105}$$

$$\varphi^{(-i,-1)}(x,z,\bar{z}) = \frac{2\sqrt{2}}{\pi\sqrt{R_1R_2}} \sum_{n,m=0}^{\infty} \varphi^{(n,m)}(x)\{[\sin]_{n,m}(z,\bar{z}) - i[\sin]_{n+1/2,m+1/2}(iz, -i\bar{z})\} \ , \tag{B.106}$$

where

$$\begin{aligned}
[\cos]_{n+\alpha,m+\beta} &= \cos\left[-\frac{1}{2\sqrt{2}}\left\{ \left(\frac{n+\alpha}{R_1} - i\frac{n+\alpha}{R_1} - i\frac{m+\beta}{R_2}\right) z \right.\right. \\
&\qquad \left.\left. + \left(\frac{n+\alpha}{R_1} + i\frac{n+\alpha}{R_1} + i\frac{m+\beta}{R_2}\right) \bar{z} \right\}\right] , \\
[\cos]_{n+\alpha,m+\beta} &= \sin\left[-\frac{1}{2\sqrt{2}}\left\{ \left(\frac{n+\alpha}{R_1} - i\frac{n+\alpha}{R_1} - i\frac{m+\beta}{R_2}\right) z \right.\right. \\
&\qquad \left.\left. + \left(\frac{n+\alpha}{R_1} + i\frac{n+\alpha}{R_1} + i\frac{m+\beta}{R_2}\right) \bar{z} \right\}\right] .
\end{aligned} \tag{B.107}$$

Upon compactification, massless mode $\varphi^{(0,0)}(x)$ appears on 4D when $\mathbb{Z}_4$ elements are $(\mathscr{P}_0, \mathscr{P}_1) = (+1, +1)$. The massive KK modes $\varphi^{(n,m)}(x)$ do not appear in our low energy world because they have heavy masses.

If the representation matrices Q_0 and P_1 are given by

$$\begin{aligned}
Q_0 &= \mathrm{diag}(\overbrace{[+1]_{p_1}, [+1]_{p_2}, [+i]_{p_3}, [+i]_{p_4}, [-1]_{p_5}, [-1]_{p_6}, [-i]_{p_7}, [-i]_{p_8}}^{N}) \ , \\
P_1 &= \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}, [+1]_{p_5}, [-1]_{p_6}, [+1]_{p_7}, [-1]_{p_8}) \ ,
\end{aligned} \tag{B.108}$$

where $[\pm 1]_{p_i}$ and $[\pm i]_{p_i}$ represent ± 1 and $\pm i$ for all elements and $N = \sum_{i=1}^{8} p_i$, the $SU(N)$ gauge group is broken down into its subgroup such as

$$SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_8) \times U(1)^{7-\kappa} \ , \tag{B.109}$$

by orbifold breaking mechanism. In this case, the gauge fields $A_M^{\alpha(\mathscr{P}_0,\mathscr{P}_1)}$ are divided as

$$\begin{aligned}
&A_\mu^{\alpha(+1,+1)} \ , \quad A_\mu^{\beta(+1,-1)} \ , \quad A_\mu^{\beta(+i,+1)} \ , \quad A_\mu^{\beta(+i,+1)} \ , \\
&A_\mu^{\beta(-1,+1)} \ , \quad A_\mu^{\beta(-1,-1)} \ , \quad A_\mu^{\beta(-i,+1)} \ , \quad A_\mu^{\beta(-i,-1)} \ , \\
&A_z^{\beta(+1,+1)} \ , \quad A_z^{\beta(+1,-1)} \ , \quad A_z^{\beta(+i,+1)} \ , \quad A_z^{\alpha(+i,-1)} \ ,
\end{aligned}$$

$$
\begin{aligned}
&A_z^{\beta(-1,+1)} \ , \quad A_z^{\beta(-1,-1)} \ , \quad A_z^{\beta(-i,+1)} \ , \quad A_z^{\beta(-i,-1)} \ , \\
&A_{\bar{z}}^{\beta(+1,+1)} \ , \quad A_{\bar{z}}^{\beta(+1,-1)} \ , \quad A_{\bar{z}}^{\beta(+i,+1)} \ , \quad A_{\bar{z}}^{\beta(+i,-1)} \ , \\
&A_{\bar{z}}^{\beta(-1,+1)} \ , \quad A_{\bar{z}}^{\beta(-1,-1)} \ , \quad A_{\bar{z}}^{\beta(-i,+1)} \ , \quad A_{\bar{z}}^{\alpha(-i,-1)} \ ,
\end{aligned}
\tag{B.110}
$$

where the index α indicates the gauge generators of unbroken gauge symmetry and the index β indicates the gauge generators of broken gauge symmetry.

B.4 $T^2/\mathbb{Z}_6$ orbifold

B.4.1 Property

Let us discuss $SU(N)$ gauge theory on $M^4 \times T^2/\mathbb{Z}_6$. On $T^2/\mathbb{Z}_6$, T^2 is constructed by G_2 lattice, its basic vectors are $e_1 = 1$ and $e_2 = (-3 + i\sqrt{3})/2$ ($|e_2| = \sqrt{3}$). The point z is equivalent to the points $z + e_1$ and $z + e_2$, and the point z is equivalent to the points ρz where $\rho^6 = 1$ ($\rho = e^{i\pi/3}$). The fixed point for the $\mathbb{Z}_6$ transformation $z \to \rho z$ is

$$
0 \ , \tag{B.111}
$$

it for the $\mathbb{Z}_3$ transformation $z \to \rho^2 z = \omega z$ are

$$
0 \ , \quad \frac{e_1}{3} \ , \quad \frac{e_2}{3} \ , \tag{B.112}
$$

and it for the $\mathbb{Z}_2$ transformation $z \to \rho^3 z = -z$ are

$$
0 \ , \quad \frac{e_1}{2} \ , \quad \frac{e_2}{2} \ , \quad \frac{e_1 + e_2}{2} \ . \tag{B.113}
$$

The transformation around those fixed points can be defined as

$$
\begin{aligned}
&s_{60} : z \to \rho z \ , \quad s_{30} : z \to \rho^2 z \ , \quad s_{32} : z \to \rho^2 z + e_1 + e_2 \ , \\
&s_{33} : z \to \rho^2 z + 2e_1 + 2e_2 \ , \quad s_{20} : z \to \rho^3 z \ , \quad s_{21} : z \to \rho^3 z + e_1 \ , \\
&s_{22} : z \to \rho^3 z + e_2 \ , \quad s_{23} : z \to \rho^3 z + e_1 + e_2 \ , \\
&t_1 : z \to z + e_1 \ , \quad t_2 : z \to z + e_2 \ ,
\end{aligned}
\tag{B.114}
$$

They satisfy the relations,

$$
\begin{aligned}
&s_{60}^6 = s_{30}^3 = s_{32}^3 = s_{33}^3 = s_{20}^2 = s_{21}^2 = s_{22}^2 = s_{23}^2 = I \ , \\
&s_{32} = t_1 t_2 s_{30} \ , \quad s_{33} = t_1^2 t_2^2 s_{30} \ , \quad s_{21} = t_1 s_{20} \ , \quad s_{22} = t_2 s_{20} \ , \\
&s_{30} s_{32} s_{33} = s_{32} s_{33} s_{30} = s_{33} s_{30} s_{32} = I \ , \\
&s_{23} = t_1 t_2 s_{20} = s_{21} s_{20} s_{22} = s_{22} s_{20} s_{21} = s_{32} s_{60} \ , \\
&s_{30} = s_{60}^2 \ , \quad s_{20} = s_{60}^3 \ , \quad t_1 t_2 = t_2 t_1
\end{aligned}
\tag{B.115}
$$

At this time, the BCs of bulk fields are characterized by matrices ($\Xi_0, \Theta_0, \Theta_2, \Theta_3, P_0, P_1, P_2, P_3, U_1, U_2$). Those matrices satisfy the relationa,

$$
\begin{aligned}
&\Xi_0^6 = \Theta_0^3 = \Theta_1^3 = \Theta_3^3 = P_0^2 = P_1^2 = P_2^2 = P_3^2 = I \ , \\
&\Theta_2 = U_1 U_2 \Theta_0 \ , \quad \Theta_3 = U_1^2 U_2^2 \Theta_0 \ , \quad P_1 = U_1 P_0 \ , \quad P_2 = U_2 P_0 \ ,
\end{aligned}
$$

$$\Theta_0\Theta_2\Theta_3 = \Theta_2\Theta_3\Theta_0 = \Theta_3\Theta_0\Theta_2 = I \ ,$$
$$P_3 = U_1U_2P_0 = P_1P_0P_2 = P_2P_0P_1 = \Theta_2\Xi_0 \ ,$$
$$\Theta_0 = \Xi_0^2 \ , \quad P_0 = \Xi_0^3 \ , \quad U_1U_2 = U_2U_1 \ . \tag{B.116}$$

Since two of those matrices is independent, we choose two matrices Ξ_0, P_1. [11]

B.4.2 Orbifold breaking by inner automophisms boundary conditions

The BCs of gauge field are determined as

$$s_{60}: \ A_\mu(x,\rho z,\rho^5\bar{z}) = \Xi_0 A_\mu(x,z,\bar{z})\Xi_0^\dagger,$$
$$A_z(x,\rho z,\rho^5\bar{z}) = \rho^5\Xi_0 A_z(x,z,\bar{z})\Xi_0^\dagger,$$
$$A_{\bar{z}}(x,\rho z,\rho^5\bar{z}) = \rho\Xi_0 A_{\bar{z}}(x,z,\bar{z})\Xi_0^\dagger, \tag{B.117}$$
$$s_{30}: \ A_\mu(x,\rho^2 z,\rho^4\bar{z}) = \Theta_0 A_\mu(x,z,\bar{z})\Theta_0^\dagger,$$
$$A_z(x,\rho^2 z,\rho^4\bar{z}) = \rho^4\Theta_0 A_z(x,z,\bar{z})\Theta_0^\dagger,$$
$$A_{\bar{z}}(x,\rho^2 z,\rho^4\bar{z}) = \rho^2\Theta_0 A_{\bar{z}}(x,z,\bar{z})\Theta_0^\dagger, \tag{B.118}$$
$$s_{32}: \ A_\mu(x,\rho^2 z+e_1+e_2,\rho^4\bar{z}+\bar{e}_1+\bar{e}_2) = \Theta_2 A_\mu(x,z,\bar{z})\Theta_2^\dagger,$$
$$A_z(x,\rho^2 z+e_1+e_2,\rho^4\bar{z}+\bar{e}_1+\bar{e}_2) = \rho^4\Theta_2 A_z(x,z,\bar{z})\Theta_2^\dagger,$$
$$A_{\bar{z}}(x,\rho^2 z+e_1+e_2,\rho^4\bar{z}+\bar{e}_1+\bar{e}_2) = \rho^2\Theta_2 A_{\bar{z}}(x,z,\bar{z})\Theta_2^\dagger, \tag{B.119}$$
$$s_{33}: \ A_\mu(x,\rho^2 z+2e_1+2e_2,\rho^4\bar{z}+2\bar{e}_1+2\bar{e}_2) = \Theta_3 A_\mu(x,z,\bar{z})\Theta_3^\dagger,$$
$$A_z(x,\rho^2 z+2e_1+2e_2,\rho^4\bar{z}+2\bar{e}_1+2\bar{e}_2) = \rho^4\Theta_3 A_z(x,z,\bar{z})\Theta_3^\dagger,$$
$$A_{\bar{z}}(x,\rho^2 z+2e_1+2e_2,\rho^4\bar{z}+2\bar{e}_1+2\bar{e}_2) = \rho^2\Theta_3 A_{\bar{z}}(x,z,\bar{z})\Theta_3^\dagger, \tag{B.120}$$
$$s_{20}: \ A_\mu(x,\rho^3 z,\rho^3\bar{z}) = P_0 A_\mu(x,z,\bar{z})P_0^\dagger,$$
$$A_z(x,\rho^3 z,\rho^3\bar{z}) = \rho^3 P_0 A_z(x,z,\bar{z})P_0^\dagger,$$
$$A_{\bar{z}}(x,\rho^3 z,\rho^3\bar{z}) = \rho^3 P_0 A_{\bar{z}}(x,z,\bar{z})P_0^\dagger, \tag{B.121}$$
$$s_{21}: \ A_\mu(x,\rho^3 z+e_1,\rho^3\bar{z}+\bar{e}_1) = P_1 A_\mu(x,z,\bar{z})P_1^\dagger,$$
$$A_z(x,\rho^3 z+e_1,\rho^3\bar{z}+\bar{e}_1) = \rho^3 P_1 A_z(x,z,\bar{z})P_1^\dagger,$$
$$A_{\bar{z}}(x,\rho^2 z+e_1,\rho^4\bar{z}+\bar{e}_1) = \rho^3 P_1 A_{\bar{z}}(x,z,\bar{z})P_1^\dagger, \tag{B.122}$$
$$s_{22}: \ A_\mu(x,\rho^3 z+e_2,\rho^3\bar{z}+\bar{e}_2) = P_2 A_\mu(x,z,\bar{z})P_2^\dagger,$$
$$A_z(x,\rho^3 z+e_2,\rho^3\bar{z}+\bar{e}_2) = \rho^3 P_2 A_z(x,z,\bar{z})P_2^\dagger,$$
$$A_{\bar{z}}(x,\rho^3 z+e_2,\rho^3\bar{z}+\bar{e}_2) = \rho^3 P_2 A_{\bar{z}}(x,z,\bar{z})P_2^\dagger, \tag{B.123}$$
$$s_{23}: \ A_\mu(x,\rho^3 z+e_1+e_2,\rho^3\bar{z}+\bar{e}_1+\bar{e}_2) = P_3 A_\mu(x,z,\bar{z})P_3^\dagger,$$
$$A_z(x,\rho^3 z+e_1+e_2,\rho^3\bar{z}+\bar{e}_1+\bar{e}_2) = \rho^3 P_3 A_z(x,z,\bar{z})P_3^\dagger,$$
$$A_{\bar{z}}(x,\rho^3 z+e_1+e_2,\rho^3\bar{z}+\bar{e}_1+\bar{e}_2) = \rho^3 P_3 A_{\bar{z}}(x,z,\bar{z})P_3^\dagger, \tag{B.124}$$
$$t_1: \ A_M(x,z+e_1,\bar{z}+\bar{e}_1) = U_1 A_M(x,z,\bar{z})U_1^\dagger, \tag{B.125}$$

[11] Though the number of independent representation matrices for $T^2/\mathbb{Z}_6$ is stated to be three in [65], it should be two because other operations are generated using $s_0 : z \to e^{\pi i/3}z$ and $r_1 : z \to e_1 - z$. For example, $t_1 : z \to z + e_1$ and $t_2 : z \to z + e_2$ are generated as $t_1 = r_1(s_0)^3$ and $t_2 = (s_0)^2 r_1 (s_0)^4 r_1$, respectively.

$$t_1: \ A_M(x, z+e_2, \bar{z}+\bar{e}_2) = U_2 A_M(x, z, \bar{z}) U_2^\dagger, \tag{B.126}$$

where $z = x^5 + ix^6$, $\bar{z} = x^5 - ix^6$, $A_z = A_5 + iA_6$ and $A_{\bar{z}} = A_5 - iA_6$. The BCs of scalar field ϕ and spinor field ψ are determined as

$$s_{60}: \ \phi(x, \rho z, \rho^5 \bar{z}) = T_\Phi[\Xi_0]\phi(x, z, \bar{z}) \ , \tag{B.127}$$
$$s_{30}: \ \phi(x, \rho^2 z, \rho^4 \bar{z}) = T_\Phi[\Theta_0]\phi(x, z, \bar{z}) \ , \tag{B.128}$$
$$s_{32}: \ \phi(x, \rho^2 z + e_1 + e_2, \rho^4 \bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Phi[\Theta_2]\psi(x, z, \bar{z}) \ , \tag{B.129}$$
$$s_{33}: \ \phi(x, \rho^2 z + 2e_1 + 2e_2, \rho^4 \bar{z} + 2\bar{e}_1 + 2\bar{e}_2) = T_\Phi[\Theta_3]\psi(x, z, \bar{z}) \ , \tag{B.130}$$
$$s_{20}: \ \phi(x, \rho^3 z, \rho^3 \bar{z}) = T_\Phi[P_0]\phi(x, z, \bar{z}) \ , \tag{B.131}$$
$$s_{21}: \ \phi(x, \rho^3 z + e_1, \rho^3 \bar{z} + \bar{e}_1) = T_\Phi[P_1]\psi(x, z, \bar{z}) \ , \tag{B.132}$$
$$s_{22}: \ \phi(x, \rho^3 z + e_2, \rho^3 \bar{z} + \bar{e}_2) = T_\Phi[P_2]\psi(x, z, \bar{z}) \ , \tag{B.133}$$
$$s_{23}: \ \phi(x, \rho^3 z + e_1 + e_2, \rho^3 \bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Phi[P_3]\psi(x, z, \bar{z}) \ , \tag{B.134}$$
$$t_1: \ \phi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Phi[U_1]\phi(x, z, \bar{z}) \ , \tag{B.135}$$
$$t_2: \ \phi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Phi[U_2]\phi(x, z, \bar{z}) \ , \tag{B.136}$$

$$s_{60}: \ \psi(x, \rho z, \rho^5 \bar{z}) = T_\Psi[\Xi_0]\phi(x, z, \bar{z}) \ , \tag{B.137}$$
$$s_{30}: \ \psi(x, \rho^2 z, \rho^4 \bar{z}) = T_\Psi[\Theta_0]\phi(x, z, \bar{z}) \ , \tag{B.138}$$
$$s_{32}: \ \psi(x, \rho^2 z + e_1 + e_2, \rho^4 \bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Psi[\Theta_2]\psi(x, z, \bar{z}) \ , \tag{B.139}$$
$$s_{33}: \ \psi(x, \rho^2 z + 2e_1 + 2e_2, \rho^4 \bar{z} + 2\bar{e}_1 + 2\bar{e}_2) = T_\Psi[\Theta_3]\psi(x, z, \bar{z}) \ , \tag{B.140}$$
$$s_{20}: \ \psi(x, \rho^3 z, \rho^3 \bar{z}) = T_\Psi[P_0]\phi(x, z, \bar{z}) \ , \tag{B.141}$$
$$s_{21}: \ \psi(x, \rho^3 z + e_1, \rho^3 \bar{z} + \bar{e}_1) = T_\Psi[P_1]\psi(x, z, \bar{z}) \ , \tag{B.142}$$
$$s_{22}: \ \psi(x, \rho^3 z + e_2, \rho^3 \bar{z} + \bar{e}_2) = T_\Phi[P_2]\psi(x, z, \bar{z}) \ , \tag{B.143}$$
$$s_{23}: \ \psi(x, \rho^3 z + e_1 + e_2, \rho^3 \bar{z} + \bar{e}_1 + \bar{e}_2) = T_\Psi[P_3]\psi(x, z, \bar{z}) \ , \tag{B.144}$$
$$t_1: \ \psi(x, z + e_1, \bar{z} + \bar{e}_1) = T_\Psi[U_1]\psi(x, z, \bar{z}) \ , \tag{B.145}$$
$$t_2: \ \psi(x, z + e_2, \bar{z} + \bar{e}_2) = T_\Psi[U_2]\psi(x, z, \bar{z}) \ , \tag{B.146}$$

where $T_{\Phi(\Psi)}[\Xi_0]$, $T_{\Phi(\Psi)}[\Theta_i]$, $T_{\Phi(\Psi)}[P_i]$ and $T_{\Phi(\Psi)}[U_i]$ represent appropriate representation matrices including arbitrary sign factors, with the matices Ξ_0, Θ_i, P_i and U_i. The representation matrices $T_\Sigma[P] (\Sigma = \Phi, \Psi, P = \Xi_0, \Theta_0, \Theta_2, \Theta_3, P_0, P_1, P_2, P_3, U_1, U_2)$ satisfy

$$\begin{aligned}
&T_\Sigma[\Xi_0]^6 = T_\Sigma[\Theta_0]^3 = T_\Sigma[\Theta_1]^3 = T_\Sigma[\Theta_3]^3 \\
&\qquad = T_\Sigma[P_0]^2 = T_\Sigma[P_1]^2 = T_\Sigma[P_2]^2 = T_\Sigma[P_3]^2 = I \ , \\
&T_\Sigma[\Theta_2] = T_\Sigma[U_1]T_\Sigma[U_2]T_\Sigma[\Theta_0] \ , \quad T_\Sigma[\Theta_3] = T_\Sigma[U_1]^2 T_\Sigma[U_2]^2 T_\Sigma[\Theta_0] \ , \\
&T_\Sigma[P_1] = T_\Sigma[U_1]T_\Sigma[P_0] \ , \quad T_\Sigma[P_2] = T_\Sigma[U_2 T_\Sigma[P_0] \ , \\
&T_\Sigma[\Theta_0]T_\Sigma[\Theta_2]T_\Sigma[\Theta_3] = T_\Sigma[\Theta_2]T_\Sigma[\Theta_3]T_\Sigma[\Theta_0] = T_\Sigma[\Theta_3]T_\Sigma[\Theta_0]T_\Sigma[\Theta_2] = I \ , \\
&T_\Sigma[P_3] = T_\Sigma[U_1]T_\Sigma[U_2]T_\Sigma[P_0] = T_\Sigma[P_1]T_\Sigma[P_0]T_\Sigma[P_2] \\
&\qquad = T_\Sigma[P_2]T_\Sigma[P_0]T_\Sigma[P_1] = T_\Sigma[\Theta_2]T_\Sigma[\Xi_0] \ , \\
&T_\Sigma[\Theta_0] = T_\Sigma[\Xi_0]^2 \ , \quad T_\Sigma[P_0] = T_\Sigma[\Xi_0]^3 \ , \quad T_\Sigma[U_1]T_\Sigma[U_2] = T_\Sigma[U_2]T_\Sigma[U_1] \ .
\end{aligned} \tag{B.147}$$

Let $\varphi^{(\mathscr{P}_0, \mathscr{P}_1)}(x, z, \bar{z})$ be a component in a multiplet and have a definite the $\mathbb{Z}_3$ elements $\mathscr{P}_0$ and $\mathscr{P}_1$ which relate the representation matrices Ξ_0 and P_1, respectively. The eigenvalue of Ξ_0 takes ρ^i $(i = 1, \cdots, 6)$ under the $\mathbb{Z}_6$ symmetry, and of P_1 takes $+1$ or -1 under the $\mathbb{Z}_2$ symmetry. Here, φ is a generic field and it is applied to scalar field ϕ, fermion field ψ or gauge field A_M. The Fourier expansion of $\varphi^{(\mathscr{P}_0,\mathscr{P}_1)}(x,z,\bar{z})$ is given by

$$\varphi^{(+1,+1)}(x,z,\bar{z}) = \frac{3^{1/4}}{\pi\sqrt{2R_1R_2}}\varphi^{(0,0)}(x) + \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(0)}_{n,m}(z,\bar{z}) \tag{B.148}$$

$$\varphi^{(+1,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(0)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.149}$$

$$\varphi^{(\rho,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(1)}_{n,m}(z,\bar{z}) \tag{B.150}$$

$$\varphi^{(\rho,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(1)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.151}$$

$$\varphi^{(\rho^2,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(2)}_{n,m}(z,\bar{z}) \tag{B.152}$$

$$\varphi^{(\rho^2,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(2)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.153}$$

$$\varphi^{(\rho^3,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(3)}_{n,m}(z,\bar{z}) \tag{B.154}$$

$$\varphi^{(\rho^3,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(3)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.155}$$

$$\varphi^{(\rho^4,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(4)}_{n,m}(z,\bar{z}) \tag{B.156}$$

$$\varphi^{(\rho^4,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(4)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.157}$$

$$\varphi^{(\rho^5,+1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{\substack{n,m=0\\(n+m\neq 0)}}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(5)}_{n,m}(z,\bar{z}) \tag{B.158}$$

$$\varphi^{(\rho^5,-1)}(x,z,\bar{z}) = \frac{1}{\pi\sqrt{12R_1R_2}}\sum_{n,m=0}^{\infty}\varphi^{(n,m)}(x)\mathscr{F}^{(5)}_{n+1/2,m+1/2}(z,\bar{z}) \tag{B.159}$$

where

$$\begin{aligned}\mathscr{F}^{(0)}_{n+\alpha,m+\beta}(z,\bar{z}) = \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) &+ \mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\ &+ \mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\ &+ \mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z})\end{aligned}$$

$$
\begin{aligned}
\mathscr{F}^{(1)}_{n+\alpha,m+\beta}(z,\bar{z}) &= \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \rho\mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\
&\quad + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \rho^3\mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\
&\quad + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \rho^5\mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z}) \\
\mathscr{F}^{(2)}_{n+\alpha,m+\beta}(z,\bar{z}) &= \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\
&\quad + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\
&\quad + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z}) \\
\mathscr{F}^{(3)}_{n+\alpha,m+\beta}(z,\bar{z}) &= \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \rho^3\mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\
&\quad + \mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \rho^3\mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\
&\quad + \mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \rho^3\mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z}) \\
\mathscr{F}^{(4)}_{n+\alpha,m+\beta}(z,\bar{z}) &= \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\
&\quad + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\
&\quad + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z}) \\
\mathscr{F}^{(5)}_{n+\alpha,m+\beta}(z,\bar{z}) &= \mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) + \rho^5\mathscr{F}_{n+\alpha,m+\beta}(\rho z,\rho^5\bar{z}) \\
&\quad + \rho^4\mathscr{F}_{n+\alpha,m+\beta}(\rho^2 z,\rho^4\bar{z}) + \rho^3\mathscr{F}_{n+\alpha,m+\beta}(\rho^3 z,\rho^3\bar{z}) \\
&\quad + \rho^2\mathscr{F}_{n+\alpha,m+\beta}(\rho^4 z,\rho^2\bar{z}) + \rho\mathscr{F}_{n+\alpha,m+\beta}(\rho^5 z,\rho\bar{z}) \\
\mathscr{F}_{n+\alpha,m+\beta}(z,\bar{z}) &= \exp\Bigg[-\frac{i}{2}\Bigg\{\frac{n+\alpha}{R_1} - i\frac{\sqrt{3}(n+\alpha)}{R_1} - i\frac{2(n+\alpha)}{\sqrt{3}R_1}z \\
&\qquad + \frac{n+\alpha}{R_1} + i\frac{\sqrt{3}(n+\alpha)}{R_1} + i\frac{2(n+\alpha)}{\sqrt{3}R_1}\bar{z}\Bigg\}\Bigg] . \qquad \text{(B.160)}
\end{aligned}
$$

Upon compactification, massless mode $\varphi^{(0,0)}(x)$ appears on 4D when $\mathbb{Z}_3$ elements are $(\mathscr{P}_0, \mathscr{P}_1) = (+1, +1)$. The massive KK modes $\varphi^{(n,m)}(x)$ do not appear in our low energy world because they have heavy masses.

If the representation matrices Ξ_0 and P_1 are given by

$$
\begin{aligned}
\Xi_0 &= \mathrm{diag}([+1]_{p_1}, [+1]_{p_2}, [\rho]_{p_3}, [\rho]_{p_4}, [\rho^2]_{p_5}, [\rho^2]_{p_6}, \\
&\qquad \times [\rho^3]_{p_7}, [\rho^3]_{p_8}, [\rho^4]_{p_9}, [\rho^4]_{p_{10}}, [\rho^5]_{p_{11}}, [\rho^5]_{p_{12}}) , \\
P_1 &= \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}, [+1]_{p_5}, [-1]_{p_6}, \\
&\qquad \times [+1]_{p_7}, [-1]_{p_8}, [+1]_{p_9}, [-1]_{p_{10}}, [+1]_{p_{11}}, [-1]_{p_{12}}) , \qquad \text{(B.161)}
\end{aligned}
$$

where $[\pm 1]_{p_i}$ and $[\rho^a]_{p_i}$ represent ± 1 and $\rho^a (= e^{i\pi a/3})$ for all elements and $N = \sum_{i=1}^{12} p_i$, the $SU(N)$ gauge group is broken down into its subgroup such as

$$
SU(N) \to SU(p_1) \times SU(p_2) \times \cdots \times SU(p_{12}) \times U(1)^{11-\kappa} , \qquad \text{(B.162)}
$$

by orbifold breaking mechanism. In this case, the gauge fields $A_M^{\alpha(\mathscr{P}_0,\mathscr{P}_1)}$ are divided as

$$
\begin{aligned}
&A_\mu^{\alpha(+1,+1)} , \quad A_\mu^{\beta(+1,-1)} , \quad A_\mu^{\beta(\rho,+1)} , \quad A_\mu^{\beta(\rho,-1)} , \quad A_\mu^{\beta(\rho^2,+1)} , \quad A_\mu^{\beta(\rho^2,+1)} , \\
&A_\mu^{\beta(\rho^3,+1)} , \quad A_\mu^{\beta(\rho^3,-1)} , \quad A_\mu^{\beta(\rho^4,+1)} , \quad A_\mu^{\beta(\rho^4,-1)} , \quad A_\mu^{\beta(\rho^5,+1)} , \quad A_\mu^{\beta(\rho^5,-1)} ,
\end{aligned}
$$

$$
\begin{aligned}
&A_z^{\beta(+1,+1)} , \quad A_z^{\beta(+1,-1)} , \quad A_z^{\beta(\rho,+1)} , \quad A_z^{\alpha(\rho,-1)} , \quad A_z^{\beta(\rho^2,+1)} , \quad A_z^{\beta(\rho^2,+1)} , \\
&A_z^{\beta(\rho^3,+1)} , \quad A_z^{\beta(\rho^3,-1)} , \quad A_z^{\beta(\rho^4,+1)} , \quad A_z^{\beta(\rho^4,-1)} , \quad A_z^{\beta(\rho^5,+1)} , \quad A_z^{\beta(\rho^5,-1)} , \\
&A_{\bar{z}}^{\beta(+1,+1)} , \quad A_{\bar{z}}^{\beta(+1,-1)} , \quad A_{\bar{z}}^{\beta(\rho,+1)} , \quad A_{\bar{z}}^{\beta(\rho,-1)} , \quad A_{\bar{z}}^{\beta(\rho^2,+1)} , \quad A_{\bar{z}}^{\beta(\rho^2,+1)} , \\
&A_{\bar{z}}^{\beta(\rho^3,+1)} , \quad A_{\bar{z}}^{\beta(\rho^3,-1)} , \quad A_{\bar{z}}^{\beta(\rho^4,+1)} , \quad A_{\bar{z}}^{\beta(\rho^4,-1)} , \quad A_{\bar{z}}^{\beta(\rho^5,+1)} , \quad A_{\bar{z}}^{\alpha(\rho^5,-1)} ,
\end{aligned}
\tag{B.163}
$$

where the index α indicates the gauge generators of unbroken gauge symmetry and the index β indicates the gauge generators of broken gauge symmetry.

C Formulas based on equivalence relations

We present several formulas concerning the combination ${}_nC_l$, derived from the dynamical rearrangement and the feature that fermion numbers are independent of the Wilson line phases.

On $S^1/\mathbb{Z}_2$, we consider the representation matrices given by

$$P_0 = \mathrm{diag}([+1]_{p_1}, [+1]_{p_2}, [-1]_{p_3}, [-1]_{p_4}) \ , \tag{C.1}$$

$$P_1 = \mathrm{diag}([+1]_{p_1}, [-1]_{p_2}, [+1]_{p_3}, [-1]_{p_4}) \ , \tag{C.2}$$

where $[\pm 1]_{p_i}$ represents ± 1 for all p_i elements. Then, the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \to SU(p_1) \times SU(p_2) \times SU(p_3) \times SU(p_4) \times U(1)^{3-m} \ . \tag{C.3}$$

The $\mathbb{Z}_2$ parities or BCs specified by integers $\{p_i\}$ are also denoted $[p_1; p_2, p_3; p_4]$.

After the breakdown of $SU(N)$, $[N, k]$ is decomposed as

$$[N, k] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \left({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, {}_{p_3}C_{l_3}, {}_{p_4}C_{l_4} \right) \ , \tag{C.4}$$

where $p_4 = N - p_1 - p_2 - p_3$, $l_4 = k - l_1 - l_2 - l_3$, and we use ${}_pC_l$ instead of $[p, l]$. Our notation is that ${}_pC_l = 0$ for $l > p$ and $l < 0$.

The $\mathbb{Z}_2$ parities of $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, {}_{p_3}C_{l_3}, {}_{p_4}C_{l_4})$ for 4D left-handed fermions are given by

$$\mathscr{P}_0 = (-1)^{l_3+l_4}\eta_k^0 = (-1)^{l_1+l_2}(-1)^k\eta_k^0 = (-1)^{l_1+l_2+\alpha} \ , \tag{C.5}$$

$$\mathscr{P}_1 = (-1)^{l_2+l_4}\eta_k^1 = (-1)^{l_1+l_3}(-1)^k\eta_k^1 = (-1)^{l_1+l_3+\beta} \ , \tag{C.6}$$

where the intrinsic $\mathbb{Z}_2$ parities (η_k^0, η_k^1) take a value $+1$ or -1 by definition and are parameterized as $(-1)^k\eta_k^0 = (-1)^\alpha$ and $(-1)^k\eta_k^1 = (-1)^\beta$.

Zero modes for the left-handed fermions and the right-handed ones are picked out by operating the projection operators,

$$P^{(1,1)} = \frac{1+\mathscr{P}_0}{2}\frac{1+\mathscr{P}_1}{2} \quad \text{and} \quad P^{(-1,-1)} = \frac{1-\mathscr{P}_0}{2}\frac{1-\mathscr{P}_1}{2} \ , \tag{C.7}$$

respectively. Note that the intrinsic $\mathbb{Z}_2$ parities for the right-handed fermions are opposite to those for the left-handed ones.

Then, the fermion number is given by

$$
\begin{aligned}
n &= n_{\rm L}^0 - n_{\rm R}^0 \\
&= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2} \left(P^{(1,1)} - P^{(-1,-1)}\right) {}_{p_1}C_{l_1}\ {}_{p_2}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4}\ . \qquad \text{(C.8)}
\end{aligned}
$$

From the dynamical rearrangement, the following equivalence relations hold,

$$
\begin{aligned}
[p_1; p_2, p_3; p_4] &\sim [p_1 - 1; p_2 + 1, p_3 + 1; p_4 - 1] \quad (\text{for}\ \ p_1, p_4 \geq 1)\ , \\
&\sim [p_1 + 1; p_2 - 1, p_3 - 1; p_4 + 1] \quad (\text{for}\ \ p_2, p_3 \geq 1)\ . \qquad \text{(C.9)}
\end{aligned}
$$

Using (C.9) and the feature that fermion numbers are independent of the Wilson line phases, the following formula is derived,

$$
\begin{aligned}
&\sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta}\right] {}_{p_1}C_{l_1}\ {}_{p_2}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4} \\
&\quad = \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2} \left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta}\right] \\
&\qquad\qquad \times {}_{p_1\mp1}C_{l_1}\ {}_{p_2\pm1}C_{l_2}\ {}_{p_3\pm1}C_{l_3}\ {}_{p_4\mp1}C_{l_4}\ , \qquad \text{(C.10)}
\end{aligned}
$$

where $p_4 = N - p_1 - p_2 - p_3$, $l_4 = k - l_1 - l_2 - l_3$, and we use the relation,

$$
P^{(1,1)} - P^{(-1,-1)} = \frac{1}{2}\left(\mathcal{P}_0 + \mathcal{P}_1\right) = \frac{1}{2}\left[(-1)^{l_1+l_2+\alpha} + (-1)^{l_1+l_3+\beta}\right]\ . \qquad \text{(C.11)}
$$

Here and hereafter, we deal with the case that the inequality $p_i - 1 \geq 0$ is fulfilled in ${}_{p_i-1}C_{l_i}$.

In the same way, the following formulas are derived from the feature of the fermion number on $T^2/\mathbb{Z}_2$,

$$
\begin{aligned}
&\sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \\
&\qquad\qquad \times {}_{p_1}C_{l_1}\ {}_{p_2}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7}C_{l_7}\ {}_{p_8}C_{l_8} \\
&= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \\
&\qquad\qquad \times {}_{p_1\mp1}C_{l_1}\ {}_{p_2\pm1}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7\pm1}C_{l_7}\ {}_{p_8\mp1}C_{l_8} \\
&= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \\
&\qquad\qquad \times {}_{p_1}C_{l_1}\ {}_{p_2\mp1}C_{l_2}\ {}_{p_3\pm1}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6\pm1}C_{l_6}\ {}_{p_7\mp1}C_{l_7}\ {}_{p_8}C_{l_8} \\
&= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \\
&\qquad\qquad \times {}_{p_1}C_{l_1}\ {}_{p_2\mp1}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4\pm1}C_{l_4}\ {}_{p_5\pm1}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7\mp1}C_{l_7}\ {}_{p_8}C_{l_8}
\end{aligned}
$$

$$= \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \times {}_{p_1}C_{l_1}\ {}_{p_2}C_{l_2}\ {}_{p_3\pm 1}C_{l_3}\ {}_{p_4\mp 1}C_{l_4}\ {}_{p_5\mp 1}C_{l_5}\ {}_{p_6\pm 1}C_{l_6}\ {}_{p_7}C_{l_7}\ {}_{p_8}C_{l_8}\ , \tag{C.12}$$

where $p_8 = N - p_1 - p_2 - \cdots - p_7$ and $l_8 = k - l_1 - l_2 - \cdots - l_7$. $P^{(a,b,c)}$ are the projection operators that pick out the $\mathbb{Z}_2$ parities $(\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2) = (a, b, c)$, defined by

$$P^{(a,b,c)} \equiv \frac{1 + a\mathcal{P}_0}{2} \frac{1 + b\mathcal{P}_1}{2} \frac{1 + c\mathcal{P}_2}{2}\ . \tag{C.13}$$

Here, a, b and c take 1 or -1. $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$\mathcal{P}_0 = (-1)^{l_5+l_6+l_7+l_8}\eta_k^0 = (-1)^{l_1+l_2+l_3+l_4}(-1)^k\eta_k^0 = (-1)^{l_1+l_2+l_3+l_4+\alpha}\ , \tag{C.14}$$
$$\mathcal{P}_1 = (-1)^{l_3+l_4+l_7+l_8}\eta_k^1 = (-1)^{l_1+l_2+l_5+l_6}(-1)^k\eta_k^1 = (-1)^{l_1+l_2+l_5+l_6+\beta}\ , \tag{C.15}$$
$$\mathcal{P}_2 = (-1)^{l_2+l_4+l_6+l_8}\eta_k^2 = (-1)^{l_1+l_3+l_5+l_7}(-1)^k\eta_k^2 = (-1)^{l_1+l_3+l_5+l_7+\gamma}\ , \tag{C.16}$$

where α, β and γ take 0 or 1. Using (C.14), (C.15) and (C.16), $P^{(1,1,1)} - P^{(-1,-1,-1)}$ is calculated as

$$\begin{aligned} &P^{(1,1,1)} - P^{(-1,-1,-1)} \\ &\quad = \frac{1}{4}\big[(-1)^{l_1+l_2+l_3+l_4+\alpha} + (-1)^{l_1+l_2+l_5+l_6+\beta} \\ &\qquad\qquad + (-1)^{l_1+l_3+l_5+l_7+\gamma} + (-1)^{l_1+l_4+l_6+l_7+\alpha+\beta+\gamma}\big]\ . \end{aligned} \tag{C.17}$$

The following formulas are derived from the feature of the fermion numbers relating representations ${}_{p_1}C_{l_1}$ and $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2})$,

$$\begin{aligned} &\sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \times {}_{p_2}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7}C_{l_7}\ {}_{p_8}C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \times {}_{p_2\mp 1}C_{l_2}\ {}_{p_3\pm 1}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6\pm 1}C_{l_6}\ {}_{p_7\mp 1}C_{l_7}\ {}_{p_8}C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \times {}_{p_2\mp 1}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4\pm 1}C_{l_4}\ {}_{p_5\pm 1}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7\mp 1}C_{l_7}\ {}_{p_8}C_{l_8} \\ &= \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_7=0}^{k-l_1-\cdots-l_6} \left(P^{(1,1,1)} - P^{(-1,-1,-1)}\right) \times {}_{p_2}C_{l_2}\ {}_{p_3\pm 1}C_{l_3}\ {}_{p_4\mp 1}C_{l_4}\ {}_{p_5\mp 1}C_{l_5}\ {}_{p_6\pm 1}C_{l_6}\ {}_{p_7}C_{l_7}\ {}_{p_8}C_{l_8} \end{aligned} \tag{C.18}$$

and

$$\sum_{l_3=0}^{k-l_1-l_2}\sum_{l_4=0}^{k-l_1-l_2-l_3}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P^{(1,1,1)}-P^{(-1,-1,-1)}\right)$$
$$\times\,{}_{p_3}C_{l_3}\;{}_{p_4}C_{l_4}\;{}_{p_5}C_{l_5}\;{}_{p_6}C_{l_6}\;{}_{p_7}C_{l_7}\;{}_{p_8}C_{l_8}$$
$$=\sum_{l_3=0}^{k-l_1-l_2}\sum_{l_4=0}^{k-l_1-l_2-l_3}\cdots\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P^{(1,1,1)}-P^{(-1,-1,-1)}\right)$$
$$\times\,{}_{p_3\pm1}C_{l_3}\;{}_{p_4\mp1}C_{l_4}\;{}_{p_5\mp1}C_{l_5}\;{}_{p_6\pm1}C_{l_6}\;{}_{p_7}C_{l_7}\;{}_{p_8}C_{l_8}\;. \quad \text{(C.19)}$$

Furthermore, by changing $(p_3,p_4,p_5,p_6,p_7,p_8)$ into $(p_7,p_8,p_3,p_4,p_5,p_6)$ in the ordering of the summation and relabeling $(p_7,p_8,p_3,p_4,p_5,p_6)$ as $(p_3,p_4,p_5,p_6,p_7,p_8)$, the following formulas are derived from the feature of the fermion numbers relating representations $({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},{}_{p_3}C_{l_3})$ and $({}_{p_1}C_{l_1},{}_{p_2}C_{l_2},{}_{p_3}C_{l_3},{}_{p_4}C_{l_4})$,

$$\sum_{l_4=0}^{k-l_1-l_2-l_3}\sum_{l_5=0}^{k-l_1-\cdots-l_4}\sum_{l_6=0}^{k-l_1-\cdots-l_5}\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P'^{(1,1,1)}-P'^{(-1,-1,-1)}\right)$$
$$\times\,{}_{p_4}C_{l_4}\;{}_{p_5}C_{l_5}\;{}_{p_6}C_{l_6}\;{}_{p_7}C_{l_7}\;{}_{p_8}C_{l_8}$$
$$=\sum_{l_4=0}^{k-l_1-l_2-l_3}\sum_{l_5=0}^{k-l_1-\cdots-l_4}\sum_{l_6=0}^{k-l_1-\cdots-l_5}\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P'^{(1,1,1)}-P'^{(-1,-1,-1)}\right)$$
$$\times\,{}_{p_4}C_{l_4}\;{}_{p_5\mp1}C_{l_5}\;{}_{p_6\pm1}C_{l_6}\;{}_{p_7\pm1}C_{l_7}\;{}_{p_8\mp1}C_{l_8} \quad \text{(C.20)}$$

and

$$\sum_{l_5=0}^{k-l_1-\cdots-l_4}\sum_{l_6=0}^{k-l_1-\cdots-l_5}\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P'^{(1,1,1)}-P'^{(-1,-1,-1)}\right){}_{p_5}C_{l_5}\;{}_{p_6}C_{l_6}\;{}_{p_7}C_{l_7}\;{}_{p_8}C_{l_8}$$
$$=\sum_{l_5=0}^{k-l_1-\cdots-l_4}\sum_{l_6=0}^{k-l_1-\cdots-l_5}\sum_{l_7=0}^{k-l_1-\cdots-l_6}\left(P'^{(1,1,1)}-P'^{(-1,-1,-1)}\right)$$
$$\times\,{}_{p_5\mp1}C_{l_5}\;{}_{p_6\pm1}C_{l_6}\;{}_{p_7\pm1}C_{l_7}\;{}_{p_8\mp1}C_{l_8}\;, \quad \text{(C.21)}$$

where $P'^{(1,1,1)}-P'^{(-1,-1,-1)}$ is given by

$$P'^{(1,1,1)}-P'^{(-1,-1,-1)}$$
$$=\frac{1}{4}\big[(-1)^{l_1+l_2+l_5+l_6+\alpha}+(-1)^{l_1+l_2+l_7+l_8+\beta}$$
$$+(-1)^{l_1+l_3+l_5+l_7+\gamma}+(-1)^{l_1+l_3+l_6+l_8+\alpha+\beta+\gamma}\big]\;. \quad \text{(C.22)}$$

In the same way, the following formulas are derived from the feature of the fermion number on $T^2/\mathbb{Z}_3$,

$$\sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}\left(P^{(1,1)}-P^{(\omega,\omega)}\right)$$
$$\times\,{}_{p_1}C_{l_1}\;{}_{p_2}C_{l_2}\;{}_{p_3}C_{l_3}\;{}_{p_4}C_{l_4}\;{}_{p_5}C_{l_5}\;{}_{p_6}C_{l_6}\;{}_{p_7}C_{l_7}\;{}_{p_8}C_{l_8}\;{}_{p_9}C_{l_9}$$

$$= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}\left(P^{(1,1)}-P^{(\omega,\omega)}\right)$$
$$\times\ {}_{p_1\pm1}C_{l_1}\ {}_{p_2}C_{l_2}\ {}_{p_3\mp1}C_{l_3}\ {}_{p_4\mp1}C_{l_4}\ {}_{p_5\pm1}C_{l_5}\ {}_{p_6}C_{l_6}\ {}_{p_7}C_{l_7}\ {}_{p_8\mp1}C_{l_8}\ {}_{p_9\pm1}C_{l_9}$$
$$= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}\left(P^{(1,1)}-P^{(\omega,\omega)}\right)$$
$$\times\ {}_{p_1\pm1}C_{l_1}\ {}_{p_2\mp1}C_{l_2}\ {}_{p_3}C_{l_3}\ {}_{p_4}C_{l_4}\ {}_{p_5\pm1}C_{l_5}\ {}_{p_6\mp1}C_{l_6}\ {}_{p_7\mp1}C_{l_7}\ {}_{p_8}C_{l_8}\ {}_{p_9\pm1}C_{l_9}$$
$$= \sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\cdots\sum_{l_8=0}^{k-l_1-\cdots-l_7}\left(P^{(1,1)}-P^{(\omega,\omega)}\right)$$
$$\times\ {}_{p_1}C_{l_1}\ {}_{p_2\pm1}C_{l_2}\ {}_{p_3\mp1}C_{l_3}\ {}_{p_4\mp1}C_{l_4}\ {}_{p_5}C_{l_5}\ {}_{p_6\pm1}C_{l_6}\ {}_{p_7\pm1}C_{l_7}\ {}_{p_8\mp1}C_{l_8}\ {}_{p_9}C_{l_9}\ , \tag{C.23}$$

where $p_9 = N - p_1 - p_2 - \cdots - p_8$ and $l_9 = k - l_1 - l_2 - \cdots - l_8$. $P^{(\xi,\eta)}$ are the projection operators that pick out the $\mathbb{Z}_3$ elements $(\Theta_0, \Theta_1) = (\xi, \eta)$, defined by

$$P^{(\xi,\eta)} \equiv \frac{1+\overline{\xi}\Theta_0+\overline{\xi}^2\Theta_0^2}{3}\frac{1+\overline{\eta}\Theta_1+\overline{\eta}^2\Theta_1^2}{3}\ . \tag{C.24}$$

Here, ξ and η take 1, $\omega(= e^{2\pi i/3})$ or $\overline{\omega}(= e^{4\pi i/3})$, and $\overline{\xi}$ and $\overline{\eta}$ are the complex conjugates of ξ and η, respectively. Θ_0 and Θ_1 are given by

$$\Theta_0 = \omega^{l_4+l_5+l_6}\overline{\omega}^{l_7+l_8+l_9}\eta_k^0$$
$$= \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)}\overline{\omega}^k\eta_k^0 = \omega^{l_1+l_2+l_3+2(l_4+l_5+l_6)+\alpha}\ , \tag{C.25}$$
$$\Theta_1 = \omega^{l_2+l_5+l_8}\overline{\omega}^{l_3+l_6+l_9}\eta_k^1$$
$$= \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)}\overline{\omega}^k\eta_k^1 = \omega^{l_1+l_4+l_7+2(l_2+l_5+l_8)+\beta}\ , \tag{C.26}$$

where α and β take 0, 1 or 2.

In the same way, we can derive similar formulas from the feature of the fermion numbers relating representations ${}_{p_1}C_{l_1}$, $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2})$ and $({}_{p_1}C_{l_1}, {}_{p_2}C_{l_2}, {}_{p_3}C_{l_3})$ on $T^2/\mathbb{Z}_3$.

D Formulas based on independence from Wilson line phases

We derive other formulas concerning the combination ${}_nC_l$, counting the numbers of fermions irrelevant to the Wilson line phases and using the independence of fermion numbers from the Wilson line phases.

On $S^1/\mathbb{Z}_2$, we consider the representation matrices given by

$$P_0 = \mathrm{diag}([+1]_p, [-1]_{N-p})\ , \quad P_1 = \mathrm{diag}([+1]_p, [-1]_{N-p})\ . \tag{D.1}$$

Then, the following breakdown of $SU(N)$ gauge symmetry occurs:

$$SU(N) \to SU(p) \times SU(N-p) \times U(1)^{1-m}\ , \tag{D.2}$$

and $[N,k]$ is decomposed as

$$[N,k] = \sum_{l=0}^{k} ({}_pC_l, {}_{N-p}C_{k-l}) \ . \tag{D.3}$$

The $\mathbb{Z}_2$ parities of $({}_pC_l, {}_sC_{k-l})$ for 4D left-handed fermions are given and parameterized by

$$\mathcal{P}_0 = (-1)^{k-l}\eta_k^0 = (-1)^{l+\alpha} \ , \quad \mathcal{P}_1 = (-1)^{k-l}\eta_k^1 = (-1)^{l+\beta} \ , \tag{D.4}$$

where α and β take 0 or 1. Then, the fermion number is given by

$$n = n_{\rm L} - n_{\rm R} = \sum_{l=0}^{k} \frac{1}{2}\left[(-1)^{l+\alpha} + (-1)^{l+\beta}\right] {}_pC_l \ {}_{N-p}C_{k-l} \ . \tag{D.5}$$

The number of the Wilson line phases is $m \equiv {\rm Min}(p, N-p)$ and, after a suitable $SU(p) \times SU(N-p)$ gauge transformation, $\langle A_y \rangle$ is parameterized as

$$\langle A_y \rangle = \frac{-i}{gR} \begin{pmatrix} 0 & \Theta \\ -\Theta^T & 0 \end{pmatrix} \ , \tag{D.6}$$

where Θ is the $p \times (N-p)$ matrix such that

$$\Theta = \begin{pmatrix} a_1 & & 0 & \\ & a_2 & & \\ & & \ddots & \\ & 0 & & a_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad ({\rm for} \ \ p \geq N-p) \ , \tag{D.7}$$

$$\Theta = \begin{pmatrix} a_1 & & 0 & & 0 & \cdots & 0 \\ & a_2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \ddots & \vdots \\ & 0 & & a_m & 0 & \cdots & 0 \end{pmatrix} \quad ({\rm for} \ \ p \leq N-p) \ . \tag{D.8}$$

Here, $2\pi a_k$ $(k = 1, \cdots, m; m \equiv {\rm Min}(p, N-p))$ are the Wilson line phases.

For the fermion with $[N,1]$, the number of components irrelevant to a_k is $p-m$ for $p \geq N-p$ and $N-p-m$ for $p \leq N-p$, and it is expressed as

$$\left. \sum_{l'=0}^{1} {}_{p-m}C_{l'} \ {}_{N-p-m}C_{1-l'} \right|_{m={\rm Min}(p,N-p)} \ . \tag{D.9}$$

For the fermion with $[N,2]$, the number of components irrelevant to a_k is ${}_{p-m}C_2 + m$ for $p \geq N-p$ and ${}_{N-p-m}C_2 + m$ for $p \leq N-p$, and it is expressed as

$$\left. \sum_{l'=0}^{2} {}_{p-m}C_{l'} \ {}_{N-p-m}C_{2-l'} + {}_mC_1 \right|_{m={\rm Min}(p,N-p)} \ , \tag{D.10}$$

where ${}_mC_1$ comes from the components constructed from the tensor products between components in $[N,1]$ with opposite values for the Wilson line phases, and the components corresponding ${}_mC_1$ have odd $\mathbb{Z}_2$ parities. In the iterative fashion, we find that the number of components irrelevant to a_k is given by

$$\left.\sum_{n=0}^{[k/2]}\sum_{l'=0}^{k-2n} {}_mC_n \ {}_{p-m}C_{l'} \ {}_{N-p-m}C_{k-2n-l'}\right|_{m=\mathrm{Min}(p,N-p)} \tag{D.11}$$

for the fermion with $[N,k]$.

Using the independence of fermion numbers from the Wilson line phases, the number of fermions is also calculated by counting the fermions irrelevant to a_k and the following formula is derived,

$$\sum_{l=0}^{k}(-1)^l {}_pC_l \ {}_{N-p}C_{k-l} = \sum_{n=0}^{[k/2]}\sum_{l'=0}^{k-2n}(-1)^{n+l'} {}_mC_n \ {}_{p-m}C_{l'} \ {}_{N-p-m}C_{k-2n-l'} \ , \tag{D.12}$$

where we use the assignment of $\mathbb{Z}_2$ parities,

$$\begin{aligned}\mathcal{P}_0 &= (-1)^{n+k-2n-l'}\eta_k^0 = (-1)^{n+l'+\alpha} \ , \\ \mathcal{P}_1 &= (-1)^{n+k-2n-l'}\eta_k^1 = (-1)^{n+l'+\beta}\end{aligned} \tag{D.13}$$

for the component corresponding ${}_mC_n \ {}_{p-m}C_{l'} \ {}_{N-p-m}C_{k-2n-l'}$, and we take $\alpha = \beta$. The above formula (D.12) holds for the integer m satisfying $0 \le m \le \mathrm{Min}(p, N-p)$, because the above argument is valid for m as the number of non-vanishing a_k even if some of a_k vanish.

Particularly, in case with $m = p$ and $m = N-p$, (D.12) reduces to

$$\sum_{l=0}^{k}(-1)^l {}_pC_l \ {}_{N-p}C_{k-l} = \sum_{n=0}^{[k/2]}\sum_{l'=0}^{k-2n}(-1)^{n+l'} {}_pC_n \ {}_{N-2p}C_{k-2n-l'} \ , \tag{D.14}$$

and

$$\sum_{l=0}^{k}(-1)^l {}_pC_l \ {}_{N-p}C_{k-l} = \sum_{n=0}^{[k/2]}\sum_{l'=0}^{k-2n}(-1)^{n+l'} {}_{N-p}C_n \ {}_{2p-N}C_{k-2n-l'} \ , \tag{D.15}$$

respectively.

Based on the representation matrices (C.1) and (C.2), the following formula is derived,

$$\begin{aligned}&\sum_{l_1=0}^{k}\sum_{l_2=0}^{k-l_1}\sum_{l_3=0}^{k-l_1-l_2}\left[(-1)^{l_1+l_2+\alpha}+(-1)^{l_1+l_3+\beta}\right] {}_{p_1}C_{l_1} \ {}_{p_2}C_{l_2} \ {}_{p_3}C_{l_3} \ {}_{p_4}C_{l_4} \\ &= \sum_{n=0}^{[k/2]}\sum_{n_1=0}^{n}\sum_{l_1'=0}^{k-2n}\sum_{l_2'=0}^{k-2n-l_1'}\sum_{l_3'=0}^{k-2n-l_1'-l_2'}\left[(-1)^{n+l_1'+l_2'+\alpha}+(-1)^{n+l_1'+l_3'+\beta}\right] \\ &\qquad\times {}_{m_1}C_{n_1} \ {}_{m_2}C_{n-n_1} \ {}_{p_1-m_1}C_{l_1'} \ {}_{p_2-m_2}C_{l_2'} \ {}_{p_3-m_2}C_{l_3'} \ {}_{p_4-m_1}C_{l_4'} \ ,\end{aligned} \tag{D.16}$$

where $p_4 = N - p_1 - p_2 - p_3$ and $l_4' = k - 2n - l_1' - l_2' - l_3'$. The above formula (D.16) holds for the integers m_1 and m_2 satisfying $0 \le m_1 \le \mathrm{Min}(p_1, p_4)$ and $0 \le m_2 \le \mathrm{Min}(p_2, p_3)$.

In the same way, we can derive similar formulas using models on $T^2/\mathbb{Z}_M$.

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